



Article

Space-Fractional Variable-Coefficient Toda Lattice Equation: Model Derivation and Inverse Scattering Transform

Yuhan Ma ¹, Sheng Zhang ^{1,*}, Sen Zhao ^{2,3} and Bo Xu ^{1,4,*}¹ School of Mathematical Sciences, Bohai University, Jinzhou 121013, China; 2024010026@bhu.edu.cn² School of Economics, Dongbei University of Finance and Economics, Dalian 116025, China; 20211168@lise.edu.cn³ College of Science, Liaoning Institute of Science and Engineering, Jinzhou 121010, China⁴ School of Educational Sciences, Bohai University, Jinzhou 121013, China

* Correspondence: szhang@bhu.edu.cn (S.Z.); bxu@bhu.edu.cn (B.X.)

Abstract

This paper systematically studies a space-fractional variable-coefficient Toda (sfvToda) lattice equation. The core innovation of this paper lies in incorporating both fractional power of the recursion operator and time-varying coefficients into the Toda lattice, as well as supplementing relevant theories. Firstly, based on the adjoint operator of a 2×2 matrix operator defined, a fractional Toda lattice hierarchy is presented. Three time-varying coefficients are introduced to combine the first three members of this hierarchy into the sfvToda lattice equation. Secondly, the orthogonality and completeness of squared eigenfunctions related to the Toda lattice spectral problem (SP) are presented, and the explicit representations of the space-fractional Toda lattice hierarchy and the sfvToda lattice equation are derived. Thirdly, within the framework of inverse scattering transform (IST) theory, the linearization of the sfvToda lattice equation is achieved through scattering data, and explicit fractional variable-velocity N -soliton solutions are obtained. Finally, numerical simulations of single-soliton solutions, double-soliton solutions, and triple-soliton solutions with variable velocity are carried out by selecting appropriate coefficient functions. The results indicate that both the fractional extension and time-varying coefficients have significant regulatory effects on the spatiotemporal structure and propagation velocity of the obtained fractional soliton solutions.

Keywords: space-fractional variable-coefficient Toda lattice equation; orthogonality; completeness; squared eigenfunction; fractional variable-velocity soliton solution; inverse scattering transform

Academic Editors: Haci Mehmet Baskonus, Lanre Akinyemi, Mehmet Senol, Solomon Manukure and Udoh Akpan

Received: 15 August 2026

Revised: 13 October 2026

Accepted: 19 October 2026

Published: 23 October 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

In 1955, Fermi, Pasta, and Ulam [1] conducted a numerical experiment to study the thermalization process of energy in a nonlinear lattice. They expected that nonlinear interactions would eventually lead to energy equipartition in the system; however, they instead observed a near-periodic recurrence phenomenon. This observation contradicted the prevailing understanding at the time and became known as the FPU paradox. In 1967, Toda [2,3] published two foundational papers: one on nonlinear vibrations in chains [2] and the other on waves propagating in anharmonic lattices [3], introducing the well-known exponential potential:

$$\phi(r) = \frac{a}{b} e^{-r} + ar + c, \quad (1)$$

where $a > 0$, $b > 0$, and c is a constant. The lattice model using this potential, later known as the Toda lattice [4], has equations of motion that can produce exact solitary wave solutions, and the system's energy exhibits stable localization characteristics. This discovery ingeniously explains, from a theoretical perspective, why the energy in the FPU experiment remained "localized" in a few modes for a long time, thus providing a key clue to understanding the FPU paradox [5].

In 1965, Zabusky and Kruskal [6] proposed the term "soliton" in their study of the KdV equation and revealed its particle-like collision characteristics. Following the introduction of the Toda lattice, it quickly became one of the core driving forces for the development of soliton theory and integrable systems. The Toda lattice was the first to confirm the existence of solitons in a discrete system and provided an exact formula for multi-soliton solutions, thereby greatly expanding the application scope of soliton theory. The year 1974 was a milestone in the development of Toda lattice theory. Flaschka [7,8] (also Manakov [9] at the same period) introduced a key variable transformation, successfully rewriting the Toda lattice equation in the form of a Lax pair [10]. This means that the motion of the system can be described by the commutation relationship of two linear operators, with the core idea being that the energy spectrum (eigenvalues) of the system remains unchanged during evolution. The establishment of the Lax pair enabled the IST method [11], see also [12], originally used to solve the Korteweg-de Vries (KdV) equation [11], to be successfully applied to the Toda lattice [8]. The IST method can be regarded as the Fourier transform of nonlinear systems [13]. By analyzing the scattering data of the system (such as reflection coefficients, discrete eigenvalues, etc.), it simplifies complex nonlinear evolution problems into linear problems, thereby solving the initial value problem of the Toda lattice and obtaining its fractional N -soliton solution [8].

Over the past seven decades, the research on Toda lattice systems has continuously expanded and deepened. Similar to the KdV equation which has a hierarchy (the KdV hierarchy), the Toda lattice also forms a Toda lattice hierarchy [14] that contains an infinite number of high-order integrable equations. These equations share the same Lax pair structure but have more complex nonlinear terms and dispersion relations. Recent studies have extended the Toda lattice to cases with time-varying coefficients [15], cases related to non-isospectral problems (i.e., eigenvalues change with time) [16], and cases related to mixed spectral problems (i.e., unifying isospectral and non-isospectral problems) [17], further enriching the model's connotation and application scope. As demonstrated by the work of Ablowitz, Been, and Fridberg [18], the research frontier of the Toda lattice has extended to include fractional operators, with the aim of exploring the behavior of this classic system in a broader fractional framework. This indicates that the Toda lattice, which originated in the 1960s, continues to provide a continuous impetus for the exploration of the frontiers of mathematical physics.

In 1967, Gardner, Greene, Kruskal, and Miura (referred to as GGKM) [11] published a pioneering paper. In it, they first proposed a method that successfully solved the initial value problem of the KdV equation by associating it with a linear Schrödinger scattering problem. This method served as the prototype of the IST. They ingeniously discovered that the solutions to the KdV equation were closely related to the eigenvalue problem of the famous Schrödinger equation in quantum mechanics. In 1972, Zakharov and Shabat [19] achieved another key breakthrough. Using a similar idea, they successfully solved another equation crucial to fields such as nonlinear optics—the nonlinear Schrödinger (NLS) equation. This demonstrated that IST was a universal method. It was Ablowitz, Kaup, Newell, and Segur (known as the AKNS group [13]) who truly systematized and

theorized the IST method. Their work proposed a unified 2×2 scattering problem framework, under which equations such as the KdV, NLS, sine-Gordon, and modified KdV (mKdV) were all unified. At this point, the IST has been established as a systematic theory for solving nonlinear partial differential equations.

Following the establishment of the AKNS framework [13], research and application of the IST method entered an unprecedentedly prosperous period. In 1976, Ablowitz and Ladik [20] quickly extended the IST method to the discrete domain, successfully solving nonlinear differential-difference equations. This paved the way for the study of discrete integrable systems such as the Toda lattice. In recent years, the research on IST has continued to deepen, constantly integrating with new ideas in mathematical physics. For example, in 2013, the IST was extended to nonlocal nonlinear wave equations [21] that satisfy PT symmetry. Even more excitingly, in 2022, Ablowitz, Been, and Carr [22–24] made another breakthrough by extending the IST to Riesz space-fractional KdV, NLS, modified KdV, sine-Gordon, and sinh-Gordon equations [22,23], as well as Riesz space-fractional semi-discrete NLS and averaged NLS equations [24]. In 2025, Ablowitz, Been, and Fridberg [18] extended the IST to fractional Toda lattice hierarchy, where “fractional” means that fractional power of the recursion operator.

The classical Toda lattice [18],

$$\frac{d^2 q_n}{dt^2} = e^{q_{n-1} - q_n} - e^{q_n - q_{n+1}}, \quad (2)$$

assumes that the system is uniform (with constant coefficients). It is easy to see that Equation (2) can be derived from the motion equation of the n th particle in the Toda chain [25]

$$m \frac{d^2 q_n}{dt^2} = -\phi'(q_{n-1} - q_n) + \phi'(q_{n+1} - q_n), \quad (3)$$

by using the substitution of Equation (1) into Equation (3), together with the replacements $\phi'(r_n) = d\phi(r_n)/dr_n$, $r_n = q_n - q_{n-1}$, $a=1$, $b=1$, and $m=1$. Further employ the transformations:

$$u_n = e^{q_{n-1} - q_n}, \quad v_n = -\frac{dq_n}{dt}, \quad (4)$$

Equation (2) can be rewritten in matrix form [25]

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = \begin{pmatrix} u_n(v_n - v_{n-1}) \\ u_{n+1} - u_n \end{pmatrix}, \quad (5)$$

which is the model that serves as the starting point for the work in this article.

The introduction of time-varying coefficients is motivated by the need to model temporal driving and time-dependent modulation of the coupling strengths, which arise naturally in nonlinear lattice systems subject to external fields or varying environmental conditions. Spatial inhomogeneity would require n -dependent coefficients and is not considered here. Meanwhile, the introduction of fractional extension seeks to extend recursion operators from integer powers to fractional powers, thereby enabling the construction of a more general nonlinear lattice model. The primary goal of the sfvToda system is to construct a more universal nonlinear lattice model that can simultaneously characterize inhomogeneity and fractional dispersion. The ultimate goal is to understand the new physics brought by these new parameters. However, to the best of our best knowledge, there is no research on the Toda lattice that is simultaneously governed by variable coefficients and fractional operators. For a more comprehensive review, we would like to mention the fractional Toda type systems [18,26–29], variable-coefficient integrable lattices [15,16,30–32], fractional discrete integrable systems [24,33–35], and

operator fractional inverse scattering formulations [18,24,33]. The research aims to address this research gap regarding inhomogeneity and fractional extension in classical Toda theory, clarify how the coefficient functions specifically affect the spatiotemporal structure, propagation velocity, and interaction process of solitons, thereby enriching the theoretical treasury of integrable systems and providing a theoretical basis for potential applications.

The other sections of this work are organized as follows: In Section 2, a defined 2×2 matrix operator $\aleph$ is introduced, and the fractional Toda lattice hierarchy, sfvToda lattice equation, and their dispersion relations are presented using the adjoint operator $\aleph^A$ and employing the Jost solutions of the Toda lattice SP. In Section 3, we present the orthogonality and completeness of squared eigenfunctions related to the Toda lattice SP, and provide the explicit representations of the Toda lattice hierarchy and the sfvToda lattice equation. In Section 4, we recall the translation transformations and discrete Gel'fand-Leviten-Marchenko (GLM) equation for the Toda lattice SP, and determine the time dependence of the scattering data. In Section 5, the formulae for fractional N -soliton solutions of the sfvToda lattice equation are obtained in the case of reflectionless potentials within the framework of IST theory. Additionally, through numerical simulations, the research visually reveals that time-varying coefficients and fractionalization have significant dual impacts on the spatiotemporal structure and propagation velocity of single-solitons, double-solitons, and triple-solitons in the sfvToda lattice. In Section 6, we summarize the article.

2. Model and Toda Lattice Spectral Problem

Let u_n be a function defined on the set of integers. The forward shift operator $\mathfrak{S}$ is introduced such that its action on u_n increments the independent variable n by one, i.e., $\mathfrak{S}u_n = u_{n+1}$. Accordingly, the inverse operator $\mathfrak{S}^{-1}$ is defined by $\mathfrak{S}^{-1}u_n = u_{n-1}$. The difference of u_n , denoted by $u_{n+1} - u_{n-1}$, can be expressed as $(\mathfrak{S} - \mathfrak{S}^{-1})u_n$, where $\mathfrak{S} - \mathfrak{S}^{-1}$ denotes the difference operator, and its inverse operator exists and is defined as [25]:

$$(\mathfrak{S} - \mathfrak{S}^{-1})^{-1}u_n = -\sum_{m=n}^{\infty} u_{2m-n+1}, \quad (6)$$

$$(\mathfrak{S} - \mathfrak{S}^{-1})^{-1}u_n = \sum_{m=-\infty}^n u_{2m-n-1}. \quad (7)$$

Assuming that the function u_n decays sufficiently fast as $|n| \rightarrow \infty$, it can be readily verified that the following operator relations are valid:

$$(\mathfrak{S} - 1)^{-1} = (\mathfrak{S} - \mathfrak{S}^{-1})^{-1}(1 + \mathfrak{S}^{-1}), \quad (1 - \mathfrak{S}^{-1})^{-1} = (\mathfrak{S} - \mathfrak{S}^{-1})^{-1}(\mathfrak{S} + 1). \quad (8)$$

We define in this paper a 2×2 matrix operator:

$$\aleph = \begin{pmatrix} \frac{1}{u_n} \sum_{m=n+1}^{\infty} v_{m-1}(\mathfrak{S}^{-1} - 1)u_m & \frac{1}{u_n} \sum_{m=n+1}^{\infty} (\mathfrak{S}^{-1}u_m \mathfrak{S}^{-1} - u_m) \\ (1 + \mathfrak{S})u_n & v_n \end{pmatrix}, \quad (9)$$

where both the summations are performed in the same manner on the functions to their right and on the functions to which they are applied.

Utilizing the inner product $\langle a, \aleph b \rangle = \langle \aleph^A a, b \rangle$ [18], where

$$\langle a, b \rangle = \sum_{n=-\infty}^{\infty} a_n \cdot b_n = \sum_{n=-\infty}^{\infty} \sum_{j=1}^2 a_{n,j} b_{n,j}, \quad (10)$$

for arbitrary two-dimensional infinite sequences:

$$a = \{a_n = (a_{n,1}, a_{n,2})^T\}_{n=-\infty}^{\infty}, \quad b = \{b_n = (b_{n,1}, b_{n,2})^T\}_{n=-\infty}^{\infty}, \quad a_{n,j}, b_{n,j} \in \mathbb{R}^2, \quad j=1,2, \quad n = \{\dots, -2, -1, 0, 1, 2, \dots\}, \quad (11)$$

that decay sufficiently fast as $|n| \rightarrow \infty$, where T is the transpose, we obtain the adjoint operator $\mathbb{S}^A$ [25],

$$\mathbb{S}^A = \begin{pmatrix} u_n(\mathfrak{T}-1)v_{n-1}(\mathfrak{T}-1)^{-1} \frac{1}{u_n} & u_n(1+\mathfrak{T}^{-1}) \\ (\mathfrak{T}u_n\mathfrak{T}-u_n)(\mathfrak{T}-1)^{-1} \frac{1}{u_n} & v_n \end{pmatrix}, \quad (12)$$

of $\mathbb{S}$ by taking the term-wise adjoint and swapping the anti-diagonal entries. Note that $\mathfrak{T}^A = \mathfrak{T}^{-1}$ and it is straightforward to deduce that $[(\mathfrak{T}-1)^{-1}]^A u_n = \sum_{m=n+1}^{\infty} u_m$.

Note that the operator $\mathbb{S}^A$ and its fractional power $(\mathbb{S}^A)^M |\mathbb{S}^A|^\varepsilon$ are defined on the weighted sequence space $X = \{F : \sum_n (1+|n|)|F_n| < \infty\}$, with domain $D(\mathbb{S}^A) = X$ (see Appendix A1). The spectral representation and branch convention, and explicitly expression of fractional power are listed in Appendices A2-A4. In addition, the boundedness of fractional power as mappings $X \rightarrow X$ is proven through Theorem A1 in Appendix A5 using weak completeness of the matrix Toda eigenfunctions. The integrability of $|\lambda|^\varepsilon$ near $\lambda=0$ ensures convergence of the spectral integral, but operator boundedness relies on valid weak spectral estimates rather than this integrability alone. These functional-analytic properties guarantee that the right-hand side of the sfvToda lattice (16) maps X into X , so Equation (16) defines a well-posed flow on X .

2.1. Model of the sfvToda Lattice and Dispersion Relation

A general evolution system containing the Toda lattice is the equation given by:

$$\begin{pmatrix} \frac{du_n}{dt} \\ \frac{dv_n}{dt} \end{pmatrix} = \Lambda(\mathbb{S}^A) \begin{pmatrix} V_n \\ U_n \end{pmatrix}, \quad V_n = u_n(v_n - v_{n-1}), \quad U_n = u_{n+1} - u_n. \quad (13)$$

Here, $\Lambda(\lambda)$ is associated with the linear dispersion relation. Note that the Toda lattice Equation (13) is characterized by $\Lambda(\lambda) = 1$, and in the context of the Toda lattice hierarchy [25]

$$\begin{pmatrix} \frac{du_n}{dt} \\ \frac{dv_n}{dt} \end{pmatrix} = (\mathbb{S}^A)^M \begin{pmatrix} u_n(v_n - v_{n-1}) \\ u_{n+1} - u_n \end{pmatrix}, \quad M = 0, 1, 2, \dots, \quad (14)$$

which associated with $\Lambda(\lambda) = \lambda^M$.

Therefore, a space-fractional extension of the Toda lattice hierarchy (14) can be achieved by $\Lambda(\mathbb{S}^A) = (\mathbb{S}^A)^M |\mathbb{S}^A|^\varepsilon$, $-1 < \varepsilon < 1$, $M = 0, 1, 2, \dots$. For the fractional operator $(\mathbb{S}^A)^M |\mathbb{S}^A|^\varepsilon$, see Appendix A for details, where the mathematical and physical

meaning of ε is described in Remark A1. Consequently, the space-fractional Toda lattice hierarchy in operator form is constructed:

$$\begin{pmatrix} \frac{du_n}{dt} \\ \frac{dv_n}{dt} \end{pmatrix} = (\mathbb{S}^A)^M |\mathbb{S}^A|^\varepsilon \begin{pmatrix} u_n(v_n - v_{n-1}) \\ u_{n+1} - u_n \end{pmatrix}, \quad M = 0, 1, 2, \dots \quad (15)$$

Then, when M takes the values 0, 1, and 2, by introducing real-valued functions $\gamma_0(t)$, $\gamma_1(t)$, and $\gamma_2(t)$, we obtain a class of sfvToda lattice as a special case:

$$\begin{pmatrix} \frac{du_n}{dt} \\ \frac{dv_n}{dt} \end{pmatrix} = [\gamma_0(t) + \gamma_1(t)\mathbb{S}^A + \gamma_2(t)(\mathbb{S}^A)^2] |\mathbb{S}^A|^\varepsilon \begin{pmatrix} u_n(v_n - v_{n-1}) \\ u_{n+1} - u_n \end{pmatrix}, \quad (16)$$

the dispersion relation of which is determined by $\Lambda(\lambda, t) = [\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2] |\lambda|^\varepsilon$.

Remark 1. (Physical origin and role of $\gamma_i(t)$) The sfvToda lattice Equation (16) can be derived from the time-dependent Hamiltonian:

$$H(t) = \gamma_0(t)H_0 + \gamma_1(t)H_1 + \gamma_2(t)H_2,$$

where H_0 , H_1 , and H_2 are the conserved Hamiltonians of the first three fractional Toda flows. The coefficients $\gamma_0(t)$, $\gamma_1(t)$, and $\gamma_2(t)$ are real-valued, smooth, and locally integrable functions of time, the roles of the three coefficients are as follows: (i) $\gamma_0(t)$: weight of the first flow, controlling the strength of the basic nonlinear interaction and the leading-order dispersion; (ii) $\gamma_1(t)$: weight of the second flow, introducing higher-order dispersion and nonlinearity, corresponding to the next member of the Toda hierarchy; (iii) $\gamma_2(t)$: weight of the third flow, providing further higher-order corrections, relevant when stronger nonlinear or dispersive effects are present. They control the relative strength of each flow at time t and have dimension T^{-1} in the standard dimensionless normalization. Note that the potential functions u_n and v_n , the spatial index n , and the spectral parameter z are dimensionless, while the time t has the dimension of time T . Physically, they represent external driving or temporal modulation of the lattice coupling. They do not model spatial inhomogeneity; n -dependent coefficients would be required for that purpose.

2.2. Toda Lattice SP and Direct Scattering

The Toda lattice SP associated with Equation (16) reads:

$$P_{n+1}(z) + u_n P_{n-1}(z) + v_n P_n(z) = \left(z + \frac{1}{z}\right) P_n(z), \quad (17)$$

where the substitution $\lambda = z + 1/z$ had been used. We next consider the following linear problems:

$$\mathfrak{L} \begin{pmatrix} P_{1,n} \\ P_{2,n} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u_n & \lambda - v_n \end{pmatrix} \begin{pmatrix} P_{1,n} \\ P_{2,n} \end{pmatrix}, \quad (18)$$

$$\frac{d}{dt} \begin{pmatrix} P_{1,n} \\ P_{2,n} \end{pmatrix} = \begin{pmatrix} W_n & Z_n \\ N_n & Q_n \end{pmatrix} \begin{pmatrix} P_{1,n} \\ P_{2,n} \end{pmatrix}, \quad (19)$$

where the potentials $u_n \rightarrow 1$ and $v_n \rightarrow 0$ decay rapidly enough as $|n| \rightarrow \infty$, the spectral parameter λ is a constant, and W_n , Z_n , N_n , and Q_n are functions of u_n , v_n , and λ . Then, the compatibility condition of Equations (18) and (19) generates the integer-order case [15] of Equation (16) corresponding to $\varepsilon = 0$. By eliminating $P_{1,n}$ and omitting the subscript 2 of $P_{2,n}$, we can reduce Equations (18) and (19) to the Toda lattice SP (17) and its time evolution equation:

$$\frac{dP_n(z)}{dt} = N_n P_{n-1}(z) + Q_n P_n(z). \quad (20)$$

Although Equation (20), when associated with Equation (16), cannot be written explicitly due to the complicity involving the participation fractional operator $|\mathbb{S}^A|^\varepsilon$, it has the asymptotic coefficients:

$$N_n = -u_n Z_{n+1} \sim -[\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2]|\lambda|^\varepsilon, \quad Q_n = W_{n+1} + (\lambda - v_n)Z_{n+1} \sim [\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2]|\lambda|^\varepsilon. \quad (21)$$

We clarify in Appendix B that such asymptotic information is derived rigorously based on the fact that the related compatibility condition admits a solution, whose asymptotic behavior is determined by the operator action of $[\gamma_0(t) + \gamma_1(t)\mathbb{S}^A + \gamma_2(t)(\mathbb{S}^A)^2]|\mathbb{S}^A|^\varepsilon$. For Equation (17), there exist two classes of unique discrete eigenfunction/Jost solutions [25], denoted as $p_n(z)$, $\tilde{p}_n(z)$, $q_n(z)$, and $\tilde{q}_n(z)$, satisfying the following asymptotic conditions. As $n \rightarrow \infty$, we have:

$$p_n(z) \sim z^n, \quad \tilde{p}_n(z) \sim z^{-n}, \quad \frac{dp_n(z)}{dz} \sim nz^{n-1}, \quad \frac{d\tilde{p}_n(z)}{dz} \sim -nz^{-n-1}; \quad (22)$$

and as $n \rightarrow -\infty$, there hold:

$$q_n(z) \sim z^{-n}, \quad \tilde{q}_n(z) \sim z^n, \quad \frac{dq_n(z)}{dz} \sim -nz^{-n-1}, \quad \frac{d\tilde{q}_n(z)}{dz} \sim nz^{n-1}, \quad (23)$$

where $q_n(z)$ and $p_n(z)$ are analytic inside the unit circle $|z| \leq 1$ on the complex z -plane, while $\tilde{q}_n(z)$ and $\tilde{p}_n(z)$ are analytic outside this unit circle $|z| > 1$, and on the circle $|z| = 1$ they satisfy the symmetries:

$$\tilde{p}_n(z) = p_n^*(z), \quad \tilde{q}_n(z) = q_n^*(z). \quad (24)$$

On the unit circle $|z| = 1$, since $p_n(z)$ and $\tilde{p}_n(z)$, or $q_n(z)$ and $\tilde{q}_n(z)$, are independent, which form a set of fundamental solutions to Equation (17), then we may set:

$$q_n(z) = \alpha(z)\tilde{p}_n(z) + \beta(z)p_n(z), \quad (25)$$

$$\tilde{q}_n(z) = \tilde{\alpha}(z)p_n(z) + \tilde{\beta}(z)\tilde{p}_n(z). \quad (26)$$

With the discrete Wronskian of $p_n(z)$ and $q_n(z)$ defined by:

$$W(p_n(z), q_n(z)) = A_{n-1}(p_n(z)q_{n-1}(z) - p_{n-1}(z)q_n(z)), \quad A_{n-1} = \prod_{j=n}^{\infty} u_j, \quad (27)$$

the coefficients $\alpha(z)$, $\tilde{\alpha}(z)$, $\beta(z)$, and $\tilde{\beta}(z)$ can be solved as follows:

$$\left(z - \frac{1}{z}\right)\alpha(z) = W(p_n(z), q_n(z)), \quad \left(z - \frac{1}{z}\right)\beta(z) = W(q_n(z), \tilde{p}_n(z)), \quad (28)$$

$$\left(z - \frac{1}{z}\right) \tilde{\alpha}(z) = W(\tilde{p}_n(z), \tilde{q}_n(z)), \quad \left(z - \frac{1}{z}\right) \tilde{b}(z) = W(p_n(z), \tilde{q}_n(z)), \quad (29)$$

which tell that $\alpha(z)$ has analyticity for $|z| \leq 1$, $\tilde{\alpha}(z)$ has analyticity for $|z| > 1$, $\beta(z)$ and $\tilde{\beta}(z)$ are generally not analytic and have definitions only for $|z| = 1$, and:

$$\alpha(z)\tilde{\alpha}(z) - \beta(z)\tilde{\beta}(z) = 1. \quad (30)$$

Due to the fact that the relationships in Equation (24) can be extended to the entire z -complex plane, we have:

$$p_n\left(\frac{1}{z}\right) = \tilde{p}_n(z) = p_n^*\left(\frac{1}{z^*}\right), \quad q_n\left(\frac{1}{z}\right) = \tilde{q}_n(z) = q_n^*\left(\frac{1}{z^*}\right). \quad (31)$$

Equations (22), (23), (28) and (29) imply:

$$\tilde{\alpha}(z) = \alpha^*\left(\frac{1}{z^*}\right), \quad \beta(z) = \beta^*\left(\frac{1}{z^*}\right). \quad (32)$$

Assuming that $z_j \neq 0$, $j=1, 2, \dots, N$, are simple zeros of $\alpha(z)$ (note that $1/z_j^*$, denoted by $\tilde{z}_j$, $j=1, 2, \dots, N$, are simple zeros of $\tilde{\alpha}(z)$), then a series of computations yield [25]:

$$\sum_{n=-\infty}^{\infty} A_n p_n^2(z_j) = - \frac{z_j d\alpha(z) / dz|_{z=z_j}}{B_j}, \quad (33)$$

where $A_n = \prod_{j=n+1}^{\infty} u_j$ and $p_n^2(z_j)$ denotes the eigenfunction corresponding to the square sum weighted by A_n , B_j is a constant that is independent of n and is related to the linear relation $q_n(z_j) = B_j p_n(z_j)$, $z = z_j$ represents the associated eigenvalue. If there exists a variable η_j independent of n satisfying $\sum_{n=-\infty}^{\infty} \eta_j^2 A_n p_n^2(z_j) = 1$, then the normalization factor of the Toda lattice SS (17) is η_j , and $\eta_j p_n(z_j)$ is termed the normalized eigenfunction with respect to the weighted square sum. Moreover, we have the following relation [25]:

$$\eta_j^2 = - \frac{B_j}{z_j d\alpha(z) / dz|_{z=z_j}}. \quad (34)$$

3. Orthogonality and Completeness of Related Squared Eigenfunctions

Kaup's work in 1976 truly proposed the "squared eigenfunctions" as an independent and core tool [36]. He studied the completeness of the eigenfunctions associated with the AKNS scattering problem and systematically explained the key role of these squared eigenfunctions in constructing the perturbation theory of integrable systems. Based on Kaup's work, the theoretical framework of squared eigenfunctions was continuously refined and extended, rigorous proofs of their orthogonality and completeness relations were developed. Sachs's analysis [37] of the completeness of the continuous Schrödinger scattering problem laid a solid foundation for the application of such functions in continuous models. The theoretical developments during this period made squared eigenfunctions a standard tool for analyzing the linearization problems, perturbation theory, and geometric structures of integrable systems. The work by Ablowitz's team [18,22–24] establishing the completeness relations of squared eigenfunctions has become a key tool for constructing and solving fractional continuous and semi-discrete integrable systems.

Through this, one can expand the fractional extensions of integrable systems under fractional operators in the function space spanned by these eigenfunctions, thereby achieving linearization.

3.1. Orthogonality of Squared Eigenfunctions

Following Ablowitz's steps [18], we define the following column vectors:

$$P_n(z) = (p_n^2(z), u_n p_{n-1}(z) p_n(z))^T, \quad (35)$$

$$P_n^A(z) = (q_n(z) q_{n+1}(z) - u_n q_{n-1}(z) q_n(z), q_{n+1}^2(z) - q_n^2(z))^T. \quad (36)$$

In view of the Toda lattice SP (17), it is easy to see that the squared eigenfunctions also satisfy:

$$\mathfrak{S} P_n(z) = \lambda(z) P_n(z), \quad \mathfrak{S}^A P_n^A(z) = \lambda(z) P_n^A(z). \quad (37)$$

Theorem 1 (Orthogonality of squared eigenfunctions). *Suppose that the potentials u_n and v_n associated with the Toda lattice SP (17) satisfy the rapid decay conditions:*

$$\sum_{n=-\infty}^{\infty} (1+|n|)|u_n-1| < \infty, \quad \sum_{n=-\infty}^{\infty} (1+|n|)|v_n| < \infty, \quad (38)$$

with the matrix operators $\mathfrak{S}$ and $\mathfrak{S}^A$ defined by Equations (9) and (12). Then, for the given real parameters k and k' , which respectively link

$$z = e^{ik} \Rightarrow \lambda = 2 \cos k, \quad z' = e^{ik'} \Rightarrow \lambda' = 2 \cos k', \quad (39)$$

the following orthogonality relations hold in the sense of distributions:

$$\sum_{n=-\infty}^{\infty} (P_n^A)^T(k) P_n(k') = -2\pi i \sin k \alpha^2(k) \delta(k' - k), \quad (40)$$

$$\sum_{n=-\infty}^{\infty} (\tilde{P}_n^A)^T(k) \tilde{P}_n(k') = 2\pi i \sin k \tilde{\alpha}^2(k) \delta(k' - k), \quad (41)$$

$$\sum_{n=-\infty}^{\infty} (P_n^A)^T(k) \tilde{P}_n(k') = -2\pi i \sin k \tilde{\alpha}^2(k) \delta(k' + k), \quad (42)$$

$$\sum_{n=-\infty}^{\infty} (\tilde{P}_n^A)^T(k) P_n(k') = 2\pi i \sin k \alpha^2(k) \delta(k' + k). \quad (43)$$

with

$$P_n(k') = (p_n^2(k'), u_n p_{n-1}(k') p_n(k'))^T, \quad P_n^A(k) = (q_n(k) q_{n+1}(k) - u_n q_{n-1}(k) q_n(k), q_{n+1}^2(k) - q_n^2(k))^T, \quad (44)$$

$$\tilde{P}_n(k') = (\tilde{p}_n^2(k'), u_n \tilde{p}_{n-1}(k') \tilde{p}_n(k'))^T, \quad \tilde{P}_n^A(k') = (\tilde{q}_n(k') \tilde{q}_{n+1}(k') - u_n \tilde{q}_{n-1}(k') \tilde{q}_n(k'), \tilde{q}_{n+1}^2(k') - \tilde{q}_n^2(k'))^T, \quad (45)$$

where $\alpha(k) = \alpha(z)|_{z=e^{ik}}$, $\tilde{\alpha}(k) = \tilde{\alpha}(z)|_{z=e^{ik}}$, and δ is the Kronecker delta function.

Proof. We verify Equation (40) here; the others (41)–(43) can be verified similarly. From Lemma A1, we have the sum

$$\sum_{n=-N}^N y_n(k, k') = [\lambda(k') - \lambda(k)] \sum_{n=-N}^N (P_n^A)^T(k) \cdot P_n(k') = Y_{N+1}(k, k') - Y_{-N}(k, k') \quad (46)$$

Since $|z| = |z'| = 1$, each term of the flux $Y_n(k, k')$ (see Appendix C) is a local bilinear combination of $p_n(k')$ and $q_n(k)$. Under the decay conditions (38), the Jost solu-

tions possess the asymptotic properties (22), (23) and (25) as $|n| \rightarrow \infty$, so the flux $Y_{\pm N}(k, k')$ consists of oscillatory exponential $e^{\pm ikN}$ and $e^{\pm ik'N}$. Taking $N \rightarrow \infty$ in the sense of distributions and using the distributional limit:

$$\lim_{N \rightarrow \infty} e^{\pm 2iNk} = \pm \pi \delta(k), \quad (47)$$

the right end of Equation (46) gives an expression containing the δ function in the distribution sense. Evaluating the boundary terms and considering the scattering coefficient $\alpha(k)$ yields the first orthogonality relation (40). $\square$

3.2. Completeness of Squared Eigenfunctions

We provide the following assumptions: the potentials u_n and v_n satisfy the rapid decay conditions (38); the transmission coefficients $\alpha(z)$ and $\tilde{\alpha}(z)$ have no zeros on the unit circle $|z|=1$ (no spectral singularities); all zeros of $\alpha(z)$ inside $|z|<1$ and $\tilde{\alpha}(z)$ outside $|z|>1$ are simple; the function F_n belongs to the weighted space X defined by Equation (A1) in Appendix A. These assumptions are standard in the IST for the Toda lattice and ensure that the completeness relations hold.

Theorem 2 (Completeness of squared eigenfunctions). *Suppose the rapid decay conditions (38) hold. Then any decaying two-dimensional ℓ^1 function $F_n = (F_{n,1}, F_{n,2})^T \in X$ satisfies*

$$F_n = \sum_{s=1}^2 \sum_{m=-\infty}^{\infty} \int_{\Omega^{(s)}} dz H^{(s)}(n, m, z) F_m, \quad (48)$$

which means that

$$F_n = \sum_{s=1}^2 \sum_{m=-\infty}^{\infty} \int_{|z|=1} dz H^{(s)}(n, m, z) F_m + \sum_{s=1}^2 \sum_{j=1}^N \sum_{m=-\infty}^{\infty} h_j^{(s)}(n, m) F_m, \quad (49)$$

with the continuous spectral kernels

$$H^{(1)}(n, m, z) = -\frac{1}{2\pi i(z^2 - 1)\alpha^2(z)} P_n^A(z) P_m^T(z), \quad H^{(2)}(n, m, z) = \frac{1}{2\pi i(z^2 - 1)\alpha^{-2}(z)} \tilde{P}_n^A(z) \tilde{P}_m^T(z); \quad (50)$$

and the discrete spectral kernels

$$h_j^{(1)}(n, m) = -\frac{1}{(z_j^2 - 1)(d\alpha(z)/dz|_{z=z_j})^2} \left[\left(\frac{2z_j}{z_j^2 - 1} + \frac{d^2\alpha(z)/dz^2|_{z=z_j}}{d\alpha(z)/dz|_{z=z_j}} P_n^A(z) P_m^T(z) - \frac{dP_n^A(z) P_m^T(z)}{dz} \right) \right]_{z=z_j}, \quad (51)$$

$$h_j^{(2)}(n, m) = -\frac{1}{(\tilde{z}_j^2 - 1)(d\tilde{\alpha}(z)/dz|_{z=\tilde{z}_j})^2} \left[\left(\frac{2\tilde{z}_j}{\tilde{z}_j^2 - 1} + \frac{d^2\tilde{\alpha}(z)/dz^2|_{z=\tilde{z}_j}}{d\tilde{\alpha}(z)/dz|_{z=\tilde{z}_j}} \tilde{P}_n^A(\tilde{z}) \tilde{P}_m^T(\tilde{z}) - \frac{d\tilde{P}_n^A(\tilde{z}) \tilde{P}_m^T(\tilde{z})}{d\tilde{z}} \right) \right]_{z=\tilde{z}_j}, \quad (52)$$

where the integral contours $\Omega^{(1)} = \Omega_r$ and $\Omega^{(2)} = \Omega_R$ are counterclockwise circles of radius $|z|=r$ and $|z|=R$, respectively, the interior of Ω_r contains any zero z_j of $\alpha(z)$, the exterior of Ω_R contains any zero $\tilde{z}_j$ of $\tilde{\alpha}(z)$, and the zeros z_j and $\tilde{z}_j$ are simple for $j=1, 2, \dots, N$.

Proof. According to [18], it is necessary to prove the following asymptotic forms:

$$p_n(z) \sim \left(\prod_{j=n}^{\infty} u_j \right) z^n, \quad q_n(z) \sim \left(\prod_{j=-\infty}^{n-1} u_j \right) z^{-n}, \quad \alpha(z) \sim \prod_{j=-\infty}^{\infty} u_j, \quad \text{as } r \rightarrow 0, \quad (53)$$

$$\tilde{p}_n(z) \sim \left(\prod_{j=n}^{\infty} u_j \right) z^{-n}, \quad \tilde{q}_n(z) \sim \left(\prod_{j=-\infty}^{n-1} u_j \right) z^n, \quad \tilde{\alpha}(z) \sim \prod_{j=-\infty}^{\infty} u_j, \quad \text{as } R \rightarrow \infty. \quad (54)$$

In fact, when $|z| = r \rightarrow 0$, the spectral parameter behaves as $\lambda = z + 1/z \sim 1/z$. The dominant term of Equation (17) yields $u_n P_{n-1} \approx \lambda P_n$, which implies $P_n \approx z u_n P_{n-1}$. Based on this iteration, the asymptotic behaviors of the Jost solutions $p_n(z)$ and $q_n(z)$ reach those in Equation (53). For the asymptotic behavior of $\alpha(z)$ in Equation (53), it can be obtained by the limit of the first Casoratian in Equation (30) at $n \rightarrow \infty$. Similarly, as $|z| = R \rightarrow \infty$, $\lambda \sim z$, and the dominant balance yields the forms in Equation (54) for $\tilde{p}_n(z)$, $\tilde{q}_n(z)$, and $\tilde{\alpha}(z)$. These asymptotic forms guarantee [25] that $p_n(z)z^{-n}$ and $q_n(z)z^n$ are analytic for $|z| < 1$, while $\tilde{p}_n(z)z^n$ and $\tilde{q}_n(z)z^{-n}$ are analytic for $|z| > 1$.

Before expanding F_n , we discuss the boundedness property of the spectral representation operator on the weighted space X . The pointwise uniform weighted bound for the continuous-spectrum kernel $H^{(s)}(n, m, z)$ for every $z \in |z| = 1$ is not required; such a pointwise estimate fails for the free potential case. Indeed, we interpret the identity (49) as a weak completeness relation on the weighted space X .

Recall X in Equation (A1), so $X \subset \ell^2(\mathbb{R})$. For any $F \in X$, the spectral expansion (49) is understood via duality pairing with arbitrary test sequences $G \in \ell^2(\mathbb{R})$ using the real inner product (10). The ℓ -type weighted condition $F \in X$ ensures that after pairing with test functions, the sum over m and contour integral over $|z| = 1$ may be interchanged. The discrete spectral kernels $h_j^{(s)}(n, m)$ still satisfy the weighted estimate

$$\sup_{1 \leq j \leq N} \sum_n (1 + |n|) |h_j^{(s)}(n, m)| \leq C_h (1 + |m|),$$

since they are finite-rank residues from simple poles and do not suffer the sivergerence present for continuous-spectrum kernels, where C_h is positive constants unrelated to m , z , and j .

We expand the two-dimensional ℓ^1 function F_n in terms of $P_n^A(z)$ and $\tilde{P}_n^A(z)$, then the completeness relation (48) can be implied through the orthogonality relations (40)–(43). Thanks to the analyticity properties of $\alpha(z)$ inside the unit circle $|z| < 1$ and $\tilde{\alpha}(z)$ outside the unit circle $|z| > 1$, $p_n(z)z^{-n}$ and $q_n(z)z^n$ for $|z| < 1$, and $\tilde{p}_n(z)z^n$ and $\tilde{q}_n(z)z^{-n}$ for $|z| > 1$, Cauchy's theorem allows us to deform the contours Ω_r and Ω_R to the unit circle $|z| = 1$. During this deformation, the contours sweep past the zeros of $\alpha(z)$ and $\tilde{\alpha}(z)$. Since the integrands contain $\alpha^2(z)$ and $\tilde{\alpha}^2(z)$ in the denominators, these zeros are double poles. Evaluating the residues at these poles yields the discrete spectral kernels $h_j^{(1)}$ and $h_j^{(2)}$ given in Equations (51) and (52), which include the derivatives of the squared eigenfunctions. The convergence of the resulting integrals and sums is guaranteed by the weak formulation on X .

When the contours Ω_R and Ω_r are deformed to the unit circle $|z| = 1$, the factor $1/(z^2 - 1)$ in Equation (50) appears to be singular at $z = \pm 1$. However, at $z = \pm 1$, we

have $P_n = \tilde{P}_n$ and $q_n = \tilde{q}_n$, so the combination $P_n^A(z)P_m^T(z) - \tilde{P}_n^A(z)\tilde{P}_m^T(z)$ vanishes at $z = \pm 1$. Since the underlying functions are smooth, the zero in the numerator is of sufficient order to cancel the simple pole from $1/(z^2 - 1)$. Thus, $z = \pm 1$ are removable singularities, and the contour integral passes through them without contribution. Under the assumption that all zeros of $\alpha(z)$ inside $|z| < 1$ and $\tilde{\alpha}(z)$ outside $|z| > 1$ are simple, the discrete spectrum is finite and isolated. This ensures that the discrete sums in Equation (49) are finite and that the kernels (51) and (52) are well-defined. The absence of zeros on the unit circle excludes spectral singularities and guarantees that the continuous spectrum is purely absolutely continuous.

In the discrete setting, the canonical basis vectors $e_k \in \ell^2$, so weak equality under all test pairings implies pointwise equality for each integer n . Thus Equation (49) holds pointwise for all $F \in X$. $\square$

3.3. Explicit Representations of the Toda Hierarchy and sfvToda Lattice

To give an explicit representation of the sfvToda lattice Equation (16), we need to further replace $(V_n, U_n)^T = (u_n(v_n - v_{n-1}), u_{n+1} - u_n)^T$ with F_n . From the second equality in Equation (37), we obtain $(\aleph^A)^M |\aleph^A|^\epsilon P_n^A(z) = \lambda^M(z) |\lambda(z)|^\epsilon P_n^A(z)$, $M = 0, 1, 2, \dots$. Then, from Equation (48) we have

$$(\aleph^A)^M |\aleph^A|^\epsilon \begin{pmatrix} V_n \\ U_n \end{pmatrix} = (\aleph^A)^M |\aleph^A|^\epsilon \sum_{s=1}^2 \sum_{m=-\infty}^{\infty} \int_{\Omega^{(s)}} dz H^{(s)}(n, m, z) \begin{pmatrix} V_m \\ U_m \end{pmatrix}, \quad (55)$$

which helps obtain the explicit representation of the fractional Toda lattice hierarchy (15)

$$\begin{pmatrix} \frac{du_n}{dt} \\ \frac{dv_n}{dt} \end{pmatrix} = \sum_{s=1}^2 \sum_{m=-\infty}^{\infty} \int_{\Omega^{(s)}} dz \lambda^M(z) |\lambda(z)|^\epsilon H^{(r)}(n, m, z) \begin{pmatrix} u_m(v_m - v_{m-1}) \\ u_{m+1} - u_m \end{pmatrix}, \quad M = 0, 1, 2, \dots, \quad (56)$$

and the explicit representation of the sfvToda lattice Equation (16):

$$\begin{pmatrix} \frac{du_n}{dt} \\ \frac{dv_n}{dt} \end{pmatrix} = \sum_{s=1}^2 \sum_{m=-\infty}^{\infty} \int_{\Omega^{(s)}} dz [\gamma_0(t) + \gamma_1(t)\lambda(z) + \gamma_2(t)\lambda^2(z)] |\lambda(z)|^\epsilon H^{(r)}(n, m, z) \begin{pmatrix} u_m(v_m - v_{m-1}) \\ u_{m+1} - u_m \end{pmatrix}. \quad (57)$$

Note that as pointed earlier, the two integrals over Ω_r and Ω_R can be converted onto the unit circle $|z| = 1$ plus the contributions from the double poles of $\alpha(z)$ and $\tilde{\alpha}(z)$. Alternatively, we use Equation (49) to obtain the equivalent explicit representation of the sfvToda lattice Equation (16):

$$\begin{pmatrix} \frac{du_n}{dt} \\ \frac{dv_n}{dt} \end{pmatrix} = \sum_{s=1}^2 \sum_{m=-\infty}^{\infty} \left\{ \int_{|z|=1} dz [\gamma_0(t) + \gamma_1(t)\lambda(z) + \gamma_2(t)\lambda^2(z)] |\lambda(z)|^\epsilon H^{(s)}(n, m, z) \right. \\ \left. + \sum_{j=1}^N [\gamma_0(t) + \gamma_1(t)\lambda(z_j^{(s)}) + \gamma_2(t)\lambda^2(z_j^{(s)})] |\lambda(z_j^{(s)})|^\epsilon h_j^{(s)}(n, m) \right\} \begin{pmatrix} u_m(v_m - v_{m-1}) \\ u_{m+1} - u_m \end{pmatrix}, \quad (58)$$

where $z_j^{(1)} = z_j$, $z_j^{(2)} = \tilde{z}_j = 1/z_j^*$.

4. Discrete GLM Equation and Time Dependence of Scattering Data

4.1. Translation Transformation and Discrete GLM Equation

A direct computation shows that the infinite series (38) constructed from the potentials u_n and v_n are convergent. Furthermore, the functions $p_n(z)$, $\tilde{p}_n(z)$, $q_n(z)$, and $\tilde{q}_n(z)$ in Equations (22) and (23) can be expanded as power series in z by the so called translation transformations of the Toda lattice SS (17):

$$p_n(z) = \sum_{j=n}^{\infty} M_{n,j} z^j, \quad \tilde{p}_n(z) = \sum_{j=n}^{\infty} M_{n,j} z^{-j}, \quad (59)$$

$$q_n(z) = \sum_{j=-\infty}^n J_{n,j} z^{-j}, \quad \tilde{q}_n(z) = \sum_{j=-\infty}^n J_{n,j} z^j. \quad (60)$$

Substituting the expansion for $p_n(z)$ into Equation (17) and equating the coefficients of same powers of z on both sides, we obtain

$$M_{n,n} = \prod_{j=n+1}^{\infty} \frac{1}{u_j}, \quad (61)$$

$$M_{n,n+1} = \frac{1}{u_{n+1}} M_{n+1,n+2} - \frac{v_{n+1}}{u_{n+1}} M_{n+1,n+1}, \quad (62)$$

$$M_{n,n+2} = \frac{1}{u_{n+1}} (M_{n+1,n+3} + M_{n+1,n+1} - M_{n+2,n+2} - v_{n+1} M_{n+1,n+2}), \quad (63)$$

and so forth. Similarly, substituting the expansion of $q_n(z)$ into Equation (17) and comparing the coefficients, we can also determine $J_{n,j}$. In view of the relations in Equation (31), the obtained $M_{n,j}$ and $J_{n,j}$ are also valid for $\tilde{p}_n(z)$ and $\tilde{q}_n(z)$. Then, the potentials u_n and v_n can be recovered as [25]

$$u_n = \frac{M_{n,n}}{M_{n-1,n-1}}, \quad v_n = \frac{M_{n,n+1}}{M_{n,n}} - \frac{M_{n-1,n}}{M_{n-1,n-1}}. \quad (64)$$

Proposition 1 (Discrete GLM equation) [25]. *If the scattering data,*

$$S(0, z) = \{ |z|, D(z) = \beta(z) / \alpha(z), z_j, \eta_j, j = 1, 2, \dots, N \}, \quad (65)$$

corresponding to the Toda lattice SP (17), are known, then the element of the associated translation transformation matrix, denoted by $\tilde{M}_{n,m} = M_{n,m} / M_{n,n}$, satisfies the discrete GLM equation:

$$\tilde{M}_{n,m} + G_{n+m} + \sum_{s=n+1}^{\infty} \tilde{M}_{n,s} G_{s+m} = 0, \quad m < n, \quad (66)$$

where the component G_m of the vector G is given by

$$G_m = \sum_{j=1}^N \eta_j^2 z_j^m + \frac{1}{2\pi i} \oint_{|z|=1} D(z) z^{m-1} dz. \quad (67)$$

We can solve for $M_{n,m}$ and $M_{n,n}$ from the GLM Equation (66), and then determine the potentials u_n and v_n from Equation (64). A similar approach can be used to

derive the equation satisfied by $J_{n,j}$, and then solve it to obtain the corresponding GLM expressions.

4.2. Time Dependence of Scattering Data

In what follows, we determine the time dependence of the scattering data:

$$S(t, z) = \{ |z| = 1, D(t, z) = \beta(t, z) / \alpha(t, z), z_j(t), \eta_j(t), j = 1, 2, \dots, N \}. \quad (68)$$

Firstly, when considering the continuous spectrum $|z| = 1$, we know from Equations (20) and (21) that $dq_n(z)/dt - N_n q_{n-1}(z) - Q_n q_n(z)$ is a solution of the Toda lattice SP (17). Thus, we may use two undetermined functions $\delta(t)$ and $\theta(t)$, with respect to t , to assume that

$$\frac{dq_n(z)}{dt} - N_n q_{n-1}(z) - Q_n q_n(z) = \delta(t) \tilde{q}_n(z) + \theta(t) q_n(z). \quad (69)$$

As $n \rightarrow -\infty$, substituting the asymptotic properties $q_n(z) \sim z^{-n}$ and $\tilde{q}_n(z) \sim z^n$ in Equation (23), together with $[\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2]|\lambda|^\epsilon$ in Equation (21) for Q_n and $-N_n$ into Equation (69) obtains

$$-n \frac{dz}{dt} + [\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2]|\lambda|^\epsilon (z^2 - 1) = \delta(t) z^{2n+1} + \theta(t) z. \quad (70)$$

Since Equation (70) holds for all n , we have:

$$\frac{dz}{dt} = 0, \quad (71)$$

$$\delta(t) = 0, \quad (72)$$

$$\theta(t) = [\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2]|\lambda|^\epsilon \left(z - \frac{1}{z} \right). \quad (73)$$

Hence, Equation (69) reduces as:

$$\frac{dq_n(z)}{dt} - N_n q_{n-1}(z) - Q_n q_n(z) = [\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2]|\lambda|^\epsilon \left(z - \frac{1}{z} \right) q_n(z). \quad (74)$$

Substituting the linear relation in Equation (25) into Equation (74), then Equation (74) reduces to:

$$\frac{d\alpha(t, z)}{dt} z^{-n} + \frac{d\beta(t, z)}{dt} z^n = [\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2]|\lambda|^\epsilon \left(z - \frac{1}{z} \right) \beta(t, z) z^n. \quad (75)$$

From Equation (75), we obtain:

$$\frac{d\alpha(t, z)}{dt} = 0, \quad (76)$$

$$\frac{d\beta(t, z)}{dt} = [\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2]|\lambda|^\epsilon \left(z - \frac{1}{z} \right) \beta(t, z). \quad (77)$$

Integrating Equations (76) and (77) yields:

$$\alpha(t, z) = \alpha(0, z), \quad (78)$$

$$\beta(t, z) = \beta(0, z) e^{\left(z - \frac{1}{z} \right) \int_0^t [\gamma_0(\tau) + \gamma_1(\tau)\lambda + \gamma_2(\tau)\lambda^2]|\lambda|^\epsilon d\tau}, \quad (79)$$

where $\alpha(0, z)$ and $\beta(0, z)$ are constants. Equations (78) and (79) indicate that $a(t, z)$ is a conserved quantity of the sfvToda lattice Equation (16), and the reflection coefficient satisfies the time dependence:

$$D(z, t) = D(0, z) e^{\left(z - \frac{1}{z} \right) \int_0^t [\gamma_0(\tau) + \gamma_1(\tau) \lambda + \gamma_2(\tau) \lambda^2] |\lambda|^\varepsilon d\tau}, \quad D(0, z) = \frac{\beta(0, z)}{\alpha(0, z)}, \quad (80)$$

where $D(0, z)$ is constant.

Secondly, if $z = z_j$ is a discrete eigenvalue of Equation (17), we take the Jost solution $p_n(z_j)$ in Equation (22). Since $dp_n(z_j)/dt - N_n p_{n-1}(z_j) - Q_n p_n(z_j)$ solves Equation (17), it can therefore be expressed by

$$\frac{dp_n(z_j)}{dt} - N_n p_{n-1}(z_j) - Q_n p_n(z_j) = \varsigma(t) p_n(z_j) + \omega(t) \tilde{p}_n(z_j), \quad (81)$$

where the two functions $\varsigma(t)$ and $\omega(t)$ of t are determined later. Since $|z_j| < 1$, $p_n(z_j) \sim z_j^n$ and $\tilde{p}_n(z_j) \sim z_j^{-n}$ as $n \rightarrow \infty$, see Equation (22), we substitute them into Equation (81) to obtain $\omega(t) = 0$. To further determine $\varsigma(t)$, we perform the transformation

$$\mathcal{G}_n(z_j) = \eta_j(t) \sqrt{A_n} p_n(z_j), \quad (82)$$

where $A_n = \prod_{j=n+1}^{\infty} u_j$ is real and $\mathcal{G}_n(z_j)$ is a square summable function with weights A_n . Thus, from Equation (17) we have:

$$\sqrt{u_{n+1}} \mathcal{G}_{n+1}(z_j) + \sqrt{u_n} \mathcal{G}_{n-1}(z_j) + v_n \mathcal{G}_n(z_j) = \lambda_j \mathcal{G}_n(z_j). \quad (83)$$

We rewrite Equation (81) as:

$$\begin{aligned} \frac{\mathcal{G}_n(z_j)}{dt} + \frac{1}{2} (Q_{n+1} - Q_n) \mathcal{G}_n(z_j) - \frac{1}{2} (\lambda_j - v_n) Z_{n+1} \mathcal{G}_n(z_j) + \sqrt{u_n} Z_{n+1} \mathcal{G}_{n-1}(z_j) \\ = \left\{ \varsigma(t) + [\gamma_0(t) + \gamma_1(t) \lambda_j + \gamma_2(t) \lambda_j^2] |\lambda_j|^\varepsilon \left(1 - \frac{1}{2} \lambda_j \right) \right\} \mathcal{G}_n(z_j). \end{aligned} \quad (84)$$

Here, the following derivative of A_n with respect to t has been employed:

$$\frac{dA_n}{dt} = -[Q_{n+1} + Q_n - (\lambda_j - v_n) Z_{n+1}] A_n. \quad (85)$$

Substituting Equation (83) into Equation (84) and performing straightforward algebraic manipulations yields:

$$\begin{aligned} \frac{d\mathcal{G}_n(z_j)}{dt} + \frac{1}{2} (Q_{n+1} - Q_n) \mathcal{G}_n(z_j) + \frac{1}{2} (Z_{n+1} - Z_n) \sqrt{u_n} \mathcal{G}_{n-1}(z_j) \\ + \frac{1}{2} [\sqrt{u_n} Z_n \mathcal{G}_{n-1}(z_j) - \sqrt{u_{n+1}} Z_{n+1} \mathcal{G}_{n+1}(z_j)] = \varsigma_1 \mathcal{G}_n(z_j). \end{aligned} \quad (86)$$

Multiplying both sides of Equation (86) by $2\mathcal{G}_n(z_j)$ and summing over n , further assuming that $\mathcal{G}_n(z_j)$ has been normalized by $\eta_j(t)$, i.e., $\mathcal{G}_n(z_j) = \eta_j(t) \sqrt{A_n} p_n(z_j)$, which approaches $\eta_j(t) z_j^n$ as $n \rightarrow \infty$, the first term (the derivative with respect to t) and the fourth term on the left-hand side vanish. It can be proven that the following sum of the second and third terms also vanishes [25]:

$$\sum_{n=-\infty}^{\infty} [(Q_{n+1} - Q_n) \mathcal{G}_n^2(z_j) + (Z_{n+1} - Z_n) \sqrt{u_n} \mathcal{G}_{n-1}(z_j) \mathcal{G}_n(z_j)] = 0, \quad (87)$$

then we obtain $\varsigma_1(t) = 0$. Further Letting $n \rightarrow \infty$ to simplify Equation (84) as:

$$\frac{d\eta_j(t)}{dt} + n \frac{1}{z_j} \frac{dz_j}{dt} \eta_j(t) - \frac{1}{2} [\gamma_0(t) + \gamma_1(t) \lambda_j + \gamma_2(t) \lambda_j^2] |\lambda_j|^\epsilon \left(z_j - \frac{1}{z_j} \right) \eta_j(t) = 0. \quad (88)$$

From Equation (88), it follows that:

$$\frac{d\eta_j(t)}{dt} = \frac{1}{2} [\gamma_0(t) + \gamma_1(t) \lambda_j + \gamma_2(t) \lambda_j^2] |\lambda_j|^\epsilon \left(z_j - \frac{1}{z_j} \right) \eta_j(t). \quad (89)$$

Consequently, the time dependence of the normalization factor $\eta_j(t)$ is given by

$$\eta_j^2(t) = \eta_j^2(0) e^{\left(z_j - \frac{1}{z_j} \right) \int_0^t [\gamma_0(\tau) + \gamma_1(\tau) \lambda_j + \gamma_2(\tau) \lambda_j^2] |\lambda_j|^\epsilon d\tau}, \quad (90)$$

where $\eta_j(0)$ is real constant. Based on the above analysis, we can summarize the result as the following theorem:

Theorem 3 (Time dependence of scattering data). If u_n and v_n evolve according to the sfvToda lattice Equation (16), then the continuous spectrum and the discrete eigenvalues in the scattering data (66) of the associated Toda lattice SP (17) are conserved quantities of Equation (16), and the reflection coefficient $D(t, z)$ and the normalization factor $\eta_j(t)$ satisfy the exponential time-dependent relations (80) and (90), respectively.

5. Multi-Soliton Solutions Under Reflectionless Potentials

5.1. Formulae of Fractional Variable-Velocity N-Soliton Solutions and Simulation

Only when we set $D(t, z) \equiv 0$ can we obtain N-soliton solutions of the sfvToda lattice Equation (16). In such case of reflectionless potentials, following the method in [25], the component G_m takes the form

$$G_m = \sum_{j=1}^N \eta_j^2(t) z_j^m, \quad (91)$$

where $\eta_j(t)$ is determined by Equation (88). Hence $\tilde{M}_{n,m}$ can be chosen as

$$\tilde{M}_{n,m}(t) = \sum_{j=1}^N \eta_j(t) z_j^m f_{n,j}(t). \quad (92)$$

Substituting Equations (91) and (92) into the discrete GLM Equation (66) yields a system of algebraic equations for $f_{n,k}(t)$, $k = 1, 2, \dots, N$:

$$f_{n,j}(t) + \eta_j(t) z_j^n + \sum_{k=1}^N \eta_j(t) \eta_k(t) \frac{z_j^{n+1} z_k^{n+1}}{1 - z_j z_k} f_{n,k}(t) = 0, \quad j = 1, 2, \dots, N. \quad (93)$$

When the scattering data z_j and $\eta_j(t)$ are all real, the above algebraic equation is solvable and then Equation (92) can be determined as

$$\tilde{M}_{n,m}(t) = -\text{tr}\{[I + C_n(t)]^{-1} s_n(t) s_m^T(t)\}, \quad m > n, \quad (94)$$

where $\text{tr}(\cdot)$ denotes the operator for calculating the trace of the matrix being acted upon, I is the $N \times N$ identity matrix, $C_n(t)$ represents the $N \times N$ matrix whose entries are

$$(C_n(t))_{j,k} = \eta_j(t) \eta_k(t) \frac{z_j^{n+1} z_k^{n+1}}{1 - z_j z_k}, \quad (95)$$

and $s_n(t) = (\eta_1(t) z_1^n, \eta_2(t) z_2^n, \dots, \eta_N(t) z_N^n)^T$.

For the case of $m = n$, we know from [25] that $M_{n,n}(t)$ can be expressed as:

$$M_{n,n}(t) = \frac{\det[I + C_n(t)]}{\det[I + C_{n-1}(t)]}. \quad (96)$$

Thus, the potentials of the sfvToda lattice Equation (16) can be obtained in the form of fractional variable-velocity N -soliton solutions through Equations (64), (94) and (96) as follows:

$$u_n = \frac{\det[I + C_n(t)] \det[I + C_{n-2}(t)]}{\det[I + C_{n-1}(t)]^2}, \quad (97)$$

$$v_n = -\text{tr}\{[I + C_n(t)]^{-1} s_n(t) s_{n+1}^T(t)\} + \text{tr}\{[I + C_{n-1}(t)]^{-1} s_{n-1}(t) s_n^T(t)\}. \quad (98)$$

5.2. Fractional Variable-Velocity Single-Soliton Solutions and Simulation

When $N=1$, we obtain the fractional variable-velocity single-soliton solutions of the sfvToda lattice Equation (16)

$$u_n = \frac{[1 - z_1^2 + \eta_1^2(0) z_1^{2n+2} e^{\sigma_1}][1 - z_1^2 + \eta_1^2(0) z_1^{2n-2} e^{\sigma_1}]}{[1 - z_1^2 + \eta_1^2(0) z_1^{2n} e^{\sigma_1}]^2}, \quad (99)$$

$$v_n = \frac{\eta_1^2(0) z_1^{2n-1} e^{\sigma_1} (1 - z_1^2)}{1 - z_1^2 + \eta_1^2(0) z_1^{2n} e^{\sigma_1}} - \frac{\eta_1^2(0) z_1^{2n+1} e^{\sigma_1} (1 - z_1^2)}{1 - z_1^2 + \eta_1^2(0) z_1^{2n+2} e^{\sigma_1}}, \quad (100)$$

through

$$\det[I + C_n(t)]|_{N=1} = 1 + \frac{\eta_1^2(0) z_1^{2n+2} e^{\sigma_1}}{1 - z_1^2}, \quad (101)$$

where

$$\sigma_1 = \int_0^t [\gamma_0(\tau) + \gamma_1(\tau) \lambda_1 + \gamma_2(\tau) \lambda_1^2] |\lambda_1|^\varepsilon d\tau \left(z_1 - \frac{1}{z_1} \right), \quad \lambda_1 = z_1 + \frac{1}{z_1}. \quad (102)$$

In Figures 1 and 2, the fractional variable-velocity single-soliton solution (99) with different fractions $\varepsilon=0.1$ and $\varepsilon=0.9$ is shown. We can see that when the selection of other parameters is the same, the larger the fraction ε is, the faster the soliton controlled by it propagates. To indicate the influence of ε , and $\gamma_0(t)$, $\gamma_1(t)$, and $\gamma_2(t)$, we derive the theoretical velocity v in Equation (A17) in Appendix A.5. Figure 3 confirms that larger fraction ε leads to faster propagation.

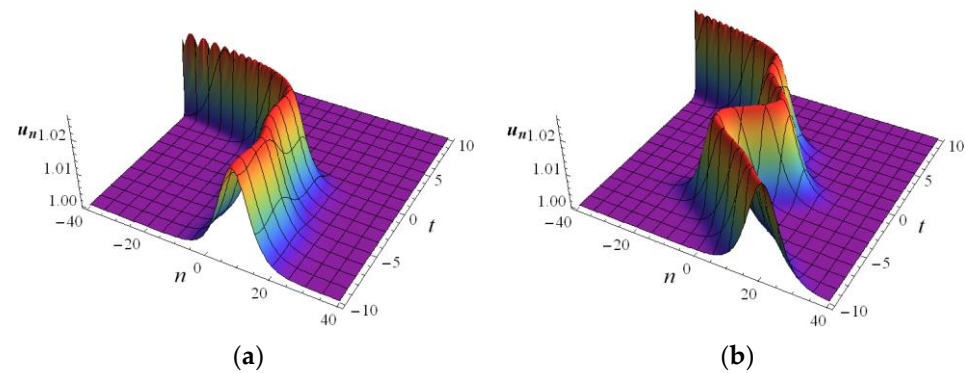


Figure 1. Fractional variable-velocity single-soliton solution (99) with $\varepsilon = 0.1$, $\eta_1(0) = 1$, and $z_1 = 0.85$: (a) $\gamma_0(t) = \sin t$, $\gamma_1(t) = e^t$, and $\gamma_2(t) = 0.1$; (b) $\gamma_0(t) = t$, $\gamma_1(t) = 0.1t^2$, and $\gamma_2(t) = \cos t$.

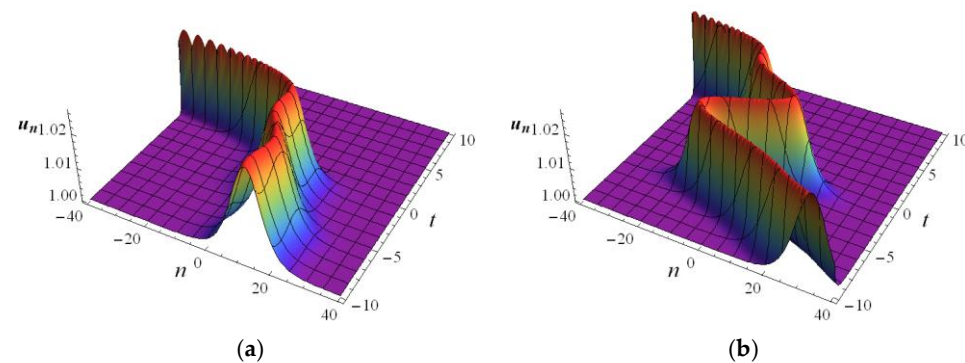


Figure 2. Fractional variable-velocity single-soliton solution (99) with $\varepsilon = 0.9$, $\eta_1(0) = 1$, and $z_1 = 0.85$: (a) $\gamma_0(t) = \sin t$, $\gamma_1(t) = e^t$, and $\gamma_2(t) = 0.1$; (b) $\gamma_0(t) = t$, $\gamma_1(t) = 0.1t^2$, and $\gamma_2(t) = \cos t$.

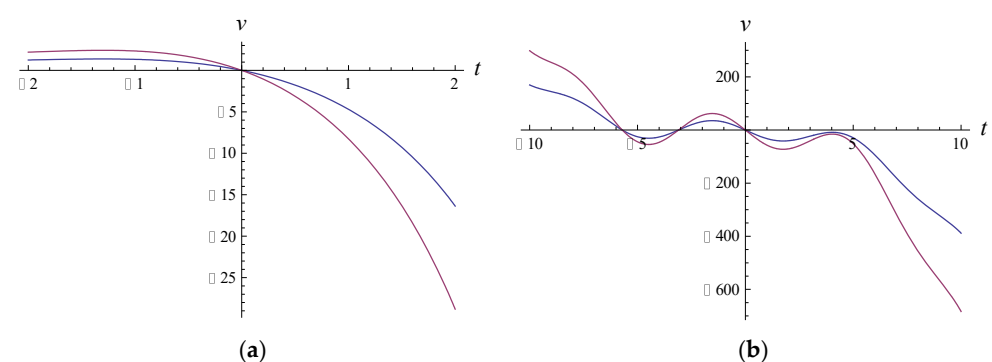


Figure 3. Velocity of fractional variable-velocity single-soliton solution (99) with $\eta_1(0) = 1$, $z_1 = 0.85$: (a) $\gamma_0(t) = \sin t$, $\gamma_1(t) = e^t$, $\gamma_2(t) = 0.1$, $\varepsilon = 0.1$ (blue line), and $\varepsilon = 0.9$ (purple line); (b) $\gamma_0(t) = t$, $\gamma_1(t) = 0.1t^2$, $\gamma_2(t) = \cos t$, $\varepsilon = 0.1$ (blue line), and $\varepsilon = 0.9$ (purple line).

For numerical verification without invoking the assumed scattering-data evolution, we define the residuals $R_n^{(u)}$ and $R_n^{(v)}$,

$$\begin{pmatrix} R_n^{(u)} \\ R_n^{(v)} \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} u_n \\ v_n \end{pmatrix} - [\gamma_0(t) + \gamma_1(t)\mathbb{S}^A + \gamma_2(t)(\mathbb{S}^A)^2] |\mathbb{S}^A|^\varepsilon \begin{pmatrix} u_n(v_n - v_{n-1}) \\ u_{n+1} - u_n \end{pmatrix},$$

and obtain them through Mathematica 8.0, which are below 10^{-13} in both the maximum norm and the norm ℓ^2 , confirming that the fractional variable-velocity single-soliton solutions (99) and (100) satisfy the sfvToda Equation (16) to high accuracy, where the parameters take those in Figure 1. By using analytical integration in Mathematica 8.0 with the grid range $n = -20, \dots, 20$, time interval $t \in [0, 5]$, and step size $\Delta t = 0.1$, the typical residual norms are obtained as:

Case of Figure 1a:

$$\max_n |R_n^{(u)}| \approx 4.13558 \times 10^{-15}, \quad \max_n |R_n^{(v)}| \approx 8.68706 \times 10^{-14}, \quad \|R_n^{(u)}\|_2 \approx 2.06098 \times 10^{-14}, \\ \|R_n^{(v)}\|_2 \approx 5.43393 \times 10^{-13};$$

Case of Figure 1b:

$$\max_n |R_n^{(u)}| \approx 3.08434 \times 10^{-15}, \quad \max_n |R_n^{(v)}| \approx 1.39821 \times 10^{-14}, \quad \|R_n^{(u)}\|_2 \approx 1.52285 \times 10^{-14}, \\ \|R_n^{(v)}\|_2 \approx 9.59409 \times 10^{-14}.$$

5.3. Fractional Variable-Velocity Double-Soliton Solutions and Simulation

When $N = 2$, the fractional variable-velocity double-soliton solutions of the sfvToda lattice Equation (16) are obtained as follows:

$$u_n = \frac{\det[I + C_n(t)] \det[I + C_{n-2}(t)]}{\det[I + C_{n-1}(t)]^2} \Big|_{N=2}, \quad (103)$$

$$v_n = -\text{tr}\{[I + C_n(t)]^{-1} s_n(t) s_{n+1}^T(t)\} + \text{tr}\{[I + C_{n-1}(t)]^{-1} s_{n-1}(t) s_n^T(t)\} \Big|_{N=2}, \quad (104)$$

through

$$\det[I + C_n(t)] \Big|_{N=2} = \begin{vmatrix} 1 + \frac{\eta_1^2(0) z_1^{2n+2} e^{\sigma_1}}{1 - z_1^2} & \frac{\eta_1(0) \eta_2(0) (z_1 z_2)^{n+1} e^{(\sigma_1 + \sigma_2)/2}}{1 - z_1 z_2} \\ \frac{\eta_2(0) \eta_1(0) (z_2 z_1)^{n+1} e^{(\sigma_1 + \sigma_2)/2}}{1 - z_2 z_1} & 1 + \frac{\eta_2^2(0) z_2^{2n+2} e^{\sigma_2}}{1 - z_2^2} \end{vmatrix}, \quad (105)$$

and $s_n(t) \Big|_{N=2} = (\eta_1(t) z_1^n, \eta_2(t) z_2^n)^T$, where $\eta_1(t)$ and $\eta_2(t)$ are constrained by Equation (90), σ_1 is in Equation (102), and σ_2 takes as

$$\sigma_2 = \int_0^t [\gamma_0(\tau) + \gamma_1(\tau) \lambda_2 + \gamma_2(\tau) \lambda_2^2] |\lambda_2|^\varepsilon d\tau \left(z_2 - \frac{1}{z_2} \right), \quad \lambda_2 = z_2 + \frac{1}{z_2}. \quad (106)$$

In Figures 4–7, the fractional variable-velocity double-soliton solution (103) is shown. We can see that during the propagation of two leftward-propagating solitons, the high soliton on the right catches up with and surpasses the soliton on the left, and the collision frequency is only once.

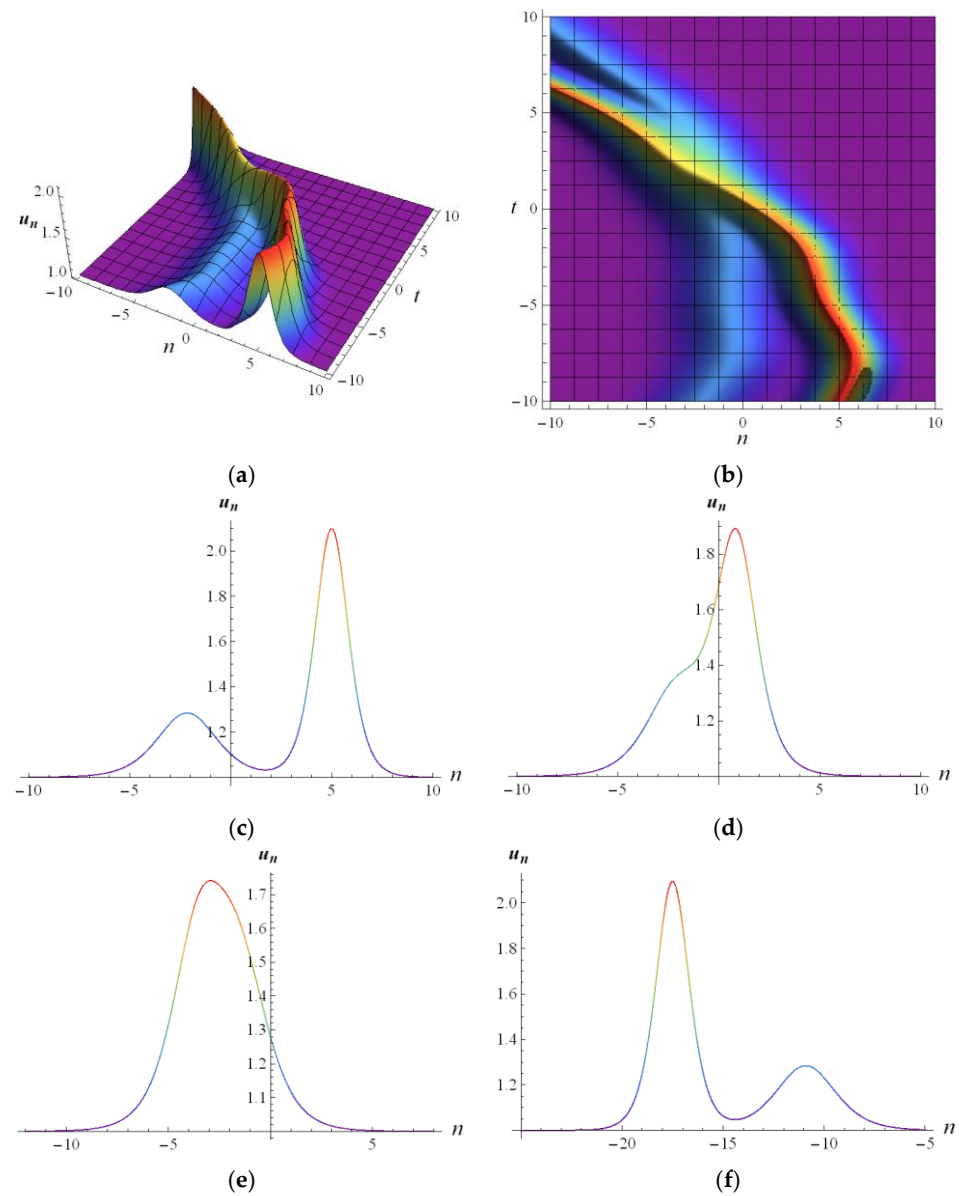
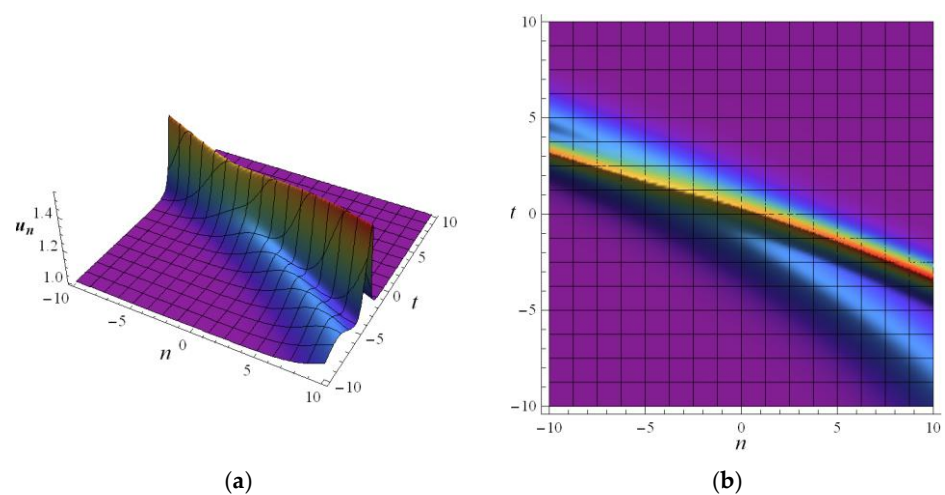


Figure 4. Fractional variable-velocity double-soliton solution (103) with $\gamma_0(t) = 0.1t$, $\gamma_1(t) = 0.1\cos t$, $\gamma_2(t) = 0.1$, $\eta_1(0) = 1$, $\eta_2(0) = 2$, $z_1 = 0.6$, $z_2 = 0.4$, and $\varepsilon = 0.1$: (a) 3D image; (b) top view; (c) $t = -10$; (d) $t = 0$; (e) $t = 2$; (f) $t = 10$.



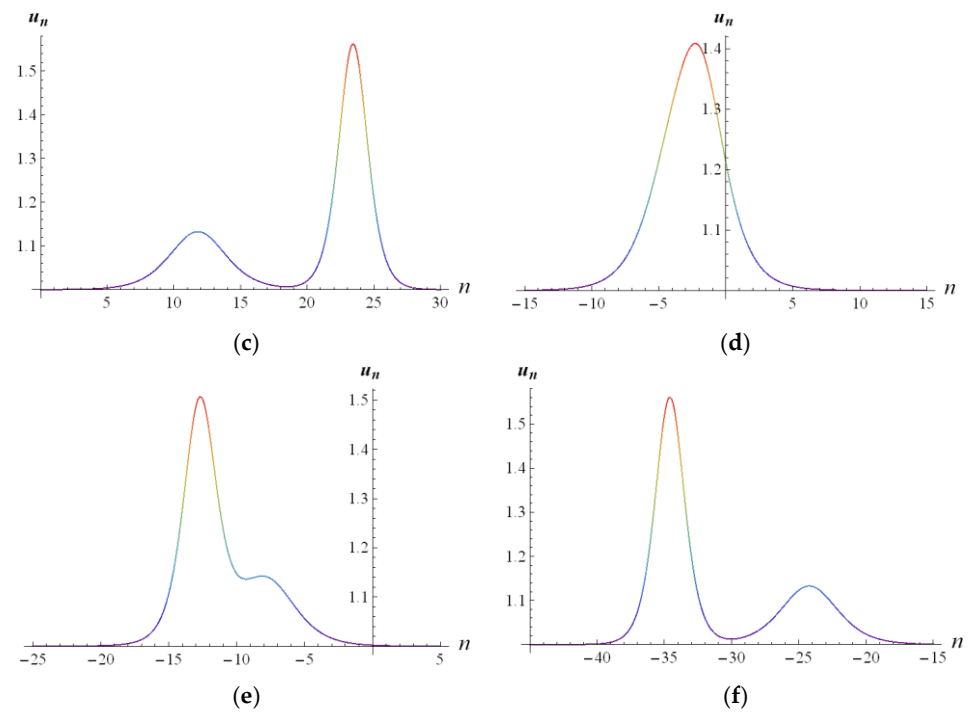
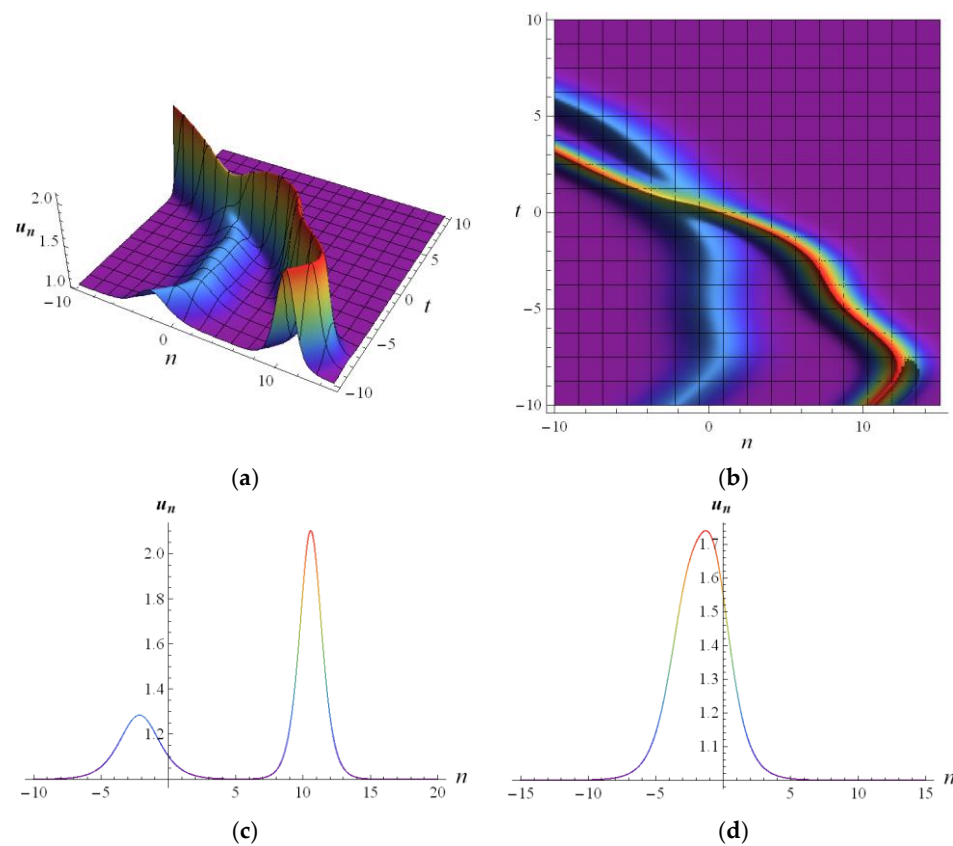


Figure 5. Fractional variable-velocity double-soliton solution (103) with $\gamma_0(t) = 0.1t$, $\gamma_1(t) = 0.2$, $\gamma_2(t) = 0.3$, $\eta_1(0) = 1$, $\eta_2(0) = 2$, $z_1 = 0.7$, $z_2 = 0.5$, and $\varepsilon = 0.1$: (a) 3D image; (b) top view; (c) $t = -10$; (d) $t = 1$; (e) $t = 4$; (f) $t = 10$.



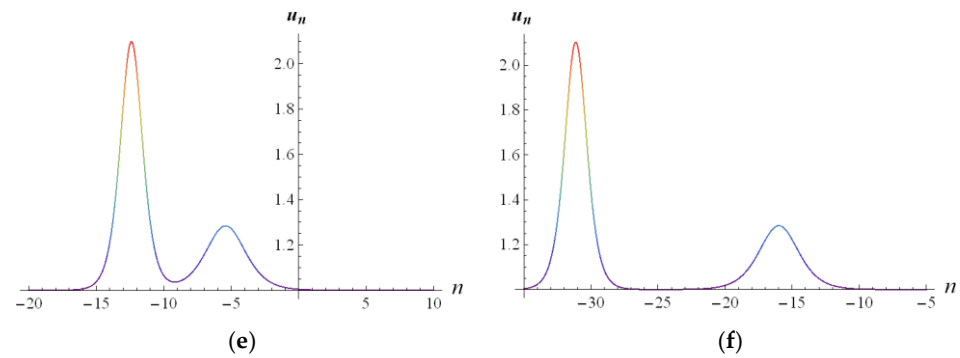


Figure 6. Fractional variable-velocity double-soliton solution (103) with $\gamma_0(t)=0.1t$, $\gamma_1(t)=0.1\cos t$, $\gamma_2(t)=0.1$, $\eta_1(0)=1$, $\eta_2(0)=2$, $z_1=0.6$, $z_2=0.4$, and $\varepsilon=0.9$: (a) 3D image; (b) top view; (c) $t=-10$; (d) $t=0.5$; (e) $t=4$; (f) $t=8$.

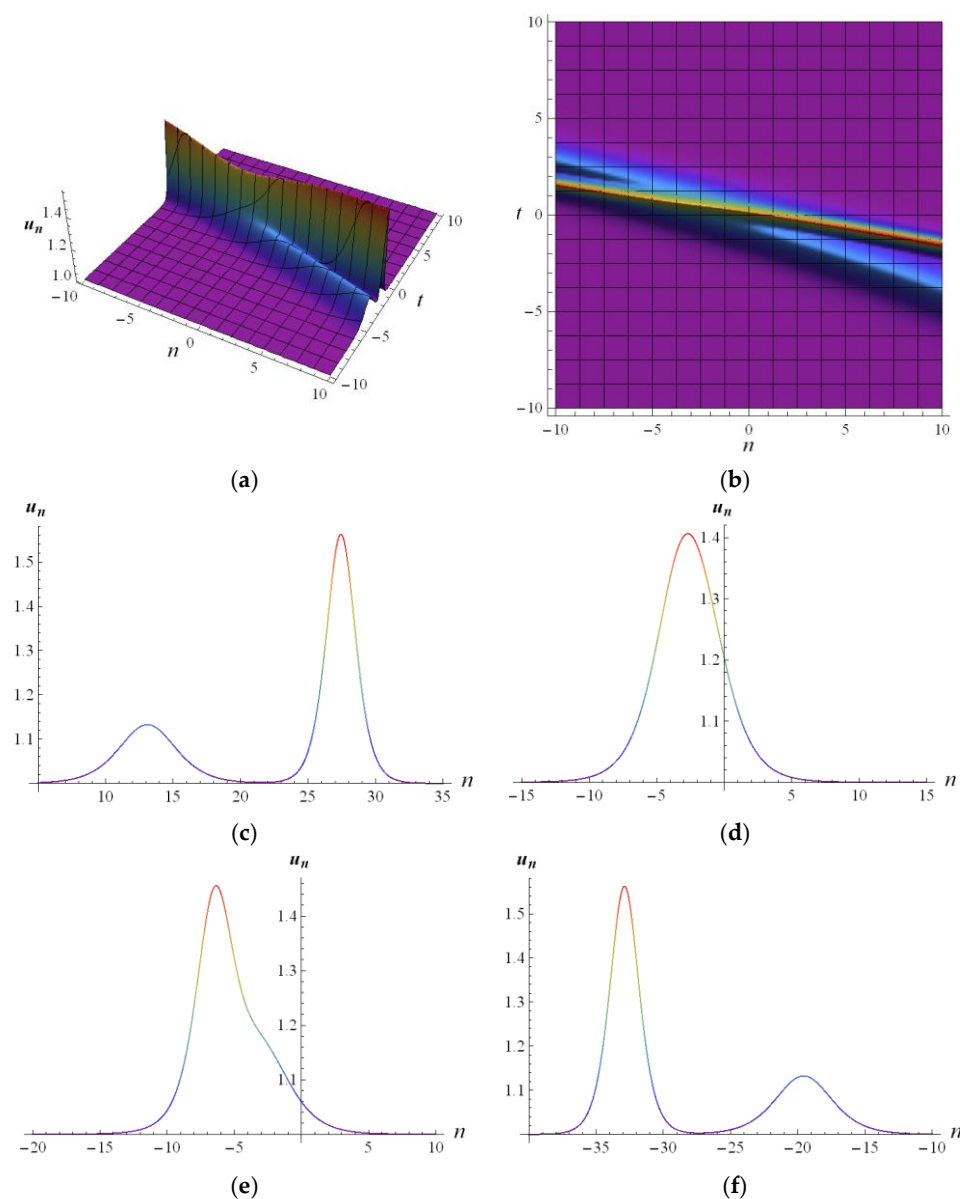


Figure 7. Fractional variable-velocity double-soliton solution (103) with $\gamma_0(t)=0.1t$, $\gamma_1(t)=0.2$, $\gamma_2(t)=0.3$, $\eta_1(0)=1$, $\eta_2(0)=2$, $z_1=0.7$, $z_2=0.5$, and $\varepsilon=0.9$: (a) 3D image; (b) top view; (c) $t=-5$; (d) $t=0.5$; (e) $t=1$; (f) $t=5$.

5.4. Fractional Variable-Velocity Triple-Soliton Solutions and Image Simulation

Similarly, when $N=3$, the fractional variable-velocity triple-soliton solutions of the sfvToda lattice Equation (16) can be obtained as follows:

$$u_n = \frac{\det[I + C_n(t)] \det[I + C_{n-2}(t)]}{\det[I + C_{n-1}(t)]^2} \Big|_{N=3}, \quad (107)$$

$$v_n = -\text{tr}\{[I + C_n(t)]^{-1} s_n(t) s_{n+1}^T(t)\} + \text{tr}\{[I + C_{n-1}(t)]^{-1} s_{n-1}(t) s_n^T(t)\} \Big|_{N=3}, \quad (108)$$

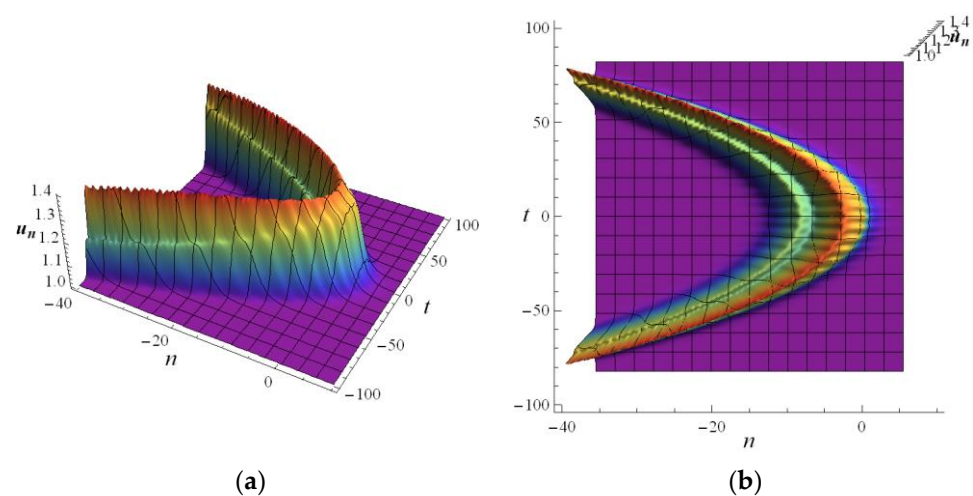
through

$$\det[I + C_n(t)] \Big|_{N=3} = \begin{vmatrix} 1 + \frac{\eta_1^2(0) z_1^{2n+2} e^{\sigma_1}}{1 - z_1^2} & \frac{\eta_1(0) \eta_2(0) (z_1 z_2)^{n+1} e^{(\sigma_1 + \sigma_2)/2}}{1 - z_1 z_2} & \frac{\eta_1(0) \eta_3(0) (z_1 z_3)^{n+1} e^{(\sigma_1 + \sigma_3)/2}}{1 - z_1 z_3} \\ \frac{\eta_2(0) \eta_1(0) (z_2 z_1)^{n+1} e^{(\sigma_1 + \sigma_2)/2}}{1 - z_2 z_1} & 1 + \frac{\eta_2^2(0) z_2^{2n+2} e^{\sigma_2}}{1 - z_2^2} & \frac{\eta_2(0) \eta_3(0) (z_2 z_3)^{n+1} e^{(\sigma_2 + \sigma_3)/2}}{1 - z_2 z_3} \\ \frac{\eta_3(0) \eta_1(0) (z_3 z_1)^{n+1} e^{(\sigma_1 + \sigma_3)/2}}{1 - z_3 z_1} & \frac{\eta_3(0) \eta_2(0) (z_3 z_2)^{n+1} e^{(\sigma_3 + \sigma_2)/2}}{1 - z_3 z_2} & 1 + \frac{\eta_3^2(0) z_3^{2n+2} e^{\sigma_3}}{1 - z_3^2} \end{vmatrix}, \quad (109)$$

and $s_n(t) \Big|_{N=3} = (\eta_1(t) z_1^n, \eta_2(t) z_2^n, \eta_3(t) z_3^n)^T$, where $\eta_1(t)$, $\eta_2(t)$, and $\eta_3(t)$ are constrained by Equation (90), σ_1 and σ_2 are in Equations (102) and (106), and σ_3 takes as

$$\sigma_3 = \int_0^t [\gamma_0(\tau) + \gamma_1(\tau) \lambda_3 + \gamma_2(\tau) \lambda_3^2] |\lambda_3|^{\epsilon} d\tau \left(z_3 - \frac{1}{z_3} \right), \quad \lambda_3 = z_3 + \frac{1}{z_3}. \quad (110)$$

In Figure 8, we show the fractional variable-velocity triple-soliton solution (107). It shows that during the propagation of the three initially rightward-propagating solitons, the middle high soliton gradually moves away from the left soliton and catches up with the right soliton. However, before all three solitons return to propagate leftward, the middle high soliton fails to surpass the right soliton. During the return process, the middle soliton gradually moves away from the right soliton and catches up with the left soliton again, with two collisions occurring throughout the entire process.



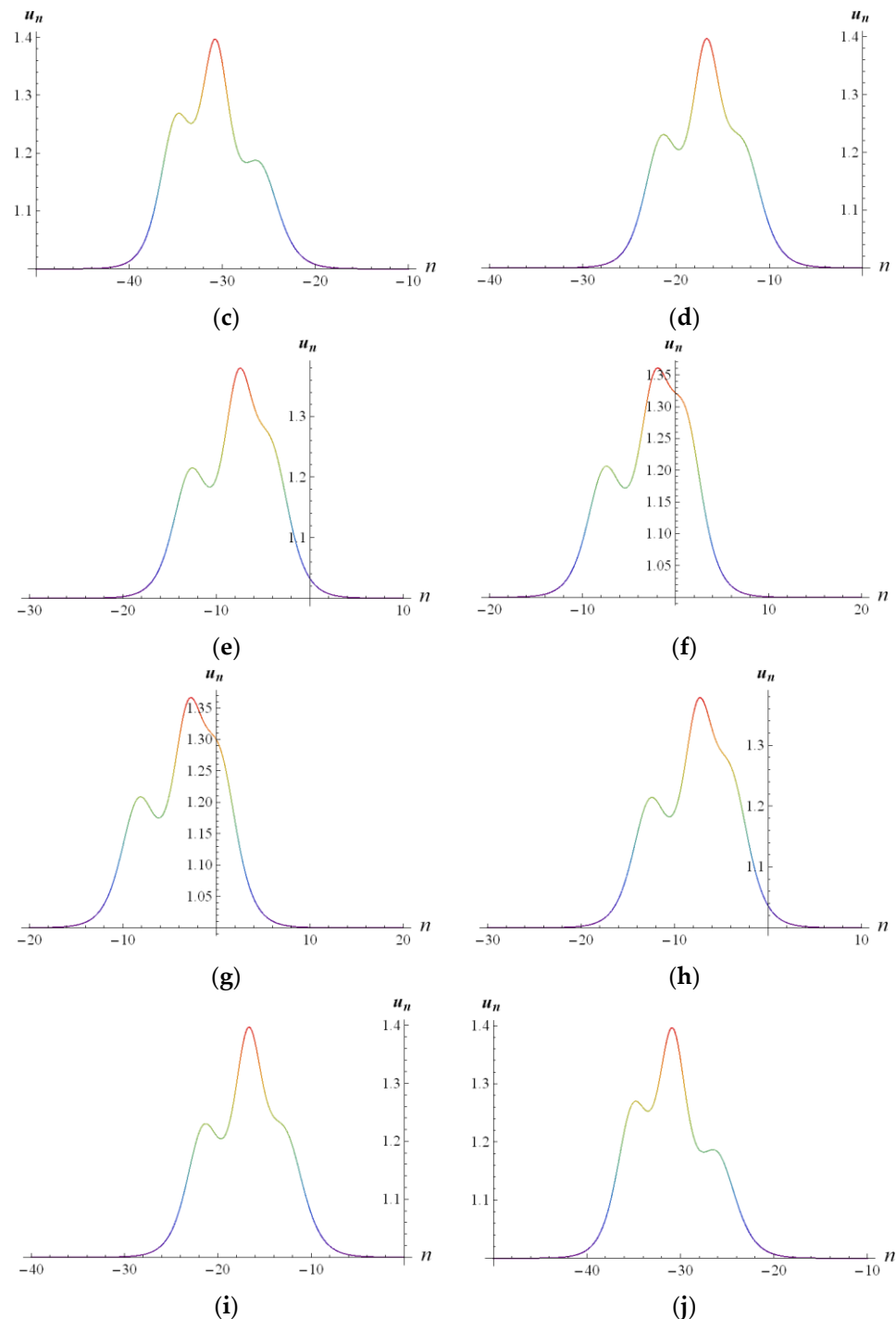


Figure 8. Fractional variable-velocity triple-soliton solution (107) with $\gamma_0(t) = 0.01t$, $\gamma_1(t) = 0.03 \cos t$, $\gamma_2(t) = 0.02 \sin t$, $\eta_1(0) = 1$, $\eta_2(0) = 1$, $\eta_3(0) = 1$, $z_1 = 0.7$, $z_2 = 0.6$, $z_3 = 0.5$, and $\varepsilon = 0.1$: (a) 3D image; (b) top view; (c) $t = -70$; (d) $t = -50$; (e) $t = -30$; (f) $t = 0$; (g) $t = 10$; (h) $t = 30$; (i) $t = 50$; (j) $t = 70$.

6. Conclusions

We have extended the methodology developed by Ablowitz, Been, and Fridberg [18] to the sfvToda lattice Equation (16), thereby establishing a systematic framework for constructing fractional variable-coefficient lattice integrable systems. A key contribution of this paper is the rigorous derivation of the completeness relation for squared eigenfunctions associated with the real tridiagonal SP (17) of the sfvToda lattice, which had

not been previously established. Based on this completeness, we show that a general lattice evolution equation in fractional operator form can be transformed into an explicit representation in physical space, leading to a well-defined sfvToda lattice and its associated hierarchy. The IST is extended to linearize the sfvToda lattice Equation (16), and explicit N -soliton solutions (97) and (98) are obtained. By setting all three coefficient functions $\gamma_0(t)$, $\gamma_1(t)$, and $\gamma_2(t)$ to 1 and the fraction ε to 0, we can verify that the resulting N -soliton solutions (97) and (98) degenerate to the results found in the literature [25]. For the fractional Toda lattice hierarchy (15), its explicit representation can be obtained by replacing $[\gamma_0(t) + \gamma_1(t)\lambda + \gamma_2(t)\lambda^2]|\lambda|^\varepsilon$ and $[\gamma_0(t) + \gamma_1(t)\lambda_j + \gamma_2(t)\lambda_j^2]|\lambda_j|^\varepsilon$ in Equations (57) and (58) with $\lambda^M|\lambda|^\varepsilon$ and $\lambda_j^M|\lambda_j|^\varepsilon$, respectively, and its N -soliton solutions can also be obtained by replacing $[\gamma_0(t) + \gamma_1(t)\lambda_j + \gamma_2(t)\lambda_j^2]|\lambda_j|^\varepsilon$ in Equations (97) and (98) with $\lambda_j^M|\lambda_j|^\varepsilon$, there $M=0,1,2,\dots$. These results demonstrate that the present framework provides a feasible extension of classical Toda lattice integrability to the fractional lattice systems, preserving the essential features of the IST with time-varying coefficient characteristics.

Author Contributions: Conceptualization, Y.M., S.Z. (Sheng Zhang), S.Z. (Shen Zhao) and B.X.; methodology, Y.M., S.Z. (Sheng Zhang), S.Z. (Shen Zhao) and B.X.; software, Y.M. and S.Z. (Sheng Zhang); validation, Y.M., S.Z. (Sheng Zhang), S.Z. (Shen Zhao) and B.X.; formal analysis, Y.M., S.Z. (Sheng Zhang), S.Z. (Shen Zhao) and B.X.; investigation, Y.M., S.Z. (Sheng Zhang), S.Z. (Shen Zhao) and B.X.; resources, S.Z. (Sheng Zhang) and B.X.; data curation, Y.M., S.Z. (Sheng Zhang) and B.X.; writing—original draft preparation, Y.M.; writing—review and editing, Y.M., S.Z. (Sheng Zhang), S.Z. (Shen Zhao) and B.X.; visualization, Y.M., S.Z. (Sheng Zhang), S.Z. (Shen Zhao) and B.X.; supervision, S.Z. (Sheng Zhang); project administration, S.Z. (Sheng Zhang), S.Z. (Shen Zhao) and B.X.; funding acquisition, S.Z. (Sheng Zhang) and B.X. All authors have read and agreed to the published version of the manuscript.

Funding: The work was funded by the Basic Scientific Research Project of Liaoning Provincial Department of Education of China (LJ212610167021, LJ212513217001, LJ212513217003) Natural Science Foundation of Liaoning Province of China (2025-MS-308) and Bohai University Graduate Innovation Fund Project (2025-13-B).

Data Availability Statement: The authors declare that all data supporting the findings of this study are available within the article.

Acknowledgments: We sincerely thank the Academic Editor and three anonymous reviewers for the valuable comments and helpful suggestions on this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

We provide here a rigorous functional-analytic framework for the fractional operator $(\mathbb{S}^A)^M |\mathbb{S}^A|^\varepsilon$ that appears throughout the paper. The construction relies on the completeness relation (Theorem 2) and the decay assumptions (38).

Appendix A.1. Function Space and Domain

Under the rapid decay conditions (38), we introduce the weighted two-dimensional $\mathcal{L}^1$ space:

$$X = \left\{ F = \{F_n\}_{n \in \mathbb{Z}} \mid F_n = (F_{n,1}, F_{n,2})^T \in \mathbb{R}^2, \|F\|_X = \sum_{n=-\infty}^{\infty} (1+|n|)(|F_{n,1}| + |F_{n,2}|) < \infty \right\}. \quad (111)$$

By Theorem 2, the completeness expansions (48) and (49) converge absolutely for every $F \in X$. Hence the operator $\mathbb{S}^A$ is densely defined on X , and we take its domain to be $D(\mathbb{S}^A) = X$. The decay conditions (38) ensure that $\mathbb{S}^A$ is a bounded linear operator on X .

Appendix A.2. Spectral Representation

The completeness theorem shows that the spectrum of $\mathbb{S}^A$ is:

$$\sigma(\mathbb{S}^A) = \sigma_c \cup \sigma_d, \quad (112)$$

where the continuous part is $\sigma_c = [-2, 2]$. On the unit circle $|z| = 1$, writing $z = e^{ik}$, $k \in [0, 2\pi)$, the spectral parameter is:

$$\lambda(z) = z + z^{-1} = 2 \cos k. \quad (113)$$

The discrete part is:

$$\sigma_d = \{\lambda_j = z_j + z_j^{-1}\}_{j=1}^N, \quad (114)$$

where z_j , $j = 1, 2, \dots, N$, are the simple zeros of the transmission coefficient $\alpha(z)$ inside $|z| < 1$ and $\tilde{z}_j$, $j = 1, 2, \dots, N$, are simple zeros of $\tilde{\alpha}(z)$ lying $|z| > 1$. For real potentials, the number N is finite.

To unify the notation in the completeness expansion, we set:

$$z_j^{(1)} = z_j, \quad z_j^{(2)} = \tilde{z}_j = \frac{1}{z_j^*}, \quad j = 1, 2, \dots, N, \quad (115)$$

so that the discrete eigenvalues are parameterized as:

$$\lambda_j^{(s)} = \lambda(z_j^{(s)}) = z_j^{(s)} + \frac{1}{z_j^{(s)}}, \quad s = 1, 2. \quad (116)$$

For real potentials, the discrete eigenvalues satisfy:

$$\lambda(z_j^{(1)}) = \lambda(z_j^{(2)}) = \lambda_j \in \mathbb{R}, \quad |\lambda_j| > 2. \quad (117)$$

Therefore, the discrete spectral part can be written equivalently as:

$$f(\lambda_j) \sum_{s=1}^2 h_j^{(s)}(n, m), \quad (118)$$

a fact that will simplify the subsequent expressions.

Appendix A.3. Branch Convention

For $\lambda \in \sigma(\mathbb{S}^A)$, we define

$$\lambda^M |\lambda|^\varepsilon = \begin{cases} \lambda^M e^{\varepsilon \ln|\lambda|}, & \lambda \neq 0, \\ 0, & \lambda = 0, \varepsilon > 0, \\ 1, & \lambda = 0, \varepsilon = 0, \\ \text{truncated principal value,} & \lambda = 0, \varepsilon < 0. \end{cases} \quad (119)$$

Here the real natural logarithm guarantees $|\lambda|^\varepsilon$ is the positive real branch. The integer power $|\lambda|^M$ requires no branch cut. When $\varepsilon < 0$, the point $\lambda = 0$ is an integrable singularity, and the operator is defined by the limit

$$\lim_{\delta \rightarrow 0^+} \int_{|z|=1, |\lambda(z)| > \delta} |\lambda(z)|^\varepsilon dz. \quad (120)$$

The limit integral converges absolutely for $|\varepsilon| < 1$; this integrability is a necessary condition, but boundedness of the associated fractional operator on X requires further weak spectral estimates developed in the Appendix A.4.

Appendix A.4. Definition of the Fractional Power

Using the completeness expansions (48) and (49) of Theorem 2, we define the operator $f(\aleph^A)$ for any function f defined on $\sigma(\aleph^A)$ by:

$$(f(\aleph^A)F)_n = \sum_{s=1}^2 \sum_{m=-\infty}^{\infty} \int_{|z|=1} dz f(\lambda(z)) H^{(s)}(n, m, z) F_m + \sum_{s=1}^2 \sum_{j=1}^N \sum_{m=-\infty}^{\infty} f(\lambda(z_j^{(s)})) h_j^{(s)}(n, m) F_m, \quad (121)$$

Taking $f(\lambda) = \lambda^M |\lambda|^\varepsilon$ gives the fractional operator $(\aleph^A)^M |\aleph^A|^\varepsilon$. This definition is consistent with the action on the squared eigenfunctions:

$$(\aleph^A)^M |\aleph^A|^\varepsilon P_n^A(z) = \lambda^M(z) |\lambda(z)|^\varepsilon P_n^A(z). \quad (122)$$

Appendix A.5. Existence and Well-Definedness

The following theorem guarantees that the operator $(\aleph^A)^M |\aleph^A|^\varepsilon$ is bounded on X .

Theorem A1 (Boundedness of fractional power). *Let the potentials satisfy the decay conditions in Equation (38). Then for every $M = 0, 1, 2, \dots$ and $|\varepsilon| < 1$, the operator $(\aleph^A)^M |\aleph^A|^\varepsilon$ defined by Equation (A11) is a bounded linear operator on X . Moreover, there exists a constant $C > 0$ such that:*

$$\left\| (\aleph^A)^M |\aleph^A|^\varepsilon F \right\|_X \leq C \left(\sum_{s=1}^2 \int_{|z|=1} |\lambda(z)|^{M+\varepsilon} dz + \sum_{s=1}^2 \sum_{j=1}^N |\lambda_j|^{M+\varepsilon} \right) \|F\|_X. \quad (123)$$

For real potentials, $\lambda(z_j^{(1)}) = \lambda(z_j^{(2)}) = \lambda_j$, $\sum_{s=1}^2 \sum_j |\lambda_j^{(s)}|^{M+\varepsilon} = 2 \sum_j |\lambda_j|^{M+\varepsilon}$ and the factor of 2 can be absorbed into constant C .

Before proving the boundedness, we recall that the continuous-spectrum kernel $H^{(s)}(n, m, z)$ does not satisfy a pointwise weighted uniform bound of the form

$$\sup_{z \in |z|=1} |H^{(s)}(n, m, z)| \leq C_H (1 + |m|) \quad (124)$$

uniformly in m , because this estimate fails for the free matrix Toda potential. Instead, the spectral representation is interpreted as a weak completeness relation on X , consistent with Theorem 2.

Proof. Let $F \in X$. From Equation (A11),

$$\left\| (\mathbb{S}^A)^M |\mathbb{S}^A|^\varepsilon F \right\|_X = \sum_n (1+|n|) \left| \sum_{s=1}^2 \int_{|z|=1} dz |\lambda(z)|^{M+\varepsilon} \sum_m H^{(s)}(n, m, z) F_m + \sum_{s=1}^2 \sum_{j=1}^N |\lambda_j|^{M+\varepsilon} \sum_m h_j^{(s)}(n, m) F_m \right| \quad (125)$$

By the triangle inequality,

$$\begin{aligned} \left\| (\mathbb{S}^A)^M |\mathbb{S}^A|^\varepsilon F \right\|_X &\leq \sum_{s=1}^2 \int_{|z|=1} dz |\lambda(z)|^{M+\varepsilon} \sum_{n,m} (1+|n|) |H^{(s)}(n, m, z) F_m| \\ &\quad + \sum_{s=1}^2 \sum_{j=1}^N |\lambda_j|^{M+\varepsilon} \sum_{n,m} (1+|n|) |h_j^{(s)}(n, m) F_m| \end{aligned} \quad (126)$$

For the discrete-spectrum part, the finite-rank residue kernels $h_j^{(s)}(n, m)$ still satisfy the weighted estimate:

$$\sum_n (1+|n|) |h_j^{(s)}(n, m)| \leq C_h (1+|m|) \quad (127)$$

because they are constructed from simple poles of the Jost matrix and are independent of the continuous-spectrum divergence. Therefore,

$$\sum_{n,m} (1+|n|) |h_j^{(s)}(n, m) F_m| \leq C_h \sum_m (1+|m|) |F_m| = C_h \|F_m\|_X \quad (128)$$

For the continuous-spectrum part, we no longer use a pointwise weighted kernel bound. Instead, we use the fact that for each fixed $z \in |z|=1$, the linear map

$$F \mapsto \sum_m H^{(s)}(\cdot, m, z) F_m \quad (129)$$

defines a bounded operator on X , uniformly in z . This operator boundedness is inherited from the weak completeness expansion established in the Theorem 2, together with the potential decay conditions (38). More precisely, for any $F \in X$,

$$\sum_{n,m} (1+|n|) |H_j^{(s)}(n, m, z) F_m| \leq C_H \|F_m\|_X \quad (130)$$

where C_H is independent of z and F . Substituting these bounds into Equation (A16) gives Equation (A13).

The integral (A13) converges because $|\varepsilon| < 1$, and the discrete sum is finite. The integral appearing in the spectral representation converges absolutely for $|\varepsilon| < 1$, since the singularity at $\lambda = 0$ is integrable, and the discrete sum is finite. This integrability is a necessary condition for the spectral integral to be well-defined. Operator boundedness on X follows from the weak-completeness framework and uniform operator bounds for the spectral representation derived above, not merely from integrability of the integrand. The case $\varepsilon < 0$ is handled by the truncated principal value, and the same estimate holds uniformly as the truncation parameter tends to zero. $\square$

This completes the rigorous definition of the fractional operator $(\mathbb{S}^A)^M |\mathbb{S}^A|^\varepsilon$ and establishes its bounded mapping property on the weighted space X . All the deriva-

tions in this paper, including the sfvToda lattice Equation (16) and its IST solution, are built upon this functional-analytic foundation.

Remark A1 (Mathematical and physical meaning of ε). Mathematically, ε is the fractional exponent in the spectral multiplier $f(\lambda) = \lambda^M |\lambda|^\varepsilon$, $\lambda \in \sigma(\mathbb{S}^A)$. The operator family $(\mathbb{S}^A)^M |\mathbb{S}^A|^\varepsilon$ continuously interpolates between the integer flows $(\mathbb{S}^A)^M$ and $(\mathbb{S}^A)^{M+1}$. In the spectral representation, ε modifies the dispersion relation according to $\Lambda(\lambda) = \lambda^M |\lambda|^\varepsilon$. The restriction $|\varepsilon| < 1$ guarantees that the singularity at $\lambda = 0$ in the continuous spectrum is integrable, which is a necessary condition for the well-definedness of the spectral integral. Boundedness of the resulting operator on the weighted space X requires the weak spectral estimates established above.

Physically, the fractional power $|\mathbb{S}^A|^\varepsilon$ introduces nonlocality and long-range spatial interactions into the lattice dynamics. Being a spatially defined fractional operator, it does not introduce time memory effects. For the sfvToda Equation (16), the fractional variable-velocity one-soliton solution (99) retains its shape while its speed becomes:

$$v = \frac{1 - z_1^2}{2z_1 \ln z_1} [\gamma_0(t) + \gamma_1(t)\lambda_1 + \gamma_2(t)\lambda_1^2] |\lambda_1|^\varepsilon, \quad \lambda_1 = z_1 + \frac{1}{z_1}. \quad (131)$$

For $z_1 > 0$, Equation (A21) indicates that ε tunes the propagation speed without altering the width or amplitude of the soliton. In the continuum limit, $|\mathbb{S}^A|^\varepsilon$ corresponds to a space-fractional operator, and the resulting equations describe anomalous spatial dispersion and nonlocal transport on the lattice. Thus, ε is the parameter that fills the gaps between the integer-order flows of the variable-coefficient Toda hierarchy, providing a continuous family of integrable equations with nonlocal spatial interactions within the same operator-based framework.

Appendix B

In this section we justify the compatibility condition for the sfvToda lattice Equation (16) and prove that the asymptotic information is sufficient to determine the time evolution of the scattering data.

We rewrite the spatial and temporal parts of the Lax pair in Equations (18) and (19) as:

$$\mathfrak{I} \begin{pmatrix} P_{1,n} \\ P_{2,n} \end{pmatrix} = L_n \begin{pmatrix} P_{1,n} \\ P_{2,n} \end{pmatrix}, \quad L_n = \begin{pmatrix} 0 & 1 \\ -u_n & \lambda - v_n \end{pmatrix}. \quad (132)$$

$$\frac{d}{dt} \begin{pmatrix} P_{1,n} \\ P_{2,n} \end{pmatrix} = M_n \begin{pmatrix} P_{1,n} \\ P_{2,n} \end{pmatrix}, \quad M_n = \begin{pmatrix} W_n & Z_n \\ N_n & Q_n \end{pmatrix}. \quad (133)$$

The compatibility condition $\mathfrak{I} d/dt = d/dt \mathfrak{I}$ yields the zero-curvature equation:

$$L_{n,t} = M_{n+1} L_n - L_n M_n. \quad (134)$$

Comparing entries, we obtain [25]:

$$\begin{cases} N_n = -u_n Z_{n+1}, \\ Q_n = W_{n+1} + (\lambda - v_n) Z_{n+1}, \\ u_{n,t} = u_n Q_{n+1} - u_n W_n + (\lambda - v_n) N_n, \\ v_{n,t} = -N_{n+1} - (\lambda - v_n) Q_{n+1} - u_n Z_n + (\lambda - v_n) Q_n. \end{cases} \quad (135)$$

We equivalently rewrite Equation (A25) as:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = \aleph_1^A \begin{pmatrix} Q_n \\ Z_n \end{pmatrix} - \lambda \aleph_2^A \begin{pmatrix} Q_n \\ Z_n \end{pmatrix} + \sum_{l=0}^2 \gamma_l(t) \lambda^{l+1} \aleph_2^A \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (136)$$

with $\aleph^A = \aleph_1^A (\aleph_2^A)^{-1}$;

$$\aleph_1^A = \begin{pmatrix} u_n(\aleph - \aleph^{-1}) & u_n(\aleph - 1)v_{n-1} \\ v_n(\aleph - 1) & \aleph u_n \aleph - u_n \end{pmatrix}, \quad \aleph_2^A = \begin{pmatrix} 0 & u_n(\aleph - 1) \\ \aleph - 1 & 0 \end{pmatrix}, \quad (\aleph_2^A)^{-1} = \begin{pmatrix} 0 & (\aleph - 1)^{-1} \\ (\aleph - 1)^{-1} \frac{1}{u_n} & 0 \end{pmatrix}. \quad (137)$$

Supposing Q_n and Z_n consist of the vector

$$\begin{pmatrix} Q_n \\ Z_n \end{pmatrix} = \sum_{j=0}^2 \begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix} \lambda^{2-j}, \quad \begin{pmatrix} d_{n,0} \\ b_{n,0} \end{pmatrix} = \gamma_2(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (138)$$

with the boundary conditions $Q_n = Z_n = \sum_{l=0}^2 \gamma_l(t) \lambda^l$ when $u_n = 1$ and $v_n = 0$. Substituting Equation (28) into Equation (26) and comparing the same powers of λ , we have

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = \aleph_1^A \begin{pmatrix} d_{n,2} \\ b_{n,2} \end{pmatrix}, \quad \begin{pmatrix} d_{n,2} \\ b_{n,2} \end{pmatrix} = (\aleph_2^A)^{-1} \aleph_1^A \begin{pmatrix} d_{n,1} \\ b_{n,1} \end{pmatrix} + \gamma_0(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} d_{n,1} \\ b_{n,1} \end{pmatrix} = (\aleph_2^A)^{-1} \aleph_1^A \begin{pmatrix} d_{n,0} \\ b_{n,0} \end{pmatrix} + \gamma_1(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (139)$$

which indicates that

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = [\gamma_0(t) + \gamma_1(t) \aleph^A + \gamma_2(t) (\aleph^A)^2] \aleph_1^A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = [\gamma_0(t) + \gamma_1(t) \aleph^A + \gamma_2(t) (\aleph^A)^2] \begin{pmatrix} u_n(v_n - v_{n-1}) \\ u_{n+1} - u_n \end{pmatrix}. \quad (140)$$

For the case of sfvToda lattice Equation (16), we take the corresponding operator $\Lambda(\aleph^A, t) = [\gamma_0(t) + \gamma_1(t) \aleph^A + \gamma_2(t) (\aleph^A)^2] |\aleph^A|^\epsilon$. In this case, the boundary conditions $Q_n = Z_n = \sum_{l=0}^2 \gamma_l(t) \lambda^l |\lambda|^\epsilon = \Lambda(\lambda, t)$ when $u_n = 1$ and $v_n = 0$, the first two equations of Equation (A25) give

$$N_n = -u_n Z_{n+1} \sim -\Lambda(\lambda, t), \quad W_n = Q_{n-1} - (\lambda - v_{n-1}) Z_n \sim \Lambda(\lambda, t)(1 - \lambda), \quad (141)$$

that is to say

$$M_n = \begin{pmatrix} W_n & Z_n \\ N_n & Q_n \end{pmatrix} \sim \Lambda(\lambda, t) \begin{pmatrix} 1 - \lambda & 1 \\ -1 & 1 \end{pmatrix}. \quad (142)$$

At the same time, the remaining equations in Equation (A25) establish the sfvToda lattice Equation (16). This confirms that the asymptotic information is sufficient to determine the exact time evolution of the scattering data (78)–(80) and (90).

Appendix C

In this section, we establish the telescoping identity. For this purpose, we let

$$y_n(k, k') = [\lambda(k') - \lambda(k)] (P_n^A)^T(k) \cdot P_n(k'), \quad (143)$$

where

$$P_n(k') = \begin{pmatrix} p_n^2(k') \\ u_n p_{n-1}(k') p_n(k') \end{pmatrix} = \begin{pmatrix} P_{n,1}(k') \\ P_{n,2}(k') \end{pmatrix}, \quad P_n^A(k) = \begin{pmatrix} q_n(k) q_{n+1}(k) - u_n q_{n-1}(k) q_n(k) \\ q_{n+1}^2(k) - q_n^2(k) \end{pmatrix}. \quad (144)$$

Introduce

$$R_n(k') = (\mathfrak{I} - 1)^{-1} P_{n,1}^A(k') \quad (145)$$

which satisfies

$$R_{n+1}(k') - R_n(k') = P_{n,1}^A(k'), \quad (\mathfrak{I} - 1)R_n(k') = P_{n,1}^A(k'). \quad (146)$$

Lemma A1 (Telescoping sum). Let $P_n = P_n(k')$ and $q_n = q_n(k)$ be two solutions of the Toda lattice SP (17) for two spectral parameters $z' = e^{ik'}$ and $z = e^{ik}$. Assume the potentials u_n and v_n satisfy the weighted rapid decay conditions (38) such that the series

$$Y_n(k, k') = - \sum_{j=n}^{\infty} y_j(k, k') \quad (147)$$

converges for each fixed lattice index n . Then for any finite integers $-N < N$, there exists the telescoping sum:

$$\sum_{n=-N}^N y_n(k, k') = Y_{N+1}(k, k') - Y_{-N}(k, k') \quad (148)$$

Pointwise for every integer n , $y_n(k, k') = Y_{n+1}(k, k') - Y_n(k, k')$.

Proof. By the relationships (37), we can easily compute the summand

$$y_n(k, k') = [\lambda(k') - \lambda(k)](P_n^A)^T(k) \cdot P_n(k') = [(P_n^A)^T(k) \cdot \mathfrak{S} P_n(k') - P_n^T(k') \cdot \mathfrak{S}^A P_n^A(k)] \quad (149)$$

Using the definitions of $\mathfrak{S}$ and $\mathfrak{S}^A$ in Equations (9) and (12) yields:

$$\begin{aligned} y_n(k, k') = & \frac{P_{n,1}^A(k')}{u_n} \sum_{m=n+1}^{\infty} \{v_{m-1}[u_{m-1}P_{m,1}(k') - u_m P_{m,1}(k')] + u_{m-1}P_{m-1,2}(k') - u_m P_{m,2}(k')\} \\ & + P_{n,2}(k)[u_n P_{n,1}(k') + u_{n+1}P_{n+1,1}(k') + v_n P_{n,2}(k)] - P_{n,1}(k')\{v_n P_{n,1}(k') + u_n[P_{n,2}(k) + P_{n-1,2}(k)]\} \\ & - P_{n,2}(k) \left[\frac{u_{n+2}R_{n+2}(k') - u_n R_n(k')}{u_n} + v_n P_{n,2}(k) \right] \end{aligned} \quad (150)$$

From Equation (A37), we have

$$Y_{n+1}(k, k') - Y_n(k, k') = - \sum_{j=n+1}^{\infty} y_j(k, k') + \sum_{j=n}^{\infty} y_j(k, k') = y_{n+1}(k, k') \quad (151)$$

Hence $y_n(k, k') = Y_{n+1}(k, k') - Y_n(k, k')$. Summing it from $n = -N$ to N , we obtain:

$$\sum_{n=-N}^N y_n(k, k') = \sum_{n=-N}^N [Y_{n+1}(k, k') - Y_n(k, k')] = Y_{N+1}(k, k') - Y_{-N}(k, k') \quad (152)$$

This completes the proof. $\square$

References

1. Fermi, E.; Pasta, J.; Ulam, S. Studies of nonlinear problems. I. Technical Report LA-1940, Los Alamos Scientific Laboratory, 1955. In *Collected Papers of Enrico Fermi*; Segré, E., Eds.; University of Chicago Press: Chicago, IL, USA 1965; Volume II, pp. 978–988.
2. Toda, M. Vibration of a chain with nonlinear interaction. *J. Phys. Soc. Jpn.* **1967**, *22*, 431–436.
3. Toda, M. Wave propagation in an harmonic lattices. *J. Phys. Soc. Jpn.* **1967**, *23*, 501–506.

4. Kodama, Y.; Shipman, B.A. Fifty years of the finite nonperiodic Toda lattice: A geometric and topological viewpoint. *J. Phys. A Math. Theor.* **2018**, *51*, 353001.
5. Ford, J. The Fermi-Pasta-Ulam problem: Paradox turns discovery. *Phys. Rep.* **1992**, *213*, 271–310.
6. Zabusky, N.J.; Kruskal, M.D. Interactions of solitons in a collisionless plasma and therecurrence of initial states. *Phys. Rev. Lett.* **1965**, *15*, 240–243.
7. Flaschka, H. The Toda lattice. I. Existence of integrals. *Phys. Rev. B* **1974**, *9*, 1924–1925.
8. Flaschka, H. On the Toda lattice. II. Inverse-scattering solution. *Prog. Theor. Phys.* **1974**, *51*, 703–716.
9. Manakov, S.V. Complete integrability and stochastization of discrete dynamical systems. *Sov. Phys. ZETP* **1975**, *40*, 269–274.
10. Lax, P.D. Integrals of nonlinear equations of evolution and solitary waves. *Commun. Pure Appl. Math.* **1968**, *21*, 467–490.
11. Gardner, C.S.; Greene, J.M.; Kruskal, M.D.; Miura, R.M. Method for solving the Korteweg-de Vries equation. *Phys. Rev. Lett.* **1967**, *19*, 1095–1097.
12. Ablowitz, M.J.; Clarkson, P.A. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*; Cambridge University Press: New York, NY, USA, 1991.
13. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. The inverse scattering transform—Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **1974**, *53*, 249–315.
14. Flashka, H.; McLaughlin, D. Some comments on Bäcklund transformations, canonical transformations, and the inverse scattering method. In *Bäcklund Transformations, the Inverse Scattering Method, Solitons and Their Applications*; Lecture Notes in Mathematics; Miura, R.M., Eds.; Springer-Verlag: Berlin/Heidelberg, Germany; New York, NY, USA, 1976; Volume 515, pp. 253–295.
15. Zhang, S.; Zhao, S.; Xu, B. Analytical insights into a generalized semidiscrete system with time-varying coefficients: Derivation, exact solutions, and nonlinear soliton dynamics. *Complexity* **2020**, *2020*, 1543503.
16. Zhang, S.; Wang, D. Variable-coefficient nonisospectral Toda lattice hierarchy and its exact solutions. *Pramana-J. Phys.* **2015**, *85*, 1143–1156.
17. Zhang, J.B.; Zhang, D.J.; Chen, D.Y. Exact solutions to a mixed Toda lattice hierarchy through the inverse scattering transform. *J. Phys. A Math. Theor.* **2011**, *44*, 115201.
18. Ablowitz, M.J.; Been, J.B.; Fridberg, K. Fractional integrable Toda lattice and hierarchy. *Proc. R. Soc. A Math. Phys.* **2025**, *481*, 20240812.
19. Zakharov, V.E.; Shabat, A.B. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP* **1972**, *34*, 62–69.
20. Ablowitz, M.J.; Ladik, J.F. Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.* **1976**, *17*, 1011–1018.
21. Ablowitz, M.J.; Musslimani, Z.H. Integrable nonlocal nonlinear Schrödinger equation. *Phys. Rev. Lett.* **2013**, *110*, 064105.
22. Ablowitz, M.J.; Been, J.B.; Carr, L.D. Fractional integrable nonlinear soliton equations. *Phys. Rev. Lett.* **2022**, *128*, 184101.
23. Ablowitz, M.J.; Been, J.B.; Carr, L.D. Integrable fractional modified Korteweg-de Vries, sine-Gordon, and sinh-Gordon equations. *J. Phys. A Math. Theor.* **2022**, *55*, 384010.
24. Ablowitz, M.J.; Been, J.B.; Carr, L.D. Fractional integrable and related discrete nonlinear Schrödinger equations. *Phys. Lett. A* **2022**, *452*, 128459.
25. Chen, D.Y. *Introduction to Solitons*; Science Press: Beijing, China, 2006. (In Chinese)
26. Bekir, A.; Güner, Ö.; Ayhan, B. Exact solutions of some systems of fractional differential-difference equations. *Math. Methods Appl. Sci.* **2015**, *38*, 3807–3817.
27. Bakkyaraj, T.; Thomas, R. Lie symmetry analysis and exact solution of (2+1)-dimensional nonlinear time-fractional differential-difference equations. *Pramana-J. Phys.* **2022**, *96*, 225.
28. Sahadevan, R.; Bakkyaraj, T. Invariant subspace method and exact solutions of certain nonlinear time fractional partial differential equations. *Fract. Calc. Appl. Anal.* **2015**, *18*, 146–162.
29. Meng, F.W. A new variable-coefficient Riccati subequation method for solving nonlinear lattice equations. *Abstr. Appl. Anal.* **2013**, *2013*, 666–686.
30. Zhang, S.; Wang, D. A Toda lattice hierarchy with variable coefficients and its multi-wave solutions. *Therm. Sci.* **2014**, *18*, 1563–1566.
31. Zhang, S.; Liu, D. Multisoliton solutions of a (2+1)-dimensional variable-coefficient Toda lattice equation via Hirota's bilinear method. *Can. J. Phys.* **2014**, *92*, 184–190.
32. Liu, Y.H.; Guo, R.; Zhang, J.W. Riemann-Hilbert approach for the inhomogeneous discrete nonlinear Schrödinger equation with nonzero boundary conditions. *Wave Motion* **2024**, *128*, 103322.

33. Zhao, S.; Zhang, S.; Xu, B. Fractional and integrable perspectives on a variable coefficient semi-discrete NLS equation: Analytical method and engineering applications. *Fractal Fract.* **2025**, *9*, 790.
34. Maheswari, C.U.; Sahadevan, R.; Yogeshwaran, M. Method of separation of variables and exact solution of time fractional nonlinear partial differential and differential-difference equations. *Fract. Calc. Appl. Anal.* **2023**, *26*, 2421–2438.
35. Feng, Q.H. Exact solutions for fractional differential-difference equations by an extended Riccati sub-ODE method. *Commun. Theor. Phys.* **2013**, *59*, 521–527.
36. Kaup, D.J. Closure of the squared Zakharov-Shabat eigenstates. *J. Math. Anal. Appl.* **1976**, *54*, 849–864.
37. Sachs, R.L. Completeness of derivatives of squared Schrödinger eigenfunctions and explicit solutions of the linearized KdV equation. *SIAM J. Math. Anal.* **1983**, *14*, 674–683.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.