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Dynamics and Stability Results for Hilfer Fractional Type Thermistor Problem

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Abstract: In this paper, we study the dynamics and stability of thermistor problem for Hilfer fractional type. Classical fixed point theorems are utilized in deriving the results.

Keywords: nonlocal thermistor problem; Hilfer fractional derivative; existence; Ulam stability; fixed point

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1. Introduction

Fractional differential equations (FDEs) occur in many engineering systems and scientific disciplines such as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, etc. FDEs also provide as an efficient tool for explanations of hereditary properties of different resources and processes. As a result, the meaning of the FDEs has been of great importance and attention, and one can refer to Kilbas [1], Podlubny [6] and the papers [2–5,7–9]. Recently, the Hilfer fractional derivative [10] for FDEs has become a very active area of research. R. Hilfer initiated the Hilfer fractional derivative. This is used to interpolate both the Riemann–Liouville and the Caputo fractional derivative for the theory and applications of the Hilfer fractional derivative (see, e.g., [5,10–16] and references cited therein). Analogously, we prefer the Hilfer derivative operator that interpolates both the Riemann–Liouville and the Caputo derivative.

English scientist Michael Faraday first discovered the concept of thermistors in 1833 while reporting on the semiconductor behavior of silver sulfide. From his research work, he noticed that the silver sulfides resistance decreased as the temperature increased. This later leads to the commercial production of thermistors in the 1930s when Samuel Ruben invented the first commercial thermistor. Ever since, technology has improved; this made it possible to improve manufacturing processes along with the availability of advanced quality material.

A thermistor is a thermally sensitive resistor that displays a precise and predictable change in resistance proportional to small changes in body temperature. How much its resistance will change is dependent upon its unique composition. Thermistors are part of a larger group of passive components. Unlike their active component counterparts, passive devices are incapable of providing power gain, or amplification to a circuit. Thermistors can be found everywhere in airplanes, air conditioners, in cars, computers, medical equipment, hair dryers, portable heaters, incubators, electrical outlets, refrigerators, digital thermostats, ovens, stove tops and in all kinds of appliances. Ice sensors and aircraft wings, if ice builds up on the wings, the thermistor senses this temperature drop and a heater will be activated to remove the ice. Flight tests need to be completed on a particular date, hence there may not be enough time to create a flight test technique on that date. However, it is possible to take a number of recommendations on the needs of any future flight plan to examine the nature of thermistor thermometer at high subsonic and supersonic speeds. In general, the unusual behaviour of the thermistor thermometer is caused by the possibility of vortices and an aerodynamic disturbance generating non-uniform flow, happening in the chamber with sensing element. The thermistors are small, which makes them very delicate to such effects [17,18].

A thermistor is a temperature dependent resistor and comes in two varieties, negative temperature coefficient (NTC) and positive temperature coefficient (PTC), although NTCs are most commonly used. With NTC, the resistance variation is inverse to the temperature change i.e.,: as temperature goes up, resistance goes down. NTC Thermistors are nonlinear, and their resistance decreases as temperature increases. A phenomenon called self-heating may affect the resistance of an NTC thermistor. When current flows through the NTC thermistor, it absorbs the heat causing its own temperature to rise. In [19], Khan et al. investigated the coupled p-Laplacian fractional differential equations with nonlinear boundary conditions. Wenjing Song and Wenjie Gao studied the existence of solutions for a nonlocal initial value problem to a p-Laplacian thermistor problems on time scales in [20]. Later, Moulay Rchid Sidi Ammi and Delfim F.M. Torres developed and applied a numerical method for the time-fractional nonlocal thermistor problem in [21]. They investigated the existence and uniqueness of a positive solution to generalized nonlocal thermistor problems with fractional-order derivatives in [22]. Recently, Moulay Rchid Sidi Ammi and Delfim F. M. Torres [23] discussed the existence and uniqueness results for a fractional Riemann–Liouville nonlocal thermistor problem on arbitrary time scales. Interested readers can refer to recent papers [22–26] treating a nonlocal thermistor problem.

Motivated by the aforementioned papers, we study the existence, uniqueness and Ulam–Hyers stability types of solutions for Hilfer type thermistor problem of the form

$$\begin{cases} D_{0^+}^{\alpha,\beta}u(t) = \frac{\lambda f(u(t))}{\left(\int_0^T f(u(x))dx\right)^2}, \quad t \in J := [0,T], \\ I_{0^+}^{1-\gamma}u(0) = u_0, \quad \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{1}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β, $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and let $J = [0,T]$, X be a Banach space, $f : J \times X \to X$ is a given continuous function. The operator $I_{0^+}^{1-\gamma}$ denotes the left-sided Riemann–Liouville fractional integral of order $1-\gamma$. Choosing λ such that $0 < \lambda < \left(\frac{LT^{\alpha+1-\gamma}}{(C_1T)^2\Gamma(\alpha+1)} + \frac{2C_2^2LT^{\alpha+3-\gamma}}{(C_1T)^2\Gamma(\alpha+1)}\right)^{-1}$ is discussed in Section 4.

It is seen that (1) is equivalent to the following nonlinear integral equation

$$u(t) = \frac{u_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2}ds. \tag{2}$$

The stability of the functional equations were first introduced in a discourse conveyed in 1940 at the University of Wisconsin. The issue made by Ulam is as per the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? [4,27–29]. The first reply to the topic of Ulam was given by Hyers in 1941 on account of Banach spaces. Ever since, this type of stability was known as the Ulam–Hyers stability. Rassias [29] gave a generalization of the Hyers theorem for linear mappings. Many mathematicians later extended the issue of Ulam in different ways. Recently, Ulam's problem was generalized for the stability of differential equations. A comprehensive interest was given to the study of the Ulam and Ulam–Hyers–Rassias stability of all kinds of functional equations [4,8,9,30]. An exhaustive interest was given to the investigation of the Ulam and Ulam–Hyers–Rassias stability of all kinds of functional Equation (1).

The paper is organized as follows. In Section 2, we introduce some definitions, notations, and lemmas that are used throughout the paper. In Section 3, we will prove existence and uniqueness results concerning problem (1). Section 4 is devoted to the Ulam–Hyers stabilities of problem (1).

2. Basic Concepts and Results

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper. For more details on Hilfer fractional derivative, interested readers can refer to [5,10,12,13,15,31].

Definition 1. *Let $C[J,X]$ denote the Banach space of all continuous functions from $[0,T]$ into X with the norm*

$$\|u\|_C := \sup\{|u(t)| : t \in J\}.$$

We denote $L^1\{R_+\}$, the space of Lebesgue integrable functions on J.

By $C_\gamma[J,X]$ and $C^1_\gamma[J,X]$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma[J,X] := \{f(t) : J \to X | t^\gamma f(t) \in C[J,X]\},$$

with the norm

$$\|f\|_{C_\gamma} = \|t^\gamma f(t)\|_C,$$

and

$$\|f\|_{C^n_\gamma} = \sum_{k=0}^{n-1} \left\|f^k\right\|_C + \left\|f^{(n)}\right\|_{C_\gamma}, \quad n \in N.$$

Moreover, $C^0_\gamma[J,X] := C_\gamma[J,X]$.

Now, we give some results and properties of fractional calculus.

Definition 2 ([1,16])**.** *The left-sided mixed Riemann–Liouville integral of order $\alpha > 0$ of a function $h \in L^1\{R_+\}$ is defined by*

$$(I^\alpha_{0^+}h)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds, \quad \text{for a.e. } t \in J,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt; \quad \xi > 0.$$

Notice that for all $\alpha, \alpha_1, \alpha_2 > 0$ and each $h \in C[J,X]$, we have $I^\alpha_{0^+}h \in C[J,X]$, and

$$(I^{\alpha_1}_{0^+}I^{\alpha_2}_{0^+}h)(t) = (I^{\alpha_1+\alpha_2}_{0^+}h)(t); \text{for a.e. } t \in J.$$

Definition 3 ([1,16])**.** *The Riemann–Liouville fractional derivative of order $\alpha \in (0,1]$ of a function $h \in L^1\{R_+\}$ is defined by*

$$\begin{aligned}(D^\alpha_{0^+}h)(t) &= \left(\frac{d}{dt}I^{1-\alpha}_{0^+}h\right)(t)\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}h(s)ds; \quad \text{for a.e. } t \in J.\end{aligned}$$

Let $\alpha \in (0,1]$, $\gamma \in [0,1)$ and $h \in C_{1-\gamma}[J,X]$. Then, the following expression leads to the left inverse operator as follows:

$$(D_{0^+}^{\alpha} I_{0^+}^{\alpha} h)(t) = h(t); \quad \textit{for all } t \in (0,T].$$

Moreover, if $I_{0^+}^{1-\alpha} h \in C_{1-\gamma}^{1}[J,X]$, then the following composition

$$(I_{0^+}^{\alpha} D_{0^+}^{\alpha} h)(t) = h(t) - \frac{(I_{0^+}^{1-\alpha} h)(0^+)}{\Gamma(\alpha)} t^{\alpha-1}; \quad \textit{for all } t \in (0,T].$$

Definition 4 ([1,16])**.** *The Caputo fractional derivative of order $\alpha \in (0,1]$ of a function $h \in L^1\{R_+\}$ is defined by*

$$\begin{aligned}({}^{c}D_{0^+}^{\alpha} h)(t) &= (I_{0^+}^{1-\alpha} \frac{d}{dt} h)(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} h(s) ds; \quad \textit{for a.e. } t \in J.\end{aligned}$$

In [10], Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases (see also [5,32]).

Definition 5 (Hilfer derivative)**.** *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $h \in L^1\{R_+\}$, $I_{0^+}^{(1-\alpha)(1-\beta)} \in C_{\gamma}^{1}[J,X]$. The Hilfer fractional derivative of order α and type β of h is defined as*

$$(D_{0^+}^{\alpha,\beta} h)(t) = \left(I_{0^+}^{\beta(1-\alpha)} \frac{d}{dt} I_{0^+}^{(1-\alpha)(1-\beta)} h \right)(t); \quad \textit{for a.e. } t \in J. \tag{3}$$

Properties. Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta - \alpha\beta$, and $h \in L^1\{R_+\}$.

1. The operator $(D_{0^+}^{\alpha,\beta} h)(t)$ can be written as

$$(D_{0^+}^{\alpha,\beta} h)(t) = \left(I_{0^+}^{\beta(1-\alpha)} \frac{d}{dt} I_{0^+}^{1-\gamma} h \right)(t) = \left(I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\gamma} h \right)(t); \quad \text{for a.e. } t \in J.$$

   Moreover, the parameter γ satisfies

$$0 < \gamma \le 1, \quad \gamma \ge \alpha, \quad \gamma > \beta, \quad 1-\gamma < 1 - \beta(1-\alpha).$$

2. The generalization (3) for $\beta = 0$ coincides with the Riemann–Liouville derivative and for $\beta = 1$ with the Caputo derivative

$$D_{0^+}^{\alpha,0} = D_{0^+}^{\alpha}, \quad \text{and} \quad D_{0^+}^{\alpha,1} = {}^{c} D_{0^+}^{\alpha}.$$

3. If $D_{0^+}^{\beta(1-\alpha)} h$ exists and in $L^1\{R_+\}$, then

$$(D_{0^+}^{\alpha,\beta} I_{0^+}^{\alpha} h)(t) = \left(I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\beta(1-\alpha)} h \right)(t); \quad \text{for a.e. } t \in J.$$

   Furthermore, if $h \in C_{\gamma}[J,X]$ and $I_{0^+}^{1-\beta(1-\alpha)} h \in C_{\gamma}^{1}[J,X]$, then

$$(D_{0^+}^{\alpha,\beta} I_{0^+}^{\alpha} h)(t) = h(t); \quad \text{for a.e. } t \in J.$$

4. If $D^{\gamma}_{0^+}h$ exists and in $L^1\{R_+\}$, then

$$\left(I^{\alpha}_{0^+}D^{\alpha,\beta}_{0^+}h\right)(t) = \left(I^{\gamma}_{0^+}D^{\gamma}_{0^+}h\right)(t) = h(t) - \frac{I^{1-\gamma}_{0^+}h(0^+)}{\Gamma(\gamma)}t^{\gamma-1}; \quad \text{for a.e. } t \in J.$$

In order to solve our problem, the following spaces are presented

$$C^{\alpha,\beta}_{1-\gamma}[J,X] = \left\{f \in C_{1-\gamma}[J,X], D^{\alpha,\beta}_{0^+}f \in C_{1-\gamma}[J,X]\right\},$$

and

$$C^{\gamma}_{1-\gamma}[J,X] = \left\{f \in C_{1-\gamma}[J,X], D^{\gamma}_{0^+}f \in C_{1-\gamma}[J,X]\right\}.$$

It is obvious that

$$C^{\gamma}_{1-\gamma}[J,X] \subset C^{\alpha,\beta}_{1-\gamma}[J,X].$$

Corollary 1 ([31])**.** *Let $h \in C_{1-\gamma}[J,X]$. Then, the linear problem*

$$D^{\alpha,\beta}_{0^+}x(t) = h(t), \quad t \in J = [0,T],$$
$$I^{1-\gamma}_{0^+}x(0) = x_0, \quad \gamma = \alpha + \beta - \alpha\beta,$$

has a unique solution $x \in L^1\{R_+\}$ given by

$$x(t) = \frac{x_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds.$$

From the above corollary, we conclude the following lemma.

Lemma 1. *Let $f : J \times X \to X$ be a function such that $f \in C_{1-\gamma}[J,X]$. Then, problem (1) is equivalent to the problem of the solutions of the integral Equation (2).*

Theorem 1 (Schauder fixed point theorem [31,33])**.** *Let B be closed, convex and nonempty subset of a Banach space E. Let $P : B \to B$ be a continuous mapping such that $P(B)$ is a relatively compact subset of E. Then, P has at least one fixed point in B.*

Now, we study the Ulam stability, and we adopt the definitions in [3,30,34] of the Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability.

Consider the following Hilfer type termistor problem

$$D^{\alpha,\beta}_{0^+}u(t) = \frac{\lambda f(u(t))}{\left(\int_0^T f(u(x))dx\right)^2}, \quad t \in J := [0,T], \tag{4}$$

and the following fractional inequalities:

$$\left| D_{0^+}^{\alpha,\beta} z(t) - \frac{\lambda f(z(t))}{\left(\int_0^T f(z(x))dx\right)^2} \right| \leq \epsilon, \quad t \in J, \tag{5}$$

$$\left| D_{0^+}^{\alpha,\beta} z(t) - \frac{\lambda f(z(t))}{\left(\int_0^T f(z(x))dx\right)^2} \right| \leq \epsilon\varphi(t), \quad t \in J, \tag{6}$$

$$\left| D_{0^+}^{\alpha,\beta} z(t) - \frac{\lambda f(z(t))}{\left(\int_0^T f(z(x))dx\right)^2} \right| \leq \varphi(t), \quad t \in J. \tag{7}$$

Definition 6. *Equation (4) is Ulam–Hyers stable if there exists a real number $C_f > 0$ such that, for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma}^{\gamma}[J, X]$ of Inequality (5), there exists a solution $u \in C_{1-\gamma}^{\gamma}[J, X]$ of Equation (4) with*

$$|z(t) - u(t)| \leq C_f \epsilon, \quad t \in J.$$

Definition 7. *Equation (4) is generalized Ulam–Hyers stable if there exists $\psi_f \in C([0, \infty), [0, \infty))$, $\psi_f(0) = 0$ such that, for each solution $z \in C_{1-\gamma}^{\gamma}[J, X]$ of Inequality (5), there exists a solution $u \in C_{1-\gamma}^{\gamma}[J, X]$ of Equation (4) with*

$$|z(t) - u(t)| \leq \psi_f \epsilon, \quad t \in J.$$

Definition 8. *Equation (4) is Ulam–Hyers–Rassias stable with respect to $\varphi \in C_{1-\gamma}[J, X]$ if there exists a real number $C_f > 0$ such that, for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma}^{\gamma}[J, X]$ of Inequality (6), there exists a solution $u \in C_{1-\gamma}^{\gamma}[J, X]$ of Equation (4) with*

$$|z(t) - u(t)| \leq C_f \epsilon \varphi(t), \quad t \in J.$$

Definition 9. *Equation (4) is generalized Ulam–Hyers–Rassias stable with respect to $\varphi \in C_{1-\gamma}[J, X]$ if there exists a real number $C_{f,\varphi} > 0$ such that, for each solution $z \in C_{1-\gamma}^{\gamma}[J, X]$ of Inequality (7), there exists a solution $u \in C_{1-\gamma}^{\gamma}[J, X]$ of Equation (4) with*

$$|z(t) - u(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in J.$$

Remark 1. *A function $z \in C_{1-\gamma}^{\gamma}[J, X]$ is a solution of Inequality (5) if and only if there exist a function $g \in C_{1-\gamma}^{\gamma}[J, X]$ (which depends on solution z) such that*

1. $|g(t)| \leq \epsilon, \quad \forall\, t \in J.$
2. $D_{0^+}^{\alpha,\beta} z(t) = \frac{\lambda f(z(t))}{\left(\int_0^T f(z(x))dx\right)^2} + g(t), \quad t \in J.$

Remark 2. *It is clear that:*

1. *Definition 6 $\Rightarrow$ Definition 7.*
2. *Definition 8 $\Rightarrow$ Definition 9.*
3. *Definition 8 for $\varphi(t) = 1 \Rightarrow$ Definition 6.*

Lemma 2 ([2]). *Let $v : [0, T] \to [0, \infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, T]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^\alpha} ds.$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^\alpha} ds,$$

for every $t \in [0, T]$.

3. Existence Results

The following existence result for Hilfer type thermistor problem (1) is based on Schauder's fixed point theorem. Let us consider the following assumptions:

Assumption 1. *Function $f : J \times X \to X$ of problem (1) is Lipschitz continuous with Lipschitz constant L such that $c_1 \leq f(u) \leq c_2$, with c_1 and c_2 two positive constants.*

Assumption 2. *There exists an increasing function $\varphi \in C_{1-\gamma}[J, X]$ and there exists $\lambda_\varphi > 0$ such that, for any $t \in J$,*

$$I_{0^+}^{\alpha} \varphi(t) \leq \lambda_\varphi \varphi(t).$$

Our main result may be presented as the following theorem.

Theorem 2 (existence). *Under the above Assumption 1, problem (1) has at least one solution $u \in X$ for all $\lambda > 0$.*

Proof. Consider the operator $P : C_{1-\gamma}[J, X] \to C_{1-\gamma}[J, X]$ is defined by

$$(Pu)(t) = \frac{u_0}{\Gamma(\gamma)} t^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2} ds. \tag{8}$$

Clearly, the fixed points of P are solutions to (1). The proof will be given in several steps.

Step 1: The operator P is continuous. Let u_n be a sequence such that $u_n \to u$ in $C_{1-\gamma}[J, X]$. Then, for each $t \in J$,

$$\begin{aligned}
&\left| t^{1-\gamma} \left((Pu_n)(t) - (Pu)(t) \right) \right| \\
&\leq \tfrac{\lambda t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| \frac{f(u_n(s))}{\left(\int_0^T f(u_n(x))dx\right)^2} - \frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2} \right| ds \\
&\leq \tfrac{\lambda t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| \frac{1}{\left(\int_0^T f(u_n(x))dx\right)^2} \left(f(u_n(s)) - f(u(s)) \right) \right. \\
&\quad \left. + f(u(s)) \left(\frac{1}{\left(\int_0^T f(u_n(x))dx\right)^2} - \frac{1}{\left(\int_0^T f(u(x))dx\right)^2} \right) \right| \\
&\leq \tfrac{\lambda t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\left(\int_0^T f(u_n(x))dx\right)^2} \left| f(u_n(s)) - f(u(s)) \right| ds \\
&\quad + \tfrac{\lambda t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(u(s)) \right| \left| \frac{1}{\left(\int_0^T f(u_n(x))dx\right)^2} - \frac{1}{\left(\int_0^T f(u(x))dx\right)^2} \right| ds \leq I_1 + I_2,
\end{aligned} \tag{9}$$

where

$$I_1 = \frac{\lambda t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\left(\int_0^T f(u_n(x))dx\right)^2} |f(u_n(s)) - f(u(s))| \, ds,$$

$$I_2 = \frac{\lambda t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(u(s))| \left| \frac{1}{\left(\int_0^T f(u_n(x))dx\right)^2} - \frac{1}{\left(\int_0^T f(u(x))dx\right)^2} \right| ds.$$

We estimate I_1 and I_2 terms separately. By Assumption 1, we have

$$\begin{aligned} I_1 &\leq \frac{\lambda t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\left(\int_0^T f(u_n(x))dx\right)^2} |f(u_n(s)) - f(u(s))| \, ds \\ &\leq \frac{\lambda t^{1-\gamma}}{(c_1 T)^2 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(u_n(s)) - f(u(s))| \, ds \\ &\leq \frac{L\lambda t^{1-\gamma}}{(c_1 T)^2 \Gamma(\alpha)} \|u_n - u\|_{C_{1-\gamma}} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{L\lambda T^{\alpha+1-\gamma}}{(c_1 T)^2 \Gamma(\alpha+1)} \|u_n - u\|_{C_{1-\gamma}}. \end{aligned}$$

Then,

$$I_1 \leq \frac{L\lambda T^{\alpha+1-\gamma}}{(c_1 T)^2 \Gamma(\alpha+1)} \|u_n - u\|_{C_{1-\gamma}}, \tag{10}$$

$$\begin{aligned} I_2 &= \frac{\lambda t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(u(s))| \left| \frac{1}{\left(\int_0^T f(u_n(x))dx\right)^2} - \frac{1}{\left(\int_0^T f(u(x))dx\right)^2} \right| ds \\ &\leq \frac{\lambda t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(u(s))| \frac{\left| \left(\int_0^T f(u_n(x))dx\right)^2 - \left(\int_0^T f(u(x))dx\right)^2 \right|}{\left(\int_0^T f(u_n(x))dx\right)^2 \left(\int_0^T f(u(x))dx\right)^2} ds \\ &\leq \frac{\lambda t^{1-\gamma} c_2}{(c_1 T)^4 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| \left(\int_0^T f(u_n(x))dx\right)^2 - \left(\int_0^T f(u(x))dx\right)^2 \right| ds \\ &\leq \frac{\lambda t^{1-\gamma} c_2}{(c_1 T)^4 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| \left(\int_0^T (f(u_n(x)) - f(u(x)))dx\right) \left(\int_0^T (f(u_n(x)) + f(u(x)))dx\right) \right| ds \\ &\leq \frac{2\lambda c_2^2 T t^{1-\gamma}}{(c_1 T)^4 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^T |f(u_n(x)) - f(u(x))| \, dx \right) ds \\ &\leq \frac{2\lambda c_2^2 T L t^{1-\gamma}}{(c_1 T)^4 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^T |u_n(x) - u(x)| \, dx \right) ds \\ &\leq \frac{2\lambda c_2^2 L T^2 t^{1-\gamma}}{(c_1 T)^4 \Gamma(\alpha)} \|u_n - u\|_{C_{1-\gamma}} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{2\lambda c_2^2 L T^{\alpha+3-\gamma}}{(c_1 T)^4 \Gamma(\alpha+1)} \|u_n - u\|_{C_{1-\gamma}}. \end{aligned}$$

It follows that

$$I_2 \leq \frac{2\lambda c_2^2 L T^{\alpha+3-\gamma}}{(c_1 T)^4 \Gamma(\alpha+1)} \|u_n - u\|_{C_{1-\gamma}}. \tag{11}$$

To substitute (10) and (11) into (9), we have

$$\left|t^{1-\gamma}\left((Pu_n)(t)-(Pu)(t)\right)\right| \le I_1+I_2 \\ \le \left(\frac{L\lambda T^{\alpha+1-\gamma}}{(c_1T)^2\Gamma(\alpha+1)}+\frac{2\lambda c_2^2LT^{\alpha+3-\gamma}}{(c_1T)^4\Gamma(\alpha+1)}\right)\|u_n-u\|_{C_{1-\gamma}}.$$

Then,

$$\|Pu_n-Pu\|_{C_{1-\gamma}} \le \left(\frac{L\lambda T^{\alpha+1-\gamma}}{(c_1T)^2\Gamma(\alpha+1)}+\frac{2\lambda c_2^2LT^{\alpha+3-\gamma}}{(c_1T)^4\Gamma(\alpha+1)}\right)\|u_n-u\|_{C_{1-\gamma}}.$$

Here, independently of λ, the right-hand side of the above inequality converges to zero as $u_n \to u$. Therefore, $Pu_n \to Pu$. This proves the continuity of P.

Step 2: The operator P maps bounded sets into bounded sets in $C_{1-\gamma}[J,X]$.

Indeed, it is enough to show that, for $r>0$, there exists a positive constant l such that $u \in B_r\left\{u \in C_{1-\gamma}[J,X] : \|u\| \le r\right\}$, we have $\|(Pu)\|_{C_{1-\gamma}} \le l$. Set $M=\sup_{B_r}\frac{f}{(c_1T)^2}$:

$$\begin{aligned}\left|t^{1-\gamma}(Pu)(t)\right| &\le \frac{|u_0|}{\Gamma(\gamma)}+\frac{\lambda t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{|f(u(s))|}{\left(\int_0^T f(u(x))dx\right)^2}ds \\ &\le \frac{|u_0|}{\Gamma(\gamma)}+\frac{\lambda M t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\ &\le \frac{|u_0|}{\Gamma(\gamma)}+\frac{\lambda M T^{1-\gamma+\alpha}}{\Gamma(\alpha)}.\end{aligned}$$

Thus,

$$\|Pu\|_{C_{1-\gamma}} \le \frac{|u_0|}{\Gamma(\gamma)}+\frac{\lambda M T^{1-\gamma+\alpha}}{\Gamma(\alpha)} :\le l.$$

Step 3: P maps bounded sets into equicontinuous set of $C_{1-\gamma}[J,X]$.

Let $t_1,t_2 \in J, t_1<t_2$, B_r be a bounded set of $C_{1-\gamma}[J,X]$ and $u \in B_r$. Then,

$$\begin{aligned}&\left|t_2^{1-\gamma}(Pu)(t_2)-t_1^{1-\gamma}(Pu)(t_1)\right| \\ &\le \frac{\lambda}{\Gamma(\alpha)}\left|t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\alpha-1}\frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2}ds - t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2}ds\right| \\ &\le \frac{\lambda t_2^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{|f(u(s))|}{\left(\int_0^T f(u(x))dx\right)^2}ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha)}\int_0^{t_2}\left|t_2^{1-\gamma}(t_2-s)^{\alpha-1}-t_1^{1-\gamma}(t_1-s)^{\alpha-1}\right|\frac{|f(u(s))|}{\left(\int_0^T f(u(x))dx\right)^2}ds \\ &\le \frac{\lambda c_2 t_2^{1-\gamma}}{(c_1T)^2\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds+\frac{\lambda c_2}{(c_1T)^2\Gamma(\alpha)}\int_0^{t_2}\left|t_2^{1-\gamma}(t_2-s)^{\alpha-1}-t_1^{1-\gamma}(t_1-s)^{\alpha-1}\right|ds \\ &\le \frac{\lambda c_2 t_2^{1-\gamma}}{(c_1T)^2\Gamma(\alpha+1)}(t_2-t_1)^{1-\alpha}+\frac{\lambda c_2}{(c_1T)^2\Gamma(\alpha)}\int_0^{t_2}\left|t_2^{1-\gamma}(t_2-s)^{\alpha-1}-t_1^{1-\gamma}(t_1-s)^{\alpha-1}\right|ds.\end{aligned}$$

Beacause the right-hand side of the above inequality does not depend on u and tends to zero when $t_2 \to t_1$, we conclude that $P(\overline{B_r})$ is relatively compact. Hence, B is compact by the Arzela–Ascoli theorem. Consequently, since P is continuous, it follows by Theorem 1 that problem (1) has a solution. The proof is completed. $\square$

4. The Ulam–Hyers–Rassias Stability

In this section, we investigate generalized Ulam–Hyers–Rassias stability for problem (1). The stability results are based on the Banach contraction principle.

Lemma 3 (Uniqueness)**.** *Assume that the Assumption 1 is hold. If*

$$\left(\frac{L\lambda T^{\alpha+1-\gamma}}{(c_1T)^2\Gamma(\alpha+1)} + \frac{2\lambda c_2^2 LT^{\alpha+3-\gamma}}{(c_1T)^2\Gamma(\alpha+1)}\right) < 1, \tag{12}$$

then problem (1) *has a unique solution.*

Proof. Consider the operator $P : C_{1-\gamma}[J, X] \to C_{1-\gamma}[J, X]$:

$$(Pu)(t) = \frac{u_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2}ds. \tag{13}$$

It is clear that the fixed points of P are solutions of problem (1).
Letting $u, v \in C_{1-\gamma}[J, X]$ and $t \in J$, then we have

$$\begin{aligned}\left|t^{1-\gamma}\left((Pv)(t) - (Pu)(t)\right)\right| &\le \frac{\lambda t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\frac{f(v(s))}{\left(\int_0^T f(v(x))dx\right)^2} - \frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2}\right| ds \\ &\le \left(\frac{L\lambda T^{\alpha+1-\gamma}}{(c_1T)^2\Gamma(\alpha+1)} + \frac{2\lambda c_2^2 LT^{\alpha+3-\gamma}}{(c_1T)^2\Gamma(\alpha+1)}\right)\|v-u\|_{C_{1-\gamma}}.\end{aligned}$$

Then,

$$\|Pv - Pu\|_{C_{1-\gamma}} \le \left(\frac{L\lambda T^{\alpha+1-\gamma}}{(c_1T)^2\Gamma(\alpha+1)} + \frac{2\lambda c_2^2 LT^{\alpha+3-\gamma}}{(c_1T)^2\Gamma(\alpha+1)}\right)\|v-u\|_{C_{1-\gamma}}.$$

Choosing λ such that $0 < \lambda < \left(\frac{LT^{\alpha+1-\gamma}}{(C_1T)^2\Gamma(\alpha+1)} + \frac{2C_2^2 LT^{\alpha+3-\gamma}}{(C_1T)^2\Gamma(\alpha+1)}\right)^{-1}$, the map $P : C_{1-\gamma}[J, X] \to C_{1-\gamma}[J, X]$ is a contraction. From (12), it follows that P has a unique fixed point, which is a solution of problem (1). $\square$

Theorem 3. *In Assumption 1 and* (12)*, problem* (1) *is Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C^{\gamma}_{1-\gamma}[J, X]$ be a function that satisfies Inequality (5) and let $u \in C^{\gamma}_{1-\gamma}[J, X]$ be the unique solution of the following Hilfer type thermistor problem

$$\begin{aligned} D_{0^+}^{\alpha,\beta}u(t) &= \frac{\lambda f(u(t))}{\left(\int_0^T f(u(x))dx\right)^2}, \quad t \in J := [0, T], \\ I_{0^+}^{1-\gamma}u(t) &= I_{0^+}^{1-\gamma}z(t) = u_0, \end{aligned}$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$. From Lemma 1, we have

$$u(t) = \frac{u_0}{\Gamma(\gamma)} t^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2} ds.$$

By integration of (5), we obtain

$$\left| z(t) - \frac{u_0}{\Gamma(\gamma)} t^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{f(z(s))}{\left(\int_0^T f(z(x))dx\right)^2} ds \right| \leq \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}, \tag{14}$$

for all $t \in J$. From the above, it follows:

$$\begin{aligned}
&|z(t) - u(t)| \\
&\leq \left| z(t) - \frac{u_0}{\Gamma(\gamma)} t^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{f(z(s))}{\left(\int_0^T f(z(x))dx\right)^2} ds \right| \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| \frac{f(z(s))}{\left(\int_0^T f(z(x))dx\right)^2} - \frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2} \right| ds \\
&\leq \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\left(\int_0^T f(z(x))dx\right)^2} \left| f(z(s)) - f(u(s)) \right| ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(u(s)) \right| \left| \frac{1}{\left(\int_0^T f(z(x))dx\right)^2} - \frac{1}{\left(\int_0^T f(u(x))dx\right)^2} \right| ds.
\end{aligned} \tag{15}$$

For computational convenience, we set

$$\begin{aligned}
K_1 &= \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\left(\int_0^T f(z(x))dx\right)^2} \left| f(z(s)) - f(u(s)) \right| ds, \\
K_2 &= \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(u(s)) \right| \left| \frac{1}{\left(\int_0^T f(z(x))dx\right)^2} - \frac{1}{\left(\int_0^T f(u(x))dx\right)^2} \right| ds.
\end{aligned}$$

We estimate K_1, K_2 terms separately. By Assumption 1, we have

$$\begin{aligned}
K_1 &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\left(\int_0^T f(z(x))dx\right)^2} \left| f(z(s)) - f(u(s)) \right| ds \\
&\leq \frac{\lambda}{(c_1 T)^2 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(z(s)) - f(u(s)) \right| ds \\
&\leq \frac{\lambda L}{(c_1 T)^2 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| z(s) - u(s) \right| ds,
\end{aligned} \tag{16}$$

$$
\begin{aligned}
K_2 &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(u(s))| \frac{\left| \left(\int_0^T f(z(x))dx\right)^2 - \left(\int_0^T f(u(x))dx\right)^2 \right|}{\left(\int_0^T f(z(x))dx\right)^2 \left(\int_0^T f(u(x))dx\right)^2} ds \\
&\leq \frac{2\lambda c_2^2 TL}{(c_1 T)^4 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^T |z(x)-u(x)|\, dx \right) ds \\
&\leq \frac{2\lambda c_2^2 TL}{(c_1 T)^4 \Gamma(\alpha)} \|z-u\|_{C_{1-\gamma}} \int_0^t (t-s)^{\alpha-1} ds \\
&\leq \frac{2\lambda c_2^2 TL}{(c_1 T)^4 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s)-u(s)|\, ds.
\end{aligned} \tag{17}
$$

To substitute (16) and (17) into (15), we get

$$
\begin{aligned}
|z(t)-u(t)| \leq& \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda L}{(c_1 T)^2 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s)-u(s)|\, ds \\
&+ \frac{2\lambda c_2^2 T^2 L}{(c_1 T)^4 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s)-u(s)|\, ds \\
\leq& \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + \left(\frac{\lambda L}{(c_1 T)^2} + \frac{2\lambda c_2^2 T^2 L}{(c_1 T)^4} \right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s)-u(s)|\, ds,
\end{aligned}
$$

and, to apply Lemma 2, we have

$$
|z(t)-u(t)| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left[1 + \frac{\nu T^\alpha}{\Gamma(\alpha+1)} \left(\frac{\lambda L}{(c_1 T)^2} + \frac{2\lambda c_2^2 T^2 L}{(c_1 T)^4} \right) \right] \epsilon := C_f \epsilon,
$$

where $\nu = \nu(\alpha)$ is a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon) = C_f \epsilon$; $\psi(0) = 0$, then problem (1) is generalized Ulam–Hyers stable. □

Theorem 4. *In Assumptions 1, 2 and (12), problem (1) is Ulam–Hyers–Rassias stable.*

Proof. Let $z \in C_{1-\gamma}^{\gamma}[J, X]$ be solution of Inequality (6) and let $z \in C_{1-\gamma}^{\gamma}[J, X]$ be the unique solution of the following Hilfer type thermistor problem

$$
\begin{aligned}
D_{0^+}^{\alpha,\beta} u(t) &= \frac{\lambda f(u(t))}{\left(\int_0^T f(u(x))dx\right)^2}, \quad t \in J := [0, T], \\
I_{0^+}^{1-\gamma} u(t) &= I_{0^+}^{1-\gamma} z(t) = u_0,
\end{aligned}
$$

where $\alpha \in (0, 1)$, $\beta \in [0, 1]$. From Lemma 1, we have

$$
u(t) = \frac{u_0}{\Gamma(\gamma)} t^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2} ds.
$$

By integration of (6) and Assumption 2, we obtain

$$
\left| z(t) - \frac{u_0}{\Gamma(\gamma)} t^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{f(z(s))}{\left(\int_0^T f(z(x))dx\right)^2} ds \right| \leq \epsilon \lambda_\varphi \varphi(t), \tag{18}
$$

for all $t \in J$. From the above, it follows:

$$
\begin{aligned}
|z(t)-u(t)| \quad &\leq \left| z(t) - \tfrac{u_0}{\Gamma(\gamma)} t^{\gamma-1} - \tfrac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{f(z(s))}{\left(\int_0^T f(z(x))dx\right)^2} ds \right| \\
&\quad + \tfrac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| \frac{f(z(s))}{\left(\int_0^T f(z(x))dx\right)^2} - \frac{f(u(s))}{\left(\int_0^T f(u(x))dx\right)^2} \right| ds \\
&\leq \epsilon \lambda_\varphi \varphi(t) + \tfrac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\left(\int_0^T f(z(x))dx\right)^2} |f(z(s)) - f(u(s))| \, ds \\
&\quad + \tfrac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(u(s))| \left| \frac{1}{\left(\int_0^T f(z(x))dx\right)^2} - \frac{1}{\left(\int_0^T f(u(x))dx\right)^2} \right| ds.
\end{aligned} \tag{19}
$$

To substitute (16) and (17) into (19), we get

$$
\begin{aligned}
|z(t)-u(t)| &\leq \epsilon \lambda_\varphi \varphi(t) + \frac{\lambda L}{(c_1 T)^2 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s)-u(s)| \, ds \\
&\quad + \frac{2\lambda c_2^2 T^2 L}{(c_1 T)^4 \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s)-u(s)| \, ds \\
&\leq \epsilon \lambda_\varphi \varphi(t) + \left(\frac{\lambda L}{(c_1 T)^2} + \frac{2\lambda c_2^2 T^2 L}{(c_1 T)^4} \right) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s)-u(s)| \, ds,
\end{aligned}
$$

and, to apply Lemma 2, we have

$$
|z(t)-u(t)| \leq \left[\left(1 + \nu_1 \lambda_\varphi \left(\frac{\lambda L}{(c_1 T)^2} + \frac{2\lambda c_2^2 T^2 L}{(c_1 T)^4} \right) \right) \lambda_\varphi \right] \epsilon \varphi(t) = C_f \epsilon \varphi(t),
$$

where $\nu_1 = \nu_1(\alpha)$ is a constant. It completes the proof of Theorem 4. □

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