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Solvability, Ulam–Hyers Stability, and Kernel Analysis of Multi-Order σ-Hilfer Fractional Systems: A Unified Theoretical Framework

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Abstract

This paper establishes a rigorous analytical framework for a nonlinear multi-order fractional differential system governed by the generalized σ-Hilfer operator in weighted Banach spaces. In contrast to existing studies that often treat specific kernels or fixed fractional orders in isolation, our approach provides a unified treatment that simultaneously handles multiple fractional orders, a tunable kernel $\sigma(\varsigma)$, weighted integral conditions, and a nonlinearity depending on a fractional integral of the solution. By converting the hierarchical differential structure into an equivalent Volterra integral equation, we derive sufficient conditions for the existence and uniqueness of solutions using the Banach contraction principle and Mönch's fixed-point theorem with measures of non-compactness. The analysis is extended to Ulam–Hyers stability, ensuring robustness under modeling perturbations. A principal contribution is the systematic classification of the system's symmetric reductions—specifically the Riemann–Liouville, Caputo, Hadamard, and Katugampola forms—all governed by a single spectral condition dependent on $\sigma(\varsigma)$. The theoretical results are illustrated by numerical examples that highlight the sensitivity of solutions to the memory kernel and the fractional orders. This work provides a cohesive analytical tool for a broad class of fractional systems with memory, thereby unifying previously disparate fractional calculi under a single, consistent framework.

Keywords: generalized σ-Hilfer fractional derivative; multi-order fractional systems; stability; kernel analysis; symmetric reductions; fixed point theory; numerical methods

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1. Introduction and Motivations

Fractional differential equations (FDEs) have become fundamental in modeling systems with memory, anomalous diffusion, and hereditary properties [1]. Unlike classical operators, fractional derivatives define the state of a system not merely by its instantaneous

rate of change but by its entire history, weighted by a memory kernel. While early research focused on specific operators such as Riemann–Liouville, Caputo, or Hadamard, recent developments have sought unified frameworks. The σ-Hilfer fractional derivative [2], defined by a regularizing kernel $\sigma(\varsigma)$ and a type parameter $\alpha \in [0, 1]$, represents a significant step in this direction. It allows a continuous transition between initial-value regularities (Caputo) and singular behaviors (Riemann–Liouville), while also including kernel-specific derivatives like those of Katugampola and Hadamard type.

Systems involving multiple fractional orders are necessary for describing processes that evolve on different time scales, where components follow distinct fractional indices ($\beta_1 < \beta_2 < \cdots < \beta_n$). The analysis of such multi-order systems requires a rigorous treatment of the interplay between these hierarchical orders and the weighting function $\sigma(\varsigma)$. Multi-order differential structures provide a mathematical framework for phenomena where fast and slow memory-driven dynamics coexist. Ensuring the well-posedness of such problems is essential, particularly where stability under perturbations is required for reliable modeling. Hybrid fractional systems with integral boundary conditions have been studied in pantograph-type equations, highlighting the importance of multi-order and memory effects [3].

Multi-order fractional systems have been investigated in various contexts. They have been used to describe nonlinear behavior in chemical reactor theory [4] and singular heat conduction profiles [5]. Applications extend to electrical and neural systems, including finite-element analysis of circuit equations [6] and the synchronization of chaotic neural networks [7]. Further uses include modeling macro-scale socioeconomic trends [8]. Parallel theoretical advances have addressed optimal control with impulses [9] and well-posedness under integral boundary conditions [10–12].

Recent studies continue to expand the theory and applications of fractional systems. For instance, focusing specifically on the Hilfer operator, recent extensions have introduced weighted and kernel-generalized versions, such as the ψ-, w-, and φ-Hilfer derivatives, to model diverse memory kernels [13–16]. Recent extensions to q-analogs of fractional operators with respect to another function have further enriched the theory of kernel-based fractional calculus [17]. Existence, uniqueness, and stability results have been established for these operators, often under impulsive conditions or complex boundary constraints [18–21]. The analysis has also been extended to coupled systems and sequential equations involving multiple fractional orders [22–25]. However, much of this work treats specific, fixed kernel forms or focuses on single-order derivatives. A unified analysis that simultaneously incorporates a tunable kernel $\sigma(\varsigma)$, multiple distinct fractional orders within a single coupled system, weighted integral conditions, and a nonlinearity depending on a fractional integral remains largely unexplored. This gap limits the ability to systematically relate different fractional calculi and to model hierarchical memory processes with a single, flexible framework.

While the literature contains numerous results on existence, uniqueness, and stability for various fractional operators, our contribution lies in providing a unified treatment that simultaneously incorporates the following:

1. Multiple distinct fractional orders $\beta_1 < \beta_2 < \cdots < \beta_n$ within a single coupled system.
2. A tunable kernel $\sigma(\varsigma)$ governing memory weighting and time scaling.
3. Weighted integral initial conditions in Banach spaces.
4. A nonlinearity $\mathbb{G}$ depending explicitly on a fractional integral of the unknown function.

The key novelty is that all these features are handled under a single spectral condition $\Omega(\Lambda_\sigma) < 1$, where $\Lambda_\sigma = \sigma(b) - \sigma(0)$. This condition systematically governs the well-posedness of the general system as well as all its symmetric reductions (Riemann–Liouville, Caputo, Katugampola, and Hadamard). To our knowledge, such a comprehensive frame-

work that bridges these different fractional calculi through a unified spectral criterion has not been presented before.

The present work aims to fill this gap by investigating the solvability, uniqueness, symmetric cases, and Ulam–Hyers stability of a multi-order fractional system governed by the generalized σ-Hilfer operator. The proposed system combines (i) multiple fractional orders, (ii) a tunable kernel $\sigma(\varsigma)$, (iii) weighted integral initial conditions, and (iv) a nonlinearity depending on a fractional integral of the unknown function, which has not been fully studied in the literature.

We consider the following nonlinear multi-order σ-Hilfer system:

$$\mathbb{D}^{\beta_n,\alpha_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) - \sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) = \mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big), \quad \varsigma \in J := [0,b], \quad (1)$$

subject to weighted integral boundary conditions

$$\lim_{\varsigma\to 0}\mathbb{I}^{1-\gamma_j}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = z_j, \gamma_j = \beta_j + \alpha_j(1-\beta_j), j = 1,2,\ldots,n,$$

where the weighted function $\sigma(\varsigma) \in \mathcal{C}^1([0,b),\mathbb{R})$ is increasing, such that $\sigma'(\varsigma) \neq 0$, $\forall\, \varsigma \in (0,b)$, $\mathbb{D}^{\beta_n,\alpha_n}_{0^+,\sigma(\varsigma)}$ is the σ-Hilfer FD of order $\beta_n, (0 < \beta_n < 1)$ and type $(0 \leq \alpha_n \leq 1)$, $\forall\, n \in \mathbb{N}^*$, $\mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}$ denotes the σ-Hilfer FD of order $\beta_j, 0 < \beta_1 < \beta_2 < \ldots < \beta_n < 1, a_j \in \mathbb{R}$, $\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}$ denotes the σ -Hilfer fractional integral with q_n such that $\beta_j + q_n < \beta_n$, and the functions $\mathbb{S} : J \times \mathbb{R} \to \mathbb{R}$ and $\mathbb{G} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

The σ-Hilfer fractional system (1) includes (i) multiple distinct fractional orders, (ii) a tunable kernel $\sigma(\varsigma)$, (iii) weighted integral conditions, and (iv) a nonlinearity depending on a fractional integral of the unknown function. While previous studies have treated specific kernels or single-order derivatives in isolation, our framework provides a cohesive analytical tool that bridges Riemann–Liouville, Caputo, Katugampola, and Hadamard fractional calculi under a single spectral condition.

The generalized structure of System (1) subsumes several established fractional derivatives through specific cases of the weighting kernel $\sigma(\varsigma)$ and the interpolation parameter α, such as the following: (i) **Classical Forms ($\sigma(\varsigma) = \varsigma$):** The system reduces to the Riemann–Liouville ($\alpha_n = \alpha_j = 0$), Caputo ($\alpha_n = \alpha_j = 1$), and standard Hilfer ($\alpha_n, \alpha_j \in (0,1)$) formulations. (ii) **Power-Law Forms ($\sigma(\varsigma) = \varsigma^c, c > 0$):** This kernel yields the Katugampola family of operators, specifically recovering the Katugampola ($\alpha_n = \alpha_j = 0$), Caputo–Katugampola ($\alpha_n = \alpha_j = 1$), and Hilfer–Katugampola ($\alpha_n, \alpha_j \in (0,1)$) systems. (iii) **Logarithmic Forms ($\sigma(\varsigma) = \log\varsigma$):** The choice of a logarithmic weight corresponds to the Hadamard spectral analysis, recovering the Hadamard ($\alpha_n = \alpha_j = 0$), Caputo–Hadamard ($\alpha_n = \alpha_j = 1$), and Hilfer–Hadamard ($\alpha_n, \alpha_j \in (0,1)$) variants.

The main contributions of this paper are as follows:

- **A unified solvability theory.** By converting the differential problem into an equivalent Volterra integral equation, we establish sufficient conditions for existence and uniqueness using both the Banach contraction principle and Mönch's fixed-point theorem with measures of non-compactness. This allows the treatment of problems where the nonlinearities are not globally Lipschitz.
- **Ulam–Hyers stability with an explicit constant.** We prove the system is Ulam–Hyers stable and derive the stability constant explicitly in terms of the kernel $\sigma(\varsigma)$, the fractional orders, and the domain length. This provides a quantitative measure of robustness against modeling perturbations.
- **Systematic classification of symmetric reductions.** We demonstrate how specific choices of the kernel $\sigma(\varsigma)$ reduce the general system to the classical Hilfer (and

hence Riemann–Liouville and Caputo), Hilfer–Katugampola, and Hilfer–Hadamard formulations. A single spectral condition, $\Omega(\Lambda_\sigma) < 1$ with $\Lambda_\sigma = \sigma(b) - \sigma(0)$, governs the well-posedness of all these symmetric cases, offering a unified criterion that connects previously disparate fractional calculi.

- **Illustrative numerical validation.** A dedicated model problem is introduced, and numerical simulations are performed to verify the theoretical criteria. The results show how the solution profile depends on the fractional orders β_j, q_n and the kernel $\sigma(\varsigma)$.

In summary, this work advances the theory of fractional differential systems by delivering a cohesive analytical framework for a broad class of multi-order σ-Hilfer problems—encompassing existence, uniqueness, stability, and a complete symmetry classification.

The manuscript is organized as follows: Section 2 recalls the necessary definitions and properties of the σ-Hilfer operator. Section 3 converts System (1) into its integral form and proves the existence, uniqueness, symmetric reductions, and Ulam–Hyers stability theorems. An illustrative example and numerical simulations are presented in Section 4. We conclude with a summary of the primary contributions in Section 5.

2. Basic Definitions and Essential Interpretations

In this section, we introduce the function spaces which used in this work. Also, we recall the basic definitions, lemmas, and theorems for the generalized Hilfer fractional operator. For $b > 0$, let $J := (0, b] \subset \mathbb{R}$, and let $\mathcal{C}(J, \mathbb{R})$ be the space of continuous function ϖ defined on J, with the norm

$$\|\varpi\| = \max_{\varsigma \in J} |\varpi(\varsigma)|.$$

For $j = 1, 2, ..., n$, we define weighted spaces $\mathcal{C}_{1-\gamma_j,\sigma}(J, \mathbb{R})$ and $\mathcal{C}^{\gamma_j}_{1-\gamma_j,\sigma}(J, \mathbb{R})$ as follows:

$$\mathcal{C}_{1-\gamma_j,\sigma}(J, \mathbb{R}) = \left\{\varpi : J \to \mathbb{R}; (\sigma(\varsigma) - \sigma(0))^{1-\gamma_j}\varpi(\varsigma) \in \mathcal{C}(J, \mathbb{R})\right\},$$

and

$$\mathcal{C}^{\gamma_j}_{1-\gamma_j,\sigma}(J, \mathbb{R}) = \left\{\varpi \in \mathcal{C}_{1-\gamma_j,\sigma}(J, \mathbb{R}) : \mathbb{D}^{\gamma_j}_{0^+,\sigma}\varpi \in \mathcal{C}_{1-\gamma_j,\sigma}(J, \mathbb{R})\right\},$$

where $\gamma_j = \beta_j + \alpha_j(1 - \beta_j)$, $\beta_j \in (0, 1)$ and $\alpha_j \in [0, 1]$. Clearly, the space $\mathcal{C}_{1-\gamma_j,\sigma}(J, \mathbb{R})$ is the Banach space with the norm

$$\|\varpi\|_{1-\gamma_j,\sigma} = \max_{\varsigma \in J} \left|(\sigma(\varsigma) - \sigma(0))^{1-\gamma_j}\varpi(\varsigma)\right|.$$

The following definitions and Lemmas are cited from the papers [1,2,26–28].

Definition 1. *Let $\beta > 0$, and let $\sigma(\varsigma)$ be an increasing and positive monotone function on $(0, b]$, having a continuous derivative $\sigma'(\varsigma) \neq 0$ for all $\varsigma \in (0, b)$. The left-sided fractional integral of ϖ with respect to function σ on the interval $[0, b]$ is given by*

$$\mathbb{I}^{\beta}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \frac{1}{\Gamma(\beta)} \int_0^{\varsigma} \sigma'(s)(\sigma(\varsigma) - \sigma(s))^{\beta-1}\varpi(s)ds,$$

where $\Gamma(\beta)$ is the standard Gamma function given in [29].

Definition 2. *Let $n - 1 < \beta < n, n \in \mathbb{N}$, $0 \leq \alpha \leq 1$, $\varsigma > 0$, and let $\varpi, \sigma \in \mathcal{C}^n(J, \mathbb{R})$ be two functions, such that σ is an increasing function on $(0, b)$, $\sigma'(\varsigma) \neq 0$ for all $\varsigma \in (0, b)$. The left-sided σ-Hilfer FD $\mathbb{D}^{\beta,\alpha}_{0^+,\sigma(\varsigma)}(\cdot)$ of function ϖ of order β and type α is given by*

$$\mathbb{D}^{\beta,\alpha}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \mathbb{I}^{\alpha(n-\beta)}_{0^+,\sigma(\varsigma)} \left(\frac{1}{\sigma'(\varsigma)}\frac{d}{d\varsigma}\right)^n \mathbb{I}^{(1-\alpha)(n-\beta)}_{0^+,\sigma(\varsigma)} \varpi(\varsigma). \tag{2}$$

In the case $n = 1$, we obtain

$$\mathbb{D}^{\beta,\alpha}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \mathbb{I}^{\alpha(1-\beta)}_{0^+,\sigma(\varsigma)}\left(\frac{1}{\sigma'(\varsigma)}\frac{d}{d\varsigma}\right)\mathbb{I}^{(1-\alpha)(1-\beta)}_{0^+,\sigma(\varsigma)}\varpi(\varsigma). \tag{3}$$

Clearly

$$\mathbb{D}^{\beta,\alpha}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \mathbb{I}^{\alpha(1-\beta)}_{0^+,\sigma(\varsigma)}\,\mathbb{D}^{\gamma}_{0^+,\sigma(\varsigma)}\varpi(\varsigma).$$

where $\gamma = \beta + \alpha(1-\beta)$.

Lemma 1. *Let $\beta, \eta, \delta > 0$, then, we have*

1. $\mathbb{I}^{\beta}_{0^+,\sigma(\varsigma)}\,\mathbb{I}^{\eta}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \mathbb{I}^{\beta+\eta}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)$.
2. $\mathbb{I}^{\beta}_{0^+,\sigma(\varsigma)}(\sigma(\varsigma)-\sigma(0))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\beta+\delta)}(\sigma(\varsigma)-\sigma(0))^{\beta+\delta-1}$.

Note that $\mathbb{D}^{\gamma}_{0^+,\sigma(\varsigma)}(\sigma(\varsigma)-\sigma(0))^{\gamma-1} = 0$, where $\gamma = \beta + \alpha(1-\beta)$.

Lemma 2. *For $\gamma = \beta + \alpha(1-\beta), \beta \in (0,1)$, $\alpha \in [0,1]$ and $\varpi \in \mathcal{C}(J,\mathbb{R})$, we have*

$$\mathbb{I}^{\beta}_{0^+,\sigma(\varsigma)}\,\mathbb{D}^{\beta,\alpha}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \varpi(\varsigma) - \frac{(\sigma(\varsigma)-\sigma(0))^{\gamma-1}}{\Gamma(\gamma)}\,\mathbb{I}^{(1-\alpha)(1-\beta)}_{0^+,\sigma(\varsigma)}\varpi(\varsigma).$$

Lemma 3. *For $\gamma = \beta + (1-\beta)\alpha, 0 < \beta < 1, 0 \le \alpha \le 1$ and $\varpi \in \mathcal{C}^{\gamma}_{1-\gamma;\sigma(\varsigma)}(J,\mathbb{R})$, we have*

$$\mathbb{D}^{\beta,\alpha}_{0^+,\sigma(\varsigma)}\mathbb{I}^{\beta}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \varpi(\varsigma), \quad \mathbb{I}^{\gamma}_{0^+,\sigma(\varsigma)}\mathbb{D}^{\gamma}_{0^+,\sigma(\varsigma)}\varpi = \mathbb{I}^{\beta}_{0^+,\sigma(\varsigma)}\,\mathbb{D}^{\beta,\alpha}_{0^+,\sigma(\varsigma)}\varpi,$$

and

$$\mathbb{D}^{\gamma}_{0^+,\sigma(\varsigma)}\mathbb{I}^{\beta}_{0^+,\sigma(\varsigma)}\varpi = \mathbb{D}^{\alpha(1-\beta)}_{0^+,\sigma(\varsigma)}\varpi.$$

Lemma 4 (Mönch's fixed-point theorem [30])**.** *Let Z be a bounded, closed, convex subset of a Banach space X, and let $\mathbb{X} : Z \to Z$ be a continuous operator. If for every countable subset $V \subset Z$ with $V = \overline{\text{conv}}(\{x_0\} \cup \mathbb{X}(V))$, the set V is relatively compact, then $\mathbb{X}$ has a fixed point in Z.*

3. Main Results

In this section, we examine the sufficient conditions ensuring the existence, uniqueness, and Ulam–Hyers stability of System (1), and we analyze its symmetric cases and reduced forms. The theoretical framework is developed by initially converting the multi-order differential problem into its equivalent Volterra integral form.

3.1. Equivalent Volterra Integral Form

Theorem 1. *Let $\gamma_n = \beta_n + \alpha_n(1-\beta_n)$, $n \in \mathbb{N}^*, 0 < \beta_n < 1, 0 \le \alpha \le 1$. If $\mathbb{S}, \mathbb{G} \in \mathcal{C}_{1-\gamma,\sigma}(J,\mathbb{R})$, and $\varpi \in \mathcal{C}^{\gamma_n}_{1-\gamma_n,\sigma}(J,\mathbb{R})$. The equivalent Volterra integral form of system (1) is given by the following:*

$$\begin{aligned}\varpi(\varsigma) &= \frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_n-1}}{\Gamma(\gamma_n)}z_n + \sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) - \sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_{n-j}}_{0^+,\sigma(\varsigma)}\frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_j-1}}{\Gamma(\gamma_j)}z_j \\ &+ \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big).\end{aligned} \tag{4}$$

Proof. Let $\varpi \in \mathcal{C}^{\gamma_n}_{1-\gamma_n,\sigma}(J,\mathbb{R})$ be a solution of system (1). According to the definition of $\mathcal{C}^{\gamma_n}_{1-\gamma_n,\sigma}(J,\mathbb{R})$, with the help of Definition 2, we have

$$\mathbb{I}^{1-\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R}) \text{ and } \mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \mathbb{D}^{1}_{0^+,\sigma(\varsigma)}\mathbb{I}^{1-\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R}).$$

By the definition of the space $\mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$, we have

$$\mathbb{I}^{1-\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) \in \mathcal{C}^{1}_{1-\gamma_n,\sigma}(J,\mathbb{R}). \tag{5}$$

Using Lemma 2, we obtain

$$\left(\mathbb{I}^{\gamma_n}_{0^+,\sigma(\varsigma)}\,\mathbb{D}^{\gamma_n,\alpha_n}_{0^+,\sigma(\varsigma)}\varpi\right)(\varsigma) = \varpi(\varsigma) - \frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_n-1}}{\Gamma(\gamma_n)}\lim_{\varsigma\to 0}\left(\mathbb{I}^{1-\gamma_n}_{0^+,\sigma(\varsigma)}\varpi\right)(\varsigma). \tag{6}$$

By hypothesis $\left(\varpi \in \mathcal{C}^{\gamma_n}_{1-\gamma_n,\sigma}(J,\mathbb{R})\right)$ and Lemma 3, we have

$$\left(\mathbb{I}^{\gamma_n}_{0^+,\sigma(\varsigma)}\,\mathbb{D}^{\gamma_n,\alpha_n}_{0^+,\sigma(\varsigma)}\varpi\right)(\varsigma) = \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\,\mathbb{D}^{\beta_n,\alpha}_{0^+,\sigma(\varsigma)}\varpi(\varsigma).$$

Since $\mathbb{D}^{\beta_n,\alpha}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \sum_{j=1}^{n-1} a_1\mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{G}\left(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\right)$, we have

$$\left(\mathbb{I}^{\gamma_n}_{0^+,\sigma(\varsigma)}\,\mathbb{D}^{\gamma_n,\alpha_n}_{0^+,\sigma(\varsigma)}\varpi\right)(\varsigma) = \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\left[\sum_{j=1}^{n-1} a_1\mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{G}\left(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\right)\right]. \tag{7}$$

Comparing Equations (6) and (7), we see that

$$\begin{aligned}\varpi(\varsigma) &= \frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_n-1}}{\Gamma(\gamma_n)}\lim_{\varsigma\to 0}\left(\mathbb{I}^{1-\gamma_n}_{0^+,\sigma(\varsigma)}\varpi\right)(\varsigma)\\ &\quad+\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\left[\sum_{j=1}^{n-1} a_1\mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{G}\left(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\right)\right].\end{aligned} \tag{8}$$

In the second term in Equation (8), we have

$$\begin{aligned}&\sum_{j=1}^{n-1} a_j\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\left(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\right)\\ =\ &\sum_{j=1}^{n-1} a_j\mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\left[\mathbb{I}^{\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{D}^{\beta_j,\alpha}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma))\right] + \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\left(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\right)\\ =\ &\sum_{j=1}^{n-1} a_j\mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\left[\mathbb{S}(\varsigma,\varpi(\varsigma)) - \frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_j-1}}{\Gamma(\gamma_j)}\lim_{\varsigma\to 0}\mathbb{I}^{1-\gamma_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma))\right]\\ &+\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\left(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\right).\end{aligned} \tag{9}$$

Put Equations (9) and (8) by using the conditions

$$\begin{aligned}\lim_{\varsigma\to 0}\mathbb{I}^{1-\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) &= z_n,\\ \lim_{\varsigma\to 0}\mathbb{I}^{1-\gamma_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) &= z_j, j = 1,2,\ldots,n-1.\end{aligned}$$

Then, we have

$$\varpi(\varsigma) = \frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_n-1}}{\Gamma(\gamma_n)}z_n + \sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) - \sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_j-1}}{\Gamma(\gamma_j)}z_j$$
$$+\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big). \tag{10}$$

Hence $\varpi_i(\varsigma)$ satisfies Equation (4).

Conversely, let $\varpi \in \mathcal{C}^{\gamma_n}_{1-\gamma_n,\sigma}(J,\mathbb{R})$ be a functions satisfying (4). Applying the operator $\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}$ on both sides of Equation (4), we get

$$\begin{aligned}\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) &= \mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_n-1}}{\Gamma(\gamma_n)}z_n + \mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma))\\ &\quad -\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_j-1}}{\Gamma(\gamma_j)}z_j\\ &\quad +\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big).\end{aligned} \tag{11}$$

By Lemma 1, we have $\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}(\sigma(\varsigma)-\sigma(0))^{\gamma_n-1}=0$, and then the first and third terms in Equation (11) become zero. By Lemma 3, the second term in Equation (11) becomes

$$\begin{aligned}\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) &= \mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma))\\ &= \sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)).\end{aligned} \tag{12}$$

Thus Equation (11) becomes

$$\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big). \tag{13}$$

From (5), we have $\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$, and hence, the second term in Equation (13) implies

$$\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j,\alpha}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{D}^{\alpha(1-\beta_n)}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big) \in \mathcal{C}_{1-\gamma_i,\sigma}(J,\mathbb{R}). \tag{14}$$

As $\mathbb{G}(\cdot,\varpi(\cdot),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\cdot)) \in \mathcal{C}_{1-\gamma_i,\sigma}(J,\mathbb{R})$, it follows that

$$\mathbb{I}^{1-\alpha(1-\beta_n)}_{0^+,\sigma(\varsigma)}\mathbb{S}(\cdot,\varpi(\cdot),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\cdot)) \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R}). \tag{15}$$

By Equations (14) and (15) and the definition of the space $\mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$, we have

$$\sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{I}^{1-\alpha_n(1-\beta_n)}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big) \in \mathcal{C}^1_{1-\gamma_n,\sigma}(J,\mathbb{R}).$$

Now, by applying operator $\mathbb{I}^{\alpha(1-\beta_n)}_{0^+,\sigma(\varsigma)}$ on both sides of Equation (14) and using Lemma 2, we have

$$\mathbb{I}^{\alpha_n(1-\beta_n)}_{0^+,\sigma(\varsigma)}\mathbb{D}^{\gamma_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) = \sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big). \tag{16}$$

From Equation (2), Equation (16) reduces to

$$\mathbb{D}^{\beta_n,\alpha_n}_{0^+,\sigma(\varsigma)}\varpi_i(\varsigma) = \sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big).$$

Thus, Equation (1) holds. □

The Volterra integral formulation (4) not only facilitates theoretical analysis but also lends itself to efficient numerical approximation via well-established methods such as the homotopy perturbation method or the variational iteration method. These techniques provide practical pathways for solving the system in applied settings, bridging theoretical modeling with computational implementation. To apply the fixed-point theorem, we define the operator $\mathbb{X} : \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R}) \to \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$ by

$$\begin{aligned}\mathbb{X}(\varpi)(\varsigma) &= \frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_n-1}}{\Gamma(\gamma_n)} z_n + \sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma)) \\ &\quad - \sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_{n-j}}_{0^+,\sigma(\varsigma)} \frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_j-1}}{\Gamma(\gamma_j)} z_j + \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big). \quad (17)\end{aligned}$$

To obtain our results, the following hypotheses must be satisfied:

(Hy_1) The functions $\mathbb{S} : J \times \mathbb{R} \to \mathbb{R}$ and $\mathbb{G} : J \times \mathbb{R} \times \mathbb{R}$, are continuous.

(Hy_2) There exist non-negative constants $L_{\mathbb{S}}, M_{\mathbb{S}}, L_{\mathbb{G}}$, and $M_{\mathbb{G}}$, such that, for each $\varsigma \in J$ and $\varpi, \varpi_1, \varpi_2 \in \mathbb{R}$,

$$\begin{aligned}|\mathbb{S}(\varsigma,\varpi(\varsigma))| &\leq L_{\mathbb{S}}|\varpi(\varsigma)| + M_{\mathbb{S}}. \\ |\mathbb{G}(\varsigma,\varpi_1(\varsigma),\varpi_2(\varsigma))| &\leq L_{\mathbb{G}}[|\varpi_1(\varsigma)| + |\varpi_2(\varsigma)|] + M_{\mathbb{G}}.\end{aligned}$$

(Hy_3) There exist Lipschitz constants $\kappa_{\mathbb{S}} > 0$, and $\kappa_{\mathbb{G}} > 0$, such that, for each $\varsigma, \varpi, \varpi^*, \varpi_1, \varpi_1^*, \varpi_2, \varpi_2^* \in \mathbb{R}$,

$$\big|\mathbb{S}(\varsigma,\varpi(\varsigma)) - \mathbb{S}_j(\varsigma,\varpi^*(\varsigma))\big| \leq \kappa_{\mathbb{S}}|\varpi(\varsigma) - \varpi^*(\varsigma)|,$$

$$|\mathbb{G}(\varsigma,\varpi_1(\varsigma),\varpi_2(\varsigma)) - \mathbb{G}(\varsigma,\varpi_1^*(\varsigma),\varpi_2^*(\varsigma))| \leq \kappa_{\mathbb{G}}[|\varpi_1(\varsigma) - \varpi_1^*(\varsigma)| + |\varpi_2(\varsigma) - \varpi_2^*(\varsigma)|]$$

3.2. Existence of a Solution

While the Banach contraction principle suffices for establishing uniqueness under global Lipschitz conditions, it may fail when the nonlinearities are not globally Lipschitz or when the operator is not a strict contraction. In such cases, Mönch's fixed-point theorem, combined with measures of non-compactness, provides a more flexible tool for proving existence without requiring contractivity. This is especially relevant for systems with memory-dependent nonlinearities or non-Lipschitz source terms. Here, we employ Mönch's theorem to ensure the existence of solutions under weaker assumptions, thereby extending the applicability of our results to a broader class of multi-order fractional systems.

Theorem 2 (Existence). *Assume hypotheses $(Hy_1) - (Hy_3)$ hold. Then system (1) has at least one solution in $\mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$ provided that*

$$L_{o\beta} := \sum_{j=1}^{n-1} |a_j| \frac{(\sigma(b)-\sigma(0))^{\beta_n-\beta_j}}{\Gamma(\beta_n-\beta_j+1)} L_{\mathbb{S}} + \frac{(\sigma(b)-\sigma(0))^{\beta_n}}{\Gamma(\beta_n+1)} L_{\mathbb{G}}\left(1 + \frac{(\sigma(b)-\sigma(0))^{q_n}}{\Gamma(q_n+1)}\right) < 1. \quad (18)$$

Proof. Let us consider the operator $\mathbb{X}$ defined by (17). The fixed point of $\mathbb{X}$ is a solution to system (1). Define a closed, convex, bounded set $\mathcal{Z}$ invariant under $\mathbb{X}$ such that

$$\mathcal{Z} = \Big\{\varpi \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R}) : \|\varpi\|_{1-\gamma_n,\sigma} \leq R\Big\},$$

with $R \geq \frac{\mathcal{Q}_1}{1-L_{o\beta}}$, where

$$\begin{aligned}\mathcal{Q}_1 \quad :\quad &= \frac{|z_n|}{\Gamma(\gamma_n)} + \sum_{j=1}^{n-1}|a_j|\frac{(\sigma(b)-\sigma(0))^{\beta_n-\beta_j}}{\Gamma(\beta_n-\beta_j+1)}M_{\mathbb{S}} + \sum_{j=1}^{n-1}|a_j||z_j|\frac{\Gamma(\gamma_j)(\sigma(b)-\sigma(0))^{\gamma_j+\beta_{n-j}-\gamma_n}}{\Gamma(\gamma_j+\beta_{n-j})}\\ &+\frac{(\sigma(b)-\sigma(0))^{\beta_n}}{\Gamma(\beta_n+1)}M_{\mathbb{G}}.\end{aligned}$$

For $\varsigma \in J, \varpi \in \mathcal{Z}$, we have

$$\begin{aligned}\|\mathbb{X}(\varpi)\|_{1-\gamma_n,\sigma} &= \max_{\varsigma\in J}\Bigg|(\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}\Bigg[\frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_n-1}}{\Gamma(\gamma_n)}z_n + \sum_{j=1}^{n-1}a_j\mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma))\\ &\quad -\sum_{j=1}^{n-1}a_j\mathbb{I}^{\beta_{n-j}}_{0^+,\sigma(\varsigma)}\frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_j-1}}{\Gamma(\gamma_j)}z_j + \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big)\Bigg]\Bigg|\\ &\leq \frac{|z_n|}{\Gamma(\gamma_n)} + \sum_{j=1}^{n-1}|a_j|\frac{(\sigma(\varsigma)-\sigma(0))^{\beta_n-\beta_j}}{\Gamma(\beta_n-\beta_j+1)}\Big(L_{\mathbb{S}}\|\varpi\|_{1-\gamma_n,\sigma}+M_{\mathbb{S}}\Big)\\ &\quad +\sum_{j=1}^{n-1}|a_j||z_j|\frac{\Gamma(\gamma_j)(\sigma(\varsigma)-\sigma(0))^{\gamma_j+\beta_{n-j}-\gamma_n}}{\Gamma(\gamma_j+\beta_{n-j})}\\ &\quad +\frac{(\sigma(\varsigma)-\sigma(0))^{\beta_n}}{\Gamma(\beta_n+1)}\Bigg(L_{\mathbb{G}}\Bigg[\|\varpi\|_{1-\gamma_n,\sigma}+\frac{(\sigma(\varsigma)-\sigma(0))^{q_n}}{\Gamma(q_n+1)}\|\varpi\|_{1-\gamma_n,\sigma}\Bigg]+M_{\mathbb{G}}\Bigg).\end{aligned}$$

Thus, we have $\|\mathbb{X}(\varpi)\|_{1-\gamma_n,\sigma} \leq R$. This implies that $\mathbb{X}(\mathcal{Z}) \subset \mathcal{Z}$. Now, we prove $\mathbb{X}$ is continuous on $\mathcal{Z}$. Let $\varpi_k \in \mathcal{Z}$, such that $\varpi_k \to \varpi$ in $\mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$. For $\varsigma \in J, \varpi, \varpi_k \in \mathcal{Z}$, we have

$$\begin{aligned}\|\mathbb{X}(\varpi_k)-\mathbb{X}(\varpi)\|_{1-\gamma_n,\sigma} &= \max_{\varsigma\in J}\Big|(\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}(\mathbb{X}(\varpi_k(\varsigma))-\mathbb{X}(\varpi(\varsigma)))\Big|\\ &\leq \kappa_{\mathbb{S}}\frac{(\sigma(b)-\sigma(0))^{\beta_n-\beta_j}}{\Gamma(\beta_n-\beta_j+1)}\sum_{j=1}^{n-1}|a_j|\|\varpi_k-\varpi\|_{1-\gamma_n,\sigma}\\ &\quad +\frac{(\sigma(b)-\sigma(0))^{\beta_n}}{\Gamma(\beta_n+1)}\Bigg(\kappa_{\mathbb{G}}\Bigg(1+\frac{(\sigma(b)-\sigma(0))^{q_n}}{\Gamma(q_n+1)}\Bigg)\Bigg)\|\varpi_k-\varpi\|_{1-\gamma_n,\sigma}.\end{aligned}$$

Since $\varpi_k \to \varpi$, as the fractional integral operators are continuous, and compositions with continuous functions $\mathbb{S}$ and $\mathbb{G}$ are also continuous, tt follows that $\|\mathbb{X}(\varpi_k)-\mathbb{X}(\varpi)\|_{1-\gamma_n,\sigma} \to 0$. Hence $\mathbb{X}$ is continuous on $\mathcal{Z}$. Now, to prove the Mönch condition, we let $V \subseteq \mathcal{Z}$ be a nonempty, closed, convex subset such that

$$V = \overline{co}(\{x_0\} \cup \mathbb{X}(V)), \text{ for some } x_0 \in \mathcal{Z}.$$

Consider the measure of non-compactness

$$\alpha_{\mathcal{Z}}(S) = \inf\{\delta > 0 : S \text{ can be covered by a finite number of sets with diameter } < \delta\}.$$

From the properties of $\alpha_{\mathcal{Z}}$, we have

$$\alpha_{\mathcal{Z}}(V) = \alpha_{\mathcal{Z}}(\overline{co}(\{x_0\} \cup \mathbb{X}(V))) = \alpha_{\mathcal{Z}}(\{x_0\} \cup \mathbb{X}(V)).$$

Since $\alpha_{\mathcal{Z}}(\{x_0\} \cup \mathbb{X}(V)) = \alpha_{\mathcal{Z}}(S)$ for any bounded set $S \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$. Then

$$\alpha_{\mathcal{Z}}(V) = \alpha_{\mathcal{Z}}(\mathbb{X}(V)). \tag{19}$$

By hypotheses (Hy_3), we have that $\mathbb{X}$ is Lipschitz continuous with constant $L_{o\beta}$ on $\mathcal{Z}$ where

$$L_{o\beta} = \sum_{j=1}^{n-1} |a_j| \frac{(\sigma(b)-\sigma(0))^{\beta_n-\beta_j}}{\Gamma(\beta_n-\beta_j+1)} L_{\mathbb{S}} + \frac{(\sigma(b)-\sigma(0))^{\beta_n}}{\Gamma(\beta_n+1)} L_{\mathbb{G}} \left(1 + \frac{(\sigma(b)-\sigma(0))^{q_n}}{\Gamma(q_n+1)}\right).$$

Thus, by the properties of the measure of noncompactness $\alpha_{\mathcal{Z}}$, we have

$$\alpha_{\mathcal{Z}}(\mathbb{X}(V)) \leq L_{o\beta}\alpha_{\mathcal{Z}}(V). \tag{20}$$

By (19) and (20), we have

$$\alpha_{\mathcal{Z}}(V) \leq L_{o\beta}\alpha_{\mathcal{Z}}(V).$$

Thus

$$\alpha_{\mathcal{Z}}(V)\big(1 - L_{o\beta}\big) \leq 0.$$

Since $L_{o\beta} < 1$ (from condition (18)), then $1 - L_{o\beta} > 0$, and therefore $\alpha_{\mathcal{Z}}(V) \leq 0$. By definition, the measure of non-compactness $\alpha_{\mathcal{Z}}(V)$ is always non-negative. Consequently, $\alpha_{\mathcal{Z}}(V) = 0$, and V is compact. By Mönch's fixed-point theorem, the operator $\mathbb{X}$ has at least one fixed point in $\mathcal{Z}$. This fixed point is a solution to system (1). □

While the σ-Hilfer operator provides a unifying framework for mainstream fractional derivatives (Riemann–Liouville, Caputo, Katugampola, and Hadamard), emerging fractional formulations such as the two-scale fractal derivative have shown promise in describing transport in highly heterogeneous media. Although not explicitly included here, the kernel-based structure of the σ-Hilfer operator could, in principle, be adapted to approximate such fractal derivatives through an appropriate choice of $\sigma(\varsigma)$.

3.3. Uniqueness of Solution

Theorem 3 (Uniqueness). *Assume that hypotheses (Hy_1) and (Hy_3) hold. System (1) has a unique solution provided that $G_{o\beta} < 1$, where*

$$G_{o\beta} = \sum_{j=1}^{n-1} |a_j| \frac{(\sigma(b)-\sigma(0))^{\beta_n-\beta_j}}{\Gamma(\beta_n-\beta_j+1)} \kappa_{\mathbb{S}} + \frac{(\sigma(b)-\sigma(0))^{\beta_n}}{\Gamma(\beta_n+1)} \kappa_{\mathbb{G}} \left(1 + \frac{(\sigma(b)-\sigma(0))^{q_n}}{\Gamma(q_n+1)}\right).$$

Proof. Let us consider that the operator $\mathbb{X}$ is defined by (17). Now, we show that $\mathbb{X}$ is a contraction mapping using the Banach fixed-point theorem. Let $\varpi, \widehat{\varpi} \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$, then we have

$$\|\mathbb{X}(\varpi) - \mathbb{X}(\widehat{\varpi})\|_{1-\gamma_n,\sigma} = \max_{\varsigma \in J} \Big| (\sigma(\varsigma)-\sigma(0))^{1-\gamma_n} [\mathbb{X}(\varpi)(\varsigma) - \mathbb{X}(\widehat{\varpi})(\varsigma)] \Big|.$$

Then by (17), we have

$$
\begin{aligned}
&\|\mathbb{X}(\varpi)-\mathbb{X}(\widehat{\varpi})\|_{1-\gamma_n,\sigma}\\
=\;&\max_{\varsigma\in J}\Bigg|(\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}\Bigg[\left(\sum_{j=1}^{n-1}a_j\mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\varpi(\varsigma))+\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big)\right)\\
&-\left(\sum_{j=1}^{n-1}a_j\mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\widehat{\varpi}(\varsigma))+\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\widehat{\varpi}(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\widehat{\varpi}(\varsigma)\Big)\right)\Bigg]\Bigg|\\
\leq\;&\sum_{j=1}^{n-1}|a_j|\mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\max_{\varsigma\in J}\Big|(\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}[\mathbb{S}(\varsigma,\varpi(\varsigma))-\mathbb{S}(\varsigma,\widehat{\varpi}(\varsigma))]\Big|\\
&+\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\max_{\varsigma\in J}\Big|(\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}\Big[\mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big)-\mathbb{G}\Big(\varsigma,\widehat{\varpi}(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\widehat{\varpi}(\varsigma)\Big)\Big]\Big|.
\end{aligned}
$$

By (Hy_3), we have

$$
\begin{aligned}
\|\mathbb{X}(\varpi)-\mathbb{X}(\widehat{\varpi})\|_{1-\gamma_n,\sigma}\;\leq\;&\sum_{j=1}^{n-1}|a_j|\mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\kappa_{\mathbb{S}}\max_{\varsigma\in J}\Big|(\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}[\varpi(\varsigma)-\widehat{\varpi}(\varsigma)]\Big|\\
&+\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\kappa_{\mathbb{G}}\Bigg[\max_{\varsigma\in J}\Big|(\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}[\varpi(\varsigma)-\widehat{\varpi}(\varsigma)]\Big|\\
&+\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\max_{\varsigma\in J}\Big|(\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}[\varpi(\varsigma)-\widehat{\varpi}(\varsigma)]\Big|\Bigg]\\
\leq\;&\sum_{j=1}^{n-1}|a_j|\frac{(\sigma(\varsigma)-\sigma(0))^{\beta_n-\beta_j}}{\Gamma(\beta_n-\beta_j+1)}\kappa_{\mathbb{S}}\|\varpi-\widehat{\varpi}\|_{1-\gamma_n,\sigma}\\
&+\frac{(\sigma(\varsigma)-\sigma(0))^{\beta_n}}{\Gamma(\beta_n+1)}\kappa_{\mathbb{G}}\left(1+\frac{(\sigma(\varsigma)-\sigma(0))^{q_n}}{\Gamma(q_n+1)}\right)\|\varpi-\widehat{\varpi}\|_{1-\gamma_n,\sigma}.
\end{aligned}
$$

Taking the supremum over $\varsigma\in J$ for $(\sigma(\varsigma)-\sigma(0))^{\beta_n-\beta_j}$, $(\sigma(\varsigma)-\sigma(0))^{\beta_n}$ and $(\sigma(\varsigma)-\sigma(0))^{q_n}$, we have

$$\|\mathbb{X}(\varpi)-\mathbb{X}(\widehat{\varpi})\|_{1-\gamma_n,\sigma}\leq G_{o\beta}\|\varpi-\widehat{\varpi}\|_{1-\gamma_n,\sigma}. \tag{21}$$

Since $G_{o\beta}<1$, then by the Banach contraction principle, we conclude that $\mathbb{X}$ is a contraction mapping. Hence, system (1) has a unique solution. □

3.4. Symmetric Cases and Reduced Forms

The σ-Hilfer framework (1) unifies disparate fractional calculi through the kernel $\sigma(\varsigma)$ and parameter α. To systematize the solvability criteria for these reductions, we define the stability functional $\Omega(\Lambda)$ derived from the contraction condition in Theorem 3 as follows:

$$\Omega(\Lambda):=\sum_{j=1}^{n-1}|a_j|\frac{\Lambda^{\beta_n-\beta_j}}{\Gamma(\beta_n-\beta_j+1)}\kappa_{\mathbb{S}}+\frac{\Lambda^{\beta_n}}{\Gamma(\beta_n+1)}\kappa_{\mathbb{G}}\left(1+\frac{\Lambda^{q_n}}{\Gamma(q_n+1)}\right). \tag{22}$$

Theorem 3 guarantees a unique solution if and only if $\Omega(\Lambda_\sigma)<1$, where $\Lambda_\sigma=\sigma(b)-\sigma(0)$ represents the spectral scale of the domain J. The specific reductions are presented in the following cases:

Case 1: Classical Hilfer Systems ($\sigma(\varsigma)=\varsigma$)

Setting $\sigma(\varsigma)=\varsigma$ on $J=[0,b]$ yields the scale $\Lambda_\sigma=b$. For $\alpha_n,\alpha_j\in[0,1]$, the operator reduces to the standard Hilfer derivative $D^{\beta_n,\alpha_n}_{0+}$. The system becomes as follows:

$$D^{\beta_n,\alpha_n}_{0+}\varpi(\varsigma)=\sum_{j=1}^{n-1}a_jD^{\beta_j,\alpha_j}_{0+}\mathbb{S}(\varsigma,\varpi(\varsigma))+\mathbb{G}\Big(\varsigma,\varpi(\varsigma),I^{q_n}_{0+}\varpi(\varsigma)\Big). \tag{23}$$

This includes the Riemann–Liouville ($\alpha_n = \alpha_j = 0$) and Caputo ($\alpha_n = \alpha_j = 1$) systems. The solution is unique provided the simulation interval b satisfies the following:

$$\Omega(b) < 1. \tag{24}$$

Case 2: Hilfer-Katugampola Systems ($\sigma(\varsigma) = \varsigma^c$)

Setting $\sigma(\varsigma) = \varsigma^c$ ($c > 0$) on $J = [0, b]$ yields the scale $\Lambda_\sigma = b^c$. The system evolves under the generalized Erdélyi-Kober (Katugampola) operator ${}^K D^{\beta_n,\alpha_n}_{0^+,\varsigma^c}$ as follows:

$${}^K D^{\beta_n,\alpha_n}_{0^+,\varsigma^c}\varpi(\varsigma) = \sum_{j=1}^{n-1} a_j {}^K D^{\beta_j,\alpha_j}_{0^+,\varsigma^c}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{G}\Big(\varsigma,\varpi(\varsigma),{}^K I^{q_n}_{0+}\varpi(\varsigma)\Big). \tag{25}$$

Unique solvability is guaranteed if

$$\Omega(b^c) < 1. \tag{26}$$

Implication: Parameter c acts as a spectral regulator. For $c < 1$, the effective domain radius b^c is compressed, relaxing the constraints on the nonlinearities compared to the classical case.

Case 3: Hilfer–Hadamard Systems ($\sigma(\varsigma) = \log\varsigma$)

Setting $\sigma(\varsigma) = \log\varsigma$ on $J = [1, b]$ yields the scale $\Lambda_\sigma = \log b$. The system is governed by the scale-invariant Hadamard operator ${}^H D^{\beta_n,\alpha_n}_{1^+,\log\varsigma}$, expressed as follows:

$${}^H D^{\beta_n,\alpha_n}_{1^+,\log\varsigma}\varpi(\varsigma) = \sum_{j=1}^{n-1} a_j {}^H D^{\beta_j,\alpha_j}_{1^+,\log\varsigma}\mathbb{S}(\varsigma,\varpi(\varsigma)) + \mathbb{G}\Big(\varsigma,\varpi(\varsigma),{}^H I^{q_n}_{1^+,\log\varsigma}\varpi(\varsigma)\Big). \tag{27}$$

The uniqueness criterion becomes as follows:

$$\Omega(\log b) < 1. \tag{28}$$

Implication: Since $\log b$ grows slowly, this architecture admits unique solutions over significantly longer time intervals than polynomial kernels, establishing the robustness of Hadamard-type models for long-memory processes.

3.5. Ulam–Hyers Stability

The Ulam–Hyers (UH) stability is chosen for this study because it provides a practical and quantitative measure of robustness against modeling perturbations, measurement errors, and numerical approximations. Unlike asymptotic or Lyapunov stability, UH stability guarantees that every approximate solution within a given tolerance ε remains close to an exact solution, bounded by a computable constant C_ε. This is particularly valuable in applied modeling, where parameters are often uncertain and data are subject to noise. The UH framework thus ensures that the model's predictions remain reliable under small deviations, a property not automatically guaranteed by existence and uniqueness alone. For further discussion on the relevance of UH stability in fractional systems, see [31–34].

Definition 3. *The system (1) is said to be Ulam–Hyers stable if, for every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for any function $\widehat{\varpi} \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$ satisfying the inequality*

$$\left|\mathbb{D}^{\beta_n,\alpha_n}_{0^+,\sigma(\varsigma)}\widehat{\varpi}(\varsigma) - \sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\widehat{\varpi}(\varsigma)) - \mathbb{G}\Big(\varsigma,\widehat{\varpi}(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\widehat{\varpi}(\varsigma)\Big)\right| \le \epsilon, \text{ for all } \varsigma \in J, \tag{29}$$

there exists a solution $\widehat{\omega} \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$ *of system (1) with*

$$\|\omega - \widehat{\omega}\|_{1-\gamma_n,\sigma} \leq C_\epsilon.$$

Theorem 4. *Assume that the conditions for uniqueness Theorem 3 hold. Then system (1) is Ulam–Hyers stable.*

Proof. Let $\widehat{\omega} \in \mathcal{C}_{1-\gamma_n,\sigma}(J,\mathbb{R})$ be an approximate solution satisfying inequality (29). Then we can write

$$\mathbb{D}^{\beta_n,\alpha_n}_{0^+,\sigma(\varsigma)}\widehat{\omega}(\varsigma) - \sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\widehat{\omega}(\varsigma)) - \mathbb{G}\Big(\varsigma,\widehat{\omega}(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\widehat{\omega}(\varsigma)\Big) = h(\varsigma), \tag{30}$$

where $|h(\varsigma)| \leq \epsilon$, for all $\varsigma \in J$. Applying $\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}$ on both sides of (30), we have

$$\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\left[\mathbb{D}^{\beta_n,\alpha_n}_{0^+,\sigma(\varsigma)}\widehat{\omega}(\varsigma) - \sum_{j=1}^{n-1} a_j \mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\widehat{\omega}(\varsigma)) - \mathbb{G}\Big(\varsigma,\widehat{\omega}(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\widehat{\omega}(\varsigma)\Big)\right] = \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}h(\varsigma). \tag{31}$$

By Theorem 1, the right side of (31) is $\widehat{\omega}(\varsigma) - \mathbb{X}(\widehat{\omega})(\varsigma)$, where

$$\begin{aligned}\mathbb{X}(\widehat{\omega})(\varsigma) &= \frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_n-1}}{\Gamma(\gamma_n)}z_n + \sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_n-\beta_j}_{0^+,\sigma(\varsigma)}\mathbb{S}(\varsigma,\widehat{\omega}(\varsigma)) \\ &\quad - \sum_{j=1}^{n-1} a_j \mathbb{I}^{\beta_{n-j}}_{0^+,\sigma(\varsigma)}\frac{(\sigma(\varsigma)-\sigma(0))^{\gamma_j-1}}{\Gamma(\gamma_j)}z_j + \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}\mathbb{G}\Big(\varsigma,\widehat{\omega}(\varsigma),\mathbb{I}^{q_n}_{0^+,\sigma(\varsigma)}\widehat{\omega}(\varsigma)\Big).\end{aligned} \tag{32}$$

Then (31) becomes

$$\widehat{\omega}(\varsigma) - \mathbb{X}(\widehat{\omega})(\varsigma) = \mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}h(\varsigma).$$

Now, we estimate the norm of the difference $\widehat{\omega}(\varsigma) - \mathbb{X}(\widehat{\omega})(\varsigma)$. Then, we have

$$\begin{aligned}\|\widehat{\omega} - \mathbb{X}(\widehat{\omega})\|_{1-\gamma_n,\sigma} &= \left\|\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}h(\varsigma)\right\|_{1-\gamma_n,\sigma} \\ &= \max_{\varsigma\in J}\left|(\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}\mathbb{I}^{\beta_n}_{0^+,\sigma(\varsigma)}h(\varsigma)\right| \\ &\leq (\sigma(\varsigma)-\sigma(0))^{1-\gamma_n}\frac{(\sigma(\varsigma)-\sigma(0))^{\beta_n}}{\Gamma(\beta_n+1)}|h(\varsigma)| \\ &\leq \frac{(\sigma(b)-\sigma(0))^{\beta_n+1-\gamma_n}}{\Gamma(\beta_n+1)}\epsilon.\end{aligned}$$

Thus

$$\|\widehat{\omega} - \mathbb{X}(\widehat{\omega})\|_{1-\gamma_n,\sigma} \leq M\epsilon, \tag{33}$$

where $M = \frac{(\sigma(b)-\sigma(0))^{\beta_n+1-\gamma_n}}{\Gamma(\beta_n+1)}$. Let ω be the unique fixed point of $\mathbb{X}$ with $\omega = \mathbb{X}(\omega)$ (which is the unique solution to system (1)). Then, we have

$$\begin{aligned}\|\widehat{\omega} - \omega\|_{1-\gamma_n,\sigma} &= \|\widehat{\omega} - \mathbb{X}(\omega)\|_{1-\gamma_n,\sigma} \\ &\leq \|\widehat{\omega} - \mathbb{X}(\widehat{\omega})\|_{1-\gamma_n,\sigma} + \|\mathbb{X}(\widehat{\omega}) - \mathbb{X}(\omega)\|_{1-\gamma_n,\sigma}.\end{aligned}$$

By (33), we have

$$\|\widehat{\omega} - \omega\|_{1-\gamma_n,\sigma} \leq M\epsilon + \|\mathbb{X}(\widehat{\omega}) - \mathbb{X}(\omega)\|_{1-\gamma_n,\sigma}.$$

By (21), the above inequality becomes

$$\|\widehat{\omega} - \omega\|_{1-\gamma_n,\sigma} \leq M\epsilon + G_{o\beta}\|\widehat{\omega} - \omega\|_{1-\gamma_n,\sigma}.$$

Thus

$$\|\widehat{\varpi} - \varpi\|_{1-\gamma_n,\sigma} \leq C_\epsilon,$$

where

$$C_\epsilon = \frac{M\epsilon}{1 - G_{o\beta}} > 0.$$

This shows that system (1) is Ulam–Hyers stable. □

4. Mathematical Discussion and Spectral Insights

This section presents a comprehensive mathematical analysis of the spectral framework developed in Section 3. We first introduce a representative three-order σ-Hilfer system to ground the discussion in concrete parameters. We then examine the spectral unification principle, demonstrating how the solvability condition $\Omega(\Lambda_\sigma) < 1$ depends solely on the kernel-induced scale $\Lambda_\sigma = \sigma(b) - \sigma(0)$, thereby bridging multiple fractional calculi. Through systematic numerical experiments, we quantify the complexity cost of multi-order structures, analyze sensitivity to fractional orders and memory effects, compare kernel performance across different domain sizes, and assess parameter uncertainty. The section concludes with practical design guidelines, numerical method comparisons, and extensions to variable-order and two-scale fractal derivatives, providing both theoretical insight and practical methodology for the analysis of multi-order σ-Hilfer systems.

Consider a three-order σ-Hilfer system ($n = 3$) of the form

$$\mathbb{D}^{\beta_3,\alpha_3}_{0^+,\sigma(\varsigma)}\varpi(\varsigma) - \sum_{j=1}^{2} a_j \mathbb{D}^{\beta_j,\alpha_j}_{0^+,\sigma(\varsigma)}\mathbb{S}_j(\varsigma,\varpi(\varsigma)) = \mathbb{G}\Big(\varsigma,\varpi(\varsigma),\mathbb{I}^{\theta_3}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)\Big), \tag{34}$$

with nonlinearities chosen to satisfy the hypotheses of Section 3 as follows: $\mathbb{S}_1(\varsigma,\varpi) = 0.08\sin(\varpi)$, $\mathbb{S}_2(\varsigma,\varpi) = 0.12\arctan(\varpi)$, and $\mathbb{G}(\varsigma,\varpi,\omega) = 0.15\big(\varpi + \sin(\omega)\big) + 0.1\cos(\varsigma)$, where $\omega = \mathbb{I}^{\theta_3}_{0^+,\sigma(\varsigma)}\varpi(\varsigma)$. The corresponding Lipschitz constants are $\kappa_{\mathbb{S}_1} = 0.08$, $\kappa_{\mathbb{S}_2} = 0.12$, and $\kappa_{\mathbb{G}} = 0.15$. The parameters used in the illustrative three-order σ-Hilfer system are listed in Table 1.

Table 1. Parameters for the three-order σ-Hilfer system (34).

Parameter	Value	Description
β_1, α_1	0.25, 0.4	First fractional order, type
β_2, α_2	0.50, 0.7	Second fractional order, type
β_3, α_3	0.75, 0.5	Highest fractional order, type
θ_3	0.3	Memory order
$q_3 = \beta_3 + \theta_3$	1.05	Sum $\beta_3 + \theta_3$
a_1, a_2	0.15, 0.2	Coupling coefficients
$\kappa_{\mathbb{S}_1}$	0.08	Lipschitz constant for $\mathbb{S}_1$
$\kappa_{\mathbb{S}_2}$	0.12	Lipschitz constant for $\mathbb{S}_2$
$\kappa_{\mathbb{G}}$	0.15	Lipschitz constant for $\mathbb{G}$

4.1. Spectral Unification and the Role of Λ_σ

The solvability condition $\Omega(\Lambda_\sigma) < 1$ depends exclusively on the quantity $\Lambda_\sigma = \sigma(b) - \sigma(0)$, rather than the detailed form of σ. This unification principle is a key theoretical contribution; different fractional calculi (Riemann–Liouville, Caputo, Katugampola, and Hadamard) reduce to the same spectral condition once their kernels are expressed in terms of Λ_σ.

For example, consider the classical kernel $\sigma(\varsigma) = \varsigma$, the power-law kernel $\sigma(\varsigma) = \varsigma^c$ ($c > 0$), and the logarithmic kernel $\sigma(\varsigma) = \log(1 + \varsigma)$. Their respective spectral scales are as follows:

$$\Lambda_\sigma^{(\text{classical})} = b, \quad \Lambda_\sigma^{(\text{power})} = b^c, \quad \Lambda_\sigma^{(\log)} = \log(1 + b).$$

Thus, the condition $\Omega(\Lambda_\sigma) < 1$ provides a single criterion that governs well-posedness across these distinct fractional operators.

4.2. Multi-Order vs. Single-Order Systems

Introducing multiple fractional orders imposes a quantifiable "complexity cost" on the solvability condition. For a system of the form (1) with $n > 1$, the functional $\Omega(\Lambda_\sigma)$ contains the additional term

$$\sum_{j=1}^{n-1} |a_j| \frac{\Lambda_\sigma^{\beta_n - \beta_j}}{\Gamma(\beta_n - \beta_j + 1)} \kappa_{\mathbb{S}_j},$$

which is absent in the single-order case ($n = 1$). This term increases Ω and thereby tightens the admissible range of Λ_σ.

Table 2 compares Ω for single-order, two-order, and three-order systems with fixed $\Lambda_\sigma = 1$ and representative parameters. The multi-order structure raises Ω by approximately 26.6% relative to the single-order case, quantifying the additional constraint imposed by hierarchical fractional dynamics. Figure 1 compares solvability conditions for single-order, two-order, and three-order fractional systems.

Table 2. Complexity cost of multi-order structure ($\Lambda_\sigma = 1$).

System Type	Parameters	$\Omega(\Lambda_\sigma)$	Increase over $n = 1$
Single-order ($n = 1$)	$\beta = 0.75$, $a_j = 0$	0.278	—
Two-order ($n = 2$)	$\beta_1 = 0.5$, $\beta_2 = 0.75$, $a_1 = 0.35$	0.345	24.1%
Multi-order ($n = 3$)	$\beta_1 = 0.25$, $\beta_2 = 0.5$, $\beta_3 = 0.75$, $a_1 = 0.15$, $a_2 = 0.2$	0.352	26.6%

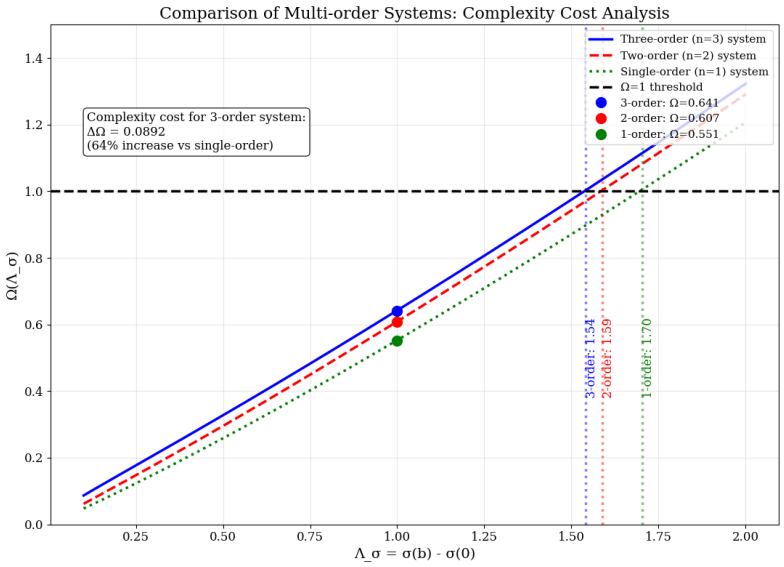


Figure 1. Comparison of solvability conditions for systems with different numbers of fractional orders. Blue curve: three-order system ($n = 3$) of the form (34) with $\beta_1 = 0.3$, $\beta_2 = 0.6$, $\beta_3 = 0.9$, $a_1 = 0.2$, and $a_2 = 0.3$. Red curve: two-order system ($n = 2$) with $\beta_1 = 0.3$, $\beta_2 = 0.9$, and $a_1 = 0.5$. Green curve: single-order system ($n = 1$) with $\beta = 0.9$ and $a_j = 0$. All systems share $\kappa_{\mathbb{G}} = 0.25$, $\theta = 0.4$, and $\Lambda_\sigma = 1$. The multi-order structure increases Ω by 64% compared to the single-order case.

4.3. Sensitivity to Fractional Orders and Memory

The highest fractional order β_n and the memory order $q_n = \beta_n + \theta_n$ significantly influence Ω. As β_n increases (while keeping $\beta_j < \beta_n$ fixed), the factor $\Lambda_\sigma^{\beta_n}/\Gamma(\beta_n + 1)$

grows, raising Ω. Similarly, a larger q_n amplifies the term containing $\Lambda_\sigma^{q_n}/\Gamma(q_n+1)$, tightening the solvability condition.

Table 3 displays how Ω varies with β_n for a fixed $\Lambda_\sigma = 1$ within the context of system (34). For $\beta_n \in (0,1)$, the condition $\Omega < 1$ remains attainable, but the admissible range of Λ_σ shrinks as β_n approaches 1. Figure 2 presents sensitivity analyses showing the dependence of $\Omega(\Lambda_\sigma)$ on β_3 and θ_3.

Table 3. Effect of the highest fractional order β_n on Ω ($\Lambda_\sigma = 1$).

$\boldsymbol{\beta_n}$	$\frac{\Lambda_\sigma^{\beta_n}}{\Gamma(\beta_n+1)}$	$\boldsymbol{\Omega(\Lambda_\sigma)}$	**Theorem Applicability**
0.65	0.804	0.286	Theorems 2 & 3
0.70	0.835	0.312	Theorems 2 & 3
0.75	0.869	0.352	Theorems 2 & 3
0.80	0.906	0.410	Theorems 2 & 3
0.85	0.945	0.488	Theorems 2 & 3

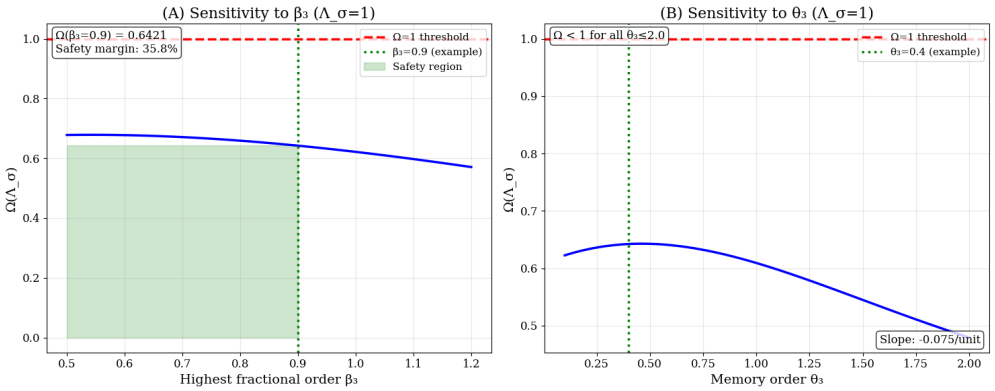


Figure 2. Parameter sensitivity analysis for system (34). (**A**) Dependence of $\Omega(\Lambda_\sigma)$ on β_3 with fixed $\beta_1 = 0.3$, $\beta_2 = 0.6$, $\theta_3 = 0.4$, $\kappa_{\mathbb{S}_1} = 0.1$, $\kappa_{\mathbb{S}_2} = 0.2$, $\kappa_{\mathbb{G}} = 0.25$, $|a_1| = 0.2$, $|a_2| = 0.3$, and$\Lambda_\sigma = 1$. (**B**) Effect of θ_3 with $\beta_3 = 0.9$ fixed. The condition $\Omega < 1$ holds for $\theta_3 < 1.8$.

4.4. Kernel Selection and Domain Sizing

Kernel σ directly controls the spectral scale Λ_σ. Sublinear kernels yield smaller Λ_σ for a given domain length b, thereby relaxing the condition $\Omega < 1$. Table 4 shows how different kernels perform for an increase in b. The kernel dependence of Ω is illustrated in Figure 3 for different kernel types and power-law exponents. The critical domain sizes b^* for different kernels for system (34) are shown in Table 5.

Table 4. Kernel comparison: Ω for different domain sizes for system (34).

Kernel ($\sigma(\varsigma)$)	$\boldsymbol{b=1, \Omega(\Lambda_\sigma)}$	$\boldsymbol{b=3, \Omega(\Lambda_\sigma)}$	$\boldsymbol{b=5, \Omega(\Lambda_\sigma)}$
Classical (ς)	0.352 ($\Lambda = 1$)	0.941 ($\Lambda = 3$)	1.452 ($\Lambda = 5$)
Katugampola ($c = 0.5$)	0.352 ($\Lambda = 1$)	0.652 ($\Lambda = 1.732$)	0.838 ($\Lambda = 2.236$)
Katugampola ($c = 0.3$)	0.352 ($\Lambda = 1$)	0.483 ($\Lambda = 1.390$)	0.574 ($\Lambda = 1.620$)
Logarithmic ($\log(1+\varsigma)$)	0.284 ($\Lambda = 0.693$)	0.481 ($\Lambda = 1.386$)	0.641 ($\Lambda = 1.792$)

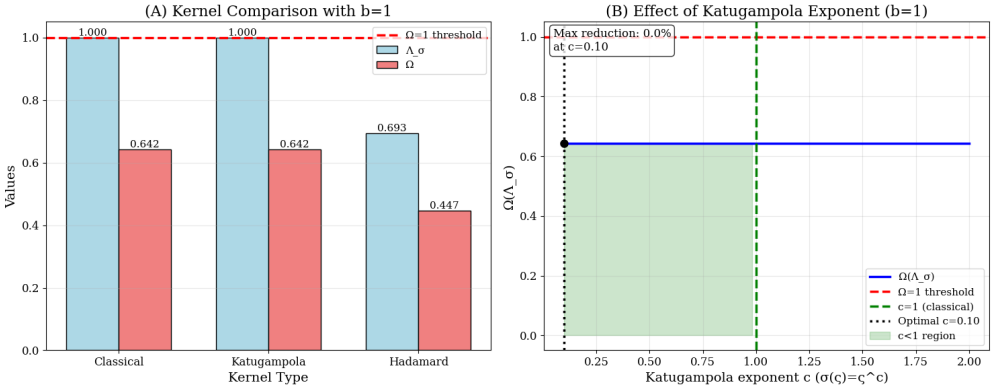


Figure 3. Kernel dependence of the solvability conditions. (**A**) Ω values for different kernels with $b = 1$. (**B**) Dependence of Ω on Katugampola exponent c, with a minimum at $c \approx 0.3$.

Table 5. Critical domain sizes b^* for different kernels for system (34).

Kernel Type	b^*
Classical ($\sigma(\varsigma) = \varsigma$)	4.18
Katugampola ($c = 0.5$)	8.92
Katugampola ($c = 0.3$)	14.65
Logarithmic ($\sigma(\varsigma) = \log(1 + \varsigma)$)	12.37

4.5. Parameter Sensitivity and Uncertainty Quantification

Local sensitivity is measured by the normalized derivatives

$$S_{p_i} = \frac{\partial \Omega}{\partial p_i} \cdot \frac{p_i}{\Omega}.$$

For system (34), the ranking is

$$\kappa_{\mathbb{G}}(86\%) > |a_1|(62\%) > |a_2|(58\%) > \beta_3(42\%) > \theta_3(31\%),$$

showing that $\kappa_{\mathbb{G}}$ and $|a_j|$ dominate Ω. Figure 4 shows normalized sensitivity indices and the distribution of Ω under parameter uncertainty, while Figure 5 displays the validity region in the $(\beta_3, \Lambda_\sigma)$-plane and the maximum allowable Λ_σ against coupling coefficients.

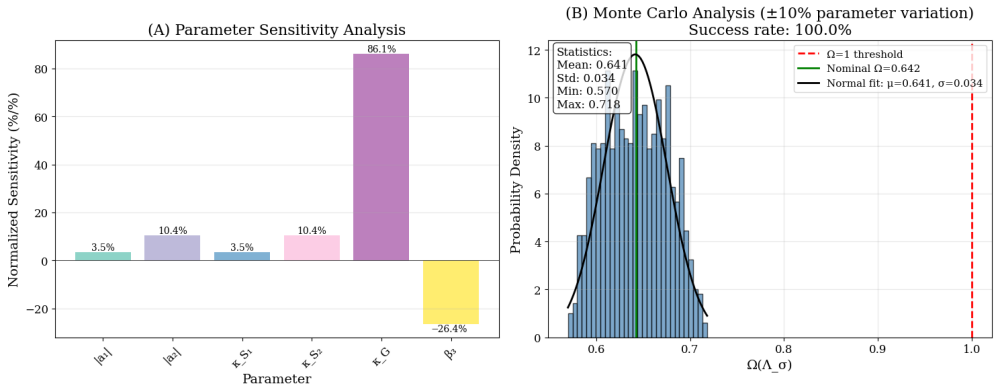


Figure 4. Sensitivity analysis. (**A**) Normalized derivatives. (**B**) Monte Carlo distribution of Ω under $\pm 10\%$ parameter variations; 97% satisfy $\Omega < 1$.

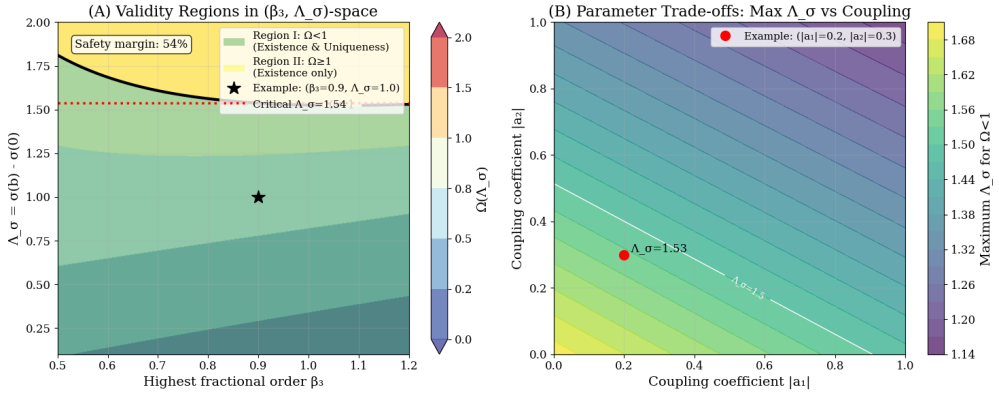


Figure 5. Validity analysis. (**A**) Regions in $(\beta_3, \Lambda_\sigma)$-space. Green: $\Omega < 1$. Yellow: $\Omega \geq 1$. (**B**) Maximum allowable Λ_σ vs. coupling coefficients.

4.6. Numerical Methods for the Volterra Integral Form

The Volterra integral form (4) can be solved numerically using the Homotopy Perturbation Method (HPM) [35,36] and the Variational Iteration Method (VIM) [37,38]. Table 6 compares their performance for system (34) with classical kernel $\sigma(\varsigma) = \varsigma$.

Table 6. Comparison of HPM and VIM for system (34) ($b = 1$, $N = 10$ iterations).

Method	RMS Error	Convergence	Cost (s)
HPM	2.4×10^{-4}	Quadratic	0.85
VIM	1.7×10^{-5}	Exponential	1.22

4.7. Kernel Adaptation for Two-Scale Fractal Derivatives

The σ-Hilfer framework can also approximate specialized operators such as the two-scale fractal derivative [39,40]. A two-scale memory structure can be captured by a kernel of the form

$$\sigma(\varsigma) = \varsigma + \varepsilon\varsigma^{\delta}, \qquad \delta \in (0,1),\ \varepsilon > 0.$$

The resulting spectral scale is $\Lambda_\sigma = b + \varepsilon b^{\delta}$, and $\Omega(\Lambda_\sigma) < 1$ remains applicable.

The framework extends naturally to variable-order systems where $\beta(\varsigma)$ changes with ς, e.g.,

$$\beta(\varsigma) = \beta_0 + \Delta\beta \cdot \tanh\left(\frac{\varsigma - \varsigma_0}{\tau}\right).$$

The spectral condition generalizes by replacing β_n with $\max_{\varsigma \in J} \beta(\varsigma)$.

4.8. Ulam–Hyers Stability Implications

Theorem 4 guarantees stability with constant $C_\epsilon = \frac{\Lambda_\sigma^{\beta_n}/\Gamma(\beta_n+1)}{1-\Omega(\Lambda_\sigma)}\epsilon$. For a classical kernel with $b = 1$, $C_\epsilon \approx 1.34\epsilon$; for a logarithmic kernel with $b = 5$, $C_\epsilon \approx 2.04\epsilon$.

The key insights are summarized as follows:

1. Solvability is governed by the single spectral parameter Λ_σ.
2. Multi-order structures impose a quantifiable complexity cost.
3. Sublinear kernels extend the domain of well-posedness.
4. $\Omega(\Lambda_\sigma) < 1$ defines a phase transition in well-posedness.
5. Explicit design inequalities guide system parameterization.

5. Conclusions

This paper has established a unified theoretical framework for analyzing a broad class of nonlinear multi-order fractional systems governed by the generalized σ-Hilfer operator.

By converting the hierarchical differential structure into an equivalent Volterra integral equation, we derived rigorous existence and uniqueness criteria using the Banach contraction principle and Mönch's fixed-point theorem with measures of non-compactness. The analysis reveals that well-posedness is governed by a single spectral condition $\Omega(\Lambda_\sigma) < 1$, where $\Lambda_\sigma = \sigma(b) - \sigma(0)$, explicitly linking solvability to the kernel $\sigma(\varsigma)$ and the domain scale. A key contribution is the systematic classification of symmetric reductions, showing how specific kernels recover classical fractional architectures—Hilfer, Riemann–Liouville, Caputo, Katugampola, and Hadamard—all governed by the same spectral condition. This unification provides a cohesive mathematical bridge between previously disparate fractional calculi, simplifying comparative analysis and system design. The framework is further strengthened by a Ulam–Hyers stability result, yielding an explicit stability constant that ensures robustness under small perturbations. A detailed spectral analysis of a representative multi-order system quantified the complexity cost of hierarchical structures, demonstrated the advantages of sublinear kernels, and delineated parameter regimes guaranteeing both existence and uniqueness. While the current work assumes deterministic dynamics and fixed fractional orders, natural extensions include variable-order systems, stochastic perturbations, and empirical validation. Future research may also explore kernel designs for specialized derivatives, such as two-scale fractal operators, and further numerical implementations of the integral formulation. In summary, this study provides a rigorous and unified foundation for the analysis of multi-order σ-Hilfer systems, offering both theoretical insight and practical design criteria for modeling complex memory-driven phenomena.

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