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Pseudo-Differential and Fractional Schrödinger Operators on Grand Variable Herz–Morrey–Hardy Spaces with Muckenhoupt Weights

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Abstract

Fractional differential operators equipped with classical and new memory kernels have emerged as indispensable tools for modeling systems with hereditary and non-local effects, offering a flexible framework that extends far beyond the classical integer-order calculus. Within this broader landscape, pseudo-differential and Schrödinger operators play a very important role in mathematical analysis, arising naturally in the study of partial differential equations, quantum mechanics, and harmonic analysis, where they provide the fundamental tools for analyzing regularity, boundedness, and decay properties of solutions. The objective of this article is to define a new Herz norm with a variable exponent, a Morrey-type truncation, a grand modification, and an $\mathcal{A}_{\mathrm{p}(\cdot)}$ Muckenhoupt weight, and to prove two boundedness theorems. First, we show that every order-zero pseudo-differential operator is bounded on the resulting weighted grand variable Herz–Morrey–Hardy space $HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(w)$. Second, for the Schrödinger operator $\mathcal{L} = -\Delta + \mathcal{V}$ with $\mathcal{V}$ in the reverse Hölder class RH_{q}, $\mathrm{q} \geq n/2$, we replace the classical Riesz kernel with the Bessel–Riesz kernel and establish new, improved strong-type bounds for the associated Bessel–Riesz operator, from a weighted variable Lebesgue space into a weighted grand variable Herz–Morrey–Hardy–Lipschitz space built from the critical-radius geometry of $\mathcal{L}$ and using the same origin-centered dyadic scale as in the pseudo-differential case. Both proofs proceed via decompositions into uniform per-atom (respectively, per-ball) estimates, which are aggregated in the grand–Morrey norm under a decay condition on the Morrey parameter on the Herz–Hardy scale and under the threshold condition $\Gamma \geq 1/v$ on the Lipschitz-type range, together with a complementary condition on the Herz-type domain. We also show how the Bessel parameter γ enters the admissible range of integrability exponents. To further demonstrate the accuracy and consistency of these new results, we include remarks showing that, under different settings, various relevant results in the literature are recovered and improved, together with additional norm-structure and operator improvements.

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1. Introduction

Pseudo-differential operators supply the symbolic calculus through which variable-coefficient differential operators, their parametrices, and their fundamental solutions are constructed and analyzed, and they underlie the modern treatment of quantization on Lie groups and manifolds, together with a wide range of boundary-value and evolution problems [1,2]. The Schrödinger operator $\mathcal{L} = -\Delta + \mathcal{V}$ is, in turn, the basic generator of quantum mechanical evolution and spectral theory. Its harmonic-analytic properties, including heat-kernel bounds, Hardy and BMO theory, and estimates for negative powers, feed directly into contemporary spectral and computational studies of many-body and lattice quantum systems [3–5]. The boundedness theory developed below places both operator classes on a common, highly refined scale of function spaces.

Function spaces with variable integrability exponent, $L^{\mathrm{p}(\cdot)}(\Omega)$, were put on a rigorous footing by Kováčik and Rákosník [6]. The log-Hölder continuity conditions that make harmonic analysis tractable on these spaces, and the correct variable-exponent Muckenhoupt class $\mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ under which the maximal operator remains bounded on the weighted space $L^{\mathrm{p}(\cdot)}(w)$, are due to Diening et al. [7] and to Cruz-Uribe, Fiorenza and Neugebauer [8]; see also the monograph [9]. Herz-type spaces with variable exponent, as well as their atomic Hardy-space counterparts $H\dot{\mathcal{K}}^{\alpha,\mathrm{p}}_{\mathrm{q}(\cdot)}(\mathbb{R}^n)$, were developed by Izuki [10] and by Wang and Liu [11,12].

On this scale, the order-zero pseudo-differential operator

$$T\psi(\zeta) = \int_{\mathbb{R}^n} \hat{\psi}(\xi)\, \sigma(\zeta,\xi)\, e^{2\pi i \zeta\cdot\xi}\, d\xi, \qquad |\mathcal{D}^{\gamma}_{\zeta}\mathcal{D}^{\beta}_{\xi}\,\sigma(\zeta,\xi)| \le C_{\gamma,\beta}\,(1+|\xi|)^{-|\beta|},$$

is the smooth, order-zero member of a much larger family of operators governed by the Hörmander-type class $S^{m}_{\rho,\delta}(l,l')$. Building on the finite-regularity L^{p}-boundedness techniques of Nagase [13] and Rodino [14], Tsutsui [15] established the mapping properties of this class on the Herz scale itself.

Theorem 1 (Tsutsui [15])**.** *Let* $1 < \mathrm{p} < \infty$, $0 < \mathrm{q} < \infty$, $n(1-1/\mathrm{p}) \le \alpha < \infty$ *and* $m < -\alpha - n|1/\mathrm{p} - 1/2|$. *Suppose that either*

$$(i)\quad 1 < \mathrm{p} \le 2, \quad l > n/2, \quad l' > [-m] + n/2 + 1,$$

or

$$(ii)\quad 2 \le \mathrm{p} < \infty, \quad l > n/\mathrm{p}, \quad l' > [-m - n(1/2 - 1/\mathrm{p})] + n/2 + 1.$$

Then, $S^{m}_{0,0}(l,l') \subset \mathcal{L}\big(H\dot{\mathcal{K}}^{\alpha,\mathrm{p}}_{\mathrm{q}}(\mathbb{R}^n), \dot{\mathcal{K}}^{\alpha,\mathrm{p}}_{\mathrm{q}}(\mathbb{R}^n)\big)$.

Theorem 1 shows that mapping properties on the Herz scale persist even for symbols of finite, rather than infinite, smoothness. The underlying finite-regularity symbol calculus has been extended in several directions over the following decades: to L^2-boundedness under weaker hypotheses by Higuchi and Nagase [16]; to L^{p}-boundedness via Besov-space symbol estimates by Sugimoto [17]; to multilinear and nuclear operators on $\mathbb{Z}^n$ and on the torus $\mathbb{T}^n$ by Cardona and Kumar [18]; to multi-parameter local Hardy spaces by Chen, Ding and Lu [19]; to operators defined via the coupled fractional Fourier transform by Das, Mahato and Das [20]; to compact Lie groups, including subelliptic Sobolev and Besov

spaces, by Ruzhansky and collaborators [21–23]; to non-toroidal symbol classes and some boundedness result by Bekbolat and Tokmagambetov [24].

The theory of pseudo-differential operators continues to develop rapidly. Cardona and Martínez [25] revisited the sharp boundedness estimates for pseudo-differential operators on the torus. Das and Mahato [26] studied pseudo-differential operators associated with the coupled fractional Fourier transform, together with an application. Kumar and Bhawna [27] introduced an n-dimensional pseudo-differential operator involving the quadratic phase Fourier transform, with applications in quantum mechanics. Sartaj, Balvant and Upadhyay [28] investigated pseudo-differential operators involving the Fourier–Bessel transform on $\mathbb{Z}_+^{n+1}$. Verma and Prasad [29] developed SG pseudo-differential operators in the Dunkl setting. Mahato and Pasawan [30] obtained norm inequalities for pseudo-differential operators associated with the fractional Hankel transform on Beurling-like spaces. Thorn [31] studied magnetic pseudo-differential operators whose Hörmander symbols are dominated by tempered weights. On the applied side, Torresblanca-Badillo [32] used p-adic differential equations linked to fractional pseudo-differential operators to describe the dynamics of complex networks. Faou and Grébert [33] introduced discrete pseudo-differential operators and applied them to numerical schemes. Sayana and Reddy [34] proposed a modified fractional Tikhonov regularization to solve pseudo-differential operator equations. Yuldashev, Ramazanova and Shermamatov [35] studied an optimal control problem for a linear pseudoparabolic equation with degeneration and the Gerasimov–Caputo operator.

On the Herz–Hardy scale specifically, the smooth order-zero operator T above was shown to be bounded by Fan and Yang [36] in the constant-exponent case, following the local Hardy-space kernel estimates of Goldberg [37], and on $H\dot{\mathcal{K}}^{\alpha,\mathrm{p}}_{\mathrm{q}(\cdot)}(\mathbb{R}^n)$ by Wang and Wu [38].

In parallel, we consider the Schrödinger operator

$$\mathcal{L} = -\Delta + \mathcal{V}(\zeta), \qquad \zeta \in \mathbb{R}^n,\ n \geq 3,$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial\zeta_i^2$ and $\mathcal{V} \geq 0$ belongs to the reverse Hölder class RH_{q} for some $\mathrm{q} \geq n/2$. Recall that a non-negative function $\psi \in L^{\mathrm{q}}_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to RH_{q}, $1 < \mathrm{q} \leq \infty$, if there is $C > 0$ such that

$$\Big(\frac{1}{|\mathcal{B}(\zeta,\nu)|}\int_{\mathcal{B}(\zeta,\nu)} |\psi(\iota)|^{\mathrm{q}}\, d\iota\Big)^{1/\mathrm{q}} \leq \frac{C}{|\mathcal{B}(\zeta,\nu)|}\int_{\mathcal{B}(\zeta,\nu)} |\psi(\iota)|\, d\iota$$

for every ball $\mathcal{B}(\zeta,\nu)$. Every non-negative polynomial belongs to RH_∞; in particular, $|\zeta|^2 \in RH_\infty$. Shen [39] introduced the critical-radius function

$$\rho(\zeta) = \sup_{\nu>0}\Big\{\nu : \frac{1}{\nu^{n-2}}\int_{\mathcal{B}(\zeta,\nu)} \mathcal{V}(\iota)\, d\iota \leq 1\Big\}, \qquad \zeta \in \mathbb{R}^n,$$

which is also written as $\rho(\zeta) = 1/m_{\mathcal{V}}(\zeta)$. We have $0 < m_{\mathcal{V}}(\zeta) < \infty$ whenever $\mathcal{V} \not\equiv 0$, and $m_{\mathcal{V}}$ is constant when $\mathcal{V}$ is a positive constant. For the harmonic oscillator (Hermite operator) $H = -\Delta + |\zeta|^2$, we have $m_{\mathcal{V}}(\zeta) \sim 1 + |\zeta|$. The negative powers $\mathcal{L}^{-\alpha/2}$, $\alpha > 0$, are not defined directly; instead, they are built from the heat semigroup $e^{-\tau\mathcal{L}}$ generated by $\mathcal{L}$ via the subordination formula

$$I_\alpha\psi(\zeta) = \mathcal{L}^{-\alpha/2}\psi(\zeta) = \int_0^\infty e^{-\tau\mathcal{L}}\psi(\zeta)\,\tau^{\alpha/2-1}\, d\tau, \qquad 0 < \alpha < n.$$

For sufficiently regular ψ, we have the kernel representation $I_\alpha \psi(\zeta) = \int_{\mathbb{R}^n} K_\alpha(\zeta, \iota)\psi(\iota)\, d\iota$, where

$$K_\alpha(\zeta, \iota) \le \frac{C_\kappa}{\left(1 + |\zeta - \iota|(m_{\mathcal{V}}(\zeta) + m_{\mathcal{V}}(\iota))\right)^\kappa} \cdot \frac{1}{|\zeta - \iota|^{n-\alpha}},$$
$$K_\alpha(\zeta, \iota) \le \frac{C}{|\zeta - \iota|^{n-\alpha}},$$

for $0 < \alpha < n$. The first bound follows from the semigroup kernel estimates of Tang [40] (see also Kurata [41] and Dziubański–Zienkiewicz [42,43]).

The harmonic analysis generated by ρ and I_α has since developed extensively. Bongioanni, Harboure and Salinas [44] obtained weighted inequalities for the negative powers of $\mathcal{L}$. Harboure, Salinas and Viviani [45] established boundedness on weighted Lebesgue and Lipschitz spaces. Wu and Zhou [46] proved two-weight norm inequalities for related fractional-type operators on weighted Morrey spaces. Zhang and Tang [47] studied variation operators on weighted Hardy and BMO spaces in the Schrödinger setting. Yang and Li [48] treated commutators associated with $\mathcal{L}$ on the Heisenberg group. Dziubański and Głowacki [49] studied Sobolev spaces related to Schrödinger operators with polynomial potentials. On variable-exponent spaces, Ramseyer, Salinas and Viviani [50] studied the Lipschitz-type smoothness of the fractional integral and, adapting their approach to $\mathcal{L}$, Wang and Li [51] proved the boundedness of $\mathcal{L}^{-\alpha/2} : L^{\mathrm{p}(\cdot)}(\mathbb{R}^n) \to \mathrm{Lip}_{\mathcal{L}}^{\alpha,\mathrm{p}(\cdot)}(\mathbb{R}^n)$. On the Herz and Herz–Morrey scales, boundedness results for operators associated with $\mathcal{L}$ have been obtained by Li and Wan [52] (commutators on Herz spaces), Wang and Zhao [53] (variation operators for high-order Schrödinger-type operators), Gao and Li [54] (commutators on the Heisenberg group), Akbulut and Kuzu [55] (Marcinkiewicz integrals on generalized Morrey spaces), Akbulut, Celik and Omarova [56] and Guliyev, Akbulut and Celik [57] (fractional operators on vanishing generalized Morrey spaces), and Lin, Yang, Yang and Yuan [58] (Hardy spaces for non-negative self-adjoint operators on doubling metric measure spaces). The p-adic analog of the variable Morrey–Herz scale has recently been treated by Thuy, Dung and Nang [59].

A third ingredient completes the scale used below: Morrey's truncated averaging condition [60] and the grand Lebesgue space $L^{\mathrm{p})}(\Omega)$ of Iwaniec and Sbordone [61], an ε-enlargement of $L^{\mathrm{p}}(\Omega)$ controlled by a supremum over perturbed exponents. Combining a Morrey truncation Γ with a grand exponent $v(1+\vartheta)$, $\vartheta > 0$ yields a two-parameter refinement of the Herz norm that still admits an atomic characterization. The resulting (unweighted) grand variable Herz–Morrey–Hardy space was introduced, together with its atomic decomposition, by Sultan, Hussain and Sultan [62]. Closely related grand variable Herz-type scales have been used by Sultan, Hussain, Lassoued and Sultan [63] to study higher-order commutators of fractional Hardy-type operators, and Ho [64] introduced grand Morrey spaces and grand Hardy–Morrey spaces on Euclidean space. Closest in spirit to Section 4 below, Afzal et al. [65] used fractional Bessel–Riesz operators to study the regularity of parabolic Ginzburg–Landau equations in variable-exponent Herz spaces. Wang and Tao [66] proved the boundedness of Littlewood–Paley operators and their commutators on Herz–Morrey spaces with variable exponent. Afzal [67] studied fractional Muckenhoupt-type weighted singular operators governing the regularity of parabolic Ornstein–Uhlenbeck equations in variable Herz spaces. Wang and Xu [68] established the boundedness of vector-valued bilinear Calderón–Zygmund operators on products of weighted Herz–Morrey spaces with variable exponents. Maeda, Mizuta, Ohno and Shimomura [69] studied the duality of central Herz–Morrey–Musielak–Orlicz spaces of variable exponents. Cen and Song [70] developed extrapolation to product Morrey–Herz spaces with applications, and Ho [71] obtained the boundedness of operators and related

inequalities on Morrey–Banach spaces. In a related direction, Tacy [72,73] obtained L^p estimates for joint quasimodes of semiclassical pseudo-differential operators, including the case where the characteristic sets have kth order contact.

Potential-type operators on such scales have also been studied in several works. Afzal et al. [74] obtained boundedness estimates for the exponentially damped Riesz potential, with applications to PDEs. Shi and Tao [75] proved the boundedness of multilinear Riesz potential operators on Herz-type spaces and generalized Morrey spaces. Afzal [76] studied the boundedness and regularity of the nonlinear system in Herz spaces. Zhang and Liu [77] proved the boundedness of multilinear commutators of singular integrals in Morrey spaces on homogeneous spaces. Afzal et al. [78] proved boundedness results for the Wolff potential in Morrey spaces over homogeneous groups. Artes et al. [79] studied nonlinear potential estimates in grand variable Herz spaces in connection with Birkhoff's ergodic theorem. Afzal et al. [80] resolved some open problems concerning summability and integrability in more general function spaces, using Orlicz–Zygmund spaces and new geometric properties of Morrey spaces. Khan et al. [81] proved the boundedness of Havin–Maz'ya potentials in Herz spaces, with applications to Laplace equations. Finally, the Bessel–Riesz operator $I_{\alpha,\gamma}$, whose kernel $K_{\alpha,\gamma}(\zeta) = |\zeta|^{\alpha-n}(1+|\zeta|)^{-\gamma}$ combines a Riesz-type local singularity with a Bessel-type global decay, was introduced by Idris, Gunawan and Eridani [82,83], who showed that it is bounded on generalized Morrey spaces; the classical Riesz potential is recovered at $\gamma = 0$.

Motivation and Contribution

Pseudo-differential operators on variable-exponent Herz–Hardy spaces, and Schrödinger fractional integrals into variable-exponent Lipschitz spaces, have each been developed with at most one of the following features present: variable exponent, Morrey truncation, or grand modification, and without Muckenhoupt weights. The theory developed here incorporates all of these features simultaneously within one framework, so that the theorems below are instances of a single method. Our contributions are as follows.

(i) We introduce the weighted grand variable Herz–Morrey–Hardy space, which extends the unweighted grand variable Herz–Morrey–Hardy space of [62] by a Muckenhoupt weight (Section 2), and we prove that every order-zero pseudo-differential operator is bounded on it (Theorem 2).

(ii) In contrast to existing work, we improve the results from two perspectives. First, we refine the norm structure by introducing a grand parameter, together with a Morrey factor and a Muckenhoupt weight. Second, whereas existing work uses the Riesz kernel, we use the more general Bessel–Riesz kernel. For the Schrödinger operator $\mathcal{L} = -\Delta + \mathcal{V}$ with associated Bessel–Riesz operator $I_{\alpha,\gamma}$, we construct the weighted grand variable Herz–Morrey–Hardy–Lipschitz space from the critical-radius geometry of $\mathcal{L}$, and we prove that $I_{\alpha,\gamma}$ is bounded into this space from a weighted variable Lebesgue space (strong type) and, in place of the weak-type Lebesgue space used in the classical theory, from a weighted grand variable Herz space (Theorems 3 and 4).

(iii) We isolate the conditions under which uniform per-atom (respectively, per-ball) estimates embed into the grand–Morrey norm, and we use them in both proofs (Section 2, Lemmas 7 and 20).

(iv) We show how the Bessel parameter γ enters the admissible range of integrability exponents in the Schrödinger results (Corollary 3).

As recorded in Corollaries 1 and 2, our theorems specialize to results of [38,51]. Section 2 fixes definitions and notation for both operator classes, Section 3 treats the pseudo-differential operator, Section 4 treats the Bessel–Riesz operator, and Section 5 concludes the paper.

2. Preliminaries

Throughout the paper, C denotes a generic positive constant, independent of the main parameters, whose value may change from one occurrence to the next. Constants whose values must be tracked within a statement or a proof are numbered consecutively as $C_1, C_2, C_3, \dots$ within that statement or proof. We write $\mathcal{B}(\zeta, \nu) = \{\iota \in \mathbb{R}^n : |\zeta - \iota| < \nu\}$, and, for a ball $\mathcal{B}$ and $t > 0$, $t\mathcal{B}$ denotes the ball with the same center and t times the radius. For a measurable set $S \subset \mathbb{R}^n$, $|S|$ is its Lebesgue measure and χ_S is its characteristic function, that is, $\chi_S(\zeta) = 1$ for $\zeta \in S$ and $\chi_S(\zeta) = 0$ otherwise. We write $\psi \lesssim \mu$ (respectively, $\psi \gtrsim \mu$) if $\psi \le C\mu$ (respectively, $\psi \ge C^{-1}\mu$) for some constant $C > 0$, and $\psi \sim \mu$ if both $\psi \lesssim \mu$ and $\psi \gtrsim \mu$ hold, that is, $C^{-1}\mu \le \psi \le C\mu$.

2.1. Variable Exponent Lebesgue Spaces

For a measurable function $\mathrm{p}(\cdot) : \mathbb{R}^n \to [1, \infty)$, the variable exponent Lebesgue space and its Luxemburg norm are defined by

$$L^{\mathrm{p}(\cdot)}(\mathbb{R}^n) = \Big\{\psi \text{ measurable} : \int_{\mathbb{R}^n} \Big|\frac{\psi(\zeta)}{s}\Big|^{\mathrm{p}(\zeta)} d\zeta < \infty \ \text{ for some } s > 0\Big\},$$
$$\|\psi\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)} = \inf\Big\{s > 0 : \int_{\mathbb{R}^n} \Big|\frac{\psi(\zeta)}{s}\Big|^{\mathrm{p}(\zeta)} d\zeta \le 1\Big\},$$

which makes $L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ a Banach function space [6]. The following notation and structural properties are used throughout.

- We set $\mathrm{p}^- = \operatorname{ess\,inf} \mathrm{p}$, $\mathrm{p}^+ = \operatorname{ess\,sup} \mathrm{p}$, and $P(\mathbb{R}^n) := \{\mathrm{p}(\cdot) : 1 < \mathrm{p}^- \le \mathrm{p}^+ < \infty\}$ [6].
- The exponent $\mathrm{p}(\cdot)$ is called *locally log-Hölder continuous* if $|\mathrm{p}(\zeta) - \mathrm{p}(\iota)| \le C/\log(e + 1/|\zeta - \iota|)$ for all $\zeta, \iota \in \mathbb{R}^n$, and *log-Hölder continuous at infinity* if $\mathrm{p}(\infty) := \lim_{|\zeta| \to \infty} \mathrm{p}(\zeta)$ exists and $|\mathrm{p}(\zeta) - \mathrm{p}(\infty)| \le C/\log(e + |\zeta|)$ for all $\zeta \in \mathbb{R}^n$. The class $P^{\log}(\mathbb{R}^n)$ consists of all $\mathrm{p}(\cdot) \in P(\mathbb{R}^n)$ satisfying both conditions. Moreover, $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n)$ implies $\mathrm{p}'(\cdot) = \mathrm{p}(\cdot)/(\mathrm{p}(\cdot) - 1) \in P^{\log}(\mathbb{R}^n)$ [7].
- $\mathfrak{B}(\mathbb{R}^n)$ denotes the class of all $\mathrm{p}(\cdot) \in P(\mathbb{R}^n)$ for which the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)$. By Diening's boundedness criterion, $P^{\log}(\mathbb{R}^n) \subset \mathfrak{B}(\mathbb{R}^n)$ [7].
- For a measurable function ψ and $\tau > 0$, we write $E_\tau(\psi) := \{\zeta \in \mathbb{R}^n : |\psi(\zeta)| > \tau\}$, and $\chi_{E_\tau(\psi)}$ denotes the characteristic function of this set. The weak-type space $L^{\mathrm{p}(\cdot),\infty}(\mathbb{R}^n)$ consists of all measurable functions ψ such that [50]

$$\|\psi\|_{L^{\mathrm{p}(\cdot),\infty}(\mathbb{R}^n)} := \inf\Big\{s > 0 : \sup_{\tau > 0} \int_{\mathbb{R}^n} \Big(\frac{\tau}{s}\Big)^{\mathrm{p}(\zeta)} \chi_{E_\tau(\psi)}(\zeta)\, d\zeta \le 1\Big\} < \infty.$$

Lemma 1 (Generalized Hölder inequality [6]). *Let* $\mathrm{p}(\cdot) \in P(\mathbb{R}^n)$, $\psi \in L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ *and* $\mu \in L^{\mathrm{p}'(\cdot)}(\mathbb{R}^n)$. *Then*

$$\int_{\mathbb{R}^n} |\psi(\zeta)\mu(\zeta)|\, d\zeta \ \le\ \nu_{\mathrm{p}}\, \|\psi\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)}\, \|\mu\|_{L^{\mathrm{p}'(\cdot)}(\mathbb{R}^n)}, \qquad \nu_{\mathrm{p}} := 1 + \frac{1}{\mathrm{p}^-} - \frac{1}{\mathrm{p}^+}.$$

Lemma 2 (Ball norm asymptotics [7]). *Let* $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n)$. *Then, for every ball* $\mathcal{B} \subset \mathbb{R}^n$,

$$\|\chi_{\mathcal{B}}\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)} \sim \begin{cases} |\mathcal{B}|^{1/\mathrm{p}(\zeta)} \ (\zeta \in \mathcal{B}), & \text{if } |\mathcal{B}| \le 2^n, \\ |\mathcal{B}|^{1/\mathrm{p}(\infty)}, & \text{if } |\mathcal{B}| \ge 1. \end{cases}$$

2.2. The Variable Muckenhoupt Class $\mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$

The classical Muckenhoupt classes A_p characterize exactly those weights for which the Hardy–Littlewood maximal operator is bounded on weighted Lebesgue spaces [84], and they are therefore the natural framework for weighted inequalities in harmonic analysis. Their variable-exponent analog $\mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ was introduced by Cruz-Uribe, Fiorenza and Neugebauer [8] and by Cruz-Uribe, Diening and Hästö [85]; see also [9]. In this subsection, we recall this class together with the weighted Hölder inequalities and ball estimates that are used throughout the paper.

Let $\mathrm{p}(\cdot) \in P(\mathbb{R}^n)$. A weight is an a.e. positive, locally integrable function w on $\mathbb{R}^n$. To measure functions in the weighted setting, we multiply them by w before taking the $L^{\mathrm{p}(\cdot)}$-norm; the weights for which this procedure is compatible with the maximal operator are described by the following condition.

Definition 1 (Variable Muckenhoupt class [8,85]). *Let $\mathrm{p}(\cdot) \in P(\mathbb{R}^n)$. The variable Muckenhoupt class $\mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ consists of all weights w such that*

$$[w]_{\mathcal{A}_{\mathrm{p}(\cdot)}} := \sup_{\mathcal{B}} \frac{1}{|\mathcal{B}|} \|w\chi_{\mathcal{B}}\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)} \|w^{-1}\chi_{\mathcal{B}}\|_{L^{\mathrm{p}'(\cdot)}(\mathbb{R}^n)} < \infty,$$

where the supremum is taken over all balls $\mathcal{B} \subset \mathbb{R}^n$. For $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$, we set $\|\psi\|_{L^{\mathrm{p}(\cdot)}(w)} := \|\psi w\|_{L^{\mathrm{p}(\cdot)}(\mathbb{R}^n)}$ and $\|\psi\|_{L^{\mathrm{p}(\cdot),\infty}(w)} := \|\psi w\|_{L^{\mathrm{p}(\cdot),\infty}(\mathbb{R}^n)}$.

Lemma 3 ([8,85]). *Let $\mathrm{p}(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ and $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$. Then the following hold:*

(a) $\|\mathcal{M}\psi\|_{L^{\mathrm{p}(\cdot)}(w)} \le C\|\psi\|_{L^{\mathrm{p}(\cdot)}(w)}$;

(b) $w^{-1} \in \mathcal{A}_{\mathrm{p}'(\cdot)}(\mathbb{R}^n)$ *and* $[w^{-1}]_{\mathcal{A}_{\mathrm{p}'(\cdot)}} = [w]_{\mathcal{A}_{\mathrm{p}(\cdot)}}$.

Lemmas 4 and 5 below are proved in the same way as their unweighted counterparts in [6], with $w\chi_{\mathcal{B}}$ in place of $\chi_{\mathcal{B}}$, by distinguishing the case $|\mathcal{B}| \le 1$ (small balls, where the local log-Hölder condition is used) from the case $|\mathcal{B}| > 1$ (large balls, where the log-Hölder condition at infinity is used).

Lemma 4 (Weighted Hölder inequalities). *Let $\mathrm{p}(\cdot) \in P(\mathbb{R}^n)$ and $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$.*

(a) *If $\psi \in L^{\mathrm{p}(\cdot)}(w)$ and $\mu w^{-1} \in L^{\mathrm{p}'(\cdot)}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |\psi(\zeta)\mu(\zeta)|\, d\zeta \le \nu_{\mathrm{p}} \|\psi\|_{L^{\mathrm{p}(\cdot)}(w)} \|\mu\|_{L^{\mathrm{p}'(\cdot)}(w^{-1})}.$$

(b) *If, in addition, $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n)$ and $\mathrm{p}(\zeta) \le \mathrm{p}(\infty)$ for $|\zeta| > \nu_0 > 1$, then for every ball $\mathcal{B}$ and every $\psi \in L^{\mathrm{p}(\cdot),\infty}(w)$,*

$$\int_{\mathcal{B}} |\psi(\zeta)|\, d\zeta \le C \|\psi\|_{L^{\mathrm{p}(\cdot),\infty}(w)} \|\chi_{\mathcal{B}}\|_{L^{\mathrm{p}'(\cdot)}(w^{-1})},$$

where $C = C(\nu_0, [w]_{\mathcal{A}_{\mathrm{p}(\cdot)}})$.

Lemma 5 (Weighted ball estimates). *Let $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n)$ and $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$. Then there are constants $C \ge 1$ and $\delta_1, \delta_2 \in (0,1)$ such that, for all balls $\mathcal{B}$ and all measurable sets $S \subset \mathcal{B}$,*

$$\frac{\|\chi_{\mathcal{B}}\|_{L^{p(\cdot)}(w)}}{\|\chi_S\|_{L^{p(\cdot)}(w)}} \le C\,\frac{|\mathcal{B}|}{|S|}, \qquad \frac{\|\chi_S\|_{L^{p(\cdot)}(w)}}{\|\chi_{\mathcal{B}}\|_{L^{p(\cdot)}(w)}} \le C\left(\frac{|S|}{|\mathcal{B}|}\right)^{\delta_1},$$
$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(w^{-1})}}{\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}} \le C\left(\frac{|S|}{|\mathcal{B}|}\right)^{\delta_2}, \qquad \frac{1}{|\mathcal{B}|}\,\|\chi_{\mathcal{B}}\|_{L^{p(\cdot)}(w)}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})} \le C.$$

In particular, $\|\chi_{2\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})} \le C\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}$.

2.3. Grand Variable Herz–Morrey–Hardy Spaces

Let $\mathcal{B}_\kappa = \{\zeta \in \mathbb{R}^n : |\zeta| \le 2^\kappa\}$, $\mathcal{A}_\kappa = \mathcal{B}_\kappa \setminus \mathcal{B}_{\kappa-1}$ and $\chi_\kappa = \chi_{\mathcal{A}_\kappa}$, $\kappa \in \mathbb{Z}$. For $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ log-Hölder continuous at the origin and at infinity, with $\alpha(0) = \lim_{\zeta\to 0}\alpha(\zeta)$ and $\alpha_\infty = \lim_{|\zeta|\to\infty}\alpha(\zeta)$, we set

$$\phi(\kappa) := \begin{cases} \alpha(0), & \kappa \le 0, \\ \alpha_\infty, & \kappa > 0. \end{cases}$$

As in the standard case [10,11], this allows us to replace $2^{\kappa\alpha(\zeta)}$, $\zeta \in \mathcal{A}_\kappa$, by $2^{\kappa\phi(\kappa)}$, up to a fixed constant.

We now introduce the two scales of spaces that are compared in this paper. In the unweighted case $w \equiv 1$, the spaces below are the grand variable Herz–Morrey space and the grand variable Herz–Morrey–Hardy space introduced by Sultan, Hussain and Sultan [62]; we denote them by $M\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(\mathbb{R}^n)$ and $HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(\mathbb{R}^n)$. For a general weight $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$, we obtain the new weighted spaces $M\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)$ and $HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)$, which are the spaces used throughout the article.

Definition 2 (Grand variable Herz–Morrey spaces, unweighted and weighted)**.** *Let* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ *be log-Hölder continuous at the origin and at infinity; that is, there exist real numbers* $\alpha(0)$, α_∞ *and a constant* $C > 0$ *such that*

$$|\alpha(\zeta) - \alpha(0)| \le \frac{C}{\log\left(e + 1/|\zeta|\right)}, \qquad |\zeta| \le \tfrac{1}{2},$$
$$|\alpha(\zeta) - \alpha_\infty| \le \frac{C}{\log\left(e + |\zeta|\right)}, \qquad \zeta \in \mathbb{R}^n.$$

Let $0 < v < \infty$, $p(\cdot) \in P(\mathbb{R}^n)$ *and* $0 \le \Gamma < \infty$, *and let* $\mathcal{B}_\kappa$, $\mathcal{A}_\kappa$, χ_κ *and* $\phi(\kappa)$ *be as above.*

(a) (Unweighted space [62].) The grand variable Herz–Morrey space $M\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(\mathbb{R}^n)$ *is the set of all measurable functions* ψ *on* $\mathbb{R}^n \setminus \{0\}$ *such that* $\psi\chi_\kappa \in L^{p(\cdot)}(\mathbb{R}^n)$ *for every* $\kappa \in \mathbb{Z}$ *and*

$$\|\psi\|_{M\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(\mathbb{R}^n)} := \sup_{\vartheta>0}\sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0} 2^{\kappa\phi(\kappa)v(1+\vartheta)}\,\|\psi\chi_\kappa\|^{v(1+\vartheta)}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/(v(1+\vartheta))} < \infty.$$

(b) *(Weighted space.) Let $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$. The weighted grand variable Herz–Morrey space $M\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)$ is the set of all measurable functions ψ on $\mathbb{R}^n \setminus \{0\}$ such that $\psi\chi_\kappa \in L^{p(\cdot)}(w)$ for every $\kappa \in \mathbb{Z}$ and*

$$\|\psi\|_{M\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)} := \sup_{\vartheta>0} \sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma} \left(\sum_{\kappa=-\infty}^{m_0} 2^{\kappa\phi(\kappa)v(1+\vartheta)} \|\psi\chi_\kappa\|^{v(1+\vartheta)}_{L^{p(\cdot)}(w)} \right)^{1/(v(1+\vartheta))} < \infty.$$

Both spaces are quasi-normed spaces (normed spaces if $v \geq 1$), and (a) is the special case $w \equiv 1$ of (b).

To pass from functions to distributions, we recall the Schwartz class and its dual. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of all $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\sup_{\zeta\in\mathbb{R}^n} |\zeta^\gamma D^\beta \varphi(\zeta)| < \infty$ for all multi-indices $\gamma, \beta \in \mathbb{N}_0^n$, endowed with the topology generated by these semi-norms. Its topological dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. For $\psi \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the convolution $\varphi * \psi(\iota) := \langle \psi, \varphi(\iota - \cdot)\rangle$ is a C^∞ function of ι.

Definition 3 (Grand maximal function [86]). *Let $N \in \mathbb{N}$ with $N > n+1$. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\tau > 0$, set $\varphi_\tau(\zeta) = \tau^{-n}\varphi(\zeta/\tau)$, $\zeta \in \mathbb{R}^n$, and define the family*

$$\mathfrak{A}_N := \Big\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{\zeta\in\mathbb{R}^n} \sup_{|\gamma|\leq N,\, |\beta|\leq N} \big|\zeta^\gamma D^\beta \varphi(\zeta)\big| \leq 1 \Big\}.$$

For $\psi \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function of ψ is defined by

$$\mathcal{G}_N\psi(\zeta) := \sup_{\varphi\in\mathfrak{A}_N} \sup_{\tau>0} \sup_{\iota\in\mathbb{R}^n:\, |\iota-\zeta|<\tau} \big|\varphi_\tau * \psi(\iota)\big|, \qquad \zeta \in \mathbb{R}^n.$$

Herz-type Hardy spaces are obtained from Herz-type spaces by measuring, instead of a function itself, the grand maximal function of a tempered distribution. This allows distributions with cancellation to be included, provides an atomic decomposition (Lemma 6 below), and yields the natural substitute for Herz spaces when $v \leq 1$ or when singular integral and pseudo-differential operators are considered. Combining this construction with Definition 2, we arrive at the central spaces of this paper.

Definition 4 (Grand variable Herz–Morrey–Hardy spaces, unweighted and weighted). *Let $\alpha(\cdot)$, v, $p(\cdot)$ and Γ be as in Definition 2, and let $N \in \mathbb{N}$, $N > n+1$ be as in Definition 3.*

(a) *(Unweighted space [62].) The grand variable Herz–Morrey–Hardy space $HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(\mathbb{R}^n)$ is the set of all $\mu \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|\mu\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(\mathbb{R}^n)} := \|\mathcal{G}_N\mu\|_{M\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(\mathbb{R}^n)} < \infty.$$

(b) *(Weighted space.) Let $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$. The weighted grand variable Herz–Morrey–Hardy space $HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)$ is the set of all $\mu \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|\mu\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)} := \|\mathcal{G}_N\mu\|_{M\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)} < \infty.$$

Remark 1 (Role of the Muckenhoupt weight). *The unweighted space in Definition 4(a) is the existing space of [62], whereas the weighted space in Definition 4(b) is new; for $w \equiv 1$, the two spaces coincide. The weighted space improves the unweighted one in the following respects.*

(a) *The class $\mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ is the natural class for weighted estimates: already in the constant-exponent case, the maximal operator is bounded on the weighted space $L^{\mathrm{p}}(w)$ precisely when the weight satisfies the Muckenhoupt condition [84], and the same role is played by $\mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ in the variable setting [8,9,85]. All estimates of this paper that rely on the maximal operator (Lemma 3(a)) therefore remain valid in the weighted spaces.*

(b) *The weighted spaces allow singular or degenerate behavior of the functions under consideration. For instance, if $\mathrm{p}(\cdot) \equiv \mathrm{p}$ is constant, then the power weight $w(\zeta) = |\zeta - \zeta_0|^a$ belongs to $\mathcal{A}_{\mathrm{p}}(\mathbb{R}^n)$ (in the normalization of Definition 1) if and only if $-n/\mathrm{p} < a < n/\mathrm{p}'$. Such weights occur naturally in PDEs with degenerate or singular coefficients.*

(c) *The weighted spaces are genuinely new. For a radial power weight centered at the origin, $w(\zeta) = |\zeta|^a$, we have $w \sim 2^{\kappa a}$ on $\mathcal{A}_\kappa$, and the weighted space coincides with the unweighted space with $\alpha(\cdot)$ replaced by $\alpha(\cdot) + a$. However, a general $\mathcal{A}_{\mathrm{p}(\cdot)}$ weight, such as $|\zeta - \zeta_0|^a$ with $\zeta_0 \neq 0$ and $a \neq 0$, is not comparable to a constant on the annuli $\mathcal{A}_\kappa$, so the weighted space cannot be obtained from the unweighted one by a shift of $\alpha(\cdot)$.*

(d) *Since every result of this paper holds uniformly for $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$, with constants depending on w only through $[w]_{\mathcal{A}_{\mathrm{p}(\cdot)}}$, the choice $w \equiv 1$ recovers the corresponding unweighted statements for the spaces of [62].*

Remark 2 (Parameter specializations). *Let $\alpha(\cdot)$, v, $\mathrm{p}(\cdot)$, Γ and w be as in Definition 4. We first observe that, for every sequence $\{c_\kappa\}_{\kappa\in\mathbb{Z}}$, the map $s \mapsto \left(\sum_\kappa |c_\kappa|^s\right)^{1/s}$ is nonincreasing on $(0,\infty)$. Hence, for fixed m_0, the supremum over $\vartheta > 0$ in Definition 2 equals the limit as $\vartheta \to 0^+$. With this observation, the space $HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(w)$ reduces to the following known spaces.*

(a) *If $w \equiv 1$, $\Gamma = 0$ and $v = \mathrm{p} \in (0,\infty)$, and we write the variable exponent as $\mathrm{q}(\cdot) := \mathrm{p}(\cdot) \in P(\mathbb{R}^n)$, then the supremum over m_0 becomes the full sum over $\kappa \in \mathbb{Z}$, and $HM\dot{\mathcal{K}}^{\alpha(\cdot),\mathrm{p})}_{0,\mathrm{q}(\cdot)}(1)$ coincides, with equal quasi-norms, with the Herz-type Hardy space with variable exponent $H\dot{\mathcal{K}}^{\alpha(\cdot),\mathrm{p}}_{\mathrm{q}(\cdot)}(\mathbb{R}^n)$ of [11,12,38].*

(b) *If, in addition to (a), $\alpha(\cdot) \equiv \alpha \in \mathbb{R}$ and $\mathrm{q}(\cdot) \equiv \mathrm{q} \in (1,\infty)$ are constant, then we obtain the classical homogeneous Herz-type Hardy space $H\dot{\mathcal{K}}^{\alpha,\mathrm{p}}_{\mathrm{q}}(\mathbb{R}^n)$, which is the unweighted case of the spaces studied in [36].*

(c) *If $w \equiv 1$, then $HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(1) = HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(\mathbb{R}^n)$ coincides with the grand variable Herz–Morrey–Hardy space $HM\dot{\mathcal{K}}^{\alpha(\cdot),v),\theta}_{\Gamma,\mathrm{p}(\cdot)}(\mathbb{R}^n)$ of [62] with $\theta = 0$.*

(d) *If $v = \mathrm{p} \in (0,\infty)$, $\Gamma > 0$, and the supremum over $\vartheta > 0$ is omitted (that is, $\vartheta = 0$), then we obtain the weighted variable Herz–Morrey–Hardy space with Morrey parameter Γ; see [63] for its grand unweighted counterpart.*

(e) *If $\Gamma = 0$, then the Morrey truncation disappears, and we obtain the weighted grand variable Herz-type Hardy space, whose grand construction follows the grand Lebesgue spaces of Iwaniec and Sbordone [61].*

In the next definition, atoms are denoted by the fraktur letter $\mathfrak{a}$, in order to distinguish them clearly from the surrounding text.

Definition 5 (Weighted central atoms). *Let $n\delta_2 \le \alpha(0)$ and $\alpha_\infty < \infty$, where δ_2 is as in Lemma 5, and let $s_0 \ge [\alpha(0) - n\delta_2]$ be an integer. A function $\mathfrak{a}$ is called a weighted central $(\alpha(\cdot),\mathrm{p}(\cdot))$-atom for $\mathcal{B}_\kappa$ if*

$$\operatorname{supp}\mathfrak{a} \subset \mathcal{B}_\kappa, \qquad \|\mathfrak{a}\|_{L^{\mathrm{p}(\cdot)}(w)} \le |\mathcal{B}_\kappa|^{-\phi(\kappa)/n}, \qquad \int_{\mathbb{R}^n} \mathfrak{a}(\zeta)\,\zeta^\beta\,d\zeta = 0 \quad \text{for all } |\beta| \le s_0.$$

Lemma 6 (Atomic decomposition [11,12,62]). *A distribution $\mu \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathfrak{p}(\cdot)}(w)$ if and only if $\mu = \sum_{\kappa=-\infty}^{\infty} \lambda_\kappa \mathfrak{a}_\kappa$ in $\mathcal{S}'(\mathbb{R}^n)$, where each $\mathfrak{a}_\kappa$ is a weighted central $(\alpha(\cdot),\mathfrak{p}(\cdot))$-atom for $\mathcal{B}_\kappa$ and*

$$\sup_{\vartheta>0}\sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma}\Big(\sum_{\kappa=-\infty}^{m_0}|\lambda_\kappa|^{v(1+\vartheta)}\Big)^{1/(v(1+\vartheta))}<\infty.$$

Moreover,

$$\|\mu\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathfrak{p}(\cdot)}(w)} \approx \inf \sup_{\vartheta>0}\sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma}\Big(\sum_{\kappa=-\infty}^{m_0}|\lambda_\kappa|^{v(1+\vartheta)}\Big)^{1/(v(1+\vartheta))},$$

where the infimum is taken over all such decompositions of μ.

The proof of Lemma 6 is the same as in [11,62], with $L^{\mathfrak{p}(\cdot)}(\mathbb{R}^n)$ replaced by $L^{\mathfrak{p}(\cdot)}(w)$ and with Lemma 5 used in place of the corresponding unweighted ball estimates.

The next lemma describes how uniform estimates for single atoms, which decay geometrically away from the diagonal, are aggregated in the grand–Morrey norm. For a sequence $\lambda = \{\lambda_j\}_{j\in\mathbb{Z}}$ of complex numbers, we write

$$\mathcal{N}_\lambda := \sup_{\vartheta>0}\sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma}\Big(\sum_{j=-\infty}^{m_0}|\lambda_j|^{v(1+\vartheta)}\Big)^{1/(v(1+\vartheta))}. \tag{1}$$

Lemma 7 (Aggregation in the grand–Morrey norm). *Let $0<v<\infty$ and $0\le\Gamma<\varepsilon<\infty$. Let $\lambda=\{\lambda_j\}_{j\in\mathbb{Z}}$ satisfy $\mathcal{N}_\lambda<\infty$, and let $\{b_{\kappa,j}\}_{\kappa,j\in\mathbb{Z}}$ be non-negative numbers such that*

$$b_{\kappa,j}\le C_1\,2^{-\varepsilon|\kappa-j|},\qquad \kappa,j\in\mathbb{Z}. \tag{2}$$

For $\vartheta>0$ and $m_0\in\mathbb{Z}$, set

$$S_{\vartheta,m_0} := 2^{-m_0\Gamma}\Big(\sum_{\kappa=-\infty}^{m_0}\Big(\sum_{j\in\mathbb{Z}}|\lambda_j|\,b_{\kappa,j}\Big)^{v(1+\vartheta)}\Big)^{1/(v(1+\vartheta))}.$$

Then there is a constant $C_2=C_2(v,\Gamma,\varepsilon)$, independent of λ, ϑ and m_0, such that

$$\sup_{\vartheta>0}\sup_{m_0\in\mathbb{Z}} S_{\vartheta,m_0} \le C_1C_2\,\mathcal{N}_\lambda. \tag{3}$$

Proof. By homogeneity, we may assume that $C_1=1$. For $t>0$, we set $c(t):=\sum_{i\in\mathbb{Z}}2^{-t|i|}=(1+2^{-t})/(1-2^{-t})$; note that $c(t)\ge1$ and that c is decreasing in t.

We first note that the coefficients satisfy a pointwise bound. Indeed, fix $j\in\mathbb{Z}$ and $\vartheta>0$; taking $m_0=j$ in (1) and keeping only the term with index j, we obtain $2^{-j\Gamma}|\lambda_j|\le\mathcal{N}_\lambda$, that is,

$$|\lambda_j|\le 2^{j\Gamma}\,\mathcal{N}_\lambda,\qquad j\in\mathbb{Z}. \tag{4}$$

Now fix $\vartheta>0$ and $m_0\in\mathbb{Z}$, and put $s:=v(1+\vartheta)>v$. For $\kappa\le m_0$, we write $\sum_{j\in\mathbb{Z}}|\lambda_j|b_{\kappa,j}=S'_\kappa+S''_\kappa$, where S'_κ is the sum over $j\le m_0$ and S''_κ is the sum over $j>m_0$, and we set

$$A:=\Big(\sum_{\kappa\le m_0}(S'_\kappa)^s\Big)^{1/s},\qquad B:=\Big(\sum_{\kappa\le m_0}(S''_\kappa)^s\Big)^{1/s}.$$

If $s \geq 1$, Minkowski's inequality gives $\big(\sum_{\kappa\leq m_0}(S'_\kappa + S''_\kappa)^s\big)^{1/s} \leq A + B$. If $s < 1$, then $(S'_\kappa + S''_\kappa)^s \leq (S'_\kappa)^s + (S''_\kappa)^s$, so that the same quantity is at most $(A^s + B^s)^{1/s} \leq 2^{1/s-1}(A + B) \leq 2^{1/v}(A + B)$. Thus, in both cases, we have

$$\Big(\sum_{\kappa\leq m_0}\Big(\sum_{j\in\mathbb{Z}}|\lambda_j|b_{\kappa,j}\Big)^s\Big)^{1/s} \leq 2^{1/v}(A + B), \tag{5}$$

and it remains to estimate A and B.

To estimate A, suppose first that $s \leq 1$. By the s-subadditivity of $t \mapsto t^s$, (2) and $s \geq v$, we have

$$A^s \leq \sum_{\kappa\leq m_0}\sum_{j\leq m_0}|\lambda_j|^s 2^{-\varepsilon s|\kappa-j|} \leq c(\varepsilon v)\sum_{j\leq m_0}|\lambda_j|^s,$$

and hence $A \leq c(\varepsilon v)^{1/s}\big(\sum_{j\leq m_0}|\lambda_j|^s\big)^{1/s} \leq c(\varepsilon v)^{1/v}\big(\sum_{j\leq m_0}|\lambda_j|^s\big)^{1/s}$. If instead $s > 1$, Young's inequality $\ell^1 * \ell^s \hookrightarrow \ell^s(\mathbb{Z})$, applied to the kernel $\{2^{-\varepsilon|i|}\}_{i\in\mathbb{Z}}$, gives $A \leq c(\varepsilon)\big(\sum_{j\leq m_0}|\lambda_j|^s\big)^{1/s}$. By (1), we therefore obtain in both cases

$$A \leq \max\{c(\varepsilon), c(\varepsilon v)^{1/v}\}\, 2^{m_0\Gamma}\,\mathcal{N}_\lambda.$$

To estimate B, note that $\kappa \leq m_0 < j$, so that $|\kappa - j| = j - \kappa$. If $s \geq 1$, Minkowski's inequality in ℓ^s, together with (2) and (4), gives

$$\begin{aligned} B &\leq \sum_{j>m_0}|\lambda_j|\Big(\sum_{\kappa\leq m_0}2^{-\varepsilon s(j-\kappa)}\Big)^{1/s} = \sum_{j>m_0}|\lambda_j|\,\frac{2^{-\varepsilon(j-m_0)}}{(1-2^{-\varepsilon s})^{1/s}} \\ &\leq \frac{\mathcal{N}_\lambda}{1-2^{-\varepsilon}}\sum_{j>m_0}2^{j\Gamma-\varepsilon(j-m_0)} = \frac{2^{m_0\Gamma}\,\mathcal{N}_\lambda}{(1-2^{-\varepsilon})(2^{\varepsilon-\Gamma}-1)}, \end{aligned}$$

where the geometric series converges because $\varepsilon > \Gamma$. If $s < 1$ (which can only happen when $v < 1$), we similarly obtain, using $s \geq v$,

$$\begin{aligned} B^s &\leq \sum_{j>m_0}|\lambda_j|^s\sum_{\kappa\leq m_0}2^{-\varepsilon s(j-\kappa)} \leq \frac{\mathcal{N}_\lambda^s}{1-2^{-\varepsilon s}}\sum_{j>m_0}2^{j\Gamma s-\varepsilon s(j-m_0)} \\ &\leq D\,2^{m_0\Gamma s}\,\mathcal{N}_\lambda^s, \qquad D := \frac{1}{(1-2^{-\varepsilon v})(2^{(\varepsilon-\Gamma)v}-1)}, \end{aligned}$$

and hence $B \leq \max\{1, D\}^{1/v}\, 2^{m_0\Gamma}\mathcal{N}_\lambda$.

Inserting the bounds for A and B into (5), we obtain $S_{\vartheta,m_0} \leq C_2\,\mathcal{N}_\lambda$ with

$$C_2 := 2^{1/v}\Big[\max\{c(\varepsilon), c(\varepsilon v)^{1/v}\} + \max\Big\{\frac{1}{(1-2^{-\varepsilon})(2^{\varepsilon-\Gamma}-1)},\ \max\{1, D\}^{1/v}\Big\}\Big],$$

which depends only on v, Γ and ε. Taking the supremum over $\vartheta > 0$ and $m_0 \in \mathbb{Z}$, we obtain (3). □

Remark 3. *(a) A bound of the form $b_{\kappa,j} \leq C_1$, uniform in κ and j but without decay in $|\kappa - j|$, is not sufficient, since the series $\sum_{\kappa=-\infty}^{m_0} C_1^{v(1+\vartheta)}$ diverges. This is why the geometric decay (2) is required in Lemma 7, and why the per-atom estimate in Section 3 is formulated with such a decay.*

(b) The condition $\Gamma < \varepsilon$ in Lemma 7 cannot be removed. Indeed, let $\Gamma > 0$, $\lambda_j = 2^{j\Gamma}$ and $b_{\kappa,j} = 2^{-\varepsilon|\kappa-j|}$. Then $\mathcal{N}_\lambda \leq \sup_{s\geq v}(1-2^{-\Gamma s})^{-1/s} < \infty$, whereas, if $\Gamma \geq \varepsilon$, we have $\sum_j \lambda_j b_{\kappa,j} \geq \sum_{j>\kappa} 2^{j\Gamma-\varepsilon(j-\kappa)} = \infty$ for every κ.

Lemma 8 (Per-annulus control by the Herz–Morrey–Hardy norm). *Let* $\mu \in HM\dot{\mathcal{K}}^{\alpha(\cdot),v}_{\Gamma,\mathfrak{p}(\cdot)}(w) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$. *Then, for every* $i \in \mathbb{Z}$, *we have*

$$\|\mu\chi_i\|_{L^{\mathfrak{p}(\cdot)}(w)} \le C\, 2^{i(\Gamma-\Phi(i))}\, \|\mu\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v}_{\Gamma,\mathfrak{p}(\cdot)}(w)}.$$

Proof. The grand maximal function dominates μ almost everywhere, that is, $|\mu(\zeta)| \le C\mathcal{G}_N\mu(\zeta)$ for a.e. ζ; this is a standard consequence of the Lebesgue differentiation theorem applied to an approximate identity in $\mathfrak{A}_N$ [86]. Hence it suffices to bound $\|(\mathcal{G}_N\mu)\chi_i\|_{L^{\mathfrak{p}(\cdot)}(w)}$. Fix $\vartheta > 0$ and take $m_0 = i$ in Definition 2. Keeping only the non-negative term with $\kappa = i$ in the sum, we obtain

$$\begin{aligned} 2^{i\Phi(i)}\|(\mathcal{G}_N\mu)\chi_i\|_{L^{\mathfrak{p}(\cdot)}(w)} &= \left(2^{i\Phi(i)v(1+\vartheta)}\|(\mathcal{G}_N\mu)\chi_i\|^{v(1+\vartheta)}_{L^{\mathfrak{p}(\cdot)}(w)}\right)^{1/(v(1+\vartheta))} \\ &\le 2^{i\Gamma}\,\|\mathcal{G}_N\mu\|_{M\dot{\mathcal{K}}^{\alpha(\cdot),v}_{\Gamma,\mathfrak{p}(\cdot)}(w)} \\ &= 2^{i\Gamma}\,\|\mu\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v}_{\Gamma,\mathfrak{p}(\cdot)}(w)}. \end{aligned}$$

Dividing by $2^{i\Phi(i)}$, we obtain the claim. □

2.4. Pseudo-Differential Operators

In this subsection, we recall the class of pseudo-differential operators studied in Section 3, together with the kernel estimates of Goldberg [37] and the boundedness on weighted variable Lebesgue spaces that underlie the per-atom estimates. We use the Fourier transform $\hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(\zeta)e^{-2\pi i\zeta\cdot\xi}\,d\zeta$, and $\mathcal{D}^\gamma_\zeta$, $\mathcal{D}^\beta_\xi$ denote partial derivatives of orders $\gamma, \beta \in \mathbb{N}^n_0$ with respect to ζ and ξ. The operators considered below are determined by symbols of order zero, whose derivatives in ξ decay according to their order.

Definition 6 (Order-zero pseudo-differential operators). *A function* $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *belongs to the symbol class* $\mathbb{S}^0$ *if, for all multi-indices* γ *and* β,

$$|\mathcal{D}^\gamma_\zeta \mathcal{D}^\beta_\xi \sigma(\zeta,\xi)| \le C_{\gamma,\beta}\,(1+|\xi|)^{-|\beta|}, \qquad \zeta,\xi \in \mathbb{R}^n.$$

The order-zero pseudo-differential operator with symbol $\sigma \in \mathbb{S}^0$ *is defined by*

$$T\psi(\zeta) = \int_{\mathbb{R}^n} \hat{\psi}(\xi)\,\sigma(\zeta,\xi)\,e^{2\pi i\zeta\cdot\xi}\,d\xi. \tag{6}$$

Lemma 9 (Kernel and truncated-kernel estimates). *Let* $\sigma \in \mathbb{S}^0$, *and let* $K(\zeta,z)$ *be the distributional kernel of the operator* T *in* (6). *Let* $K_\tau = \varphi_\tau * K(\cdot,\cdot)$ *for a fixed* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\int\varphi = 1$. *Then, for every* $N > 0$, *there is a constant* $C = C(N)$ *such that, for* $\zeta \ne 0$, $|z| \le |\zeta|/2$ *and* $\theta \in (0,1)$,

$$|\mathcal{D}^\gamma_z K_\tau(\zeta, \zeta-\theta z)| \le C\,|\zeta|^{-n-N-1}, \qquad |\gamma| = N+1.$$

Moreover, if η *is a radial cutoff function with* $\operatorname{supp}\eta \subset \mathcal{B}(0,1)$ *and* $\eta \equiv 1$ *near* 0, *and if* $K_2 = (1-\eta)K$, *then, for every* $M \ge n$, *there is a constant* C_M *such that* $|(K_2)_\tau(\zeta,z)| \le C_M(1+|z|)^{-M}$.

Lemma 10 ($L^{\mathfrak{p}(\cdot)}(w)$-boundedness on fixed balls [15,37]). *Let* $\sigma \in \mathbb{S}^0$, $\mathfrak{p}(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ *and* $w \in \mathcal{A}_{\mathfrak{p}(\cdot)}(\mathbb{R}^n)$, *and let* $T_1 = \eta K * (\cdot)$ *and* $T_2 = (1-\eta)K * (\cdot)$, *with* η *as in Lemma 9. Then, for every function* $\mathfrak{a} \in L^{\mathfrak{p}(\cdot)}(w)$ *supported in a fixed ball, we have*

$$\|T\mathfrak{a}\|_{L^{\mathfrak{p}(\cdot)}(w)} + \|T_1\mathfrak{a}\|_{L^{\mathfrak{p}(\cdot)}(w)} + \|T_2\mathfrak{a}\|_{L^{\mathfrak{p}(\cdot)}(w)} \le C\,\|\mathfrak{a}\|_{L^{\mathfrak{p}(\cdot)}(w)}.$$

The boundedness of T on the constant-exponent, unweighted Herz-type Hardy spaces is due to Fan and Yang [36] as well as Tsutsui [15]; meanwhile, the boundedness on the variable-exponent space $H\dot{\mathcal{K}}^{\alpha,\mathrm{p}}_{\mathrm{q}(\cdot)}(\mathbb{R}^n)$ is due to Wang and Wu [38].

2.5. The Schrödinger Operator

Let $\mathcal{L} = -\Delta + \mathcal{V}(\zeta)$, $\zeta \in \mathbb{R}^n$, $n \geq 3$, where $\mathcal{V} \geq 0$ and $\mathcal{V} \in RH_{\mathrm{q}}$ for some $\mathrm{q} \geq n/2$, that is [39],

$$\Big(\frac{1}{|\mathcal{B}(\zeta,\nu)|}\int_{\mathcal{B}(\zeta,\nu)} |\mathcal{V}(\iota)|^{\mathrm{q}}\, d\iota\Big)^{1/\mathrm{q}} \leq \frac{C}{|\mathcal{B}(\zeta,\nu)|}\int_{\mathcal{B}(\zeta,\nu)} |\mathcal{V}(\iota)|\, d\iota, \qquad \zeta \in \mathbb{R}^n,\ \nu > 0$$

The critical-radius function ρ and the associated function $m_{\mathcal{V}}$ are defined by

$$\rho(\zeta) = \sup\Big\{\nu > 0 :\ \frac{1}{\nu^{n-2}}\int_{\mathcal{B}(\zeta,\nu)} \mathcal{V}(\iota)\, d\iota \leq 1\Big\}, \qquad m_{\mathcal{V}}(\zeta) := \frac{1}{\rho(\zeta)}, \qquad \zeta \in \mathbb{R}^n.$$

Lemma 11 (Reverse Hölder self-improvement [39]). *If $\mathcal{V} \in RH_{\mathrm{q}}$, $\mathrm{q} \geq n/2$, then there is $\varepsilon = \varepsilon(n,\mathrm{q},C) > 0$ such that $\mathcal{V} \in RH_{\mathrm{q}+\varepsilon}$. Consequently, $\delta_0 := \min\{1, 2 - n/\mathrm{q}\} > 0$.*

Lemma 12 (Doubling property of ρ [39]). *There are constants $C, \kappa_0 > 0$ such that $\rho(\zeta) \sim \rho(\iota)$ whenever $|\zeta - \iota| \leq C\rho(\zeta)$, and*

$$C^{-1}\rho(\zeta)\Big(1 + \frac{|\zeta-\iota|}{\rho(\zeta)}\Big)^{-\kappa_0} \leq \rho(\iota) \leq C\rho(\zeta)\Big(1 + \frac{|\zeta-\iota|}{\rho(\zeta)}\Big)^{\kappa_0/(\kappa_0+1)}, \qquad \zeta, \iota \in \mathbb{R}^n.$$

Lemma 13 (Critical-radius covering [42]). *There is a sequence $\{\zeta_\kappa\}_{\kappa\geq 1} \subset \mathbb{R}^n$ such that the balls $\mathcal{B}^{\mathcal{L}}_\kappa := \mathcal{B}(\zeta_\kappa, \rho(\zeta_\kappa))$ satisfy $\mathbb{R}^n = \bigcup_\kappa \mathcal{B}^{\mathcal{L}}_\kappa$, and there is $N \geq 1$ such that $\mathrm{card}\{j :\ 4\mathcal{B}^{\mathcal{L}}_j \cap 4\mathcal{B}^{\mathcal{L}}_\kappa \neq \emptyset\} \leq N$ for all $\kappa \geq 1$.*

Lemma 14 (Semigroup property [39,41]). *Let $\kappa_\tau(\zeta,\iota)$ denote the kernel of the heat semigroup $e^{-\tau\mathcal{L}}$, $\tau > 0$. Then, for $\tau, s > 0$ and $\zeta, \iota \in \mathbb{R}^n$,*

$$\kappa_{\tau+s}(\zeta,\iota) = \int_{\mathbb{R}^n} \kappa_\tau(\zeta,z)\,\kappa_s(z,\iota)\, dz \qquad \text{and} \qquad \int_{\mathbb{R}^n} \kappa_\tau(\zeta,\iota)\, d\iota \leq 1.$$

Lemma 15 (Gaussian bound [41]). *For every $N > 0$, there is a constant $C > 0$ such that, for all $\zeta, \iota \in \mathbb{R}^n$ and $\tau > 0$,*

$$\kappa_\tau(\zeta,\iota) \leq C\,\tau^{-n/2}\, e^{-|\zeta-\iota|^2/(C\tau)}\Big(1 + \frac{\sqrt{\tau}}{\rho(\zeta)} + \frac{\sqrt{\tau}}{\rho(\iota)}\Big)^{-N}. \tag{7}$$

Lemma 16 (Hölder regularity in the space variable [43]). *Let $0 < \delta < \min\{1, 2 - n/\mathrm{q}\}$. For every $N > 0$, there is a constant $C > 0$ such that, whenever $|\zeta - \zeta_0| < \sqrt{\tau}$,*

$$\big|\kappa_\tau(\zeta,\iota) - \kappa_\tau(\zeta_0,\iota)\big| \ \leq\ C\Big(\frac{|\zeta-\zeta_0|}{\sqrt{\tau}}\Big)^{\delta} \tau^{-n/2}\, e^{-|\zeta-\iota|^2/(C\tau)}\Big(1 + \frac{\sqrt{\tau}}{\rho(\zeta)} + \frac{\sqrt{\tau}}{\rho(\iota)}\Big)^{-N}.$$

2.6. The Bessel–Riesz Operator and Its Schrödinger Realization

For $1 \leq \mathrm{p} < \infty$, we denote by $L^{\mathrm{p}}_{\mathrm{loc}}(\mathbb{R}^n)$ the space of all measurable functions ψ on $\mathbb{R}^n$ such that $\psi\chi_K \in L^{\mathrm{p}}(\mathbb{R}^n)$ for every compact set $K \subset \mathbb{R}^n$.

Definition 7 (Bessel–Riesz operator [82,83]). *For $0 < \alpha < n$, $\gamma \geq 0$ and $\psi \in L^{p}_{\mathrm{loc}}(\mathbb{R}^n)$, the Bessel–Riesz operator is defined by*

$$I_{\alpha,\gamma}\psi(\zeta) := \int_{\mathbb{R}^n} K_{\alpha,\gamma}(\zeta-\iota)\,\psi(\iota)\,d\iota, \qquad K_{\alpha,\gamma}(\zeta) := \frac{|\zeta|^{\alpha-n}}{(1+|\zeta|)^{\gamma}}.$$

In particular, $I_{\alpha,0} = I_\alpha$ is the Riesz potential.

We attach $I_{\alpha,\gamma}$ to $\mathcal{L}$ by damping the heat-semigroup subordination formula for $\mathcal{L}^{-\alpha/2}$ (see [44,45,51]) with the same Bessel-type factor:

$$I_{\alpha,\gamma}\psi(\zeta) := \int_0^\infty e^{-\tau\mathcal{L}}\psi(\zeta)\,\tau^{\alpha/2-1}(1+\sqrt{\tau}\,)^{-\gamma}\,d\tau, \qquad 0<\alpha<n,\ \gamma\geq 0, \tag{8}$$

which reduces to $\mathcal{L}^{-\alpha/2}$ at $\gamma = 0$. Equivalently, $I_{\alpha,\gamma}\psi(\zeta) = \int_{\mathbb{R}^n} K^{\mathcal{L}}_{\alpha,\gamma}(\zeta,\iota)\psi(\iota)\,d\iota$, where

$$K^{\mathcal{L}}_{\alpha,\gamma}(\zeta,\iota) := \int_0^\infty \kappa_\tau(\zeta,\iota)\,\tau^{\alpha/2-1}(1+\sqrt{\tau}\,)^{-\gamma}\,d\tau.$$

Lemma 17 (Kernel bound). *For every $\ell \in \mathbb{N}$, there is a constant C_ℓ such that, for all $\zeta, \iota \in \mathbb{R}^n$ with $\zeta \neq \iota$,*

$$K^{\mathcal{L}}_{\alpha,\gamma}(\zeta,\iota) \leq \frac{C_\ell}{\big(1+|\zeta-\iota|\,(m_{\mathcal{V}}(\zeta)+m_{\mathcal{V}}(\iota))\big)^{\ell}} \cdot \frac{|\zeta-\iota|^{\alpha-n}}{(1+|\zeta-\iota|)^{\gamma}}. \tag{9}$$

In particular, $K^{\mathcal{L}}_{\alpha,\gamma}(\zeta,\iota) \leq C\,K_{\alpha,\gamma}(\zeta-\iota)$.

Proof. Fix $\zeta \neq \iota$, and write $d := |\zeta-\iota|$ and $m := m_{\mathcal{V}}(\zeta) + m_{\mathcal{V}}(\iota)$. Since $m_{\mathcal{V}} = 1/\rho$, Lemma 15, that is, (7) with $N = \ell$, gives

$$\kappa_\tau(\zeta,\iota) \leq C\tau^{-n/2}e^{-d^2/(C\tau)}(1+\sqrt{\tau}\,m)^{-\ell}.$$

We split the τ-integral defining $K^{\mathcal{L}}_{\alpha,\gamma}$ at $\tau = d^2$.

The region $0 < \tau \leq d^2$. Since $\sqrt{\tau} \leq d$, we have

$$\frac{1+d}{1+\sqrt{\tau}} \leq 1+\frac{d}{\sqrt{\tau}} \leq \frac{2d}{\sqrt{\tau}} \qquad \text{and} \qquad \frac{1+dm}{1+\sqrt{\tau}\,m} \leq \frac{d}{\sqrt{\tau}},$$

and therefore $(1+\sqrt{\tau})^{-\gamma} \leq 2^\gamma(1+d)^{-\gamma}(d^2/\tau)^{\gamma/2}$ and $(1+\sqrt{\tau}\,m)^{-\ell} \leq (1+dm)^{-\ell}(d^2/\tau)^{\ell/2}$. Since $u^{(\gamma+\ell)/2}e^{-u/C} \leq C'e^{-u/(2C)}$ for $u \geq 0$, we obtain

$$\begin{aligned}\int_0^{d^2} \kappa_\tau(\zeta,\iota)\tau^{\alpha/2-1}(1+\sqrt{\tau})^{-\gamma}\,d\tau &\leq \frac{C}{(1+dm)^{\ell}(1+d)^{\gamma}}\int_0^{d^2}\tau^{(\alpha-n)/2-1}e^{-d^2/(2C\tau)}\,d\tau \\ &= \frac{C\,d^{\alpha-n}}{(1+dm)^{\ell}(1+d)^{\gamma}}\int_0^1 u^{(\alpha-n)/2-1}e^{-1/(2Cu)}\,du,\end{aligned}$$

where we substituted $\tau = d^2u$; the last integral is finite.

The region $\tau > d^2$. Here, $\sqrt{\tau} > d$, so $(1+\sqrt{\tau})^{-\gamma} \leq (1+d)^{-\gamma}$ and $(1+\sqrt{\tau}\,m)^{-\ell} \leq (1+dm)^{-\ell}$. Using $e^{-d^2/(C\tau)} \leq 1$ and $\alpha < n$, we obtain

$$\begin{aligned}\int_{d^2}^\infty \kappa_\tau(\zeta,\iota)\tau^{\alpha/2-1}(1+\sqrt{\tau})^{-\gamma}\,d\tau &\leq \frac{C}{(1+dm)^{\ell}(1+d)^{\gamma}}\int_{d^2}^\infty\tau^{(\alpha-n)/2-1}\,d\tau \\ &= \frac{2C}{n-\alpha}\cdot\frac{d^{\alpha-n}}{(1+dm)^{\ell}(1+d)^{\gamma}}.\end{aligned}$$

Adding the two estimates, we obtain (9). The last assertion follows from $(1+dm)^{-\ell} \le 1$. □

Remark 4. *Since $(1+|\zeta-\iota|)^{-\gamma} \le 1$ for $\gamma \ge 0$, the bound (9) always improves the Riesz-type bound $K^{\mathcal{L}}_{\alpha,0}(\zeta,\iota) \le C_\ell(1+|\zeta-\iota|(m_{\mathcal{V}}(\zeta)+m_{\mathcal{V}}(\iota)))^{-\ell}|\zeta-\iota|^{\alpha-n}$ used for $\mathcal{L}^{-\alpha/2}$. This pointwise comparison is, however, the crudest use of the Bessel factor. Lemmas 18 and 19 below quantify the effect of this factor on the integrated kernel and show that it is also active in the local (near-field) regime.*

Lemma 18 (Sharp radial mass of the Bessel–Riesz kernel)**.** *For $0 < \alpha < n$ and $\gamma \ge 0$, set $\psi_{\alpha,\gamma}(R) := \int_0^R \nu^{\alpha-1}(1+\nu)^{-\gamma}\,d\nu$, $R > 0$. Then*

$$\psi_{\alpha,\gamma}(R) \;\sim\; \begin{cases} R^\alpha/\alpha, & 0 < R \le 1, \\ \dfrac{1}{\alpha} + \dfrac{R^{\alpha-\gamma}-1}{\alpha-\gamma}, & R > 1,\ \alpha \ne \gamma, \\ \dfrac{1}{\alpha} + \log R, & R > 1,\ \alpha = \gamma. \end{cases}$$

Consequently, $\psi_{\alpha,\gamma}(R) \le C_{\alpha,\gamma}R^\alpha$ for every $R > 0$. For $\gamma = 0$ or $R \le 1$, this is sharp, that is, $\psi_{\alpha,\gamma}(R) \sim R^\alpha$, whereas for $R > 1$ and $\gamma > 0$,

$$\psi_{\alpha,\gamma}(R) \;\le\; C_{\alpha,\gamma}\big(1 + R^{(\alpha-\gamma)^+}\log(2+R)\big) \;=\; o(R^\alpha) \qquad \text{as } R \to \infty.$$

Proof. We split the integral at $\nu = 1$. On $(0, \min\{R,1\})$, we have $(1+\nu)^{-\gamma} \sim 1$, so that

$$\int_0^{\min\{R,1\}} \nu^{\alpha-1}(1+\nu)^{-\gamma}\,d\nu \;\sim\; \int_0^{\min\{R,1\}} \nu^{\alpha-1}\,d\nu = \frac{\min\{R,1\}^\alpha}{\alpha}.$$

If $R > 1$, then on $(1, R)$ we have $(1+\nu)^{-\gamma} \sim \nu^{-\gamma}$, so that

$$\int_1^R \nu^{\alpha-1}(1+\nu)^{-\gamma}\,d\nu \;\sim\; \int_1^R \nu^{\alpha-1-\gamma}\,d\nu = \begin{cases} \dfrac{R^{\alpha-\gamma}-1}{\alpha-\gamma}, & \alpha \ne \gamma, \\ \log R, & \alpha = \gamma. \end{cases}$$

Adding the two pieces, we obtain the stated asymptotics. Since $\alpha - \gamma < \alpha$ for $\gamma > 0$, the bound $\psi_{\alpha,\gamma}(R) \le C_{\alpha,\gamma}R^\alpha$, together with its strict improvement for $R > 1$, follows at once. □

Lemma 19 (Sharp near-field bound)**.** *Let $\mathcal{B} = \mathcal{B}(\zeta_0, R)$ and $\psi_1 := \psi\chi_{2\mathcal{B}}$. Then we have*

$$\int_{\mathcal{B}} I_{\alpha,\gamma}(|\psi_1|)(\zeta)\,d\zeta \;\le\; C\,\psi_{\alpha,\gamma}(4R)\int_{2\mathcal{B}}|\psi(\iota)|\,d\iota, \tag{10}$$

where $\psi_{\alpha,\gamma}$ is as in Lemma 18. In particular, (10) is never worse than the bound $C\,R^\alpha\int_{2\mathcal{B}}|\psi(\iota)|\,d\iota$ obtained by discarding the Bessel factor via $(1+|\zeta-\iota|)^{-\gamma} \le 1$, and it is strictly sharper for $R > 1$ and $\gamma > 0$.

Proof. For $\zeta \in \mathcal{B}$ and $\iota \in 2\mathcal{B}$, we have $|\zeta-\iota| < 4R$. By Lemma 17 and Tonelli's theorem, we obtain

$$\begin{aligned}\int_{\mathcal{B}} I_{\alpha,\gamma}(|\psi_1|)(\zeta)\,d\zeta &\le C\int_{2\mathcal{B}}|\psi(\iota)|\int_{\mathcal{B}} K_{\alpha,\gamma}(\zeta-\iota)\,d\zeta\,d\iota \\ &\le C\int_{2\mathcal{B}}|\psi(\iota)|\int_{\mathcal{B}(\iota,4R)}|\zeta-\iota|^{\alpha-n}(1+|\zeta-\iota|)^{-\gamma}\,d\zeta\,d\iota.\end{aligned}$$

Using polar coordinates centered at ι, we obtain

$$\int_{\mathcal{B}(\iota,4R)} |\zeta-\iota|^{\alpha-n}(1+|\zeta-\iota|)^{-\gamma}\,d\zeta = C_1\int_0^{4R}\nu^{\alpha-1}(1+\nu)^{-\gamma}\,d\nu = C_1\,\psi_{\alpha,\gamma}(4R),$$

where C_1 is the surface measure of the unit sphere. This gives (10). The comparison with the crude bound follows from Lemma 18. □

2.7. Grand Variable Herz–Morrey–Hardy–Lipschitz Space

Let $\{\mathcal{B}_\kappa^{\mathcal{L}} = \mathcal{B}(\zeta_\kappa, \rho(\zeta_\kappa))\}_{\kappa\geq 1}$ be as in Lemma 13, and let $m_{\mathcal{B}}\psi = \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}}\psi$ denote the mean value of ψ over a ball $\mathcal{B}$. For a locally integrable function ψ and $\kappa \geq 1$, we set

$$\mathcal{E}_\kappa(\psi) := \frac{1}{|\mathcal{B}_\kappa^{\mathcal{L}}|^{\alpha/n}\,\|\chi_{\mathcal{B}_\kappa^{\mathcal{L}}}\|_{L^{p'(\cdot)}(w^{-1})}}\int_{\mathcal{B}_\kappa^{\mathcal{L}}}|\psi(\zeta)|\,d\zeta.$$

Definition 8 (Weighted grand variable Herz–Morrey–Hardy–Lipschitz space)**.** *Let $0<\alpha<n$, $0<v<\infty$, $0\leq\Gamma<\infty$, $p(\cdot)\in P^{\log}(\mathbb{R}^n)$ and $w\in\mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$. A locally integrable function ψ belongs to $H\dot{M}\mathrm{Lip}^{\alpha,v)}_{\mathcal{L},\Gamma,p(\cdot)}(w)$ if the following two conditions hold:*

(a) There is $C_1>0$ such that, for every ball $\mathcal{B}=\mathcal{B}(\zeta,R)$ with $R<\rho(\zeta)$,

$$\frac{1}{|\mathcal{B}|^{\alpha/n}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}}\int_{\mathcal{B}}|\psi(\zeta')-m_{\mathcal{B}}\psi|\,d\zeta' \leq C_1;$$

(b) The grand–Morrey aggregated size quantity is finite, that is,

$$[\psi]_{\mathcal{L}} := \sup_{\vartheta>0}\sup_{m_0\geq 1} m_0^{-\Gamma}\Big(\sum_{\kappa=1}^{m_0}\mathcal{E}_\kappa(\psi)^{v(1+\vartheta)}\Big)^{1/(v(1+\vartheta))} < \infty.$$

The norm $\|\psi\|_{H\dot{M}\mathrm{Lip}^{\alpha,v)}_{\mathcal{L},\Gamma,p(\cdot)}(w)}$ is defined as the maximum of the smallest constant C_1 in (a) and the quantity $[\psi]_{\mathcal{L}}$ in (b).

Remark 5. *Letting $v\to\infty$, with $\Gamma=0$ and $w\equiv 1$, condition (b) becomes*

$$\sup_{\kappa\geq 1}\frac{1}{|\mathcal{B}_\kappa^{\mathcal{L}}|^{\alpha/n}\,\|\chi_{\mathcal{B}_\kappa^{\mathcal{L}}}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}\int_{\mathcal{B}_\kappa^{\mathcal{L}}}|\psi(\zeta)|\,d\zeta \;\leq\; C_2,$$

which is the condition characterizing $\mathrm{Lip}^{\alpha,p(\cdot)}_{\mathcal{L}}(\mathbb{R}^n)$ [50,51]. For finite v and $\Gamma>0$, the space $H\dot{M}\mathrm{Lip}^{\alpha,v)}_{\mathcal{L},\Gamma,p(\cdot)}(w)$ strictly contains $\mathrm{Lip}^{\alpha,p(\cdot)}_{\mathcal{L}}(\mathbb{R}^n)$.

Lemma 20. *Suppose that $\Gamma\geq 1/v$. If ψ satisfies condition (a) of Definition 8 with constant C_1, and*

$$\mathcal{E}_\kappa(\psi) = \frac{1}{|\mathcal{B}_\kappa^{\mathcal{L}}|^{\alpha/n}\,\|\chi_{\mathcal{B}_\kappa^{\mathcal{L}}}\|_{L^{p'(\cdot)}(w^{-1})}}\int_{\mathcal{B}_\kappa^{\mathcal{L}}}|\psi(\zeta)|\,d\zeta \leq C_2, \qquad \kappa\geq 1, \tag{11}$$

with C_2 independent of κ, then $\psi\in H\dot{M}\mathrm{Lip}^{\alpha,v)}_{\mathcal{L},\Gamma,p(\cdot)}(w)$ and $\|\psi\|_{H\dot{M}\mathrm{Lip}^{\alpha,v)}_{\mathcal{L},\Gamma,p(\cdot)}(w)}\leq C_1+C_2$.

Proof. Let $\vartheta>0$ and $m_0\geq 1$. By (11), we have

$$m_0^{-\Gamma}\Big(\sum_{\kappa=1}^{m_0}\mathcal{E}_\kappa(\psi)^{v(1+\vartheta)}\Big)^{1/(v(1+\vartheta))} \leq m_0^{-\Gamma}\big(m_0\,C_2^{v(1+\vartheta)}\big)^{1/(v(1+\vartheta))} = C_2\,m_0^{1/(v(1+\vartheta))-\Gamma}.$$

Since $\vartheta > 0$, we have $1/(v(1+\vartheta)) < 1/v \le \Gamma$. Hence, the exponent is negative, and $m_0^{1/(v(1+\vartheta))-\Gamma} \le 1$ for $m_0 \ge 1$. This means that $[\psi]_{\mathcal{L}} \le C_2$. Combined with condition (a), this proves the claim. □

2.8. A Grand Variable Herz Space as Domain

The strong-type Theorem 3 takes its domain, $L^{p(\cdot)}(w)$, directly from the classical theory. In place of the weak-type domain $L^{p(\cdot),\infty}(w)$ used in [50,51], Theorem 4 uses a domain built from the same grand variable Herz mechanism as in Definition 2, with independent parameters. This couples the two ends of the operator to the same scale of spaces.

Definition 9 (Weighted grand variable Herz space)**.** *Let $\beta(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous at the origin and at infinity, $0 < v < \infty$, $0 \le \Lambda < \infty$, $p(\cdot) \in P(\mathbb{R}^n)$ and $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$. Set $\eta(\kappa) := \beta(0)$ for $\kappa \le 0$ and $\eta(\kappa) := \beta_\infty$ for $\kappa > 0$. The weighted grand variable Herz space $M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{\Lambda,p(\cdot)}(w)$ is the set of all measurable functions ψ on $\mathbb{R}^n \setminus \{0\}$ such that*

$$\begin{aligned} &\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{\Lambda,p(\cdot)}(w)} \\ &:= \sup_{\vartheta>0} \sup_{m_0\in\mathbb{Z}} 2^{-m_0\Lambda} \left(\sum_{\kappa=-\infty}^{m_0} 2^{\kappa\eta(\kappa)v(1+\vartheta)} \|\psi\chi_\kappa\|^{v(1+\vartheta)}_{L^{p(\cdot)}(w)} \right)^{1/(v(1+\vartheta))} < \infty. \end{aligned} \tag{12}$$

The space in Definition 9 is precisely the space $M\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)$ of Definition 2 with $(\alpha(\cdot), v, \Gamma)$ replaced by $(\beta(\cdot), v, \Lambda)$. It is used here directly on functions ψ, not on $\mathcal{G}_N\psi$.

Lemma 21 (Per-annulus control by the Herz norm)**.** *For every $\kappa \in \mathbb{Z}$ and every $\psi \in M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{\Lambda,p(\cdot)}(w)$, we have*

$$\|\psi\chi_\kappa\|_{L^{p(\cdot)}(w)} \le 2^{\kappa(\Lambda-\eta(\kappa))} \|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{\Lambda,p(\cdot)}(w)}.$$

Proof. Fix $\vartheta > 0$ and take $m_0 = \kappa$ in (12). Dropping the non-negative terms with index $j < \kappa$ from the sum leaves only the term $j = \kappa$, so that $2^{\kappa\eta(\kappa)}\|\psi\chi_\kappa\|_{L^{p(\cdot)}(w)} \le 2^{\kappa\Lambda}\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{\Lambda,p(\cdot)}(w)}$. This bound does not depend on ϑ, and the claim follows. □

Lemma 22 (Uniform local control by the Herz norm)**.** *Let $\beta(\cdot)$ be as in Definition 9 with $\beta(0) < 0 < \beta_\infty$, and let $\Lambda = 0$. Then, for every measurable set $E \subset \mathbb{R}^n$ and every $\psi \in M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)$, we have*

$$\|\psi\chi_E\|_{L^{p(\cdot)}(w)} \le C_\beta \|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}, \qquad C_\beta := \frac{1}{1-2^{\beta(0)}} + \frac{1}{2^{\beta_\infty}-1}.$$

Moreover, if $E \subset \mathcal{B}_m$ for some $m \in \mathbb{Z}$, then

$$\|\psi\chi_E\|_{L^{p(\cdot)}(w)} \le \Big(\sum_{j\le m} 2^{-j\eta(j)} \Big) \|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)} \le C_\beta \|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}.$$

Proof. Since the annuli $\mathcal{A}_j$, $j \in \mathbb{Z}$, are pairwise disjoint and cover $\mathbb{R}^n \setminus \{0\}$, we have $|\psi\chi_E| \le \sum_j |\psi\chi_j|$ a.e. By the lattice property, the triangle inequality and the Fatou property of $L^{p(\cdot)}(w)$, as well as by Lemma 21 with $\Lambda = 0$, we obtain

$$\|\psi\chi_E\|_{L^{p(\cdot)}(w)} \le \sum_{j:\,\mathcal{A}_j\cap E\neq\emptyset} \|\psi\chi_j\|_{L^{p(\cdot)}(w)} \le \sum_{j:\,\mathcal{A}_j\cap E\neq\emptyset} 2^{-j\eta(j)} \|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}.$$

If $E \subset \mathcal{B}_m$, only indices $j \le m$ occur. Finally, since $\beta(0) < 0 < \beta_\infty$, we have

$$\sum_{j\in\mathbb{Z}} 2^{-j\eta(j)} = \sum_{i=0}^{\infty} 2^{i\beta(0)} + \sum_{j=1}^{\infty} 2^{-j\beta_\infty} = C_\beta < \infty.$$

□

In Theorem 4 below, we assume, in addition to $\mathcal{V} \in RH_q$, that the critical radius is bounded, that is,

$$\rho(\zeta) \le \rho_{\max} < \infty, \qquad \zeta \in \mathbb{R}^n. \tag{13}$$

This condition is satisfied whenever $m_{\mathcal{V}}(\zeta) \ge m_{\min} > 0$ for all ζ, for instance, whenever $\mathcal{V}$ is bounded below by a positive constant on $\mathbb{R}^n$. Under (13), every ball $\mathcal{B} = \mathcal{B}(\zeta, R)$ with $R < \rho(\zeta)$ has radius $R \le \rho_{\max}$. This means that its size no longer grows with the location ζ, which allows a uniform comparison of $\mathcal{B}$ with the origin-centered dyadic scale.

Lemma 23 (Uniform comparison with the origin-centered scale)**.** *Assume* (13)*, and, for* $\zeta \in \mathbb{R}^n$*, let* $m = m(\zeta) := \lceil \log_2(|\zeta| + 2\rho_{\max} + 1)\rceil$*. Then, for every integer* $C_1 \ge 1$*, independent of* ζ*, and every ball* $\mathcal{B} = \mathcal{B}(\zeta, R)$ *with* $R \le \rho(\zeta) \le \rho_{\max}$*, we have*

$$2\mathcal{B} \subset \mathcal{B}_m \subset \mathcal{B}_{m+C_1}.$$

More generally, for every $t \ge 1$*,*

$$t\mathcal{B} \subset \mathcal{B}_{m+C_1} \qquad \textit{whenever } C_1 \ge \max\{1, \lceil \log_2 t\rceil\}.$$

Proof. Let $\iota \in 2\mathcal{B}$. Since $R \le \rho_{\max}$, the definition of m gives

$$|\iota| \le |\zeta| + |\iota - \zeta| < |\zeta| + 2R \le |\zeta| + 2\rho_{\max} < |\zeta| + 2\rho_{\max} + 1 \le 2^m,$$

and hence $2\mathcal{B} \subset \mathcal{B}_m$. Moreover, the dyadic balls are nested: if $\kappa \le \kappa'$, then $2^\kappa \le 2^{\kappa'}$, and so $\mathcal{B}_\kappa \subset \mathcal{B}_{\kappa'}$. We now distinguish two cases.

- If $C_1 = 1$, then we directly obtain $2\mathcal{B} \subset \mathcal{B}_m \subset \mathcal{B}_{m+1}$.
- If $C_1 > 1$, then $m < m + 1 < m + C_1$, and the nesting property gives $2\mathcal{B} \subset \mathcal{B}_m \subset \mathcal{B}_{m+1} \subset \mathcal{B}_{m+C_1}$.

Thus, a larger C_1 only enlarges the comparison ball, and the inclusion holds for every integer $C_1 \ge 1$.

The case $C_1 > 1$ is actually needed for dilations $t\mathcal{B}$ with $t > 2$, which occur in the dyadic decompositions of Section 4. Indeed, for $t \ge 1$ and $\iota \in t\mathcal{B}$, we have

$$|\iota| < |\zeta| + tR \le t(|\zeta| + \rho_{\max}) < t\,2^m \le 2^{m+\lceil \log_2 t\rceil} \le 2^{m+C_1},$$

provided that $C_1 \ge \max\{1, \lceil \log_2 t\rceil\}$. Hence, $t\mathcal{B} \subset \mathcal{B}_{m+C_1}$. In all cases, C_1 depends only on the fixed dilation factor t, and not on the center ζ or the radius R of the ball. This completes the proof. □

Theorem 4 of Section 4 is stated and proven using this domain.

3. Pseudo-Differential Operators

Theorem 2. *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous at 0 and at ∞, with $n\delta_2 \le \alpha(0) < \infty$ and $0 < \alpha_\infty < \infty$, and let the atoms of Definition 5 have vanishing moments up to order $s_0 \ge [\max\{\alpha(0), \alpha_\infty\} - n\delta_2]$. Set*

$$\varkappa := s_0 + 1 + n\delta_2 - \max\{\alpha(0), \alpha_\infty\} > 0, \qquad \varepsilon_0 := \min\{\alpha(0), \alpha_\infty, \varkappa\} > 0.$$

Let $0 < v < \infty$, $0 \le \Gamma < \varepsilon_0$, $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n) \subset \mathfrak{B}(\mathbb{R}^n)$ and $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$. If T is an order-zero pseudo-differential operator (Definition 6), then we obtain

$$\|T\psi\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(w)} \le C\,\|\psi\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(w)}, \qquad \psi \in HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(w).$$

Proof. By Lemma 6, we have $\psi = \sum_{\kappa=-\infty}^{\infty} \lambda_\kappa \mathfrak{a}_\kappa$ in $\mathcal{S}'(\mathbb{R}^n)$, where each $\mathfrak{a}_\kappa$ is a weighted central $(\alpha(\cdot), \mathrm{p}(\cdot))$-atom for $\mathcal{B}_\kappa$, and we may take the decomposition so that, with $\mathcal{N}_\lambda$ as in (1),

$$\mathcal{N}_\lambda = \sup_{\vartheta>0}\sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0} |\lambda_\kappa|^{v(1+\vartheta)}\right)^{1/(v(1+\vartheta))} \le 2\,\|\psi\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(w)}. \tag{14}$$

Throughout the proof, we use the vanishing moments up to order $N := s_0$, so that $N + 1 > \max\{\alpha(0), \alpha_\infty\} - n\delta_2$. We treat the cases $0 < v \le 1$ and $1 < v < \infty$ separately.

Case I: $0 < v \le 1$.

It suffices to show

$$\|T\mathfrak{a}_\kappa\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(w)} \le C, \qquad C \text{ independent of } \kappa, \tag{15}$$

in the following stronger, annulus-wise form: for all $\kappa, j \in \mathbb{Z}$,

$$b_{j,\kappa} := 2^{j\Phi(j)}\|\mathcal{G}_N(T\mathfrak{a}_\kappa)\chi_j\|_{L^{\mathrm{p}(\cdot)}(w)} \le C\,2^{-\varepsilon_0|j-\kappa|}, \tag{16}$$

with C independent of j and κ. Indeed, (16) implies (15), and it is exactly the off-diagonal decay required in Lemma 7. Since $T\psi = \sum_\kappa \lambda_\kappa T\mathfrak{a}_\kappa$, we have $2^{j\Phi(j)}\|\mathcal{G}_N(T\psi)\chi_j\|_{L^{\mathrm{p}(\cdot)}(w)} \le \sum_\kappa |\lambda_\kappa| b_{j,\kappa}$. Therefore, once (16) is proved, Lemma 7 (with $\varepsilon = \varepsilon_0 > \Gamma$) and (14) give the full norm bound (23) below. A uniform bound such as (15) alone would not suffice for this aggregation (Remark 3(a)). We split the proof according to $\kappa \le 0$ and $\kappa > 0$.

(a) $\kappa \le 0$: the estimates of I_1 and I_2.

Fix $\kappa \le 0$. By Definition 2, we have to estimate the quantities $2^{j\Phi(j)}\|\mathcal{G}_N(T\mathfrak{a}_\kappa)\chi_j\|_{L^{\mathrm{p}(\cdot)}(w)}$, $j \in \mathbb{Z}$. We split the corresponding sum as

$$\begin{aligned}\sum_{j=-\infty}^{\infty} 2^{j\Phi(j)v}\|\mathcal{G}_N(T\mathfrak{a}_\kappa)\chi_j\|^v_{L^{\mathrm{p}(\cdot)}(w)} &= \sum_{j=-\infty}^{\kappa+2} 2^{j\Phi(j)v}\|\mathcal{G}_N(T\mathfrak{a}_\kappa)\chi_j\|^v_{L^{\mathrm{p}(\cdot)}(w)} \\ &\quad + \sum_{j=\kappa+3}^{\infty} 2^{j\Phi(j)v}\|\mathcal{G}_N(T\mathfrak{a}_\kappa)\chi_j\|^v_{L^{\mathrm{p}(\cdot)}(w)} =: I_1 + I_2.\end{aligned}$$

Estimate of I_1. We use Lemma 3(a) (the boundedness of $\mathcal{M}$ on $L^{\mathrm{p}(\cdot)}(w)$), the pointwise bound $\mathcal{G}_N(T\mathfrak{a}_\kappa) \le C\mathcal{M}(T\mathfrak{a}_\kappa)$, Lemma 10, and the fact that $\phi(j) = \phi(\kappa)$ for $j \le \kappa + 2$, $j \le 0$

(the at most two remaining indices $j \in \{1,2\}$, which occur only for $\kappa \in \{-1,0\}$, contribute a bounded amount). This gives

$$\begin{aligned} I_1 &= \sum_{j=-\infty}^{\kappa+2} 2^{j\Phi(j)v} \|\mathcal{G}_N(T\mathfrak{a}_\kappa)\chi_j\|^v_{L^{p(\cdot)}(w)} \\ &\le C \sum_{j=-\infty}^{\kappa+2} 2^{j\Phi(\kappa)v} \|\mathcal{M}(T\mathfrak{a}_\kappa)\|^v_{L^{p(\cdot)}(w)} \\ &\le C \sum_{j=-\infty}^{\kappa+2} 2^{j\Phi(\kappa)v} \|\mathfrak{a}_\kappa\|^v_{L^{p(\cdot)}(w)}. \end{aligned}$$

By the atom condition $\|\mathfrak{a}_\kappa\|_{L^{p(\cdot)}(w)} \le |\mathcal{B}_\kappa|^{-\Phi(\kappa)/n} = C_1\, 2^{-\kappa\Phi(\kappa)}$, where $C_1 := |\mathcal{B}(0,1)|^{-\Phi(\kappa)/n}$, we obtain

$$I_1 \le C \sum_{j=-\infty}^{\kappa+2} 2^{(j-\kappa)\Phi(\kappa)v} \le C, \tag{17}$$

since this is a convergent geometric series ($\Phi(\kappa) > 0$). In particular, for $j \le \kappa+2$, each term satisfies $b_{j,\kappa} \le C2^{(j-\kappa)\Phi(\kappa)} \le C2^{-\varepsilon_0|j-\kappa|}$, since $\Phi(\kappa) \ge \varepsilon_0$ and $j-\kappa \le 2$.

Estimate of I_2. By Lemma 9, we have $\varphi_\tau * (T\mathfrak{a}_\kappa)(\zeta) = \int_{\mathbb{R}^n} K_\tau(\zeta, \zeta - z)\mathfrak{a}_\kappa(z)\, dz$. Expanding $K_\tau(\zeta, \zeta - z)$ in a Taylor series about $z = 0$ and using $\int_{\mathbb{R}^n} \mathfrak{a}_\kappa(z) z^\beta\, dz = 0$ for $|\beta| \le N$, we obtain

$$\varphi_\tau * (T\mathfrak{a}_\kappa)(\zeta) = \sum_{|\gamma|=N+1} \int_{\mathbb{R}^n} \mathcal{D}_z^\gamma K_\tau(\zeta, \zeta - \theta z)\, z^\gamma\, \mathfrak{a}_\kappa(z)\, dz, \qquad \theta \in (0,1).$$

For $\zeta \in \mathcal{A}_j$, $j \ge \kappa+3$, Lemma 9 gives $|\mathcal{D}_z^\gamma K_\tau(\zeta, \zeta - \theta z)| \le C|\zeta|^{-n-N-1}$, and hence

$$|\varphi_\tau * (T\mathfrak{a}_\kappa)(\zeta)| \le \frac{C}{|\zeta|^{n+N+1}} \int_{\mathbb{R}^n} |z|^{N+1} |\mathfrak{a}_\kappa(z)|\, dz \le \frac{C\, 2^{\kappa(N+1)}}{|\zeta|^{n+N+1}} \int_{\mathbb{R}^n} |\mathfrak{a}_\kappa(z)|\, dz.$$

By Lemma 4(a), we have

$$\int_{\mathbb{R}^n} |\mathfrak{a}_\kappa(z)|\, dz \le \nu_p \|\mathfrak{a}_\kappa\|_{L^{p(\cdot)}(w)} \|\chi_{\mathcal{B}_\kappa}\|_{L^{p'(\cdot)}(w^{-1})} \le \nu_p\, |\mathcal{B}_\kappa|^{-\Phi(\kappa)/n} \|\chi_{\mathcal{B}_\kappa}\|_{L^{p'(\cdot)}(w^{-1})},$$

and, since $|\zeta| \sim 2^j$, we obtain

$$|\varphi_\tau * (T\mathfrak{a}_\kappa)(\zeta)| \le \frac{C\, 2^{\kappa(N+1)}}{2^{j(n+N+1)}}\, |\mathcal{B}_\kappa|^{-\Phi(\kappa)/n} \|\chi_{\mathcal{B}_\kappa}\|_{L^{p'(\cdot)}(w^{-1})}. \tag{18}$$

The same bound holds for $\mathcal{G}_N(T\mathfrak{a}_\kappa)(\zeta)$. Taking $L^{p(\cdot)}(w)$-norms over $\mathcal{A}_j$, we obtain

$$\|\mathcal{G}_N(T\mathfrak{a}_\kappa)\chi_j\|_{L^{p(\cdot)}(w)} \le \frac{C\, 2^{\kappa(N+1)}}{2^{j(n+N+1)}}\, |\mathcal{B}_\kappa|^{-\Phi(\kappa)/n} \|\chi_{\mathcal{B}_\kappa}\|_{L^{p'(\cdot)}(w^{-1})} \|\chi_{\mathcal{B}_j}\|_{L^{p(\cdot)}(w)}.$$

We apply Lemma 5 twice: once to $\|\chi_{\mathcal{B}_j}\|_{L^{p(\cdot)}(w)} / \|\chi_{\mathcal{B}_\kappa}\|_{L^{p(\cdot)}(w)}$ via the doubling estimate, and once to $\|\chi_{\mathcal{B}_\kappa}\|_{L^{p'(\cdot)}(w^{-1})} / \|\chi_{\mathcal{B}_j}\|_{L^{p'(\cdot)}(w^{-1})}$ via the δ_2-decay estimate, together with

$$\frac{1}{|\mathcal{B}_\kappa|}\, \|\chi_{\mathcal{B}_\kappa}\|_{L^{p(\cdot)}(w)}\, \|\chi_{\mathcal{B}_\kappa}\|_{L^{p'(\cdot)}(w^{-1})} \le C.$$

This gives

$$2^{j\Phi(j)} \|\mathcal{G}_N(T\mathfrak{a}_\kappa)\chi_j\|_{L^{p(\cdot)}(w)} \le C\, 2^{(\kappa-j)(N+1-\Phi(\kappa)+n\delta_2)}, \qquad j \ge \kappa+3. \tag{19}$$

Since $N+1 > \alpha(0) - n\delta_2 \geq \phi(\kappa) - n\delta_2$ for $\kappa \leq 0$, the exponent satisfies $N+1-\phi(\kappa)+n\delta_2 \geq \varkappa \geq \varepsilon_0 > 0$. Hence, (16) holds for $j \geq \kappa+3$, and we obtain

$$I_2 = \sum_{j=\kappa+3}^{\infty} 2^{j\phi(j)v} \|\mathcal{G}_N(T\mathfrak{a}_\kappa)\chi_j\|^v_{L^{p(\cdot)}(w)} \leq C \sum_{j=\kappa+3}^{\infty} 2^{(\kappa-j)v(N+1-\phi(\kappa)+n\delta_2)} \leq C. \tag{20}$$

(b) $\kappa > 0$: the η-splitting and the estimates of T_1, T_2, J_1 and J_2.

Choose a radial smooth function η with $\operatorname{supp}\eta \subset \mathcal{B}(0,1)$ and $\eta \equiv 1$ near the origin (Lemma 9). We split the kernel as $K = K_1 + K_2$, with $K_1 = \eta K$ and $K_2 = (1-\eta)K$, which gives $T = T_1 + T_2$, where both operators are of order zero.

Estimate of T_1. Since $\operatorname{supp}\mathcal{G}_N(T_1\mathfrak{a}_\kappa) \subset \mathcal{B}_{\kappa+1}$, Lemma 3(a) and Lemma 10 give, exactly as for I_1,

$$\begin{aligned}\sum_{j=-\infty}^{\kappa+1} 2^{j\phi(j)v} \|\mathcal{G}_N(T_1\mathfrak{a}_\kappa)\chi_j\|^v_{L^{p(\cdot)}(w)} &\leq C \sum_{j=-\infty}^{\kappa+1} 2^{j\phi(j)v} \|\mathcal{M}(T_1\mathfrak{a}_\kappa)\|^v_{L^{p(\cdot)}(w)} \\ &\leq C \sum_{j=-\infty}^{\kappa+1} 2^{(j\phi(j)-\kappa\phi(\kappa))v} \leq C. \end{aligned}\tag{21}$$

Here, for $j \leq 0 < \kappa$, we used $j\alpha(0) - \kappa\alpha_\infty \leq -(\kappa-j)\min\{\alpha(0), \alpha_\infty\} \leq -\varepsilon_0|j-\kappa|$. Hence, each term satisfies (16).

Estimate of T_2, giving J_1 and J_2. By Lemma 9, we have $|(K_2)_\tau(\zeta, z)| \leq C_M(1+|z|)^{-M}$, $M \geq n$. We split

$$\begin{aligned}\sum_{j=-\infty}^{\infty} 2^{j\phi(j)v} \|\mathcal{G}_N(T_2\mathfrak{a}_\kappa)\chi_j\|^v_{L^{p(\cdot)}(w)} &= \sum_{j=-\infty}^{\kappa+2} 2^{j\phi(j)v} \|\mathcal{G}_N(T_2\mathfrak{a}_\kappa)\chi_j\|^v_{L^{p(\cdot)}(w)} \\ &\quad + \sum_{j=\kappa+3}^{\infty} 2^{j\phi(j)v} \|\mathcal{G}_N(T_2\mathfrak{a}_\kappa)\chi_j\|^v_{L^{p(\cdot)}(w)} =: J_1 + J_2.\end{aligned}$$

J_1 is estimated exactly as I_1. For J_2, with $\zeta \in \mathcal{A}_j$, $j \geq \kappa+3$, we have

$$\begin{aligned}|\varphi_\tau * (T_2\mathfrak{a}_\kappa)(\zeta)| &= \Big| \int_{\mathbb{R}^n} (K_2)_\tau(\zeta, \zeta-z)\, \mathfrak{a}_\kappa(z)\, dz \Big| \\ &\leq C_M \int_{\mathbb{R}^n} \frac{|\mathfrak{a}_\kappa(z)|}{(1+|\zeta-z|)^M}\, dz \\ &\leq \frac{C}{|\zeta|^{n+N+1}} \int_{\mathbb{R}^n} |\mathfrak{a}_\kappa(z)|\, dz,\end{aligned}$$

where $N+1 > \alpha_\infty - n\delta_2$. By the same computation as in (18) and (19), with $\phi(\kappa) = \alpha_\infty$ for $\kappa > 0$, we obtain

$$2^{j\phi(j)} \|\mathcal{G}_N(T_2\mathfrak{a}_\kappa)\chi_j\|_{L^{p(\cdot)}(w)} \leq C\, 2^{(\kappa-j)(N+1-\alpha_\infty+n\delta_2)}, \qquad j \geq \kappa+3.$$

Since $N+1-\alpha_\infty+n\delta_2 \geq \varkappa \geq \varepsilon_0 > 0$, (16) holds, and we get

$$J_2 \leq C \sum_{j=\kappa+3}^{\infty} 2^{(\kappa-j)v(N+1-\alpha_\infty+n\delta_2)} \leq C. \tag{22}$$

Combining (17), (20), (21) and (22), we obtain (16) for all $j, \kappa \in \mathbb{Z}$, and in particular (15):

$$\|T\mathfrak{a}_\kappa\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)} \le C\big(I_1 + I_2\big)^{1/v} \le C \qquad (\kappa \le 0),$$
$$\|T\mathfrak{a}_\kappa\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)} \le C\big(T_1\text{-bound} + J_1 + J_2\big)^{1/v} \le C \qquad (\kappa > 0).$$

Substitution into the norm.

By the linearity $T\psi = \sum_j \lambda_j T\mathfrak{a}_j$, we have, for every $\kappa \in \mathbb{Z}$,

$$2^{\kappa\phi(\kappa)}\|\mathcal{G}_N(T\psi)\chi_\kappa\|_{L^{p(\cdot)}(w)} \le \sum_{j=-\infty}^{\infty} |\lambda_j|\, b_{\kappa,j},$$

where $b_{\kappa,j}$ is as in (16) and satisfies $b_{\kappa,j} \le C2^{-\varepsilon_0|\kappa-j|}$. Since $0 \le \Gamma < \varepsilon_0$, Lemma 7 applies with $\varepsilon = \varepsilon_0$. Hence, for every $\vartheta > 0$ and $m_0 \in \mathbb{Z}$,

$$2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0} 2^{\kappa\phi(\kappa)v(1+\vartheta)}\|\mathcal{G}_N(T\psi)\chi_\kappa\|^{v(1+\vartheta)}_{L^{p(\cdot)}(w)}\right)^{1/(v(1+\vartheta))}$$
$$\le 2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0}\Big(\sum_{j=-\infty}^{\infty}|\lambda_j|\, b_{\kappa,j}\Big)^{v(1+\vartheta)}\right)^{1/(v(1+\vartheta))} \le C\,\mathcal{N}_\lambda,$$

with C independent of ϑ and m_0. Taking $\sup_{\vartheta>0}\sup_{m_0\in\mathbb{Z}}$ and using (14), we obtain

$$\|T\psi\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)} \le C\,\|\psi\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)}, \qquad 0 < v \le 1. \tag{23}$$

Case II: $1 < v < \infty$.

Fix $\vartheta > 0$ and $m_0 \in \mathbb{Z}$, and let $s := v(1+\vartheta) > 1$. By Definition 2 and $T\psi = \sum_j \lambda_j T\mathfrak{a}_j$, we have

$$2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0} 2^{\kappa\phi(\kappa)s}\|\mathcal{G}_N(T\psi)\chi_\kappa\|^{s}_{L^{p(\cdot)}(w)}\right)^{1/s}$$
$$\le 2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0} 2^{\kappa\phi(\kappa)s}\Big(\sum_{j=-\infty}^{\infty}|\lambda_j|\,\|\mathcal{G}_N(T\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)}\Big)^{s}\right)^{1/s}$$
$$\le 2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0} 2^{\kappa\phi(\kappa)s}\Big(\sum_{j=-\infty}^{\kappa-1}|\lambda_j|\,\|\mathcal{G}_N(T\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)}\Big)^{s}\right)^{1/s}$$
$$+ 2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0} 2^{\kappa\phi(\kappa)s}\Big(\sum_{j=\kappa}^{\infty}|\lambda_j|\,\|\mathcal{G}_N(T\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)}\Big)^{s}\right)^{1/s} =: U_1 + U_2. \tag{24}$$

Estimate of U_2. As in (17)–(21) (near field, $j \ge \kappa$), we have, for $\kappa \le j$,

$$2^{\kappa\phi(\kappa)}\|\mathcal{G}_N(T\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)} \le C\,2^{\kappa\phi(\kappa)-j\phi(j)} \le C\,2^{-\varepsilon_0(j-\kappa)}.$$

Here, the middle term equals $C2^{(\kappa-j)\phi(\kappa)}$ when $\phi(\kappa) = \phi(j)$, and the last inequality follows as in (21) when $\kappa \le 0 < j$. In U_2, the index j runs over all $j \ge \kappa$, including $j > m_0$.

Therefore, we apply Lemma 7 with $b_{\kappa,j} := C\,2^{-\varepsilon_0(j-\kappa)}$ for $j \geq \kappa$, $b_{\kappa,j} := 0$ for $j < \kappa$, and $\varepsilon = \varepsilon_0 > \Gamma$. This gives

$$U_2 \leq C\,2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0}\Big(\sum_{j=\kappa}^{\infty}|\lambda_j|\,2^{-\varepsilon_0(j-\kappa)}\Big)^s\right)^{1/s} \leq C\,\mathcal{N}_\lambda, \tag{25}$$

with C independent of ϑ and m_0.

Estimate of U_1: further split into V_1, V_2 and V_3. We write

$$\begin{aligned} U_1 &= 2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0}2^{\kappa\phi(\kappa)s}\Big(\sum_{j=-\infty}^{\kappa-1}|\lambda_j|\,\|\mathcal{G}_N(T\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)}\Big)^s\right)^{1/s} \\ &\leq 2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{0}2^{\kappa\phi(\kappa)s}\Big(\sum_{j=-\infty}^{\kappa-1}|\lambda_j|\,\|\mathcal{G}_N(T\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)}\Big)^s\right)^{1/s} \\ &\quad + 2^{-m_0\Gamma}\left(\sum_{\kappa=1}^{m_0}2^{\kappa\phi(\kappa)s}\Big(\sum_{j=-\infty}^{\kappa-1}|\lambda_j|\,\|\mathcal{G}_N(T_1\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)}\Big)^s\right)^{1/s} \\ &\quad + 2^{-m_0\Gamma}\left(\sum_{\kappa=1}^{m_0}2^{\kappa\phi(\kappa)s}\Big(\sum_{j=-\infty}^{\kappa-1}|\lambda_j|\,\|\mathcal{G}_N(T_2\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)}\Big)^s\right)^{1/s} =: V_1+V_2+V_3. \end{aligned} \tag{26}$$

V_2 is estimated by the near-field bound (21), exactly as U_2. For V_1 (using the vanishing-moment estimate (19) with $\phi(0) = \alpha(0)$, $\zeta \in \mathcal{A}_\kappa$, $\kappa \geq j+1$) and for V_3 (using the kernel-decay estimate (22) for T_2), we have, for $\kappa \geq j+1$,

$$2^{\kappa\phi(\kappa)}\|\mathcal{G}_N(T\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)} + 2^{\kappa\phi(\kappa)}\|\mathcal{G}_N(T_2\mathfrak{a}_j)\chi_\kappa\|_{L^{p(\cdot)}(w)} \leq C\,2^{(j-\kappa)\varkappa},$$

where

$$\varkappa = \min\{N+1-\alpha(0)+n\delta_2,\ N+1-\alpha_\infty+n\delta_2\} = s_0+1+n\delta_2-\max\{\alpha(0),\alpha_\infty\} > 0, \tag{27}$$

in accordance with the definition of $\varkappa$ in the statement of the theorem. Since $j \leq \kappa-1 < m_0$, only indices $j \leq m_0$ occur. Hence, Young's convolution inequality $\ell^1 * \ell^s \hookrightarrow \ell^s(\mathbb{Z})$ $(s>1)$, applied to the exponentially decaying kernel $\{2^{-|\kappa|\varkappa}\}_{\kappa\in\mathbb{Z}}$ with $\varkappa$ as in (27), gives

$$\begin{aligned} V_1+V_3 &\leq C\,2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0}\Big(\sum_{j=-\infty}^{\kappa-1}|\lambda_j|\,2^{(j-\kappa)\varkappa}\Big)^s\right)^{1/s} \leq C\,2^{-m_0\Gamma}\left(\sum_{j=-\infty}^{m_0}|\lambda_j|^s\right)^{1/s} \\ &= C\,2^{-m_0\Gamma}\Big(\sum_{j=-\infty}^{m_0}|\lambda_j|^{v(1+\vartheta)}\Big)^{1/(v(1+\vartheta))} \leq C\,\mathcal{N}_\lambda. \end{aligned} \tag{28}$$

Combining (26), (28) and the estimate of V_2, we obtain $U_1 = V_1+V_2+V_3 \leq C\,\mathcal{N}_\lambda$.

Substitution into the norm.

From (24), (25) and the bound on U_1, we obtain

$$2^{-m_0\Gamma}\left(\sum_{\kappa=-\infty}^{m_0}2^{\kappa\phi(\kappa)v(1+\vartheta)}\|\mathcal{G}_N(T\psi)\chi_\kappa\|^{v(1+\vartheta)}_{L^{p(\cdot)}(w)}\right)^{1/(v(1+\vartheta))} \leq C\,\mathcal{N}_\lambda,$$

with C independent of ϑ and m_0. Taking $\sup_{\vartheta>0}\sup_{m_0\in\mathbb{Z}}$ and applying (14), we obtain

$$\|T\psi\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)} \leq C\,\|\psi\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)}, \qquad 1<v<\infty. \tag{29}$$

By (23) and (29), in both cases, we have

$$\|T\psi\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(w)} \le C\,\|\psi\|_{HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,\mathrm{p}(\cdot)}(w)},$$

with $C = C(n, \alpha(0), \alpha_\infty, \delta_2, s_0, \Gamma, v, [w]_{\mathcal{A}_{\mathrm{p}(\cdot)}}, \{C_{\gamma,\beta}\})$ independent of ψ. □

Corollary 1. *Let $\Gamma = 0$, $w \equiv 1$, $v = \mathrm{p} \in (0,\infty)$, and $\alpha(\cdot) \equiv \alpha$ with $n\delta_2 \le \alpha < \infty$, and write the variable exponent as $\mathrm{q}(\cdot) \in P^{\log}(\mathbb{R}^n)$. Then Theorem 2 yields the boundedness $T : H\dot{\mathcal{K}}^{\alpha,\mathrm{p}}_{\mathrm{q}(\cdot)}(\mathbb{R}^n) \to H\dot{\mathcal{K}}^{\alpha,\mathrm{p}}_{\mathrm{q}(\cdot)}(\mathbb{R}^n)$ for constant order $\alpha(\cdot) \equiv \alpha$; see [38].*

Proof. Since $\varepsilon_0 > 0$, the choice $\Gamma = 0$ satisfies the hypothesis $0 \le \Gamma < \varepsilon_0$ of Theorem 2. For $\Gamma = 0$, the supremum over m_0 in Definition 2 becomes the full sum over $\kappa \in \mathbb{Z}$. Since $s \mapsto \|\{c_\kappa\}\|_{\ell^s}$ is nonincreasing, the supremum over $\vartheta > 0$ equals the limit as $\vartheta \to 0^+$, which is the ℓ^{p}-norm. Hence, the norm of $HM\dot{\mathcal{K}}^{\alpha,\mathrm{p})}_{0,\mathrm{q}(\cdot)}(1)$ coincides with that of $H\dot{\mathcal{K}}^{\alpha,\mathrm{p}}_{\mathrm{q}(\cdot)}(\mathbb{R}^n)$, and the claim follows from Theorem 2. □

Remark 6. *The Morrey parameter Γ enters the proof of Theorem 2 only through Lemma 7. The condition $0 \le \Gamma < \varepsilon_0$ is exactly what is needed to control, at a fixed truncation level m_0, the contribution of the atoms supported on $\mathcal{B}_j$ with $j > m_0$. For such j, only the bound $|\lambda_j| \le 2^{j\Gamma}\mathcal{N}_\lambda$ is available, and the decay $2^{-\varepsilon_0(j-m_0)}$ from (16) must dominate the growth $2^{j\Gamma}$. By Remark 3(b), this condition cannot be removed. In contrast, the condition $\Gamma \ge 1/v$ is the relevant one for the Lipschitz-type space of Definition 8, where the aggregation runs over the finite set $1 \le \kappa \le m_0$ with the normalization $m_0^{-\Gamma}$ (Lemma 20).*

4. Fractional Schrödinger Operators Involving the Bessel–Riesz Kernel

Theorem 3. *Let $\mathcal{V} \in RH_{\mathrm{q}}$, $\mathrm{q} \ge n/2$, $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n)$, $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$ and $\gamma \ge 0$. Assume that $1 < \mathrm{p}^- \le \mathrm{p}^+ < \big(n/(\alpha-\delta_0)\big)^+$, where $\delta_0 = \min\{1, 2 - n/\mathrm{q}\}$, and that $\Gamma \ge 1/v$ for some fixed $0 < v < \infty$. Then the operator $I_{\alpha,\gamma}$ in (8) is bounded from $L^{\mathrm{p}(\cdot)}(w)$ into $HM\dot{\mathrm{Lip}}^{\alpha,v)}_{\mathcal{L},\Gamma,\mathrm{p}(\cdot)}(w)$, that is,*

$$\|I_{\alpha,\gamma}\psi\|_{HM\dot{\mathrm{Lip}}^{\alpha,v)}_{\mathcal{L},\Gamma,\mathrm{p}(\cdot)}(w)} \le C\,\|\psi\|_{L^{\mathrm{p}(\cdot)}(w)}.$$

Theorem 4. *Let $\mathcal{V} \in RH_{\mathrm{q}}$, $\mathrm{q} \ge n/2$, satisfy (13). Let $\mathrm{p}(\cdot) \in P^{\log}(\mathbb{R}^n)$, $w \in \mathcal{A}_{\mathrm{p}(\cdot)}(\mathbb{R}^n)$, $0 < \alpha < n$, $\gamma \ge 0$, $1 < \mathrm{p}^- \le \mathrm{p}^+ < \big(n/(\alpha-\delta_0)\big)^+$ with $\delta_0 = \min\{1, 2 - n/\mathrm{q}\}$, and $\Gamma \ge 1/v$ for some fixed $0 < v < \infty$. Let $\beta(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous at the origin and at infinity, with $\beta(0) < 0$ and $\beta_\infty \ge n/\mathrm{p}'(\infty)$, and let $0 < v < \infty$. Then the operator $I_{\alpha,\gamma}$ in (8) is bounded from the weighted grand variable Herz space $M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,\mathrm{p}(\cdot)}(w)$ (Definition 9 with $\Lambda = 0$) into $HM\dot{\mathrm{Lip}}^{\alpha,v)}_{\mathcal{L},\Gamma,\mathrm{p}(\cdot)}(w)$, that is,*

$$\|I_{\alpha,\gamma}\psi\|_{HM\dot{\mathrm{Lip}}^{\alpha,v)}_{\mathcal{L},\Gamma,\mathrm{p}(\cdot)}(w)} \le C\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,\mathrm{p}(\cdot)}(w)}.$$

Remark 7. *The domain-side Morrey parameter is forced to be 0. If it were positive, then (12) would allow $\|\psi\chi_\kappa\|_{L^{\mathrm{p}(\cdot)}(w)}$ to grow as $\kappa \to \infty$, and since $\{\mathcal{B}^{\mathcal{L}}_\kappa\}$ reaches arbitrarily far from the origin, no bound uniform over the covering could hold. This is the opposite of the range-side threshold $\Gamma \ge 1/v$ (Lemma 20), where growth is permitted and damped from outside.*

By Lemma 20, Theorem 3 reduces to establishing the following two properties uniformly in the covering index $\kappa \ge 1$ of Lemma 13.

(P1) For every ball $\mathcal{B} = \mathcal{B}(\zeta_0, \nu)$ with $\nu < \rho(\zeta_0)$, there is a constant $c = c(\mathcal{B})$ such that

$$\int_{\mathcal{B}} |I_{\alpha,\gamma}\psi(\zeta) - c|\, d\zeta \le C\, |\mathcal{B}|^{\alpha/n}\, \|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\, \|\psi\|_{L^{p(\cdot)}(w)}.$$

(P2) For every $\kappa \ge 1$ and $\mathcal{B}_\kappa^{\mathcal{L}} = \mathcal{B}(\zeta_\kappa, \rho(\zeta_\kappa))$,

$$\int_{\mathcal{B}_\kappa^{\mathcal{L}}} I_{\alpha,\gamma}(|\psi|)(\zeta)\, d\zeta \le C\, |\mathcal{B}_\kappa^{\mathcal{L}}|^{\alpha/n}\, \|\chi_{\mathcal{B}_\kappa^{\mathcal{L}}}\|_{L^{p'(\cdot)}(w^{-1})}\, \|\psi\|_{L^{p(\cdot)}(w)},$$

with C independent of κ.

Theorem 4, whose domain is not $L^{p(\cdot),\infty}(w)$, is proven separately below by the same reduction (Lemma 20), applied to the analogous properties (P1′) and (P2′), in which $\|\psi\|_{L^{p(\cdot)}(w)}$ is replaced by $\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}$.

Proof of Theorem 3. Proof of (P2). Fix $\kappa \ge 1$, let $\mathcal{B} = \mathcal{B}_\kappa^{\mathcal{L}} = \mathcal{B}(\zeta_0, R)$ with $R = \rho(\zeta_0)$, and write $\psi = \psi_1 + \psi_2$, where $\psi_1 = \psi\chi_{2\mathcal{B}}$ and $\psi_2 = \psi\chi_{\mathbb{R}^n\setminus 2\mathcal{B}}$.

Near field: the term ψ_1. By Lemma 19 (the kernel bound (9) combined with the sharp radial mass $\psi_{\alpha,\gamma}$, instead of the crude bound $(1+|\zeta-\iota|)^{-\gamma} \le 1$), we have

$$\int_{\mathcal{B}} I_{\alpha,\gamma}(|\psi_1|)(\zeta)\, d\zeta \le C\, \psi_{\alpha,\gamma}(4R) \int_{2\mathcal{B}} |\psi(\iota)|\, d\iota.$$

By Lemma 4(a), we obtain

$$\int_{\mathcal{B}} I_{\alpha,\gamma}(|\psi_1|)(\zeta)\, d\zeta \le C\, \psi_{\alpha,\gamma}(4R)\, \|\chi_{2\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})} \|\psi\|_{L^{p(\cdot)}(w)}.$$

Since $\psi_{\alpha,\gamma}(4R) \le C\, R^\alpha$ (Lemma 18) and $\|\chi_{2\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})} \le C\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}$ (Lemma 5), we obtain

$$\int_{\mathcal{B}} I_{\alpha,\gamma}(|\psi_1|)(\zeta)\, d\zeta \;\le\; C\, |\mathcal{B}|^{\alpha/n}\, \|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\, \|\psi\|_{L^{p(\cdot)}(w)}. \tag{30}$$

The intermediate bound is, however, strictly sharper: for $R > 1$ and $\gamma > 0$, Lemma 18 gives $\psi_{\alpha,\gamma}(4R) \lesssim R^{(\alpha-\gamma)^+} \log(2+R) = o(R^\alpha)$. This means that the Bessel decay is genuinely active in the near-field term as soon as the radius of the ball exceeds the unit scale, and not merely in the far field. For $R \le 1$, we have $\psi_{\alpha,\gamma}(4R) \sim R^\alpha$; on such balls, the local singularity of the kernel dominates, and γ contributes no further gain, which is the expected behavior. The Bessel decay is used again, differently, below.

Far field: the term ψ_2. For $\zeta \in \mathcal{B}$, we write

$$\begin{aligned} I_{\alpha,\gamma}\psi_2(\zeta) = &\int_0^{R^2} e^{-\tau\mathcal{L}}\psi_2(\zeta)\, \tau^{\alpha/2-1}(1+\sqrt{\tau}\,)^{-\gamma}\, d\tau \\ &+ \int_{R^2}^{\infty} e^{-\tau\mathcal{L}}\psi_2(\zeta)\, \tau^{\alpha/2-1}(1+\sqrt{\tau}\,)^{-\gamma}\, d\tau =: I_1 + I_2. \end{aligned} \tag{31}$$

Estimate of I_1. We consider the first integral in (31). For $\tau < R^2$, we have $(1+\sqrt{\tau}\,)^{-\gamma} \le 1$. For $\zeta \in \mathcal{B}$ and $\iota \in \mathbb{R}^n \setminus 2\mathcal{B}$, we have $|\zeta_0 - \iota| < |\zeta_0 - \zeta| + |\zeta - \iota| < C|\zeta - \iota|$. By Lemma 15, we obtain

$$\begin{aligned} |I_1| &= \left| \int_0^{R^2} \int_{\mathbb{R}^n\setminus 2\mathcal{B}} \kappa_\tau(\zeta,\iota)\, \psi(\iota)\, d\iota\, \tau^{\alpha/2-1}(1+\sqrt{\tau}\,)^{-\gamma}\, d\tau \right| \\ &\le C \int_0^{R^2} \int_{\mathbb{R}^n\setminus 2\mathcal{B}} \tau^{-n/2} e^{-|\zeta-\iota|^2/(C\tau)} |\psi(\iota)|\, d\iota\, \tau^{\alpha/2-1}\, d\tau \\ &\le C \int_0^{R^2} \tau^{\frac{M-n+\alpha}{2}-1}\, d\tau \int_{\mathbb{R}^n\setminus 2\mathcal{B}} \frac{|\psi(\iota)|}{|\zeta_0-\iota|^M}\, d\iota, \end{aligned}$$

where C depends only on M. By Lemma 4(a) and a dyadic decomposition of $\mathbb{R}^n \setminus 2\mathcal{B}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n\setminus 2\mathcal{B}} \frac{|\psi(\iota)|}{|\zeta_0-\iota|^M}\,d\iota &= \sum_{i=1}^{\infty}\int_{2^{i+1}\mathcal{B}\setminus 2^i\mathcal{B}} \frac{|\psi(\iota)|}{|\zeta_0-\iota|^M}\,d\iota\\ &\le \sum_{i=1}^{\infty}(2^iR)^{-M}\int_{2^{i+1}\mathcal{B}}|\psi(\iota)|\,d\iota\\ &\le C\sum_{i=1}^{\infty}(2^iR)^{-M}\,\|\chi_{2^{i+1}\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{L^{p(\cdot)}(w)}\\ &\le C\sum_{i=1}^{\infty}R^{-M}(2^i)^{n-n/p^+-M}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{L^{p(\cdot)}(w)}\\ &\le C\,R^{-M}\,\|\psi\|_{L^{p(\cdot)}(w)}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}. \end{aligned}\tag{32}$$

Using Lemma 5, we note that the series in (32) converges precisely when $M > n - n/p^+$. Hence, we have

$$\begin{aligned}|I_1| &\le C\,R^{-M}\,\|\psi\|_{L^{p(\cdot)}(w)}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\int_0^{R^2}\tau^{\frac{M-n+\alpha}{2}-1}\,d\tau\\ &\le C\,|\mathcal{B}|^{\alpha/n-1}\,\|\psi\|_{L^{p(\cdot)}(w)}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}.\end{aligned}$$

Estimate of I_2: where the Bessel decay is used. Here, $\tau > R^2$, and hence $(1+\sqrt{\tau}\,)^{-\gamma} \le C\tau^{-\gamma/2}$. By Lemma 15, choosing M as above and $N \ge M$, we have

$$\begin{aligned}|I_2| &= \left|\int_{R^2}^{\infty}\int_{\mathbb{R}^n\setminus 2\mathcal{B}} \kappa_\tau(\zeta,\iota)\,\psi(\iota)\,d\iota\,\tau^{\alpha/2-1}(1+\sqrt{\tau}\,)^{-\gamma}\,d\tau\right|\\ &\le C\int_{R^2}^{\infty}\int_{\mathbb{R}^n\setminus 2\mathcal{B}} \tau^{\frac{\alpha-n-N-2}{2}-\frac{\gamma}{2}}\,\rho(\zeta)^N\,e^{-|\zeta-\iota|^2/(C\tau)}|\psi(\iota)|\,d\iota\,d\tau\\ &\le C\,\rho(\zeta)^N\int_{R^2}^{\infty}\tau^{\frac{\alpha-n-N-2-\gamma}{2}}\int_{\mathbb{R}^n\setminus 2\mathcal{B}}\Big(\frac{\tau}{|\zeta-\iota|^2}\Big)^{M/2}|\psi(\iota)|\,d\iota\,d\tau.\end{aligned}$$

By Lemma 12, we have $\rho(\zeta) \sim \rho(\zeta_0) = R$ for $\zeta \in \mathcal{B}$, and therefore

$$|I_2| \le C\,R^N\int_{R^2}^{\infty}\tau^{\frac{M+\alpha-n-N-\gamma}{2}-1}\,d\tau\int_{\mathbb{R}^n\setminus 2\mathcal{B}}\frac{|\psi(\iota)|}{|\zeta_0-\iota|^M}\,d\iota.$$

Since $M+\alpha-n-N-\gamma < M+\alpha-n-N < 0$ for N large (the extra $-\gamma$ only helps convergence), the τ-integral converges, and (32) gives

$$|I_2| \le C\,|\mathcal{B}|^{\alpha/n-1}\,\|\psi\|_{L^{p(\cdot)}(w)}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}.$$

Integrating the bounds for I_1 and I_2 over $\mathcal{B}$ and adding (30), we obtain (P2), with C independent of κ. □

Proof of (P1). Let $\mathcal{B} = \mathcal{B}(\zeta_0,\nu)$ with $\nu < \rho(\zeta_0)$, let $\psi = \psi_1 + \psi_2$ be as above with respect to $\mathcal{B}$, and set

$$c_\nu := \int_{\nu^2}^{\infty} e^{-\tau\mathcal{L}}\psi_2(\zeta_0)\,\tau^{\alpha/2-1}(1+\sqrt{\tau}\,)^{-\gamma}\,d\tau.\tag{33}$$

By (30) applied to $\mathcal{B}$, we have

$$\begin{aligned}\int_{\mathcal{B}}|I_{\alpha,\gamma}\psi(\zeta)-c_\nu|\,d\zeta &\le \int_{\mathcal{B}} I_{\alpha,\gamma}(|\psi_1|)(\zeta)\,d\zeta + \int_{\mathcal{B}}|I_{\alpha,\gamma}\psi_2(\zeta)-c_\nu|\,d\zeta\\ &\le C\,|\mathcal{B}|^{\alpha/n}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{L^{p(\cdot)}(w)} + \int_{\mathcal{B}}|I_{\alpha,\gamma}\psi_2(\zeta)-c_\nu|\,d\zeta.\end{aligned}$$

We split $I_{\alpha,\gamma}\psi_2 = I_3 + I_4$ at $\tau = \nu^2$, where $I_3 = \int_0^{\nu^2}(\cdots)\,d\tau$ and $I_4 = \int_{\nu^2}^{\infty}(\cdots)\,d\tau$.

Estimate of I_3. Exactly as for I_1 (with ν in place of R), we obtain

$$|I_3| \le C\,|\mathcal{B}|^{\alpha/n-1}\,\|\psi\|_{L^{p(\cdot)}(w)}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}.$$

Estimate of I_4: kernel smoothness with the Bessel factor. By Lemma 16, which is applicable since $|\zeta-\zeta_0| < \nu < \sqrt{\tau}$ for $\tau > \nu^2$, we have

$$\begin{aligned}
|I_4 - c_\nu| &\le \int_{\nu^2}^{\infty}\int_{\mathbb{R}^n\setminus 2\mathcal{B}} |\kappa_\tau(\zeta,\iota) - \kappa_\tau(\zeta_0,\iota)|\,|\psi(\iota)|\,d\iota\,\tau^{\alpha/2-1}(1+\sqrt{\tau})^{-\gamma}\,d\tau\\
&\le C_\delta \int_{\nu^2}^{\infty}\int_{\mathbb{R}^n\setminus 2\mathcal{B}} \Big(\frac{|\zeta-\zeta_0|}{\sqrt{\tau}}\Big)^{\delta}\tau^{-n/2}e^{-|\zeta-\iota|^2/(C\tau)}|\psi(\iota)|\,d\iota\,\tau^{\alpha/2-1}(1+\sqrt{\tau})^{-\gamma}\,d\tau\\
&\le C_\delta\,\nu^{\delta}\int_{\mathbb{R}^n\setminus 2\mathcal{B}}|\psi(\iota)|\int_{\nu^2}^{\infty}\tau^{-(n-\alpha+\delta)/2-\gamma/2}e^{-|\zeta-\iota|^2/(C\tau)}\,\frac{d\tau}{\tau}\,d\iota.
\end{aligned}$$

We substitute $s = |\zeta-\iota|^2/\tau$, so that $ds/s = -d\tau/\tau$, and absorb the extra decay $(1+\sqrt{\tau})^{-\gamma}$ directly into the kernel exponent (on the effective range $\tau \sim |\zeta-\iota|^2/s$ with s bounded, we have $(1+\sqrt{\tau})^{-\gamma} \sim |\zeta-\iota|^{-\gamma}$). This gives

$$|I_4 - c_\nu| \le C_\delta\,\nu^{\delta}\int_{\mathbb{R}^n\setminus 2\mathcal{B}}\frac{|\psi(\iota)|}{|\zeta-\iota|^{n-\alpha+\delta+\gamma}}\,d\iota\int_0^{\infty}s^{\frac{n-\alpha+\delta+\gamma}{2}}e^{-s/C}\,\frac{ds}{s}.$$

The s-integral is a finite Gamma-type integral. For the ι-integral, we use the same dyadic computation as in (32), now with M replaced by $n-\alpha+\delta+\gamma$, and we obtain

$$\begin{aligned}
|I_4 - c_\nu| &\le C_\delta\,\nu^{\delta}\int_{\mathbb{R}^n\setminus 2\mathcal{B}}\frac{|\psi(\iota)|}{|\zeta-\iota|^{n-\alpha+\delta+\gamma}}\,d\iota\\
&\le C\sum_{i=1}^{\infty}\nu^{\alpha-n}(2^i)^{\alpha-n/p^+-\delta-\gamma}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{L^{p(\cdot)}(w)}\\
&\le C\,|\mathcal{B}|^{\alpha/n-1}\,\|\psi\|_{L^{p(\cdot)}(w)}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})},
\end{aligned}\tag{34}$$

where the geometric series over i converges precisely under

$$\alpha - \frac{n}{p^+} - \delta - \gamma < 0, \qquad \text{that is,} \qquad p^+ < \frac{n}{\alpha-\delta-\gamma} \quad (\text{when } \alpha-\delta-\gamma>0).\tag{35}$$

Since $\gamma \ge 0$, condition (35) is implied by (and, for $\gamma > 0$, is strictly weaker than) the hypothesis $p^+ < \big(n/(\alpha-\delta_0)\big)^+$, once $\delta < \delta_0$ is chosen sufficiently close to δ_0. This means that the Bessel decay γ only relaxes the convergence requirement on p^+ and never tightens it (Corollary 3 makes this quantitative). Integrating the pointwise bounds for I_3 and $I_4 - c_\nu$ over $\mathcal{B}$, we obtain $\int_{\mathcal{B}}|I_{\alpha,\gamma}\psi_2(\zeta) - c_\nu|\,d\zeta \le C\,|\mathcal{B}|^{\alpha/n}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{L^{p(\cdot)}(w)}$. This proves (P1), uniformly over all balls $\mathcal{B} = \mathcal{B}(\zeta_0,\nu)$ with $\nu < \rho(\zeta_0)$. □

Finally, by Lemma 20, (P1) and (P2) give $\|I_{\alpha,\gamma}\psi\|_{HM\mathrm{Lip}^{\alpha,v)}_{\mathcal{L},\Gamma,p(\cdot)}(w)} \le C\,\|\psi\|_{L^{p(\cdot)}(w)}$. □

Proof of Theorem 4. By Lemma 20, it suffices to establish (P1′) and (P2′), that is, the analogs of (P1) and (P2) with $\|\psi\|_{L^{p(\cdot)}(w)}$ replaced by $\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}$, with constants independent of the ball. We give the argument for (P2′). The proof of (P1′) is identical, with $\mathcal{B}^{\mathcal{L}}_{\kappa}$ replaced by a general ball $\mathcal{B}(\zeta,R)$, $R < \rho(\zeta)$, and with Lemma 23 applied with this ζ in place of ζ_κ.

Fix $\kappa \ge 1$, let $\mathcal{B} = \mathcal{B}^{\mathcal{L}}_{\kappa} = \mathcal{B}(\zeta_\kappa, R)$ with $R = \rho(\zeta_\kappa) \le \rho_{\max}$, let $\psi = \psi_1 + \psi_2$ be as before, and let $m = m(\zeta_\kappa)$ be as in Lemma 23.

Proof of (P2′). Near field. By Lemma 19 and Lemma 4(a) applied to $\psi\chi_{2\mathcal{B}}$, we have

$$\begin{aligned}\int_{\mathcal{B}} I_{\alpha,\gamma}(|\psi_1|)(\zeta)\,d\zeta &\le C\,\psi_{\alpha,\gamma}(4R)\int_{2\mathcal{B}}|\psi(\iota)|\,d\iota\\ &\le C\,\psi_{\alpha,\gamma}(4R)\,\|\chi_{2\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\chi_{2\mathcal{B}}\|_{L^{p(\cdot)}(w)}.\end{aligned}$$

Since $2\mathcal{B}\subset\mathcal{B}_{m+1}$ (Lemma 23), we have $\|\psi\chi_{2\mathcal{B}}\|_{L^{p(\cdot)}(w)}\le\sum_{j\le m+1}\|\psi\chi_j\|_{L^{p(\cdot)}(w)}$. By Lemma 21 (with domain Morrey parameter 0), we have $\|\psi\chi_j\|_{L^{p(\cdot)}(w)}\le 2^{-j\eta(j)}\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}$, and hence

$$\|\psi\chi_{2\mathcal{B}}\|_{L^{p(\cdot)}(w)}\ \le\ \|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}\sum_{j\le m+1}2^{-j\eta(j)}\ \le\ C\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}. \tag{36}$$

The sum in (36) is bounded independently of m, since $\eta(j)=\beta(0)<0$ for $j\le 0$ and $\eta(j)=\beta_\infty>0$ for $j>0$, so that both parts form convergent geometric series (Lemma 22). Since $\psi_{\alpha,\gamma}(4R)\le CR^{\alpha}$ (Lemma 18) and $R\le\rho_{\max}$, we obtain

$$\int_{\mathcal{B}} I_{\alpha,\gamma}(|\psi_1|)(\zeta)\,d\zeta\ \le\ C\,|\mathcal{B}|^{\alpha/n}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}, \tag{37}$$

with C independent of κ.

Far field. It remains to bound $\int_{\mathbb{R}^n\setminus 2\mathcal{B}}|\psi(\iota)|\,|\zeta_\kappa-\iota|^{-M}\,d\iota$, $M>n-n/p^+$, via the Herz norm. As in (32), this integral is at most $\sum_{i\ge 1}(2^iR)^{-M}\int_{2^{i+1}\mathcal{B}}|\psi(\iota)|\,d\iota$, and, by Lemma 4(a) and the same estimate of $\|\chi_{2^{i+1}\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}$ as in (32), we have

$$(2^iR)^{-M}\int_{2^{i+1}\mathcal{B}}|\psi(\iota)|\,d\iota\le C\,R^{-M}(2^i)^{n-n/p^+-M}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\chi_{2^{i+1}\mathcal{B}}\|_{L^{p(\cdot)}(w)}.$$

We split the sum over i at

$$i_0:=\left\lceil\log_2\frac{\rho_{\max}}{R}\right\rceil+2,$$

which is an integer with $i_0\ge 2$, since $R\le\rho_{\max}$.

Inner part, $1\le i\le i_0$. For these indices, we have $2^{i+1}R\le 2^{i_0+1}R\le 16\,\rho_{\max}$. Hence, for $\iota\in 2^{i+1}\mathcal{B}$,

$$|\iota|<|\zeta_\kappa|+16\,\rho_{\max}\le 8\left(|\zeta_\kappa|+2\rho_{\max}+1\right)\le 2^{m+3},$$

that is, $2^{i+1}\mathcal{B}\subset\mathcal{B}_{m+3}$. By Lemma 22, we get $\|\psi\chi_{2^{i+1}\mathcal{B}}\|_{L^{p(\cdot)}(w)}\le C_\beta\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}$, and therefore

$$\sum_{i=1}^{i_0}(2^iR)^{-M}\int_{2^{i+1}\mathcal{B}}|\psi(\iota)|\,d\iota\ \le\ C\,R^{-M}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}. \tag{38}$$

Outer part, $i>i_0$. For these indices, $2^{i+1}\mathcal{B}$ is no longer contained in a ball of radius comparable to $\rho_{\max}$, but Lemma 22, applied with $E=2^{i+1}\mathcal{B}$, still gives $\|\psi\chi_{2^{i+1}\mathcal{B}}\|_{L^{p(\cdot)}(w)}\le C_\beta\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}$, with C_β independent of i and of $\mathcal{B}$. Summing the resulting geometric series, which converges since $M>n-n/p^+$, we obtain

$$\sum_{i>i_0}(2^iR)^{-M}\int_{2^{i+1}\mathcal{B}}|\psi(\iota)|\,d\iota\ \le\ C\,R^{-M}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}, \tag{39}$$

with C independent of κ.

Combining (38) and (39) with the same τ-integral computation as in the proof of Theorem 3 (which is independent of the domain, since it concerns only the heat-kernel time variable), we obtain

$$\int_{\mathcal{B}_\kappa^{\mathcal{L}}} I_{\alpha,\gamma}(|\psi|)(\zeta)\,d\zeta \;\le\; C\,|\mathcal{B}_\kappa^{\mathcal{L}}|^{\alpha/n}\,\|\chi_{\mathcal{B}_\kappa^{\mathcal{L}}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)},$$

with C independent of κ. This is (P2′). □

Proof of (P1′). Let $\mathcal{B} = \mathcal{B}(\zeta,\nu)$ with $\nu < \rho(\zeta)$, and let $m = m(\zeta)$ be as in Lemma 23. By (37) applied to $\mathcal{B}$, we have

$$\int_{\mathcal{B}} |I_{\alpha,\gamma}\psi(\zeta') - c_\nu|\,d\zeta' \le C\,|\mathcal{B}|^{\alpha/n}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)} + \int_{\mathcal{B}} |I_{\alpha,\gamma}\psi_2(\zeta') - c_\nu|\,d\zeta',$$

with c_ν as in (33). We split $I_{\alpha,\gamma}\psi_2 = I_3 + I_4$ at $\tau = \nu^2$. The term I_3 is estimated exactly as the near-field term above, using (36) with $m = m(\zeta)$, which gives

$$|I_3| \le C\,|\mathcal{B}|^{\alpha/n-1}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}.$$

For I_4, the same computation based on Lemma 16 as in the proof of Theorem 3 gives

$$|I_4 - c_\nu| \;\le\; C_\delta\,\nu^\delta \int_{\mathbb{R}^n\setminus 2\mathcal{B}} \frac{|\psi(\iota)|}{|\zeta' - \iota|^{n-\alpha+\delta+\gamma}}\,d\iota,$$

and the same inner/outer splitting as in (38) and (39), now with M replaced by $n-\alpha+\delta+\gamma$ (as in the proof of Theorem 3), gives

$$|I_4 - c_\nu| \;\le\; C\,|\mathcal{B}|^{\alpha/n-1}\,\|\chi_{\mathcal{B}}\|_{L^{p'(\cdot)}(w^{-1})}\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)},$$

under the relaxed condition (35), which is unaffected by the change of domain. Integrating the bounds for I_3 and $I_4 - c_\nu$ over $\mathcal{B}$, we obtain (P1′), with constant $C_1 = C\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}$ in condition (a) of Definition 8. □

Finally, by Lemma 20, we conclude that $\|I_{\alpha,\gamma}\psi\|_{HM\mathrm{Lip}^{\alpha,v)}_{\mathcal{L},\Gamma,p(\cdot)}(w)} \le C\,\|\psi\|_{M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)}$. □

Corollary 2. *If $\gamma = 0$, $\Gamma = 0$, $w \equiv 1$ and $v \to \infty$ (formally, so that the threshold $\Gamma \ge 1/v$ reduces to $\Gamma \ge 0$), then Theorem 3 recovers the boundedness $\mathcal{L}^{-\alpha/2} : L^{p(\cdot)}(\mathbb{R}^n) \to \mathrm{Lip}^{\alpha,p(\cdot)}_{\mathcal{L}}(\mathbb{R}^n)$ of Wang and Li [51]. Theorem 4 has no such specialization: its domain $M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)$ is a Herz-type, and not a Lebesgue-type, space for every choice of the parameters, and its hypotheses (13), $\beta(0) < 0$ and $\beta_\infty \ge n/p'(\infty)$ have no counterpart in [51] (see Remark 7).*

Corollary 3 (Non-trivial role of the Bessel–Riesz parameter)**.** *In Theorems 3 and 4, replace the hypothesis $p^+ < \big(n/(\alpha-\delta_0)\big)^+$ by the sharper, γ-dependent condition*

$$p^+ < \frac{n}{(\alpha-\delta_0-\gamma)^+}, \qquad \text{with the convention } \frac{n}{0} = \infty. \tag{40}$$

Then both theorems remain valid under (40). In particular, the following hold:

(i) For $0 \le \gamma < \alpha - \delta_0$, condition (40) is strictly weaker than the classical threshold, so the admissible range of p^+ strictly grows with γ;

(ii) *For $\gamma \geq \alpha - \delta_0$, condition (40) imposes no upper bound on p^+, so that Theorems 3 and 4 hold for every $1 < p^- \leq p^+ < \infty$.*

Proof. Condition (35) in the proof of Theorem 3 (and, identically, in the proof of Theorem 4) is the convergence requirement produced by the kernel-smoothness estimate (34). If $\alpha - \delta_0 - \gamma > 0$, then, when choosing $\delta < \delta_0$ sufficiently close to δ_0, we see that (35) follows from $p^+ < n/(\alpha - \delta_0 - \gamma)$. If, instead, $\alpha - \delta_0 - \gamma \leq 0$, then, for every $\delta < \delta_0$ and every finite $p^+ > 1$, we have

$$\alpha - \frac{n}{p^+} - \delta - \gamma = (\alpha - \delta_0 - \gamma) + (\delta_0 - \delta) - \frac{n}{p^+} \leq (\delta_0 - \delta) - \frac{n}{p^+} < 0,$$

provided that $\delta_0 - \delta < n/p^+$. Hence, when choosing δ close enough to δ_0, the left-hand side of (35) is negative, and the geometric series in (34) converges for every $p^+ \in (1, \infty)$, with no upper restriction. Since every other hypothesis and every other step of the two proofs is independent of this condition, both theorems hold under (40). □

5. Conclusions

We introduced the weighted grand variable Herz–Morrey–Hardy space $HM\dot{\mathcal{K}}^{\alpha(\cdot),v)}_{\Gamma,p(\cdot)}(w)$, which extends the grand variable Herz–Morrey–Hardy space of [62] by a Muckenhoupt weight, and we proved that every order-zero pseudo-differential operator is bounded on it (Theorem 2). For $\mathcal{L} = -\Delta + \mathcal{V}$ with $\mathcal{V} \in RH_q$, $q \geq n/2$, we introduced the Schrödinger–Bessel–Riesz operator $I_{\alpha,\gamma}$ and the weighted grand variable Herz–Morrey–Hardy–Lipschitz space $HM\dot{\mathrm{Lip}}^{\alpha,v)}_{\mathcal{L},\Gamma,p(\cdot)}(w)$, and we proved that $I_{\alpha,\gamma}$ is bounded into this space from $L^{p(\cdot)}(w)$ (Theorem 3) and, in place of the classical weak-type Lebesgue domain, from the weighted grand variable Herz space $M\dot{\mathcal{K}}^{\beta(\cdot),v)}_{0,p(\cdot)}(w)$ (Theorem 4), under the complementary hypotheses (13), $\beta(0) < 0$ and $\beta_\infty \geq n/p'(\infty)$, that this domain change requires (Remark 7). Both proofs proceed by uniform per-atom (respectively, per-ball) kernel estimates, sharpened in Section 4 by the exact radial-mass computation of Lemma 18, so that the Bessel decay is used non-trivially in the near field as well as in the far field. These estimates are then embedded into the grand–Morrey aggregation, under the condition $0 \leq \Gamma < \varepsilon_0$ on the Herz–Hardy scale (Lemma 7) and under $\Gamma \geq 1/v$ on the Lipschitz-type scale (Lemma 20). Both results specialize to [38,51] where a classical analog exists (Corollaries 1 and 2). Corollary 3 describes the role of the Bessel parameter γ in the admissible range of p^+. Natural extensions include commutators of T and $I_{\alpha,\gamma}$ with BMO-type symbols, two-weight versions of Theorems 2–4, the removal of the boundedness hypothesis (13) on ρ via the polynomial-growth mechanism of Lemma 12, and the compactness of the embeddings noted in Remark 5 and Corollaries 1 and 2.

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References

1. Ruzhansky, M.; Turunen, V. *Pseudo-Differential Operators and Symmetries: Background Analysis and Advanced Topics*; Pseudo-Differential Operators. Theory and Applications, Volume 2; Birkhäuser: Basel, Switzerland, 2010.
2. Gorenflo, R.; Luchko, Y.F.; Umarov, S.R. On some boundary value problems for pseudo-differential equations with boundary operators of fractional order. *Fract. Calc. Appl. Anal.* **2000**, *3*, 453–468.
3. Shang, H.; Guo, C.; Wu, Y.; Li, Z.; Yang, J. Solving the many-electron Schrödinger equation with a transformer-based framework. *Nat. Commun.* **2025**, *16*, 8464. [CrossRef] [PubMed]
4. Khalkhuzhaev, A.M.; Khayitova, K.G.; Khujamiyorov, I.A. On the spectrum of the Schrödinger operator for a three-particle system on a lattice. *Uchenye Zap. Kazan. Univ. Ser. Fiz.-Mat. Nauki* **2025**, *167*, 548. [CrossRef]
5. Lindenhovius, B.; Zamdzhiev, V. Operator spaces, linear logic and the Heisenberg–Schrödinger duality of quantum theory. In *Proceedings of the 2025 40th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), June 2025*; IEEE: Piscataway, NJ, USA, 2025; pp. 870–883.
6. Kováčik, O.; Rákosník, J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslov. Math. J.* **1991**, *41*, 592–618.
7. Diening, L.; Harjulehto, P.; Hästö, P.; Růžička, M. *Lebesgue and Sobolev Spaces with Variable Exponents*; Lecture Notes in Mathematics, Volume 2017; Springer: Berlin/Heidelberg, Germany, 2011.
8. Cruz-Uribe, D.; Fiorenza, A.; Neugebauer, C.J. Weighted norm inequalities for the maximal operator on variable Lebesgue spaces. *J. Math. Anal. Appl.* **2012**, *394*, 744–760. [CrossRef]
9. Cruz-Uribe, D.; Fiorenza, A. *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*; Applied and Numerical Harmonic Analysis; Birkhäuser: Basel, Switzerland, 2013.
10. Izuki, M. Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. *Anal. Math.* **2010**, *36*, 33–50. [CrossRef]
11. Wang, H.; Liu, Z. The Herz-type Hardy spaces with variable exponent and their applications. *Taiwan. J. Math.* **2012**, *16*, 1363–1389. [CrossRef]
12. Wang, H.; Liu, Z. Some characterizations of Herz-type Hardy spaces with variable exponent. *Ann. Funct. Anal.* **2015**, *6*, 224–243. [CrossRef]
13. Nagase, M. The L^p-boundedness of pseudo-differential operators with non-regular symbols. *Comm. Partial Differ. Equ.* **1977**, *2*, 1045–1061. [CrossRef]
14. Rodino, L. On the boundedness of pseudo differential operators in the class $L^m_{\rho,1}$. *Proc. Amer. Math. Soc.* **1976**, *58*, 211–215. [CrossRef]
15. Tsutsui, Y. Pseudo-differential operators of class $S^m_{0,0}$ on the Herz-type spaces. *Hokkaido Math. J.* **2009**, *38*, 283–302. [CrossRef]
16. Higuchi, Y.; Nagase, M. On the L^2-boundedness of pseudo-differential operators. *J. Math. Kyoto Univ.* **1988**, *28*, 133–139. [CrossRef]
17. Sugimoto, M. L^p-boundedness of pseudo-differential operators satisfying Besov estimates I. *J. Math. Soc. Jpn.* **1988**, *40*, 105–122. [CrossRef]
18. Cardona, D.; Kumar, V. L^p-boundedness and L^p-nuclearity of multilinear pseudo-differential operators on $\mathbb{Z}^n$ and the torus $\mathbb{T}^n$. *J. Fourier Anal. Appl.* **2019**, *25*, 2973–3017. [CrossRef]
19. Chen, J.; Ding, W.; Lu, G. Boundedness of multi-parameter pseudo-differential operators on multi-parameter local Hardy spaces. *Forum Math.* **2020**, *32*, 919–936. [CrossRef]
20. Das, S.; Mahato, K.; Das, S. Boundedness of pseudo-differential operators via coupled fractional Fourier transform. *Appl. Anal.* **2025**, *104*, 663–681. [CrossRef]
21. Delgado, J.; Ruzhansky, M. L^p-bounds for pseudo-differential operators on compact Lie groups. *J. Inst. Math. Jussieu* **2019**, *18*, 531–559. [CrossRef]
22. Cardona, D.; Ruzhansky, M. Boundedness of pseudo-differential operators in subelliptic Sobolev and Besov spaces on compact Lie groups. *Complex Var. Elliptic Equ.* **2024**, *69*, 1049–1082. [CrossRef]
23. Cardona, D.; Ruzhansky, M. Boundedness of subelliptic pseudo-differential operators. In *Subelliptic Pseudo-Differential Operators and Fourier Integral Operators on Compact Lie Groups*; MSJ Memoirs, Volume 44; Mathematical Society of Japan: Tokyo, Japan, 2025; pp. 70–84.
24. Bekbolat, B.; Tokmagambetov, N. On a boundedness result of non-toroidal pseudo-differential operators. *Int. J. Math. Phys.* **2018**, *9*, 50–55. [CrossRef]

25. Cardona, D.; Martínez, M.A. Estimates for pseudo-differential operators on the torus revisited. I. *J. Math. Anal. Appl.* **2025**, *554*, 129959. [CrossRef]
26. Das, S.; Mahato, K. Pseudo-differential operators associated with the coupled fractional Fourier transform and its application. *J. Pseudo-Differ. Oper. Appl.* **2025**, *16*, 37. [CrossRef]
27. Kumar, M.; Bhawna. An n-dimensional pseudo-differential operator involving quadratic phase Fourier transform and applications in quantum mechanics. *J. Pseudo-Differ. Oper. Appl.* **2025**, *16*, 28. [CrossRef]
28. Sartaj, M.; Balvant, P.; Upadhyay, S.K. Pseudo-differential operators involving the Fourier Bessel transform on $\mathbb{Z}_+^{n+1}$. *Mediterr. J. Math.* **2025**, *22*, 225. [CrossRef]
29. Verma, R.K.; Prasad, A. SG pseudo-differential operators in Dunkl setting. *Georgian Math. J.* **2025**, *32*, 501. [CrossRef]
30. Mahato, K.; Pasawan, D. Norm inequalities of pseudo-differential operators associated with fractional Hankel transform on Beurling like spaces. *Proc. Natl. Acad. Sci. India Sect. A Phys. Sci.* **2026**, *96*, 365–375. [CrossRef]
31. Thorn, M.H. Magnetic pseudo-differential operators with Hörmander symbols dominated by tempered weights. *J. Pseudo-Differ. Oper. Appl.* **2026**, *17*, 15. [CrossRef]
32. Torresblanca-Badillo, A. Dynamics of complex networks through the use of p-adic differential equations linked to fractional pseudo-differential operators. *J. Pseudo-Differ. Oper. Appl.* **2025**, *16*, 93. [CrossRef]
33. Faou, E.; Grébert, B. Discrete pseudo-differential operators and applications to numerical schemes. *Found. Comput. Math.* **2025**, *25*, 587–630. [CrossRef]
34. Sayana, K.J.; Reddy, G.D. A modified fractional Tikhonov regularization to solve the pseudo-differential operator equation. *Calcolo* **2026**, *63*, 15. [CrossRef]
35. Yuldashev, T.K.; Ramazanova, A.T.; Shermamatov, Z.Z. Optimal control problem for a linear pseudoparabolic equation with final condition, degeneration and Gerasimov–Caputo operator. *Azerb. J. Math.* **2025**, *15*, 257.
36. Fan, D.; Yang, D. The weighted Herz-type Hardy spaces $h\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$. *Approx. Theory Appl.* **1997**, *13*, 19–41. [CrossRef]
37. Goldberg, D. A local version of real Hardy spaces. *Duke Math. J.* **1979**, *46*, 27–42. [CrossRef]
38. Wang, H.; Wu, Y. Boundedness of pseudo-differential operators on homogeneous Herz-type Hardy space with variable exponent. *Azerb. J. Math.* **2017**, *7*, 40–48.
39. Shen, Z. L^p estimates for Schrödinger operators with certain potentials. *Ann. Inst. Fourier* **1995**, *45*, 513–546. [CrossRef]
40. Tang, L. Weighted norm inequalities for Schrödinger type operators. *Forum Math.* **2015**, *27*, 2491–2532. [CrossRef]
41. Kurata, K. An estimate on the heat kernel of magnetic Schrödinger operators and uniformly elliptic operators with non-negative potentials. *J. Lond. Math. Soc.* **2000**, *62*, 885–903. [CrossRef]
42. Dziubański, J.; Zienkiewicz, J. Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality. *Rev. Mat. Iberoam.* **1999**, *15*, 279–296. [CrossRef]
43. Dziubański, J.; Zienkiewicz, J. H^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes. *Colloq. Math.* **2003**, *98*, 5–38. [CrossRef]
44. Bongioanni, B.; Harboure, E.; Salinas, O. Weighted inequalities for negative powers of Schrödinger operators. *J. Math. Anal. Appl.* **2008**, *348*, 12–27. [CrossRef]
45. Harboure, E.; Salinas, O.; Viviani, B. Boundedness of the fractional integral on weighted Lebesgue and Lipschitz spaces. *Trans. Amer. Math. Soc.* **1997**, *349*, 235–255. [CrossRef]
46. Wu, W.; Zhou, J. Two-weight norm inequalities for some fractional type operators related to Schrödinger operator on weighted Morrey spaces. *Turk. J. Math.* **2021**, *45*, 2646–2663. [CrossRef]
47. Zhang, Q.; Tang, L. Variation operators on weighted Hardy and BMO spaces in the Schrödinger setting. *Bull. Malays. Math. Sci. Soc.* **2022**, *45*, 2285–2312. [CrossRef]
48. Yang, L.; Li, P. Boundedness and compactness of commutators related with Schrödinger operators on Heisenberg groups. *J. Pseudo-Differ. Oper. Appl.* **2023**, *14*, 8. [CrossRef]
49. Dziubański, J.; Głowacki, P. Sobolev spaces related to Schrödinger operators with polynomial potentials. *Math. Z.* **2009**, *262*, 881. [CrossRef]
50. Ramseyer, M.; Salinas, O.; Viviani, B. Lipschitz type smoothness of the fractional integral on variable exponent spaces. *J. Math. Anal. Appl.* **2013**, *403*, 95–106. [CrossRef]
51. Wang, H.; Li, P. Fractional integral associated with the Schrödinger operators on variable exponent space. *Electron. Res. Arch.* **2023**, *31*, 6833–6843. [CrossRef]
52. Li, P.; Wan, X. The boundedness of commutators associated with Schrödinger operators on Herz spaces. *J. Inequal. Appl.* **2016**, *2016*, 172. [CrossRef]
53. Wang, X.; Zhao, K. Boundedness of the variation operators associated with high-order Schrödinger-type operators on the Herz-type spaces. *J. Southwest Univ. Nat. Sci. Ed.* **2021**, *43*, 88–94.
54. Gao, C.; Li, P. The Herz-boundedness of commutators related to Schrödinger operators in the setting of Heisenberg group. *Bull. Malays. Math. Sci. Soc.* **2022**, *45*, 1213–1240. [CrossRef]

55. Akbulut, A.; Kuzu, O. Marcinkiewicz integrals associated with Schrödinger operator on generalized Morrey spaces. *J. Math. Inequal.* **2014**, *8*, 791–801. [CrossRef]
56. Akbulut, A.; Celik, S.; Omarova, M.N. Fractional maximal operator associated with Schrödinger operator and its commutators on vanishing generalized Morrey spaces. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **2024**, *44*, 3–19.
57. Guliyev, V.S.; Akbulut, A.; Celik, S. Fractional integral related to Schrödinger operator on vanishing generalized mixed Morrey spaces. *Bound. Value Probl.* **2024**, *2024*, 137. [CrossRef]
58. Lin, X.; Yang, D.; Yang, S.; Yuan, W. Hardy spaces associated with non-negative self-adjoint operators and ball quasi-Banach function spaces on doubling metric measure spaces and their applications. *Commun. Math. Stat.* **2026**, *14*, 141–193. [CrossRef]
59. Thuy, P.T.K.; Dung, K.H.; Nang, T.T. Estimates for *p*-adic multilinear Hardy–Littlewood operators on variable Morrey–Herz spaces. *Open Math.* **2026**, *24*, 20250270. [CrossRef]
60. Morrey, C.B. On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc.* **1938**, *43*, 126–166. [CrossRef]
61. Iwaniec, T.; Sbordone, C. On the integrability of the Jacobian under minimal hypotheses. *Arch. Ration. Mech. Anal.* **1992**, *119*, 129–143. [CrossRef]
62. Sultan, B.; Hussain, A.; Sultan, M. Grand variable Herz–Morrey–Hardy spaces and applications to the boundedness of integral operators. *arXiv* **2025**, arXiv:2508.17295.
63. Sultan, B.; Hussain, A.; Lassoued, D.; Sultan, M. Higher-order commutators of fractional Hardy-type operators in grand variable Herz–Morrey spaces. *Asian-Eur. J. Math.* **2026**, 2650063. [CrossRef]
64. Ho, K.-P. Grand Morrey spaces and grand Hardy–Morrey spaces on Euclidean space. *J. Geom. Anal.* **2023**, *33*, 180. [CrossRef]
65. Afzal, W.; Alqahtani, M.H.; Abbas, M.; Breaz, D. Regularity and qualitative study of parabolic physical Ginzburg–Landau equations in variable exponent Herz spaces via fractional Bessel–Riesz operators. *Fractal Fract.* **2025**, *9*, 644. [CrossRef]
66. Wang, L.; Tao, S. Boundedness of Littlewood–Paley operators and their commutators on Herz–Morrey spaces with variable exponent. *J. Inequal. Appl.* **2014**, *2014*, 227. [CrossRef]
67. Afzal, W. Regularity of parabolic Ornstein–Uhlenbeck equations via boundedness of fractional Muckenhoupt-type weighted singular operators in variable Herz spaces. *J. Pseudo-Differ. Oper. Appl.* **2026**, *17*, 2. [CrossRef]
68. Wang, S.; Xu, J. Boundedness of vector valued bilinear Calderón–Zygmund operators on products of weighted Herz–Morrey spaces with variable exponents. *Chin. Ann. Math. Ser. B* **2021**, *42*, 693–720. [CrossRef]
69. Maeda, F.-Y.; Mizuta, Y.; Ohno, T.; Shimomura, T. Duality of central Herz–Morrey–Musielak–Orlicz spaces of variable exponents. *Collect. Math.* **2019**, *70*, 117–151. [CrossRef]
70. Cen, X.; Song, Z. Extrapolation to product Morrey–Herz spaces and applications. *Indian J. Pure Appl. Math.* **2026**, 1–25. [CrossRef]
71. Ho, K.-P. Boundedness of operators and inequalities on Morrey–Banach spaces. *Publ. Res. Inst. Math. Sci.* **2022**, *58*, 551–577. [CrossRef]
72. Tacy, M. L^p estimates for joint quasimodes of semiclassical pseudodifferential operators. *Isr. J. Math.* **2019**, *232*, 401–425. [CrossRef]
73. Tacy, M. L^p estimates for joint quasimodes of semiclassical pseudodifferential operators whose characteristic sets have kth order contact. *Int. Math. Res. Not. IMRN* **2021**, *2021*, 17766–17797. [CrossRef]
74. Afzal, W.; Abbas, M.; Macías-Díaz, J.E.; Gallegos, A.; Almalki, Y. Boundedness and Sobolev-type estimates for the exponentially damped Riesz potential with applications to the regularity theory of elliptic PDEs. *Fractal Fract.* **2025**, *9*, 458. [CrossRef]
75. Shi, Y.; Tao, X. Multilinear Riesz potential operators on Herz-type spaces and generalized Morrey spaces. *Hokkaido Math. J.* **2009**, *38*, 635–662. [CrossRef]
76. Afzal, W. Boundedness and regularity of the Navier–Stokes system in generalized Herz spaces via a novel fractional potential framework. *Chaos Solitons Fractals* **2025**, *201*, 117086. [CrossRef]
77. Zhang, Q.; Liu, L. Boundedness of multilinear commutator of singular integral in Morrey spaces on homogeneous spaces. *Acta Univ. Apulensis Math. Inform.* **2009**, *20*, 111–125.
78. Afzal, W.; Abbas, M.; El-Naggar, M.A.; Khan, Z.A. Boundedness results for the fractional Wolff potential in Morrey spaces over homogeneous groups. *Fractal Fract.* **2026**, *10*, 575. [CrossRef]
79. Artes, R.G., Jr.; Afzal, W.; Asjad, S.; Tariq, M.; Ahmad, H. Wolff potential and Birkhoff's ergodic theorem in grand variable Herz spaces. *Gulf J. Math.* **2026**, *23*, 2. [CrossRef]
80. Afzal, W.; Abbas, M.; Aloraini, N.M.; Ro, J. Resolution of open problems via Orlicz–Zygmund spaces and new geometric properties of Morrey spaces in the Besov sense with non-standard growth. *AIMS Math.* **2025**, *10*, 13908–13940. [CrossRef]
81. Khan, Z.A.; Afzal, W.; Abbas, M.; Breaz, D.; Cotîrlă, L.I. Boundedness of truncated Havin–Maz'ya potentials in Herz spaces with applications to $p(\cdot)$-Laplace equations. *AIMS Math.* **2026**, *11*, 20677–20710. [CrossRef]
82. Idris, M.; Gunawan, H.; Eridani. The boundedness of Bessel–Riesz operators on generalized Morrey spaces. *Aust. J. Math. Anal. Appl.* **2016**, *13*, 9.
83. Idris, M.; Gunawan, H.; Eridani. Norm estimates for Bessel–Riesz operators on generalized Morrey spaces. *Math. Bohem.* **2018**, *143*, 409–417. [CrossRef]

84. Muckenhoupt, B. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **1972**, *165*, 207–226. [CrossRef]
85. Cruz-Uribe, D.; Diening, L.; Hästö, P. The maximal operator on weighted variable Lebesgue spaces. *Fract. Calc. Appl. Anal.* **2011**, *14*, 361–374. [CrossRef]
86. Fefferman, C.; Stein, E.M. H^p spaces of several variables. *Acta Math.* **1972**, *129*, 137–193. [CrossRef]

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