fractal and fractional

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Some Nonlocal Operators in the First Heisenberg Group

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Received: 1 November 2017; Accepted: 23 November 2017; Published: 27 November 2017

Abstract: In this paper we construct some nonlocal operators in the Heisenberg group. Specifically, starting from the Grünwald-Letnikov derivative and Marchaud derivative in the Euclidean setting, we revisit those definitions with respect to the one of the fractional Laplace operator. Then, we define some nonlocal operators in the non-commutative structure of the first Heisenberg group adapting the approach applied in the Euclidean case to the new framework.

Keywords: Marchaud derivative; Grünwald–Letnikov derivative; Heisenberg group; nonlocal operators

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1. Introduction

In 1927, in his PhD thesis André Marchaud (see [1], p. 47, Section 27, (23), or the published paper [2], p. 383, (23)), defined the following fractional differentiation for sufficiently regular real functions $f:\mathbb{R}\to\mathbb{R}$ for every $\alpha\in(0,1)$:

$$\mathbf{D}^{\alpha}f(x)=c\int_0^{+\infty}\frac{f(x)-f(x-t)}{t^{1+\alpha}}dt,$$

where c is a suitable normalizing constant depending on α only.

There exist two Marchaud derivatives: one from the right and the other from the left. They are respectively defined for functions defined for $\mathbb{R}$ and $\alpha\in(0,1)$ in such a way that

$$\mathbf{D}^{\alpha}_{+}f(x)=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{+\infty}\frac{f(x)-f(x-t)}{t^{1+\alpha}}dt$$

and

$$\mathbf{D}^{\alpha}_{-}f(x)=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{+\infty}\frac{f(x)-f(x+t)}{t^{1+\alpha}}dt,$$

where Γ is the usual Euler Gamma function.

We remark that Marchaud derivative $\mathbf{D}^{\alpha}f$, for functions f that are sufficiently "good", coincides with the Riemann–Liouville derivative:

$$\mathcal{D}^{\alpha}f(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x}\frac{f(t)}{(x-t)^{\alpha}}.$$

Nevertheless, the definition given by Marchaud can be used even for functions that make growth at infinity less than $|x|^{\alpha}$, while in the definition of the Riemann–Liouville derivative this behavior is not admitted (see e.g., [3], the remark at p. XXXIII). For instance, by considering only the case of the Marchaud derivative $\mathbf{D}^{\alpha}_{+}f$, and recalling the Riemann–Liouville derivative, $\alpha\in(0,1)$,

$$\mathcal{D}^{\alpha}_{+}f(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^{+\infty}\frac{f(x-t)}{t^{\alpha}}dt,$$

assuming that $f \in C^1(\mathbb{R})$ and $f = o(|x|^{\alpha-1-\epsilon})$, $x \to +\infty$ for $\epsilon > 0$, then, by invoking the Lebesgue dominated convergence theorem first and then integrating by parts, we obtain:

$$\begin{aligned}
\mathcal{D}^{\alpha}_{+} f(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f'(x-t)}{t^{\alpha}} dt = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} f'(x-t) \left(\int_t^{\infty} \tau^{-\alpha-1} d\tau \right) dt \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\epsilon \to 0^+} \left\{ [-f(x-t) \left(\int_t^{+\infty} \tau^{-\alpha-1} d\tau \right)]_{t=\epsilon}^{+\infty} - \int_{\epsilon}^{+\infty} \frac{f(x-t)}{t^{1+\alpha}} dt \right\} \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\epsilon \to 0^+} \left\{ f(x-\epsilon) \left(\int_{\epsilon}^{+\infty} \tau^{-\alpha-1} d\tau \right) - \int_{\epsilon}^{+\infty} \frac{f(x-t)}{t^{1+\alpha}} dt \right\} \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\epsilon \to 0^+} \left\{ \int_{\epsilon}^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt + (f(x-\epsilon) - f(x)) \int_{\epsilon}^{+\infty} \tau^{-\alpha-1} d\tau \right\} \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt = \mathbf{D}^{\alpha}_{+} f(x),
\end{aligned} \tag{1}$$

because there exists $\eta \in]x-\epsilon, x[$ such that, as $\epsilon \to 0$

$$|\alpha(f(x-\epsilon) - f(x)) \int_{\epsilon}^{+\infty} \tau^{-\alpha-1} d\tau| = |\alpha f'(x-\eta)| \le \sup_{\tau \in [x-\epsilon, x]} |f'(\tau)| \epsilon^{1-\alpha} \to 0.$$

In this way, it is clear that the Marchaud derivative is a sort of weaker version of the Riemann–Liouville derivative. For example, constants satisfy $\mathbf{D}^{\alpha}_{+} f(x) = 0$ in the Marchaud sense, but we can not consider in all of $\mathbb{R}$ the Riemann–Liouville derivative of a constant, and this fact is of course unpleasant.

It is in addition worthwhile remarking that the sum of the two Marchaud derivatives morally gives the Riesz derivative in one dimension, namely the fractional Laplace operator in dimension 1. More precisely,

$$\mathbf{D}^{\alpha}_{+} f(x) + \mathbf{D}^{\alpha}_{-} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{2f(x) - f(x-t) - f(x+t)}{t^{1+\alpha}} dt$$

or by giving more details:

$$\begin{aligned}
\mathbf{D}^{\alpha}_{+} f(x) + \mathbf{D}^{\alpha}_{-} f(x) &= \frac{\alpha}{\Gamma(1-\alpha)} \left(\int_0^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt + \int_0^{+\infty} \frac{f(x) - f(x+t)}{t^{1+\alpha}} dt \right) \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \left(\int_0^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt + \int_{-\infty}^{0} \frac{f(x) - f(x-t)}{|t|^{1+\alpha}} dt \right) \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{+\infty} \frac{f(x) - f(x-t)}{|t|^{1+\alpha}} dt = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{+\infty} \frac{f(x) - f(\xi)}{|x-\xi|^{1+\alpha}} d\xi \\
&= \frac{\alpha}{c(\alpha)\Gamma(1-\alpha)} (-\frac{d^2}{dx^2})^{\frac{\alpha}{2}} f(x),
\end{aligned}$$

where $c(\alpha)$ is the normalization constant associated with $(-\frac{d^2}{dx^2})^{\frac{\alpha}{2}}$, (see Section 3 for further details).

Hence, for $\alpha \in (0, 1)$,

$$\frac{c(\alpha)\Gamma(1-\alpha)}{\alpha} (\mathbf{D}^{\alpha}_{+} f(x) + \mathbf{D}^{\alpha}_{-} f(x)) = (-\frac{d^2}{dx^2})^{\frac{\alpha}{2}} f(x),$$

even if the right-hand side makes sense also for $\alpha \in (0, 2)$. We want to introduce now a new player in our storytelling: the Grünwald–Letnikov derivative.

Let

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

be the hypergeometric function where $(z)_n$ denotes the Pocchammer symbol, that is, for every $z \in \mathbb{C}$ and $n \in \mathbb{N} \cup \{0\}$

$$(z)_0 = 1, \quad (z)_n = z(z+1)\dots(z+n-1), \quad n \in \mathbb{N}.$$

We remark in particular that (see e.g., [4] (Formula (1.6.8))),

$$ {}_2F_1(a, b; b; z) = (1-z)^{-a}.$$

Hence

$$ {}_2F_1(a, b; b; z) = \sum_{k=0}^{\infty} (a)_k \frac{z^k}{k!} = (1-z)^{-a}$$

and, as a consequence for $\alpha \in (0,1)$ we get:

$$ {}_2F_1(-\alpha, b; b; 1) = \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} = 0$$

that implies

$$- \sum_{k=1}^{\infty} \frac{(-\alpha)_k}{k!} = 1. \tag{2}$$

In addition, the binomial coefficients are defined for each $\alpha \in \mathbb{C}$ and $n \in \mathbb{N} \cup \{0\}$ as

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \frac{(-1)^n(-\alpha)_n}{n!}, \quad n \in \mathbb{N}. \tag{3}$$

It is also true that

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha+n-1)}$$

for $\alpha \in \mathbb{C} \setminus -\mathbb{N}$ and and $n \in \mathbb{N}$. Recalling (2) and (3) we obtain, in particular, that for every $\alpha \in (0,1)$

$$\sum_{k=1}^{\infty} (-1)^k \binom{\alpha}{k} = \sum_{k=1}^{\infty} \frac{(-\alpha)_k}{k!} = -1. \tag{4}$$

Following [4], we need to introduce the notion of *non-centered differences* of fractional order $\alpha \in \mathbb{R}$ for a function f, namely

$$(\Delta_h^\alpha f)(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x - kh).$$

Thus, the Grünwald–Letnikov derivative is defined as follows (see [3,5,6]). Let $\alpha \in (0,1)$ and $f : \mathbb{R} \to \mathbb{R}$ be a function. The Grünwald–Letnikov derivative of order α of f is, by definition and separating the two cases:

$$f_+^{\alpha)}(x) = \lim_{h \to 0^+} \frac{(\Delta_h^\alpha f)(x)}{h^\alpha}$$

and

$$f_-^{\alpha)}(x) = \lim_{h \to 0^+} \frac{(\Delta_{-h}^\alpha) f(x)}{h^\alpha},$$

whenever these limits exist in a pointwise sense.

We are interested in a construction of nonlocal operators in the first Heisenberg group $\mathbb{H}^1$ and possibly compare them with the fractional operator of the sub-Laplacian already known in the literature (see [7–9]).

We recall briefly that the Heisenberg group has a particular importance in many physical aspects concerning quantum mechanics. Moreover, this group has been also studied, from a mathematical point of view, for its interesting properties associated with non-commutative biological structures (see [10]).

In applications, nonlocal phenomena sometimes appear for example in brain activity. Thus, it seems natural to improve our knowledge of operators like intrinsic fractional derivatives or intrinsic fractional Laplace. The adjective intrinsic here is related to the geometric structure of the non-commutative group. To assist the reader, in Section 5 we recall the main definitions concerning the simplest Heisenberg group.

In particular, we reach our target introducing in Section 6 a quite natural definition of the Marchaud derivative of order $\alpha \in (0,1)$, on the right, in the Heisenberg group, along $v = a_1 X + a_2 Y$. The vector v belongs to the first stratum of the Lie algebra $\mathfrak{h}_1 = \text{span}\{X, Y\}$ of the Heisenberg group. Here (a_1, a_2) is fixed and $a_1^2 + a_2^2 = 1$, where $X(P) \equiv (1, 0, 2p_2)$, $Y(P) \equiv (0, 1, -2p_1)$, and $P = (p_1, p_2, p_3) \in \mathbb{H}$. This definition is correctly given for every smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ with a nice behavior at infinity, let us say $f \in \mathcal{S}(\mathbb{R}^3)$ where $\mathbb{R}^3 \equiv \mathbb{H}^1$, as:

$$\mathcal{D}^{\alpha}_{v,\mathbb{H}^1,+} f(P) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(P) - f(-(ta_1, ta_2, 0) \circ P)}{t^{1+\alpha}} dt$$

where $\circ$ denotes the non-commutative inner law in $\mathbb{H}^1$. In this case $\mathbb{H}^1 \equiv \mathbb{R}^3$, and for every $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{H}^1$, the non-commutative inner law is defined as

$$x \circ y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(y_1 x_2 - y_2 x_1)).$$

For details, see Section 5.

In addition, we also define the following nonlocal operator:

$$(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P) = c_{\mathbb{H}^1}(s) \int_{\mathbb{R}^2} \frac{2f(P) - f((-x, -y, 0) \circ P) - f((x, y, 0) \circ P)}{(x^2 + y^2)^{\frac{2+s}{2}}} dxdy,$$

where $s \in (0,2)$, $c_{\mathbb{H}^1}(s)$ is a normalizing function depending on s such that $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P) \to -\Delta_{\mathbb{H}^1} f(P)$, $s \to 2^-$ and $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P) \to f(P)$, as $s \to 0^+$. Here $\Delta_{\mathbb{H}^1}$ denotes the sub-Laplacian in the Heisenberg group, that is:

$$\Delta_{\mathbb{H}^1} u := (\frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3})^2 u + (\frac{\partial}{\partial x_1} - 2x_1 \frac{\partial}{\partial x_3})^2 u.$$

We prove that that fixing $c_{\mathbb{H}^1}(s) = \frac{c_E(\frac{s}{2}, 2)}{2}$, where $c_E(\frac{s}{2}, 2)$ denotes the analogous normalizing constant for the fractional Laplace operator in $\mathbb{R}^2$ (see Section 3 where we recall some topics about the fractional Laplace operator in the Euclidean setting) we obtain that $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P) \to -\Delta_{\mathbb{H}^1} f(P)$, $s \to 2^-$ and $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P) \to f(P)$, as $s \to 0^+$.

Moreover, the relationship between $\mathcal{D}^s_{v,\mathbb{H}^1}$ and $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}$ is analogous to the one existing in the Euclidean setting between the Marchaud derivative and the fractional Laplace operator, that is, for every $s \in (0,1)$,

$$\int_{\mathbb{S}^1} \mathcal{D}^s_{v,\mathbb{H}^1,+} f(P) d\mathcal{H}^1(a) = \frac{s}{2\Gamma(1-s)} \int_{\mathbb{R}^2} \frac{2f(P) - f((-x, -y, 0) \circ P) - f((x, y, 0) \circ P)}{(x^2 + y^2)^{\frac{2+s}{2}}} dxdy$$

or, equivalently,

$$\frac{2\Gamma(1-s) c_{\mathbb{H}^1}(s)}{s} \int_{\mathbb{S}^1} \mathcal{D}^s_{v,\mathbb{H}^1,+} f(P) d\mathcal{H}^1(a) = (-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P).$$

This part can be improved and it is object of a further research paper. To conclude the introduction of this note, we describe the remaining sections that we have not cited yet. In Section 2 we introduce some notations about fractional derivatives. In Section 4 we examine the relationship between the Marchaud derivative and the fractional Laplace operator in the Euclidean setting.

2. Further Details about the Grünwald Derivative and the Marchaud Derivative of Higher Order

In order to better understand the reason for this definition we introduce the following definition that is fundamental in our approach and that we will handle later on in the framework of the Heisenberg group.

Definition 1. *The non-centered difference of increment h on $f : \mathbb{R} \to \mathbb{R}$ is defined as:*

$$(I - \tau^{-t})f(x) = f(x) - f(x-t).$$

Then we obtain for every $m \in \mathbb{N}$ that

$$(I - \tau^{-t})^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (\tau^{-t})^k$$

and

$$(I - \tau^{-t})^m f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (\tau^{-t})^k f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(x-kt).$$

On the other hand, taking the Taylor expansion of the function $t \to (1+t)^\alpha$ in the center $t_0 = 0$ and $\alpha \in (0,1)$, we get

$$(1+t)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} t^k.$$

Thus, it is possible to extend the previous definition to the fractional case as follows for $\alpha \in (0,1)$:

$$(I - \tau^{-t})^\alpha f(x) = \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha}{k} (\tau^{-t})^k f(x) = \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha}{k} f(x-kt).$$

Moreover, the following result holds (see e.g., [4]):

Proposition 1. *Let $\alpha, \beta > 0$. Then, for every bounded function:*

$$\Delta_h^\alpha \Delta_h^\beta f = \Delta_h^{\alpha+\beta} f.$$

In addition (see [4]), we have that:

Proposition 2. *Let $\alpha > 0$. Then, for every $f \in L^1(\mathbb{R})$*

$$\mathcal{F}(\Delta_h^\alpha f)(x) = (1 - e^{ixh})^\alpha \mathcal{F}(f)(x).$$

In particular, the Grünwald–Letnikov derivative of order $\alpha \in (0,1)$ coincides with the Marchaud derivative of the same order. Indeed, in consideration of the two previous trivial properties the following result is true (the proof is quite long and can be found in [3] (Theorem 20.4)). We simplify that theorem below.

Proposition 3. *Let $f \in L^p(\mathbb{R})$, $p \geq 1$. Then, for every $q \geq 1$ there exists*

$$f_\pm^{\alpha)}(x) = \lim_{h \to 0,\, in\ L^q} \frac{\Delta_{\pm h}^\alpha f(x)}{h^\alpha}$$

and

$$\mathbf{D}_\pm^\alpha f(x) = \lim_{\epsilon \to 0,\, in\ L^q} C(\alpha) \int_\epsilon^{+\infty} \frac{f(x) - f(x \mp h)}{h^{1+\alpha}} dh.$$

Moreover,

$$f_\pm^{\alpha)}(x) = \mathbf{D}_\pm^\alpha f(x),$$

independent of p and q.

The proof is quite long and can be found in [3]. Propositions 1 and 2 encode many facts. The first concerns the commutativity of the Grünwald–Letnikov derivative as well as the Marchaud derivative, namely $(f^{\alpha)})^{\beta)} = (f^{\beta)})^{\alpha)} = f^{\alpha+\beta)}$ and $\mathbf{D}^\alpha\mathbf{D}^\beta = \mathbf{D}^\beta\mathbf{D}^\alpha = \mathbf{D}^{\alpha+\beta}$. On this aspect we need to recall also that the definition of the Marchaud derivative can be extended to the case of $\alpha > 0$ in the following way (see [3,11]). For every $l \in \mathbb{N}$, $l \geq 1$ and for every $\alpha < l$ we set

$$\mathbf{D}^\alpha_\pm f(x) = \frac{1}{\chi(\alpha,l)}\int_0^{+\infty}\frac{\Delta^l_{\pm\tau}f(x)}{\tau^{1+\alpha}}d\tau,$$

where

$$\chi(\alpha,l) = \Gamma(-\alpha)A_l(\alpha) = \int_0^{+\infty}\frac{(1-e^{-t})^l}{t^{1+\alpha}}dt,$$

$A_l(\alpha) = \sum_{k=0}^l(-1)^k\binom{l}{k}k^\alpha$, and

$$\Delta^l_{\pm\tau}f(x) = \sum_{k=0}^l(-1)^k\binom{l}{k}f(x\mp k\tau).$$

Of course this definition can be generalized also to the case of functions $f:\mathbb{R}^n\to\mathbb{R}$ simply by taking for every $\xi\in\mathbb{R}^n$, $\xi\neq 0$ for $l\in\mathbb{N}$, $l\geq 1$, and $\alpha<l$,

$$\mathcal{D}^{\alpha)}_{\pm,\xi}f(x) = \frac{1}{\chi(\alpha,l)}\int_0^{+\infty}\frac{\Delta^l_{\pm\tau,\xi}f(x)}{\tau^{1+\alpha}}d\tau, \tag{5}$$

where

$$\Delta^l_{\pm\tau,\xi}f(x) = \sum_{k=0}^l(-1)^k\binom{l}{k}f(x\mp k\tau\xi).$$

It is worth mentioning that $\lim_{\alpha\to l^-}\mathcal{D}^{\alpha)}_{\pm,\xi}f(x) = \pm\mathbf{D}^l_\xi f(x)$, in the local (classical sense) and $\lim_{\alpha\to(l-1)^+}\mathcal{D}^{\alpha)}_{\pm,\xi}f(x) = \mathcal{D}^{l-1)}_{\pm,\xi}f(x) = \pm\mathbf{D}^{l-1}_\xi f(x)$, where $\mathbf{D}^0_\xi = I$ and whenever f has a "good" behavior (for example we suppose working in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$).

About previous notation, we remark that we use the symbol $\mathbf{D}^\alpha$ for denoting Marchaud fractional differentiation in one variable, possibly also for integer cases. While for denoting Marchaud fractional differentiation in several variables, along the vector ξ, we use the symbol $\mathcal{D}^{\alpha)}_{\pm,\xi}$ when $\alpha\notin\mathbb{N}$, otherwise we write $\mathbf{D}^l_\xi$ if $l\in\mathbb{N}$.

3. Fractional Laplace Operator in the Euclidean Setting

We are reminded that the fractional Laplace operator can be defined in several ways. In particular, using the Fourier transform we define for every $s\in(0,1)$ and for every $u\in\mathcal{S}(\mathbb{R}^n)$

$$(-\Delta)^s u = \mathcal{F}^{-1}(\|\xi\|^{2s}\mathcal{F})u.$$

The domain of definition of the fractional Laplace operator may be extended, but essentially we have to ask that $u\in L^2(\mathbb{R}^n)$ and $\|\xi\|^{2s}\mathcal{F}u\in L^2(\mathbb{R}^n)$. On the other hand, defining for every $u\in\mathcal{S}(\mathbb{R}^n)$ and $s\in(0,1)$

$$\mathcal{L}_s u(x) = c_E(\alpha,n)\int_{\mathbb{R}^n}\frac{f(x)-f(y)}{\|x-y\|^{n+2s}}dy := \lim_{\epsilon\to 0}c_E(s,n)\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{f(x)-f(y)}{\|x-y\|^{n+2s}}dy,$$

where $c(\alpha, n)$ is a normalizing constant, then $\mathcal{L}_s = (-\Delta)^s$ and

$$c_E(s,n) = \left(\int_{\mathbb{R}^n} \frac{1-\cos(\xi_1)}{\|\xi\|^{n+2s}} d\xi \right)^{-1}.$$

In addition (see [12]), if $n > 1$, then

$$\lim_{s \to 1^-} \frac{\omega_{n-1} c_E(s,n)}{4ns(1-s)} = 1$$

and

$$\lim_{s \to 0^+} \frac{\omega_{n-1} c_E(s,n)}{2s(1-s)} = 1.$$

Following [3] we can also define, by using a different expression of the constant of normalization, and considering a more general situation for $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha > 0$, $l \in \mathbb{N}$, $l \geq 1$, and $\alpha < l$, the following representation of the fractional Laplace operator:

$$(-S)^{\frac{\alpha}{2}} f(x) = \frac{\sin(\alpha \frac{\pi}{2})}{\beta_n(\alpha) A_l(\alpha)} \int_{\mathbb{R}^n} \frac{\Delta_y^l f(x)}{\|y\|^{n+\alpha}} dy,$$

where

$$\beta_n(\alpha) = \frac{\pi^{1+\frac{n}{2}}}{2^\alpha \Gamma(1+\frac{\alpha}{2}) \Gamma(\frac{n+\alpha}{2})},$$

and

$$\Delta_y^l f(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x-ky)$$

denotes the non-centered differences. Then, in [3] it was proved that $(-S)^\alpha = (-\Delta)^\alpha$.

In addition, if $U : \mathbb{R}^n \times]0,+\infty[\to \mathbb{R}$ is a solution to the following non-local problem, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$

$$\begin{cases} \operatorname{div}_{(x,y)}(y^{1-2s} \nabla U(x,y)) = 0, & \text{in} \quad \mathbb{R}^n \times]0,+\infty[\\ U(\cdot,0) = u, \quad x \in \mathbb{R}^n, \end{cases}$$

then defining

$$\mathcal{N}_s u := \lim_{y \to 0} y^{1-2s} \frac{\partial U(\cdot,y)}{\partial y}$$

it results (possibly up to a multiplicative factor depending only on s and n to $\mathcal{N}_s$,) that $(-\Delta)^s = \mathcal{L}_s = \mathcal{N}_s$ for every $u \in \mathcal{S}(\mathbb{R}^n)$. See [8,13] for the Carnot group setting. In addition, we take the opportunity to recall here that this extension approach has been recently applied also in defining the Marchaud derivative in [14]. For the sake of the completeness we also add to this list the definition of fractional Laplace operator obtained by using the semigroup properties. More precisely, let us consider the operator

$$\mathcal{A}_s = -\frac{s}{\Gamma(1-s)} \int_0^{+\infty} (e^{t\Delta} - Id) \frac{dt}{t^{1+s}}$$

where $e^{t\Delta}$ denotes the semigroup generated by the Laplace operator Δ. In concluding this section we recall also the well known approach of introducing the fractional operator by using spectral theory (see [15]), defining

$$\mathcal{B}_s = c(s,n) \int_0^{+\infty} \lambda^s dE(\lambda).$$

In any case it is possible to conclude that $(-\Delta)^s = (-S)^s = \mathcal{L}_s = \mathcal{N}_s = \mathcal{A}_s = \mathcal{B}_s$, at least for functions belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

The research about fractional Laplace operator has recently increased in popularity and many papers are appearing on this subject. See for instance [16] or even [17] for a nonlinear system case.

4. Relationship between the Marchaud Derivative and the Fractional Laplace Operator in the Euclidean Setting

First, let us fix our attention to the case $0 < \alpha < 1$ considering

$$\mathcal{D}^{\alpha)}_{\xi,\pm} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(x) - f(x \mp t\xi)}{t^{1+\alpha}} dt,$$

where $\xi \in \mathbb{S}^{n-1}$ and $f \in \mathcal{S}(\mathbb{R}^n)$. We define the operator $\mathcal{M}_{\frac{\alpha}{2}}$, let us say one more time for functions belonging to $\mathcal{S}(\mathbb{R}^n)$ and for $\alpha \in (0,1)$, as follows:

$$\mathcal{M}_{\frac{\alpha}{2}} f(x) = \int_{\partial B_1(0)} \mathcal{D}^{\alpha)}_{\xi} f(x) d\mathcal{H}^{n-1}(\xi).$$

Then, switching the order of integration we get

$$\begin{aligned} \int_{\partial B_1(0)} \mathcal{D}^{\alpha)}_{\xi} f(x) d\mathcal{H}^{n-1}(\xi) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \left(\int_{\partial B_1(0)} \frac{f(x) - f(x - t\xi)}{t^{1+\alpha}} d\mathcal{H}^{n-1}(\xi) \right) dt \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \left(\int_{\partial B_t(x)} \frac{f(x) - f(y)}{|x-y|^{1+\alpha}} d\mathcal{H}^{n-1}(y) \right) \frac{dt}{t^{n-1}} \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \left(\int_{\partial B_t(x)} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}} d\mathcal{H}^{n-1}(y) \right) dt = \frac{\alpha}{\Gamma(1-\alpha)} \frac{\beta_n(\alpha)}{\sin(\alpha\frac{\pi}{2})} (-\Delta)^{\frac{\alpha}{2}} f(x). \end{aligned} \tag{6}$$

In general, as already remarked [3] (Lemma 26.2), we get

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = -\frac{\Gamma(-\alpha)\sin(\alpha\frac{\pi}{2})}{\beta_n(\alpha)} \int_{\partial B_1(0)} \mathcal{D}^{\alpha)}_{\xi} f(x) d\mathcal{H}^{n-1}(\xi), \tag{7}$$

where, recalling the definitions introduced in the previous section, $\alpha > 0$, $l \in \mathbb{N}$, $l \geq 1$, $\alpha < l$, and

$$\mathcal{D}^{\alpha)}_{\xi} f(x) = \frac{1}{\chi(\alpha, l)} \int_0^{+\infty} \frac{\Delta^l_{\xi} f(x)}{\tau^{1+\alpha}} d\tau.$$

In particular, if $\alpha \in]0,1[$, and $l = 1$

$$\mathbf{D}^{\alpha,+} f(t) = \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \frac{f(t) - f(t-s)}{s^{1+\alpha}} ds,$$

where

$$C_{\alpha,1} = \frac{\Gamma(1-\alpha)}{\alpha}.$$

Moreover,

$$\mathbf{D}^{\alpha,-} f(t) = \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \frac{f(t) - f(t+s)}{s^{1+\alpha}} ds.$$

As a consequence we can play a little with this relationship. In fact, for every $\alpha \in (0,1)$, we obtain

$$(\mathbf{D}^{\alpha,+} + \mathbf{D}^{\alpha,-}) f(t) = \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \frac{2f(t) - f(t+\tau) - f(t-\tau)}{\tau^{1+\alpha}} d\tau.$$

By the way, we remark here that, in particular, if f is a function of one sufficiently smooth variable, then the following integral

$$\frac{\alpha}{2\Gamma(1-\alpha)\cos(\alpha\frac{\pi}{2})}\int_0^{+\infty}\frac{2f(x)-f(x+\tau)-f(x-\tau)}{\tau^{1+\alpha}}d\tau$$

is known in literature as the *Grünwald–Letnikov–Riesz fractional derivative* of order α and it is denoted by $f^{\alpha)}$ (see [3], (24.8′)). The generalization of this idea has been already introduced in Section 2 (see (5)). In that case we write, for every $\xi \in \mathbb{R}^n$, $\xi \neq 0$, for every $l \in \mathbb{N}$, $l \geq 1$, and $\alpha < l$, and for every $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\mathcal{D}_{\xi}^{\alpha)} f = \frac{1}{\chi(\alpha,l)}\int_0^{+\infty}\frac{\Delta^l_{\tau\xi}f(x)}{\tau^{1+\alpha}}d\tau.$$

Thus, for every $e \in \partial B_1(0)$ and for every $f \in \mathcal{S}(\mathbb{R}^n)$ we define

$$(\mathcal{D}_e^{\alpha)}+\mathcal{D}_{-e}^{\alpha)})f(x) = \frac{1}{C_{\alpha,1}}\int_0^{+\infty}\frac{2f(x)-f(x+e\tau)-f(x-e\tau)}{\tau^{1+\alpha}}d\tau.$$

As a consequence, integrating $\partial B_1(0)$ we get

$$\begin{aligned}
\int_{\partial B_1(0)}(\mathcal{D}_e^{\alpha)}+\mathcal{D}_{-e}^{\alpha)})f(x)d\mathcal{H}^{n-1}(e) &= \frac{1}{C_{\alpha,1}}\int_{\partial B_1(0)}\left(\int_0^{+\infty}\frac{2f(x)-f(x+e\tau)-f(x-e\tau)}{\tau^{1+\alpha}}d\tau\right)d\mathcal{H}^{n-1}(e)\\
&= \frac{1}{C_{\alpha,1}}\int_0^{+\infty}\left(\int_{\partial B_1(0)}\frac{2f(x)-f(x+e\tau)-f(x-e\tau)}{\tau^{1+\alpha}}d\mathcal{H}^{n-1}(e)\right)d\tau\\
&= \frac{1}{C_{\alpha,1}}\int_0^{+\infty}\left(\int_{\partial B_\tau(0)}\frac{2f(x)-f(x+\xi)-f(x-\xi)}{\tau^{n+\alpha}}d\mathcal{H}^{n-1}(\xi)\right)d\tau\\
&= \frac{1}{C_{\alpha,1}}\int_0^{+\infty}\left(\int_{\partial B_\tau(0)}\frac{2f(x)-f(x+\xi)-f(x-\xi)}{|x-\xi|^{n+\alpha}}d\mathcal{H}^{n-1}(\xi)\right)d\tau\\
&= \frac{1}{C_{\alpha,1}}\int_{\mathbb{R}^n}\frac{2f(x)-f(x+\xi)-f(x-\xi)}{|x-\xi|^{n+\alpha}}d\xi = \frac{2}{C_{\alpha,1}c(\frac{\alpha}{2},n)}(-\Delta)^{\frac{\alpha}{2}}f(x).
\end{aligned}$$

In this way we have proved that

$$\frac{C_{\alpha,1}c(\frac{\alpha}{2},n)}{2}\int_{\partial B_1(0)}(\mathcal{D}_e^{\alpha)}+\mathcal{D}_{-e}^{\alpha)})f(x)d\mathcal{H}^{n-1}(e) = (-\Delta)^{\frac{\alpha}{2}}f(x).$$

5. Some Information about the First Heisenberg Group

In this section, we recall some basic facts about of the simplest non-trivial case of the stratified Carnot group, the Heisenberg group $\mathbb{H}^1$. Given a group $\mathbb{G} \equiv \mathbb{R}^n$ endowed with the inner non-commutative group law $\circ$ and the Lie algebra $\mathfrak{g} \equiv \mathbb{R}^n$, we say that $\mathbb{G}$ is a stratified Carnot group if there exist $\{\mathfrak{g}_i\}_{1\leq i\leq m}$ vector spaces of $\mathfrak{g}$ such that

$$\mathfrak{g} = \bigoplus_{k=1}^m \mathfrak{g}_k,$$

and for $k = 1, \dots, m-1$

$$[\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1},$$

where $[X, Y]$ denotes the commutator of two vector fields belonging to the algebra $\mathfrak{g}$.

The simplest nontrivial case of Carnot group is given by the Heisenberg group $\mathbb{H}^1$. Indeed, in this case $\mathbb{H}^1 \equiv \mathbb{R}^3$, $\mathfrak{h}^1 \equiv \mathbb{R}^3$, and for every $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{H}^1$, we define the non-commutative inner law

$$x \circ y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(y_1x_2 - y_2x_1)).$$

Moreover, for every $x \in \mathbb{H}^1$, $-x = (-x_1, -x_2, -x_3)$ is the opposite of x. In this case, the algebra is $\mathfrak{h} = \mathfrak{h}_1 \bigoplus \mathfrak{h}_2$, where

$$\mathfrak{h}_1 = \operatorname{span}\{X, Y\}, \quad \mathfrak{h}_2 = \operatorname{span}\{T\},$$

$X = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}$, $Y = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}$ and $T = \frac{\partial}{\partial x_3}$. In particular,

$$[X, Y] = -4T$$

and

$$[\mathfrak{h}_1, \mathfrak{h}_1] = \mathfrak{h}_2.$$

In this framework a semigroup dilation δ_t, $t > 0$ is also defined such that $\delta_t(x_1, x_2, x_3) = (tx_1, tx_2, t^2 x_3)$. If $x_3 = 0$ the dilation acts as the usual Euclidean dilation. The vector fields X and Y are identified, with the vectors $(1, 0, 2x_2)$ and $(0, 1, -2x_1)$. In this case we write $X(x) = (1, 0, 2x_2)$ and $Y = (0, 1, -2x_1)$, respectively. We remark, for instance, that taking the solution of the Cauchy problem

$$\begin{cases} \varphi' = X(\varphi) \\ \varphi(0) = x, \end{cases} \tag{8}$$

then for every function u sufficiently smooth we get $(u \circ \varphi)'(0) = \frac{\partial u}{\partial x_1}(x) + 2x_2 \frac{\partial u}{\partial x_3}(x) = Xu(x)$ and an analogous computation may be done for Y. We denote by

$$\nabla_{\mathbb{H}^1} u(x) = Xu(x)X(x) + Yu(x)Y(x) = (Xu(x), Yu(x))$$

the intrinsic gradient. It is also possible to define a second-order object analogous to the Hessian matrix, even if the structure of $\mathbb{H}^1$ is not commutative. Moreover, we define the symmetrised horizontal Hessian matrix of u at x as follows:

$$D^{2,*}_{\mathbb{H}^1} u(x) = \begin{bmatrix} X^2 u(x), & \frac{(XY+YX)u(x)}{2} \\ \frac{(XY+YX)u(x)}{2}, & Y^2 u(x) \end{bmatrix}. \tag{9}$$

It is important to remark the differences with respect to the classical ∇u and the classical Hessian matrix $D^2 u(x)$ in $\mathbb{R}^3$ that is of course a 3×3 matrix. Indeed, $(Xu(x), Yu(x)) \in \mathbb{R}^2$, while $\nabla u(x) \in \mathbb{R}^3$ and $D^{2,*}_{\mathbb{H}^1} u(x)$ is a 2×2 matrix instead of being a 3×3 matrix.

6. Non-Centered Differences and Construction of Some Fractional Operators in the Heisenberg Group

We begin introducing some general ideas. Let $\mathbb{G}$ be a Carnot group endowed with the multiplicative inner law $\circ$ and a semigroup of dilation δ_t, $t > 0$.

Definition 2. *Let $\mathbb{G}$ be a Carnot group. Let $T^v_t f(P) = f(\varphi^v_P(t))$ where φ is the solution of the Cauchy problem*

$$\begin{cases} \varphi' = v(\varphi) \\ \varphi(0) = P \end{cases} \tag{10}$$

and $v \in \mathfrak{g}$.

Definition 3. *Let $\mathbb{G}$ be a group. We define a non-centered difference of increment $v \in \mathfrak{g}$ on $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{G}$ as*

$$(I - T^{-v}_t) f(x) = f(x) - f(\varphi^{-v}_P(t)),$$

where φ is the solution of the Cauchy problem

$$\begin{cases} \varphi' = v(\varphi) \\ \varphi(0) = x \end{cases} \tag{11}$$

and $v \in \mathfrak{g}$.

Thus, we define

$$\Delta^m_{v,t,\mathbb{G}} f(x) = (I - T_t^{-v})^m f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (T_t^{-v})^k f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(\varphi_x^{-v}(kt)).$$

It is interesting to remark that $\varphi_P^{-v}(t) = P \circ \varphi_0^{-v}(t)$ where $\circ$ is the inner law of the group.
Thus we define for every $\alpha \in (0,1)$

$$(I - T_t^{-v})^\alpha f(P) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(\varphi_P^{-v}(kt))$$

and we set, for $\alpha \in (0,1)$,

$$\Delta^\alpha_{v,t,\mathbb{G}} f(P) = (I - T_t^{-v})^\alpha f(P) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(\varphi_P^{-v}(kt)).$$

As a consequence, the Grünwald–Letinkov derivative of order α in a Carnot group along a vector field v of the first stratum of the Lie algebra may be defined as

$$\lim_{t \to 0^+} \frac{\Delta^\alpha_{v,t,\mathbb{G}} f(P)}{t^\alpha} = \mathcal{D}^{\alpha)}_{\mathbb{G},v} f(P),$$

whenever the pointwise limit exists.

From now on we are dealing with only the particular case of the Heisenberg group $\mathbb{H}^1$.

More precisely, if we fix $v = X \equiv (1,0,2x_2)$ at the point $P = (x_1,x_2,x_3)$, (or $v = Y \equiv (0,1,-2x_1)$), where $X, Y \in \mathfrak{h}_1$, the first stratum of the Lie algebra in the Heisenberg group, then $\varphi_P^{-X(P)} = P \circ \varphi_0^{-X(0)}$ can be explicitly computed.

In fact we obtain:

$$\varphi_P^{-X(P)}(t) = \delta_t(-1,0,0) \circ P = (-t + x_1, x_2, x_3 - 2tx_2) = P - t(1,0,2x_2) = P - t\sqrt{1+4x_2^2}\, w(P),$$

where $w(P) = \frac{(1,0,2x_2)}{\sqrt{1+4x_2^2}}$.

As a consequence

$$\lim_{t \to 0^+} \frac{\Delta^\alpha_{X(P),t,\mathbb{H}^1} f(P)}{t^\alpha} = \lim_{t \to 0^+} \frac{\Delta^\alpha_{t(1,0,2x_2)} f(P)}{t^\alpha} = \mathcal{D}^{\alpha)}_{(1,0,2x_2)} f(P),$$

that is the right Marchaud derivative of f at the point P along the direction $X(P)$, in the classical Euclidean meaning, that can be written, using a different notation, as:

$$\mathcal{D}^{\alpha)}_{X(P),\mathbb{H}^1} f(P) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(P) - f(P - t\sqrt{1+4x_2^2}\, w(P))}{t^{1+\alpha}} dt.$$

Now considering $v = a_1 X + a_2 Y$ and $a_1^2 + a_2^2 = 1$, we define in general

$$\mathcal{D}^{\alpha)}_{v,\mathbb{H}^1,+} f(P) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(P) - f(-\delta_t(a_1,a_2,0) \circ P)}{t^{1+\alpha}} dt.$$

and

$$\mathcal{D}^{\alpha)}_{v,\mathbb{H}^1,-}f(P) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{+\infty}\frac{f(P)-f(\delta_t(a_1,a_2,0)\circ P)}{t^{1+\alpha}}dt,$$

keeping in mind that $\delta_t(a_1,a_2,0) = (ta_1,ta_2,0)$.

We remark in this case that if $0<\eta<\epsilon$ then

$$\begin{aligned}\frac{\alpha}{\Gamma(1-\alpha)}\int_\eta^{+\infty}\frac{f(P)-f((-ta_1,-ta_2,0)\circ P)}{t^{1+\alpha}}dt &= \frac{\alpha}{\Gamma(1-\alpha)}\int_\eta^{\epsilon}\frac{f(P)-f((-ta_1,-ta_2,0)\circ P)}{t^{1+\alpha}}dt \\ + \frac{\alpha}{\Gamma(1-\alpha)}\int_\epsilon^{+\infty}\frac{f(P)-f((-ta_1,-ta_2,0)\circ P)}{t^{1+\alpha}}dt &= \langle\nabla_{\mathbb{H}^1}u(P),(a_1,a_2)\rangle_{\mathbb{R}^2}\int_\delta^\epsilon t^{-\alpha}+o(\epsilon^{1-\alpha}) \\ + \frac{\alpha}{\Gamma(1-\alpha)}\int_\epsilon^{+\infty}\frac{f(P)-f((-ta_1,-ta_2,0)\circ P)}{t^{1+\alpha}}dt.\end{aligned} \tag{12}$$

As a consequence,

$$\begin{aligned}&\left|\frac{\alpha}{\Gamma(1-\alpha)}\int_\eta^{+\infty}\frac{f(P)-f((-ta_1,-ta_2,0)\circ P)}{t^{1+\alpha}}dt - \frac{\alpha}{\Gamma(1-\alpha)}\frac{\langle\nabla_{\mathbb{H}^1}u(P),(a_1,a_2)\rangle}{1-\alpha}(\epsilon^{1-\alpha}-\delta^{1-\alpha})\right| \\ &\leq \frac{\alpha}{\Gamma(1-\alpha)}o(\epsilon^{1-\alpha}) + 2\frac{\alpha}{\Gamma(1-\alpha)}|f|_{L^\infty}\epsilon^{-\alpha}\to 0,\end{aligned}$$

as $\alpha\to 1^-$ because $(1-\alpha)\Gamma(1-\alpha)\to 1$ as $\alpha\to 1^-$.

Thus,

$$\lim_{\alpha\to 1^-}\mathcal{D}^{\alpha)}_{v,\mathbb{H}^1,+}f(P) = \langle\nabla_{\mathbb{H}^1}u(P),(a_1,a_2)\rangle$$

and with analogous computation

$$\lim_{\alpha\to 1^-}\mathcal{D}^{\alpha)}_{v,\mathbb{H}^1,-}f(P) = -\langle\nabla_{\mathbb{H}^1}u(P),(a_1,a_2)\rangle.$$

Now by integrating on $\mathbb{S}^1$ we get, for $0<\alpha<1$,

$$\begin{aligned}\int_{\mathbb{S}^1}\mathcal{D}^{\alpha)}_{v,\mathbb{H}^1,+}f(P)d\mathcal{H}^1(a) &= C\int_{\mathbb{S}^1}\int_0^{+\infty}\frac{f(P)-f((-ta_1,-ta_2,0)\circ P)}{t^{1+\alpha}}dtd\mathcal{H}^1(a) \\ &= C\int_0^{+\infty}\int_{\sqrt{h_1^2+h_2^2}=t}\frac{f(P)-f((-h_1,-h_2,0)\circ P)}{t^{2+\alpha}}d\mathcal{H}^1(h)dt \\ &= C\int_{\mathbb{R}^2}\frac{f(P)-f((-x,-y,0)\circ P)}{(x^2+y^2)^{\frac{2+\alpha}{2}}}dxdy\end{aligned} \tag{13}$$

It is important to remark that, thanks to a cancelation of the integral of the first-order term in a neighborhood of P, we get:

$$
\begin{aligned}
&\int_{\{x^2+y^2<\epsilon^2\}} \frac{f(P)-f((-x,-y,0)\circ P)}{(x^2+y^2)^{\frac{2+\alpha}{2}}}dxdy \\
&= \int_{\{x^2+y^2<\epsilon^2\}} \frac{\langle \nabla_{\mathbb{H}^1} f(P),(x,y)\rangle}{(x^2+y^2)^{\frac{2+\alpha}{2}}}dxdy \\
&- \int_{\{x^2+y^2<\epsilon^2\}} \frac{2^{-1}\langle D^2 f(P)(xX(P)+yY(P)),(xX(P)+yY(P))\rangle + o(x^2+y^2)}{(x^2+y^2)^{\frac{2+\alpha}{2}}}dxdy \\
&= -\int_{\{x^2+y^2<\epsilon^2\}} \frac{2^{-1}\left(x^2X^2f(P)+2xy\langle D^2f(P)X(P),Y(P))\rangle + y^2X^2f(P)\right)+o(x^2+y^2)}{(x^2+y^2)^{\frac{2+\alpha}{2}}}dxdy \qquad (14)\\
&= -\int_{\{x^2+y^2<\epsilon^2\}} \frac{2^{-1}x^2}{(x^2+y^2)^{\frac{2+\alpha}{2}}}dxdy(X^2f(P)+Y^2f(P)) + o\int_{\{x^2+y^2<\epsilon^2\}} \frac{x^2+y^2}{(x^2+y^2)^{\frac{2+\alpha}{2}}}dxdy \\
&= -2^{-1}\int_0^{2\pi}\cos^2\theta d\theta \int_0^{\epsilon}\rho^{3-2-\alpha}d\rho + o(\epsilon^{2-\alpha}) \\
&= -\frac{\pi}{2}\int_0^{\epsilon}\rho^{1-\alpha}d\rho\Delta_{\mathbb{H}^1}f(P)+o(\epsilon^{2-\alpha}) = -\frac{\pi}{2(2-\alpha)}\epsilon^{2-\alpha}\Delta_{\mathbb{H}^1}f(P)+o(\epsilon^{2-\alpha}).
\end{aligned}
$$

On the other hand, if $f \in L^\infty$ then

$$
\begin{aligned}
&\int_{\{x^2+y^2\geq\epsilon^2\}} \frac{f(P)-f((-x,-y,0)\circ P)}{(x^2+y^2)^{\frac{2+\alpha}{2}}}dxdy \\
&\leq 3||f||_{L^\infty}\int_{\{x^2+y^2\geq\epsilon^2\}} \frac{1}{(x^2+y^2)^{\frac{2+\alpha}{2}}}dxdy = 3||f||_{L^\infty}\int_{\{x^2+y^2\geq\epsilon^2\}}\frac{\rho}{\rho^{2+\alpha}}dxdy \qquad (15)\\
&= \frac{3}{\alpha}||f||_{L^\infty}\epsilon^{-\alpha}.
\end{aligned}
$$

As a consequence, we get for every $s \in (0,1)$

$$\int_{\mathbb{S}^1} \mathcal{D}^{s)}_{v,\mathbb{H}^1,+}f(P)d\mathcal{H}^1(a) = \frac{s}{\Gamma(1-s)}\int_{\mathbb{R}^2}\frac{f(P)-f((-x,-y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}}dxdy$$

If we repeat this construction using $\mathcal{D}^{s)}_{v,\mathbb{H}^1,-}$ for every $s \in (0,1)$ we obtain:

$$\int_{\mathbb{S}^1} \mathcal{D}^{s)}_{v,\mathbb{H}^1,-}f(P)d\mathcal{H}^1(a) = \frac{s}{\Gamma(1-s)}\int_{\mathbb{R}^2}\frac{f(P)-f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}}dxdy$$

and moreover with a change of variables, it results that

$$\int_{\mathbb{S}^1} \mathcal{D}^{s)}_{v,\mathbb{H}^1,+}f(P)d\mathcal{H}^1(a) = \int_{\mathbb{S}^1} \mathcal{D}^{s)}_{v,\mathbb{H}^1,-}f(P)d\mathcal{H}^1(v).$$

Then for every $s \in (0,1)$

$$
\begin{aligned}
2\int_{\mathbb{S}^1} \mathcal{D}^{s)}_{v,\mathbb{H}^1,+}f(P)d\mathcal{H}^1(a) &= \int_{\mathbb{S}^1} \mathcal{D}^{s)}_{v,\mathbb{H}^1,+}f(P)+\mathcal{D}^{s)}_{v,\mathbb{H}^1,-}f(P)d\mathcal{H}^1(a) \\
&= \frac{s}{\Gamma(1-s)}\int_{\mathbb{R}^2}\frac{2f(P)-f((-x,-y,0)\circ P)-f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}}dxdy
\end{aligned}
$$

In particular,

$$\begin{aligned}&\lim_{s\to 1^-}\int_{\mathbb{S}^1}\mathcal{D}^{s)}_{v,\mathbb{H}^1,+}f(P)d\mathcal{H}^1(a)\\&=\lim_{s\to 1^-}\frac{s}{2\Gamma(1-s)}\int_{\mathbb{R}^2}\frac{2f(P)-f((-x,-y,0)\circ P)-f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}}dxdy,\end{aligned}\tag{16}$$

but

$$\lim_{s\to 1^-}\int_{\mathbb{S}^1}\mathcal{D}^{s)}_{v,\mathbb{H}^1,+}f(P)d\mathcal{H}^1(a)=\int_{\mathbb{S}^1}\langle\nabla_{\mathbb{H}^1}f(P),(a_1,a_2)\rangle d\mathcal{H}^1(a)=0,$$

by the Gauss Theorem.

Hence, it results that

$$\lim_{s\to 1^-}\frac{s}{2\Gamma(1-s)}\int_{\mathbb{R}^2}\frac{2f(P)-f((-x,-y,0)\circ P)-f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}}dxdy=0.$$

Moreover, in analogy with the the Euclidean case, if $s\in(0,2)$ we may define the following operator:

$$(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}f(P)=c_{\mathbb{H}^1}(s)\int_{\mathbb{R}^2}\frac{2f(P)-f((-x,-y,0)\circ P)-f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}}dxdy$$

where $c(s)$ is a normalizing constant that has to be fixed.

In addition, the operator $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}$ is intrinsically homogeneous, in the sense that for every $s\in(0,2)$:

$$(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}(f(\delta_\lambda(P))=\lambda^s((-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}f)(\delta_\lambda(P)).$$

In fact,

$$\begin{aligned}&(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}(f(\delta_\lambda(P))=c_{\mathbb{H}^1}(s)\int_{\mathbb{R}^2}\frac{2f(\delta_\lambda(P))-f(\delta_\lambda((-x,-y,0)\circ P))-f(\delta_\lambda((x,y,0)\circ P))}{(x^2+y^2)^{\frac{2+s}{2}}}dxdy\\&=c_{\mathbb{H}^1}(s)\int_{\mathbb{R}^2}\frac{2f(P)-f(\delta_\lambda(-x,-y,0)\circ\delta_\lambda(P))-f(\delta_\lambda(x,y,0)\circ\delta_\lambda(P))}{(x^2+y^2)^{\frac{2+s}{2}}}dxdy.\end{aligned}$$

Thus, performing the change of variables $\lambda x=u,\quad \lambda y=v$, we get

$$\begin{aligned}&=c_{\mathbb{H}^1}(s)\int_{\mathbb{R}^2}\frac{2f(\delta_\lambda(P))-f((-u,-v,0)\circ\delta_\lambda(P))-f((u,v,0)\circ\delta_\lambda(P))}{((\frac{u}{\lambda})^2+(\frac{v}{\lambda})^2)^{\frac{2+s}{2}}}\lambda^{-2}dudv\\&=\lambda^s c_{\mathbb{H}^1}(s)\int_{\mathbb{R}^2}\frac{2f(\delta_\lambda(P))-f((-u,-v,0)\circ\delta_\lambda(P))-f((u,v,0)\circ\delta_\lambda(P))}{(u^2+v^2)^{\frac{2+s}{2}}}dudv\\&=\lambda^s((-\mathcal{L})^{\frac{s}{2}}f)(\delta_\lambda(P)).\end{aligned}$$

We come back to the point concerning the normalizing constant $c_{\mathbb{H}^1}$ since it is fundamental, as remarked in [3] in the Euclidean setting. In fact, in the Euclidean case, starting from the non-local operator defined via the the Fourier transform for every $u\in\mathcal{S}(\mathbb{R}^n)$ as

$$(-\Delta)^{\frac{\alpha}{2}}u=\mathcal{F}^{-1}(\|\xi\|^\alpha\mathcal{F}u),$$

different types of representations of $(-\Delta)^{\frac{\alpha}{2}}u$ can be determined. In [3] this problem is explicitly treated in the Euclidean framework. More precisely, it has been settled by considering the non-centered differences (and the centered differences too). In analogy to this, the choice of the constant c in the case of $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}$ should be done carefully. For example, if $c_{\mathbb{H}^1}(s)=\frac{8\sin(s\frac{\pi}{2})}{\pi^2}$, then: $-(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}f(P)\to\Delta_{\mathbb{H}^1}f(P)$ whenever $s\to 2^-$. Moreover, if $f\in C_0^\infty(\mathbb{R}^3)$ then there exists a sufficiently large Euclidean ball of radius R such that $\mathrm{supp}(f) \subset B_R(0)$. As a consequence, considering the Euclidean ball $B'_R(0)$ in $\mathbb{R}^2$ then for every $(x,y) \in \mathbb{R}^2 \setminus B'_R(0)$ we get $2f(P) - f((-x,-y,0)\circ P) - f((x,y,0)\circ P) = 2f(P)$. Thus,

$$\begin{aligned}&\int_{\mathbb{R}^2\setminus B'_R(0)} \frac{2f(P) - f((-x,-y,0)\circ P) - f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}} dxdy \\ &= 2f(P)\int_{\mathbb{R}^2\setminus B'_R(0)} \frac{1}{(x^2+y^2)^{\frac{2+s}{2}}} dxdy = 4\pi f(P)\int_R^{+\infty} \rho^{-1-s} d\rho = \frac{4\pi}{s} f(P) R^{-s}\end{aligned}$$

so that as $s \to 0^+$ we get

$$\begin{aligned}&\frac{8\sin(s\frac{\pi}{2})}{\pi^2}\int_{\mathbb{R}^2\setminus B'_R(0)} \frac{2f(P) - f((-x,-y,0)\circ P) - f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}} dxdy \\ &= \frac{8\sin(s\frac{\pi}{2})}{\pi^2}\frac{4\pi}{s} f(P) R^{-s} \to 16 f(P)\end{aligned}$$

and since as $s \to 0^+$ we have also

$$\frac{8\sin(s\frac{\pi}{2})}{\pi^2}\frac{4\pi}{s}\int_{B'_R(0)} \frac{2f(P) - f((-x,-y,0)\circ P) - f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}} dxdy \to 0,$$

concluding that

$$(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P) \to 16 f(P).$$

We can improve this result as follows. Let us consider $g : \mathbb{R}^2 \to \mathbb{R}$ and define $f : \mathbb{R}^3 \to \mathbb{R}$ such that for every $P \in \mathbb{R}^3$, $f(P) = g(P')$, where $P' = (P_1, P_2)$. Then we want to check the behavior of $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}$ on functions depending only on the first two variables. In particular

$$\begin{aligned}&(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P) = c_{\mathbb{H}^1}(s)\int_{\mathbb{R}^2} \frac{2g(P') - f((-x,-y,0)\circ P) - f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}} dxdy \\ &= c_{\mathbb{H}^1}(s)\int_{\mathbb{R}^2} \frac{2g(P') - f(-x+P_1, -y+P_2, P_3 + 2(P_1 y - P_2 x)) - f(x+P_1, y+P_2, P_3 - 2(P_1 y - P_2 x))}{(x^2+y^2)^{\frac{2+s}{2}}} dxdy \\ &= c_{\mathbb{H}^1}(s)\int_{\mathbb{R}^2} \frac{2g(P') - g(-x+P_1, -y+P_2) - g(x+P_1, y+P_2)}{(x^2+y^2)^{\frac{2+s}{2}}} dxdy = \frac{2c_{\mathbb{H}^1}(s)}{c_E(\frac{s}{2},2)}(-\Delta_2)^{\frac{s}{2}} g(P')\end{aligned} \quad (17)$$

and since in our hypothesis we know that $\Delta_{\mathbb{H}^1} f(P) = \Delta_2 g(P')$, where Δ_2, denotes the classical Laplace operator in $\mathbb{R}^2$, we ask that for this type of functions ($f(P) = g(P')$) and for every $s \in (0,1)$ it has to be true that

$$(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P) = (-\Delta_2)^{\frac{s}{2}} g(P').$$

Thus, if we fix $c_{\mathbb{H}^1}(s) = \frac{c_E(\frac{s}{2},2)}{2}$ we obtain a normalizing constant satisfying our requests.

Eventually, it would be interesting to check if, up to the normalizing constant $c_{\mathbb{H}^1}$, the operator $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}$ has some relationships with the fractional Laplace operator in the Heisenberg group $\mathbb{H}^1$ constructed using the approach described in [7,15] and then revisited in [8] following the extension presented in [13]. See also [18] for research about the fractional Sobolev norms associated with Hörmander vector fields. In [8] the fractional Laplace operator has been written, for every $s \in (0,2)$ and for every $u \in \mathcal{S}(\mathbb{R}^n)$, as follows:

$$(-\Delta_{\mathbb{H}^1})^{s/2} u(P) = \mathrm{P.V.}\int_{\mathbb{H}^1} (u(Q) - u(P))\widetilde{R}_{-s}(-Q\circ P)\, dQ, \quad (18)$$

where for $\beta < 0$, $\beta \notin \{0, -2, -4, \cdots\}$

$$\widetilde{R}_\beta(P) = \frac{\frac{\beta}{2}}{\Gamma(\beta/2)}\int_0^\infty t^{\frac{\beta}{2}-1} h(t,P)\, dt,$$

and h is the fundamental solution of the heat operator $-\Delta_{\mathbb{H}^1} h(t,P) + \frac{\partial h(t,P)}{\partial t} = 0$.

For an application of the fractional Laplace operator in the Heisenberg group to the geometric measure theory in this noncommutative framework, see [9].

7. Conclusions

The relationship between the two operators $\mathcal{D}^s_{v,\mathbb{H}^1}$ and $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}$, constructed in previous sections, is analogous to the one existing in the Euclidean setting between the Marchaud derivative and the fractional Laplace operator, that is: for every $s \in (0,1)$,

$$\int_{\mathbb{S}^1} \mathcal{D}^s_{v,\mathbb{H}^1,+} f(P) d\mathcal{H}^1(a) = \frac{s}{2\Gamma(1-s)} \int_{\mathbb{R}^2} \frac{2f(P) - f((-x,-y,0)\circ P) - f((x,y,0)\circ P)}{(x^2+y^2)^{\frac{2+s}{2}}} dxdy.$$

Equivalently, we can write:

$$\frac{2\Gamma(1-s) c_{\mathbb{H}^1}(s)}{s} \int_{\mathbb{S}^1} \mathcal{D}^s_{v,\mathbb{H}^1,+} f(P) d\mathcal{H}^1(a) = (-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}} f(P).$$

The operator $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}$ fulfills very nice properties, as remarked in Section 6. On the other hand, in the literature, the fractional Laplace operator in the Heisenberg group already exists and is represented, as recalled before, in (18)—see also [7]—with an expression that, at first glance, does not seem so close to the one of $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}$. Thus, in further research, we would like to study whether any relationship between $(-\mathcal{L}_{\mathbb{H}^1})^{\frac{s}{2}}$ and $(-\Delta_{\mathbb{H}^1})^{\frac{s}{2}}$ exists.

Acknowledgments: The author is supported by MURST, Italy, and INDAM-GNAMPA project 2017: Regolarità delle soluzioni viscose per equazioni a derivate parziali non lineari degeneri.

Conflicts of Interest: The author declares no conflict of interest.

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