



Article

The Potentials of Tangible Technologies for Learning Linear Equations

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Received: 13 August 2020; Accepted: 21 October 2020; Published: 23 October 2020;
Corrected: 10 January 2022



Abstract: Tangible technologies provide interactive links between the physical and digital worlds, thereby merging the benefits of physical and virtual manipulatives. To explore the potentials of tangible technologies for learning linear equations, a tangible manipulative (TM) was designed and developed. A prototype of the initial TM was implemented and evaluated using mixed methods (i.e., classroom interventions, paper-based tests, thinking aloud sessions, questionnaires, and interviews) in real classroom settings. Six teachers, 24 primary school students, and 65 lower secondary school students participated in the exploratory study. The quantitative and qualitative analysis revealed that the initial TM supported student learning at various levels and had a positive impact on their learning achievement. Moreover, its overall usability was also accepted. Some minor improvements with regard to its pedagogy and usability could be implemented. These findings indicate that the initial TM is likely to be beneficial for linear equation learning in pre-primary to lower secondary schools and be usable in mathematics classrooms. Theoretical and practical implications are discussed.

Keywords: manipulatives; multimodality; tangible user interface; educational technology; mathematics learning; linear equations; basic education

1. Introduction

Algorithmic skills have typically been the main focus of school mathematics [1,2]. Unfortunately, mathematics teaching that focuses only on procedural skills usually results in learning to memorise without understanding [2,3]. Linear equation solving, one of the important algebraic concepts, has commonly been taught in terms of rules and procedures rather than encouraging an understanding of the concepts leading to those rules [2,4]. Research has indicated that rule-based rote learning leads to misconceptions, inability to transfer procedures to other contexts, and forgetting rules [5].

Physical manipulatives (i.e., concrete objects that enable students to explore mathematical concepts through various senses) have traditionally been used to promote students' understanding of abstract mathematical concepts. Physical manipulatives have various benefits, including representing an abstract concept [6], encouraging physical action to facilitate learning [7], enhancing memory through physical action [8], and supporting embodied cognition [9]. Nevertheless, students who learn through physical manipulatives may not be able to connect the concrete representation with the symbolic representation of the same concept [10]. Over the last two decades, the use of virtual manipulatives (i.e., interactive pictorial representations of virtual objects on computers or tablets) has gained

attention in mathematics education. Different advantages of virtual manipulatives include providing the simultaneous link between pictorial and symbolic representations, step-by-step scaffolding, immediate feedback [11], and drawing learners' attention to the to-be-learned mathematics [12]. However, there has been concern about the disadvantage of replacing rich physical interactions using physical manipulatives with clicking a mouse or tapping and scrolling a touch screen when manipulating virtual manipulatives [13,14]. A body of research signals that the combination of physical and virtual manipulatives may facilitate students' mathematics learning [11,12,15].

Recently, tangible manipulatives (TMs, i.e., a combination of physical and digital manipulatives) have been introduced. TMs offer a new form of interaction by allowing learners to naturally manipulate physical objects, which then provide output, typically through a graphical user interface (GUI). Thus, TMs may be the possible solution to the disagreement regarding the advantages and disadvantages of physical and virtual manipulatives.

To date, a considerable amount of research has focused on usability and engagement of TMs from the learners' perspective, thereby leaving the contribution of TMs to learning as well as teachers' perspective on TMs under-researched [16]. Additionally, the available manipulatives for equation solving are usually either physical (e.g., algebra tiles and Hands-On Equations [17]) or virtual (e.g., the Hands-On Equations applet [18] and virtual algebra balance scale applet on the National Library of Virtual Manipulatives website [19]). Recently, the Multimodal Algebra Learning (MAL) project [20] has attempted to develop virtual and tangible manipulatives for solving equations. Nevertheless, the system primarily focuses on pedagogy.

To holistically explore the potentials of TMs in mathematics classrooms, we proposed an initial TM for primary students to learn about linear equations. We developed the TM by taking pedagogy, usability, and practicality into account to ensure its successful classroom adoption. Then, we conducted a mixed-methods study in real classrooms in Finland to evaluate a prototype of the TM based on students' and teachers' feedback. This paper reports the design, development, and implementation of the initial TM as well as the classroom evaluation of its prototype. The classroom evaluation addresses the following research questions in particular:

1. What are the impacts of the proposed manipulative on students' learning achievement?
2. Does the proposed manipulative promote students' learning in terms of understanding equation-solving concepts, languaging, and learning through discovery and social interaction, and, if so, then how?
3. How do students perceive the usability of the proposed manipulative, and how well can they use it?

In the following section, we highlight the theoretical background of the study. Then, we discuss the design, development, and implementation of the initial TM. Next, we describe the classroom evaluation of the TM prototype. After that, we report and discuss our findings, reflect on the limitations of this research, and provide suggestions for future development and research. Finally, we conclude our study and explore its implications.

2. Theoretical Background

2.1. Equation Solving

Mastering equation solving is often challenging for students, e.g., [2,21–23]. To learn how to solve equations, students need to understand several concepts. Understanding mathematical equivalence includes an understanding of an equal sign as a relational symbol, of each side of the equation as an entity, and of various interchangeable ways of representing numbers and expressions [21]. An equation is composed of different terms (i.e., constants, variables, and coefficients). Thus, understanding the meaning of the mathematical symbols that represent those terms is essential [22]. Moreover, it is important to understand that the purpose of equation solving is to find the values of the variables that

make the equation true [23] or to show that the equation has no real number-value solution. All these concepts informed our TM design for promoting students' understanding of equation solving.

2.2. Multimodal Mathematics

Mathematics is inherently multimodal. In mathematics, natural language (i.e., verbal and written), mathematical symbols (i.e., numbers and symbols), and visual representations (e.g., pictures, graphs, and diagrams) are typically used for meaning making. While the three resources intertwine to construct mathematical meanings as a whole, each resource has its own unique task [24–26]. According to O'Halloran [25,27], language assists in reasoning about the mathematical process and its results, symbols describe mathematical relations, and visuals present images to concretise mathematical relations. Thus, multimodality (i.e., multiple ways of communication) plays an important role in mathematics learning. Students are required to be able to interpret and make use of all these resources simultaneously [26]. This meaning-making process contributes to students' mathematical thinking and knowledge construction [26,28–30].

According to Bruner [6], a person constructs their own knowledge through physical actions, images, and abstract symbols. He proposed three modes of representation: (1) enactive representation (e.g., direct manipulation of objects), (2) iconic representation (e.g., pictures and graphics), and (3) symbolic representation (e.g., language and mathematical symbols) [6]. The Multimodal languaging model (referred to as languaging in this paper) was informed by Bruner's three modes of representation. Languaging can be defined as one's expression of their own mathematical thinking through four languages: natural, mathematical symbolic, pictorial, and tactile (e.g., manipulative manipulation) [30,31]. The use of manipulatives as mathematical representations is also recommended, for example, by Lesh et al. [32]. Using different languages to solve a mathematical problem or present the solution to a mathematical problem assists a student and their peer group in organising their own mathematical thinking and eventually gaining a better understanding of that mathematical concept or procedure [30,31,33].

Representational fluency—the ability to understand and construct multiple external representations of the same mathematical content and the ability to connect different modes of representation with each other—plays an important role in mathematics learning [32,34,35]. Research has indicated that representational fluency can contribute to mathematical knowledge and understanding [2,11,36]. Thus, representational fluency provides support for the languaging model. Languaging was used to guide our TM design and study (e.g., what modes of representations the TM should provide and how it should be used in classrooms) to ensure that the TM benefits students' equation learning.

2.3. Tangible Technologies for Learning

Tangible user interfaces (TUIs) emerged in the 1990s from Ishii and Ullmer's [37] pioneering work. TUIs enhance human–computer interaction by enabling users to physically operate (i.e., input) digital information through the manipulation of physical objects, thereby seamlessly interlinking the physical and digital world [38]. TUIs have been utilised in various application domains to date, including education.

The application of TUIs for learning is underlined by the integrated advantages of physical and virtual manipulatives. TUIs enable physical interaction with concrete objects, which provides a sense of physicality and embodiment, allows for natural bodily interaction [38], and engages multiple senses [39]. The multimodal interaction of TUIs enables mappings between physical and digital representations (i.e., physical touch and gestures with pictorial, symbolic, and other representations) [40] and thereby promotes knowledge transfer. TUIs allow parallel multi-user interaction, which encourages co-located and distanced collaborations [16,38,40]. Additionally, TUIs also provide immediate feedback; encourage independent exploration; promote facial, gestural, and verbal communication [16]; allow accessibility to various learners; and motivate learning [39].

The embedded embodied cognition perspectives provide support regarding the potential benefits of learning with TMs. From the embedded cognition viewpoint, manipulatives may work as students' external working memory, thereby allowing them to allocate their cognitive resources to learning during their interaction with manipulatives (i.e., online cognition) [9,15]. Moreover, the theory of Physically Distributed Learning (PDL) [7] suggests that a physical environment (e.g., manipulatives) that enables students' exploration can benefit their learning. From the embodied cognition viewpoint, students' previous sensorimotor experiences of interacting with manipulatives (i.e., offline cognition) may support their transfer of learning [9,41]. Gradually, students' dependence on manipulatives often decreases, while their internal cognition increases [9].

Studies have shown how TUIs have enhanced mathematics learning in different content domains and various educational levels. For example, number composition for primary school [14], fractions for primary school [42], geometry for pre-primary and primary school [43,44], trigonometry for undergraduate school [45], and algebra for lower and upper secondary school [20] have been studied. The potential benefits of TMs were used to inform our TM design and to guide our result analysis and discussion.

2.4. Learning through Discovery and Social Interaction

The use of manipulatives as hands-on learning tools for constructing abstract mathematical concepts has been recommended based on the work of Piaget, Bruner, and Montessori among the early theorists [8,10]. However, simply using manipulatives does not automatically contribute to mathematics learning [13,46,47]. Meaningful learning using manipulatives requires students to think and reflect on what they have experienced [47] and discuss their discoveries with others [46].

Discovery learning, proposed by Bruner [48], provides the theoretical foundations for the use of manipulatives to support mathematics learning through first-hand experience and reflection. Discovery learning is a process in which learners interact with the environment (e.g., manipulatives) and actively construct their own knowledge through inductive reasoning [48]. During discovery learning, learners are not left unaided but rather are assisted through guidance or scaffolding [49]. Previous studies [11,45] found that educational technology (e.g., virtual manipulatives and TMs) can be used to provide learners with first-hand experiences and guide or scaffold their inductive reasoning process during discovery learning [49].

According to Vygotsky [50], learning is a collaborative process in which learners co-construct knowledge through social interaction within the zone of proximal development. Thus, learners should work in small groups that have heterogeneous members [51] and should be encouraged to share and listen to one another's thoughts [52,53]. From the cognitive perspective, explaining the material to peers allows learners to retain that information in their memory and relate it to prior information stored in their memory [53]. Therefore, peer tutoring can benefit both the tutor and the tutee [54]. Hence, learning through social interaction not only supports languaging in mathematics classrooms [30] but also promotes meaningful learning with manipulatives (cf. discovery learning) [36,43,45]. The design of our TM and its pedagogic utilisation in classrooms was built on social constructivism, e.g., [55], particularly learning through discovery and social interaction, to ensure that the TM benefits students' mathematics learning.

2.5. The Finnish National Core Curriculum for Basic Education (NCC) 2014

The current Finnish National Core Curriculum (NCC) for Basic Education [56] has placed emphasis on teaching and learning for understanding mathematical concepts. Equation solving has been included in the NCC [56] as one of the key areas of mathematical content for Grades 3–9. Student of Grades 3–6 should be introduced to the concept of the unknown as well as be made to examine and solve linear equations through reasoning and experimentation (i.e., trial-and-error substitution of values for the unknown). Students of Grades 7–9 should be able to form and solve linear equations algebraically.

Multimodality and languaging have been incorporated into mathematics instruction in the NCC [56]. Students are encouraged to use concrete tools, spoken and written natural language, and drawings to present their own conclusions and solutions to others [56]. Thus, students should be provided with the opportunity to make mathematical meaning in different ways. For example, instead of only doing exercises by writing mathematical symbols on a worksheet, they can also do similar exercises by drawing, using manipulatives, and/or having discussions with peers.

The NCC [56] has emphasised learning through exploration and discovery, collaboration and social interaction skills, and the use of information and communication technology to enhance learning. Additionally, the differentiation of instruction based on students' personal needs and developmental differences has been highlighted to support their diversity (i.e., personal needs and developmental differences) [56]. Our TM and how it was to be used in classrooms were designed to align with the NCC. Later, its prototype was tested in real Finnish classrooms to evaluate whether the design conforms with the NCC.

3. Design, Development, and Implementation of the X-is Tangible Manipulative (TM)

3.1. Design Objectives and Principles

X-is ('X is equal to') is an initial TM designed and developed as a learning tool for primary school students to learn the concepts of linear equation solving. For this purpose, a set of design principles (DPs) was established based on the theoretical background (Section 2) as well as the literature review and empirical results derived from our initial research [57] and design concept evaluation [58]. Previously, we conducted research in primary schools to evaluate existing manipulatives [57] and our manipulative design concepts [58] in terms of their practicality.

- DP1. *Promote understanding of equation solving:* A manipulative should assist students in learning equation solving by concretising the concepts of mathematical equivalence [21], different terms in an equation [22], and equation solving [23].
- DP2. *Be in agreement with school curriculum:* Finnish teachers of basic education plan their teaching based on the Finnish NCC. Therefore, a manipulative should conform to the NCC [56] to ensure its use in classrooms.
- DP3. *Support multimodality and languaging:* A manipulative should help students to link multiple representations of equation concepts and to express their mathematical thinking through various modes of meaning making [30,31].
- DP4. *Enable learning through discovery:* A manipulative should enable students to learn through their first-hand experience and provide them with appropriate guidance and scaffolding [48,49].
- DP5. *Assist social interaction:* A manipulative should encourage students to co-construct their knowledge through peer interaction while suppressing silent and individual activities [50–53].
- DP6. *Be suitable for diverse learners:* A manipulative should provide differentiation of instruction based on students' diversity [56] by assisting students who are at different achievement levels in learning equation solving.
- DP7. *Be easy to use:* An easy-to-use manipulative is more likely to be adopted in classrooms. According to our empirical research [57,58], the ease of use of a TM can be optimised through the following:
 - DP7.1. *Single point of interaction:* The input and output of a TM should occur at the same point of interaction (i.e., a 'co-located' design [16]) to allow students to manipulate physical objects and look at a GUI without moving their sight back and forth.
 - DP7.2. *Use of base-10 blocks as physical objects:* Base-10 blocks are widely used manipulatives for learning of number sense, place value, and operation in various countries, including Finland. All the schools that participated in our previous

studies [57,58] were familiar with the base-10 system. Thus, it can be conveniently used as physical manipulative objects to reduce students' cognitive friction [59].

- DP7.3. Straightforward user interface (UI): A simple UI enables teachers and students to use it with ease. Consequently, it saves time spent on its utilisation and prevents frustration.
- DP 8. *Be feasible for classroom and school practice:* A manipulative design that takes the following factors related to classroom and school practice into account is more likely to be adopted and used in classrooms:
- DP8.1. Affordability: Our empirical research [57,58] and previous studies [60–62] have revealed that schools' tight budgets have a highly negative impact on the acquisition of manipulatives. Therefore, an affordable manipulative is more likely to be acquired under this financial pressure.
- DP8.2. Practicality and convenience: According to our research [57,58] and that of others [60–62], time constraints, manipulative preparation and organisation issues, and limited storage space are among the possible hindrances to manipulative acquisition and utilisation. Based on our concept evaluation [58], the following properties can increase the practicality and convenience of manipulatives:
- A straightforward manipulative requires less time and effort spent on preparation, instruction, operation, and clean-up.
 - A portable manipulative can be easily circulated in the classroom and around the school.
 - A proper size and sensible number of parts allows for easy storage and prevents pieces from becoming lost or mixed up.
 - A manipulative should be compatible with Android tablets or iPads due to (1) the increasing number of these devices in Finnish schools as a result of the digitalisation of learning environments that is encouraged by the current NCC [56] and (2) the growing number of Finnish teachers and students who have these devices.
- DP8.3. Durability: According to our concept evaluation [58] and previous research [61], teachers are concerned that manipulatives could get broken when used in classrooms. The durability of manipulatives increases when there are no fragile or moving parts.
- DP8.4. Utility: Our concept evaluation [58] has revealed that high utility is one of the teachers' criteria for acquiring manipulatives. A manipulative that can be used for different grade levels or content areas is preferable. Its compatibility with schools' existing infrastructure and equipment is also important.

X-is was developed with the goal of meeting the above-mentioned design principles to ensure its success in classrooms. Sections 3.2–3.4 present the X-is system and the design principles that guided its design.

3.2. The Implemented Architecture

X-is is comprised of two parts: physical objects and a tablet application (DP 8.2, 8.4). The application working prototype was developed from 2018–2019 by a team of undergraduate and graduate students and their supervisor at the Faculty of Information Technology and Communication Sciences in close cooperation with the first author.

Our current solution is rather complicated in terms of components and set-up. The reasons for this are clarified in this section, while the associated limitations and possibilities for improvement are discussed in Section 5.4. The input interactions with the tablet application occur through the placement and removal of physical objects (Figure 1i) on a tablet screen (Figure 1ii) that functions as an external

display of a computer running the application (Figure 1iii), which was developed using the Unity development platform. The free software Spacedesk [63] was used to enable a wireless video and audio connection (Figure 1iv) between the computer and the tablet. An external USB web camera that supports a 720 p resolution (Figure 1v) is connected to the computer and positioned using a tripod (Figure 1vi) so that the physical objects can be seen on the tablet screen. Image recognition algorithms making use of OpenCV for Unity Library [64] use the web camera image to detect the objects' positions and their amounts. This detection is based on distinction of the objects' sizes and colours. According to the information derived by the recognition algorithms, the computer provides visual and audio outputs to the tablet.

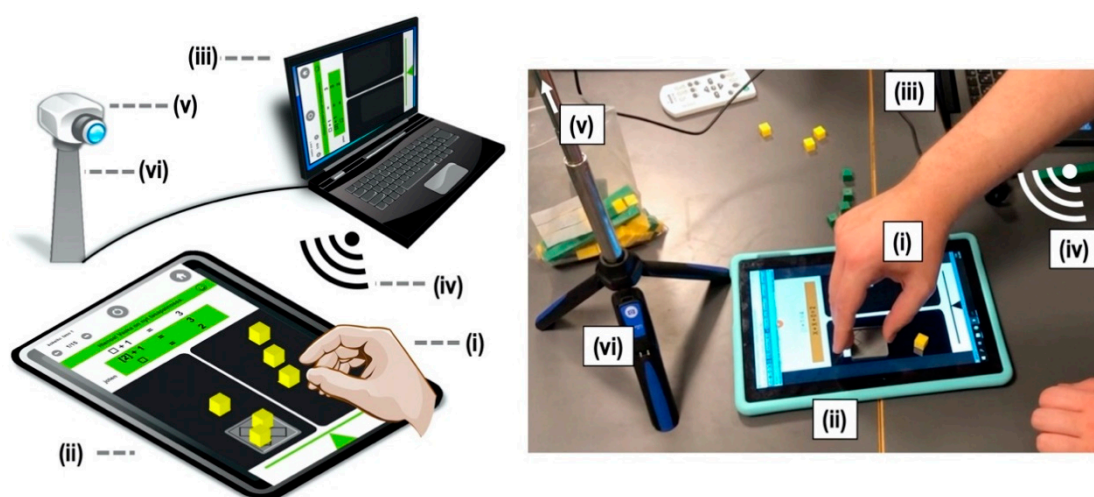


Figure 1. Architecture and components of X-is. (i) Placement of physical objects; (ii) Tablet screen; (iii) Computer; (iv) Wireless connection; (v) Web camera; (vi) Tripod.

The original conceptualisation involved the use of only an Android tablet with a minimum display size of 10 inches, which is large enough to enable two students to work at the same time (DP 5) and to place physical objects on the screen, combined with a web camera. However, during development, the running platform was changed to Windows 10 64-bit due to an issue with external devices running on Android. For an external web camera to work directly with an Android tablet, it must comply with a specification for external image devices. Android then recognises those devices and make them available to apps through its internal application programming interface (API), which is different on every version of Android and may break compatibility. However, OpenCV for the Unity Library was not able to access external image devices with the Android API used during the development period. Thus, a computer that runs Microsoft Windows or macOS and is able to connect the camera to the Unity Library was needed. In that fashion, an iPad running Spacedesk Software could also be used as a tablet because it works as an output device (i.e., secondary machine) and the actual processing happens in the computer (i.e., primary machine).

Another alternative set-up is to run the application directly on a tablet with a Windows 10 64-bit system, which is then connected to an external web camera (Figure 2). This would be more straightforward for teachers (DP 8.2). However, we did not have access to a tablet running Windows during our research.



Figure 2. Alternative set-up of the system with a tablet running Windows.

3.3. Object Tracking Alternatives

During development, various possibilities for object tracking were considered to determine the best balance between our target design principles and technological constraints. The initial design approach was to directly detect objects (i.e., types, amounts, and positions) on the tablet touch screen without requiring an external camera. This solution would not only enable seamless and direct interaction but also increase the product portability and practicality as well as decrease error factors. However, implementation proved to be unfeasible due to the limitations of the technology currently employed in touch screens. First, the vast majority of commercial touch screens currently available use projected capacitive sensing technology [65]. This technology can detect touch by capacitance variations in the screen surface, which occur, for example, with human finger contact but would not sense our plastic objects without conductive markers attached to them. Furthermore, it is not possible for the touch screens to distinguish features such as colour or size through touch. Second, the screen touch controller is optimised for finger contact and uses techniques such as ‘grip suppression’ and ‘palm rejection’ to filter out noise and undesired touches [66]. In our case, these would probably disable detection of stationary objects, limit the number of detectable objects, or merge them incorrectly as an adaptation to large fingertips.

Other tracking techniques—such as magnetic sensing (e.g., GaussSense [67]), near-field communication (e.g., Nintendo Amiibo [68]), and smart objects (e.g., MAL-Smart tiles [69])—would at least require embedding magnets, tags, or circuit boards to detectable objects. Embedment would be challenging in our case because a base-10 unit—our smallest object—is a 1 cm cube. These techniques would also increase the cost and fragility of our manipulative. Moreover, they typically require custom hardware for a specific purpose, thereby limiting utility. Neither optical tabletop systems (e.g., tanGible Augmented Interaction for Edutainment (GAINE) [70]) nor mixed reality tabletop systems (e.g., Augmented Reality (AR) enhanced tabletop system [44]) were possible because these typically require a large amount of stationary installation (e.g., computer, camera, projector, and screen) beneath or over the tabletop. Thus, tabletop systems are neither portable nor affordable. Another possibility was a vision-based tracking mixed reality system, where a mirror is placed in front of a tablet camera and enables the system to detect objects on a flat surface in front of the tablet (e.g., Ceibal Tangible (CETA) [14]). This system is inexpensive, portable, and would be able to detect our objects. However, its TUI input happens in front of a tablet, while its GUI output is displayed on a tablet screen (i.e., a ‘discrete’ design [16]). Thus, it does not allow for a single point of interaction, which is one of our most important design principles.

All in all, our selected object tracking solution enables the application system to meet many of our target principles (DP 7.1, 8.1–8.4). However, practicality and convenience are somewhat limited compared to the initial design approach.

3.4. Features and Interactions

X-is uses two types of physical objects (Figure 3a,b)—base-10 blocks representing constants (DP 7.2) and X-Boxes, specially designed for X-is, representing unknowns—for its input to assist students in recognising the distinction between different terms in an equation (DP 1). The application contains exercises divided into two levels for primary students in different grades (DP 8.4) to learn the concepts of linear equation solving using different strategies aligned with the Finnish NCC (DP 2). The goal of Level 1 is to solve equations by substituting values for the unknown so that both sides of an equation are equal (Video S1). The goal of Level 2 is to isolate a single unknown on one side of an equation and the constants on the other side by subtracting the same quantity of constants and unknowns from both sides of an equation (Video S2). This algebraic strategy was chosen because of its emphasis on the equivalence principle (DP 1). The balance model is used as a didactic model to assist students' understanding of linear equation solving [23]. Either side of the scale represents either side of the equation, while the movement of the scale (tilting or balance) represents the equality of the mathematical expressions on either side of the equation.

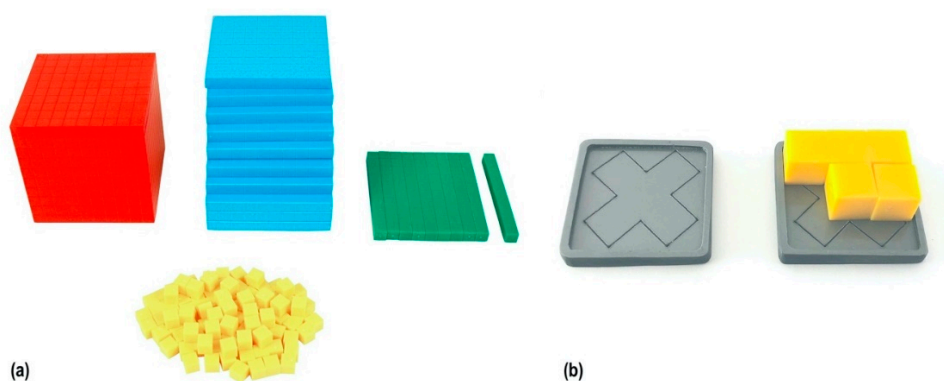


Figure 3. Physical objects used in X-is. (a) A base-10 block system, an existing and widely used manipulative, consisting of four different sizes and colours representing their individual place values: units (one's place), rods (ten's place), flats (hundred's place), and cubes (thousand's place). The system is familiar to teachers and students. Reprinted with permission [71]; (b) X-Boxes (left) specially designed for X-is, which can be used with base-10 units when solving Level 1 equations (right).

Each level starts with an introductory animation demonstrating how to model and solve an equation using the specific strategies for that level (DP 4). After that, there are exercises with gradually increasing difficulty. First, students have to model the given equation by placing base-10 blocks and X-Boxes on either side of the scale on the tablet screen (Figures 4a and 5a). This task requires students to translate the symbolic representation of the equation into a physical representation, thereby developing their representational fluency. Then, they have to solve the equation (i.e., find the value of an X-Box) by adding base-10 blocks to the X-Boxes to balance the scale in Level 1 (Figure 4b) or by removing the same number of physical objects from both sides of the scale to maintain its balance in Level 2 (Figure 5b). Solving the equations by balancing the scale emphasises the equal sign as a relational rather than operational symbol (DP 1).

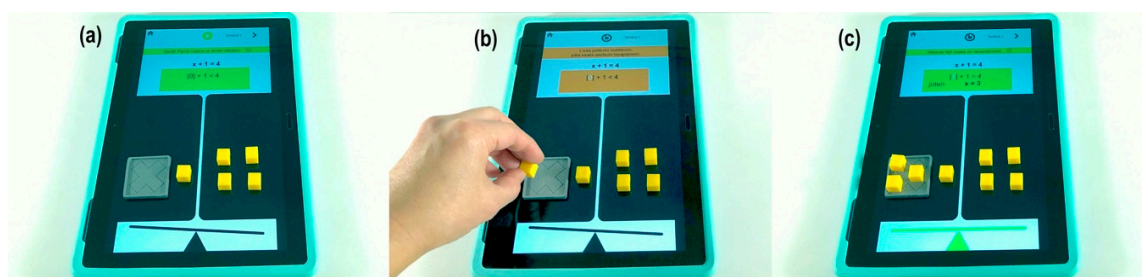


Figure 4. The steps to complete the Level 1 exercise $x + 1 = 4$. (a) Modelling the given equation; (b) Solving the equation by substituting the unknown's value; (c) The solved equation and balanced scale.

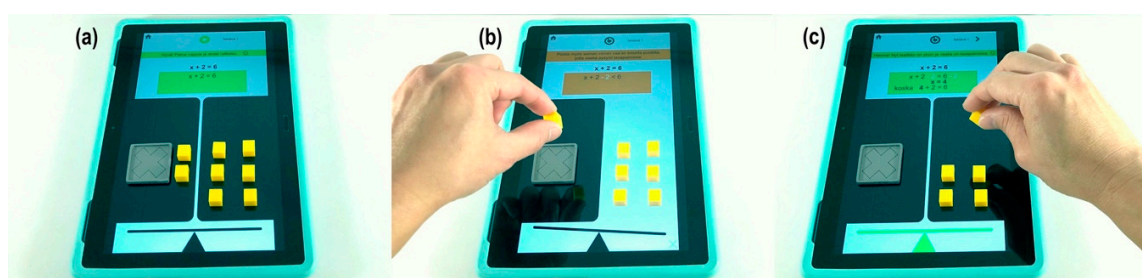


Figure 5. The steps to complete the Level 2 exercise $x + 2 = 6$: (a) Modelling the given equation; (b) Solving the equation by doing the same operation on both sides of the scale; (c) The solved equation and balanced scale.

The X-is TUI is comprised of multimodal inputs and outputs to help students link multimodal representations of equations and express their mathematical thinking through various modes of meaning making (DP 3). Moreover, the application provides instant scaffolding (i.e., guidance and feedback) in the form of pictures, text, mathematical symbols, and sounds to guide students on what to do or to inform them of the correctness of their actions (DP 4). Instead of tapping and scrolling a touch screen, students interact with the application by placing physical objects on or removing them from a working zone (Figure 6ii) above a scale (Figure 6i). The working zone is separated into the left and right sections, which are reserved for each side of the scale. When the 'weights' are unequal, the scale tilts towards the heavier side; in contrast, when both sides of the scale are equal, the scale is balanced. To emphasise that an equation is solved, in addition to being balanced, the scale turns green, which is accompanied by a ding sound and a textual compliment (Figures 4c and 5c). The given equation is situated at the top of a mathematical symbol zone, while the math sentence for the current equation-solving process is presented in a math expression window below the given equation (Figure 6iii). A text window provides textual instructions, guidance, and feedback (Figure 6iv). For example, when solving the equation $x + 2 = 6$, after students have removed two base-10 units from the left side of the scale, the text window provides the textual message: 'Remove the same quantities from the other side of the scale to keep the scale balanced!' The right side of the working zone blinks, and the scale tilts towards the right side (Figure 6).

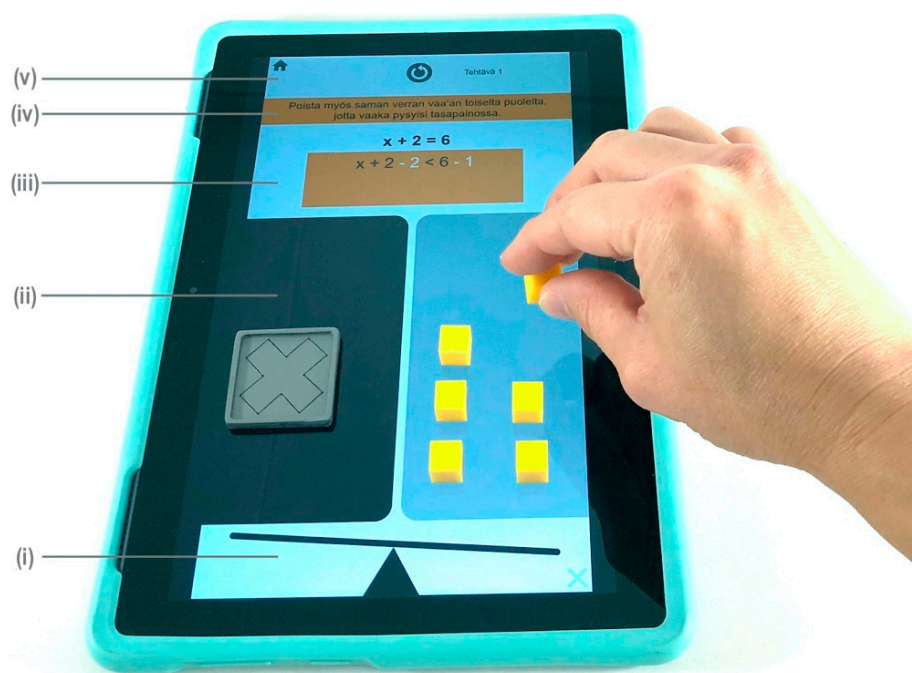


Figure 6. GUIs on the screen from bottom to top. (i) Scale; (ii) Working zone; (iii) Mathematical symbol zone; (iv) Text window; (v) Navigation and operation menu.

X-is employs a user-friendly UI to enable students' natural interactions (DP 7.3). The overall graphic design is clear and simple. The screen contains only the necessary elements to avoid students' distraction from their learning process. The navigation and operations menu uses minimal, easy-to-understand graphical icons (Figure 6v). Concise and clear sentences appear one at a time. Button tapping is kept to a minimum. After a delay, if the physical objects on the screen have not been moved, the application responds according to the last manipulation (rather than requiring the student to tap in the affirmative). Occasionally, students are required to tap buttons on the screen, for example, to start solving the equation after modelling it or to proceed to the next exercise after solving an equation.

3.5. Prototyping

We initially intended to present a working (fully interactive) prototype in the classroom to validate its technical feasibility and evaluate its potential. However, due to time constraints, the development team was only able to get Level 1 of the application fully functional. This made it problematic to use the working prototype as a task-oriented prototype [72] for classroom evaluation because the evaluation requires students to learn to solve equations at both levels. Moreover, at the moment, the performance of the image recognition algorithm depends largely on external factors, including environmental lighting conditions and the position setting of the external camera during system calibration. If the current prototype were used in a real classroom setting, it is likely that the image recognition may not function reliably. Thus, another prototype that allows students to uninterruptedly perform both levels of equation solving was required.

Wizard of Oz [72] is a rapid prototype that was created using Microsoft PowerPoint for the classroom evaluation. The prototype contains all the needed UIs for Levels 1 and 2. To create the illusion of a functioning manipulative, while the user interacted with physical objects and the tablet (i.e., secondary machine), the researcher (i.e., the Wizard) unnoticeably operated a PowerPoint slideshow by hand on a computer (i.e., primary machine) to respond to the user's actions.

4. Classroom Evaluation of X-is

4.1. Participants

We recruited participants from primary schools and lower secondary schools in southern Finland, and the equity of the Finnish education system [73] allowed for convenience sampling. Ethical practices were assured throughout the study in accordance with the European Code of Conduct for Research Integrity [74]. For instance, the participants participated in the study on their own wills, informed consent was obtained from all participants or their legal guardians prior the study, and the participants' privacy was protected. The student participants included 12 fourth graders (ages 10–11), 12 fifth graders (ages 11–12), 35 seventh graders (ages 13–14), and 30 eighth and ninth graders (ages 14–16). Additionally, one fourth- and one fifth-grade class teacher (teaching experience: 10–11 years; moderate experience using physical and virtual manipulatives), three lower secondary school mathematics teachers (teaching experience: 10–27 years), and a special education teacher (teaching experience: 4 years in primary and lower secondary schools) participated in the study. The students at each grade level had mixed achievement in mathematics.

4.2. Research Design and Procedures

The classroom evaluation was conducted to examine the potentials of X-is in classrooms in terms of leaning achievement, learning support, and usability. The convergent design (previously known as concurrent or parallel design) of the mixed methods approach [75] was used as the strategy of inquiry. First, both qualitative and quantitative data were collected simultaneously from students and teachers using various methods (see Table 1). The purposes and research design of the classroom evaluation are outlined in Table 1.

Table 1. Purposes and research design of the classroom evaluation.

Purposes	Methods			
	Class Intervention	Paper-Based Test	Questionnaire and Interview	Thinking Aloud
Learning Achievement		<ul style="list-style-type: none"> Both groups of fourth graders ($n = 12$) Both groups of fifth graders ($n = 12$) Seventh graders ($n = 35$) Eighth and ninth graders ($n = 30$) 		<ul style="list-style-type: none"> X-is group of fourth graders ($n = 6$) X-is group of fifth graders ($n = 6$)
Learning Support	<ul style="list-style-type: none"> Both groups of fourth graders ($n = 12$) Both groups of fifth graders ($n = 12$) Fourth- and fifth-grade class teachers ($n = 2$) 		<ul style="list-style-type: none"> X-is group of fourth graders ($n = 6$) X-is group of fifth graders ($n = 6$) Fourth- and fifth-grade class teachers ($n = 2$) Lower secondary school mathematics teachers ($n = 3$) Special education teacher ($n = 1$) 	<ul style="list-style-type: none"> X-is group of fourth graders ($n = 6$) X-is group of fifth graders ($n = 6$)
Usability	<ul style="list-style-type: none"> Both groups of fourth graders ($n = 12$) Both groups of fifth graders ($n = 12$) Fourth- and fifth-grade class teachers ($n = 2$) 		<ul style="list-style-type: none"> X-is group of fourth graders ($n = 6$) X-is group of fifth graders ($n = 6$) 	<ul style="list-style-type: none"> X-is group of fourth graders ($n = 6$) X-is group of fifth graders ($n = 6$)

Note: Fourth and fifth graders were divided into two groups: the paper-based intervention and the X-is intervention.

The fourth and fifth graders, who had never received any formal instruction regarding equation solving before this study, participated in one of the languaging-based class interventions, either paper-and-pencil or X-is. After the intervention, both groups individually completed a

paper-based test. Typically, Finnish students of basic education (first to ninth grades) only obtain their academic knowledge in school. For this reason, it could be assumed that the fourth and fifth graders in this study had no prior knowledge of equation solving. Therefore, a pre-test was unnecessary, and only the paper-based test was conducted after the intervention to evaluate the learning achievement of new knowledge. After the test, each X-is student was asked to think aloud as they solved equations using X-is. At the end, a questionnaire, accompanied by an interview, was completed with individual students from the X-is groups.

The seventh through ninth graders took the same paper-based test, but did not participate in any class intervention. Prior to the test, they had received several equation-solving lessons (their first, second, or third equation course respectively) as a part of their normal school curricula, thereby serving as a comparison group (i.e., students in traditional classrooms). For the same reason mentioned in the previous paragraph, it could be assumed that the seventh through ninth graders in this study had only obtained their knowledge of equation solving in school. Thus, a pre-test was unessential, and the paper-based test would evaluate their learning achievement after attending traditional classrooms. This research design was employed, because the Finnish basic education is organised according to the NCC [56], where algebraic linear equation solving is only taught in Grades 7–9. Therefore, a control group from fourth and fifth grades that had received traditional classroom instruction in algebraic equation solving could not be found. Students from lower secondary schools in the same area were recruited as the comparison group to ensure the participants' homogenous socioeconomic and academic background.

A lesson plan and a worksheet developed for the class interventions were provided to the fourth- and fifth-grade class teachers before the interventions. After the interventions, a questionnaire and an interview were administered for both class teachers. To discover the possible utilisation of X-is in other classroom contexts, X-is was also evaluated by three lower secondary school mathematics teachers and a special education teacher. First, the X-is Wizard of Oz prototype was used to demonstrate to the teachers how to solve equation exercises at both levels (see Section 3.4). Then, the teachers tried X-is. Next, they were asked to complete the same teacher questionnaire and interview.

Finally, the collected data were concurrently analysed before being compared and combined to cross-validate the findings and holistically understand the results with regard to the following aspects:

- To determine the impacts of the languaging-based instruction (with or without X-is) on students' learning achievement based on the results from paper-based test and thinking aloud sessions.
- To discover whether and how X-is promoted students' learning (i.e., their understanding of equation-solving concepts, languaging, and learning through discovery and social interaction) based on the results from class interventions, student and teacher questionnaires and interviews, and thinking aloud sessions.
- To investigate how students perceived the usability of X-is and how well they could use it based on the results from class interventions, student questionnaire and interview, and thinking aloud sessions.

4.3. Data Collection and Analysis

4.3.1. Class Intervention

The class interventions were carried out to compare between the two languaging-based instructional conditions—paper-and-pencil ($n = 12$) or X-is ($n = 12$) interventions—with regard to their usefulness in terms of supporting students' learning. Each fourth and fifth grader participated in one 45-min class intervention led by their class teachers in their classrooms during regular school hours. For each condition, the teachers divided their students into three pairs—one high- and one medium-attaining student, two medium-attaining students, and one medium- and one low-attaining student—based on the students' mathematics achievement throughout the school years. The pair

assignments also took into account students' relationships and ability to work together to ensure collaboration within pairs. Pair work was used for the intervention instead of group work to reduce group collaboration development time, while the combination of students in each pair was employed to assure their learning through social interaction [51,76].

At the beginning of the intervention, the teachers taught the concepts of equations, equivalence, the unknown, and equation solving (by substituting values of the unknown in the fourth-grade interventions and by doing the same operation on both sides of an equation in the fifth-grade interventions) to the entire class for 10–15 min. After that, the students worked in pairs to learn to solve equations under their teacher's supervision. During the pair work, the paper-and-pencil groups used only the provided worksheet (Figure 7a), whereas the X-is groups used the X-is Wizard of Oz prototype in addition to the worksheet (Figure 7b).

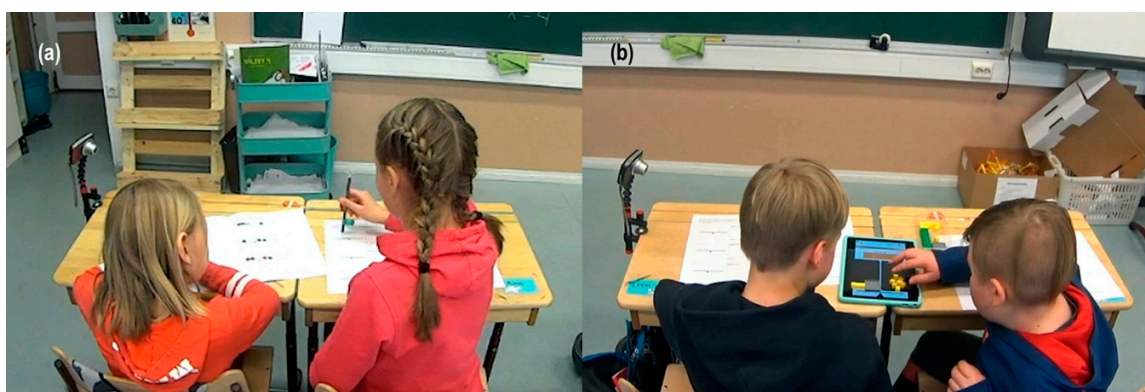


Figure 7. Pair work during the classroom interventions. (a) Paper-and-pencil condition; (b) X-is condition.

The worksheet was designed to promote students' understanding of equation concepts through translation and connections between multiple representations (i.e., verbal and written, pictorial, and mathematical symbolic) of equations. It was developed based on mathematics textbooks used in Finland and then evaluated by the third and fourth authors, who are experienced teacher educators. The worksheet contained eight equations presented in one of the following formats: $x + A = B$, $A + x = B$, $A = x + B$, or $A = B + x$, where A and B are positive integers and x is the unknown.

All 12 pair work sessions were video recorded from two different angles (530 min in total) to warrant well-captured data. The video materials were transcribed in terms of the students' actions (i.e., who did what, with whom). Due to the noisy background of the classrooms, the students' dialogues could not be transcribed. The pair work analysis particularly focused on how each instructional condition supported students' on-task peer communication (i.e., multi-representation translation, equation solving, and providing the unknown value) regarding directions and types. Communication directions and types were used as indicators of languaging and peer interaction, which encouraged learning through social interaction. Students' off-task communication (e.g., chitchatting, laughing, or cleaning up physical objects from the tablet screen before proceeding to the next exercise) and communication with the teacher were not included in the analysis. The video transcription was analysed using a qualitative deductive content analysis method [77]. A categorisation matrix was developed, and all the data were coded according to the categories in the matrix. Communication directions were categorised into one-way or two-way communication, while communication types were categorised into verbalisation, physical actions, or verbalisation and physical actions. Table 2 lists the coding categories with examples.

Table 2. Categories for communication directions and types.

Category	Description	Examples
Communication Directions		
One-way	Sending information through speaking, writing, or gestures without response from peer	<ul style="list-style-type: none"> • Speaking out loud while doing something • Asking for help, no peer response • Giving advice, no peer response
Two-way	Sending and receiving information through speaking, writing, or gestures	<ul style="list-style-type: none"> • Discussing or negotiating • Giving and taking advice/assistance • Manipulating X-is together
Communication Types		
Verbalisation	Communicating through speech	Asking or discussing
Physical actions	Communicating through gestures	Pointing or showing
Verbalisation and physical actions	Communicating through speech and gestures	Manipulating X-is and explaining at the same time

Then, the coded data were merged into communication episodes. One episode is a unit of complete actions for a specific purpose, for example, a discussion about how to solve an equation or the whole process of solving the equation. In total, 287 episodes of on-task peer communication were discovered. The frequency of individual episodes was counted (i.e., quantified) and divided into categories for descriptive statistical analysis. A Pearson's chi-squared test was performed to statistically investigate the relationship between instructional conditions and peer communication.

Additionally, the video data were analysed to seek evidence of support for each instructional condition for students' learning (i.e., understanding of equation-solving concepts, languaging, and learning through discovery and social interaction) and the usability of X-is. The discovered evidence was used to complement the findings from the teachers' questionnaires and interviews.

4.3.2. Students' Paper-Based Tests

The same 45-min paper-based test (Appendix A) was administered for all students to compare the learning achievement of students in languaging-based intervention conditions (the fourth and fifth graders in the paper-and-pencil and X-is groups: $n = 24$) to that of students in typical classrooms (the seventh through ninth graders: $n = 65$). The test was almost identical to the intervention worksheet. The only difference was that the test contained only six equations. Students could earn a maximum of three points for each equation for the correct representation translation, equation solving, and value of the unknown, for a total score of 18. Cronbach's alpha of the test was 0.82. As it is appropriate for small sample sizes, the Mann–Whitney U test was used to investigate the difference in students' learning achievement for languaging-based intervention groups (with or without manipulatives) and the comparison groups. The learning achievement difference between the two intervention groups (paper-and-pencil and X-is) and between each intervention group and the comparison groups was not statistically analysed due to the sample size being too small.

4.3.3. Students' Thinking Aloud

Thinking aloud [78] was conducted to assess the usability of X-is and its contribution to students' learning process and achievement. Only the students in the X-is groups ($n = 12$) were individually asked to model and then solve one to two equations and at the same time verbalise their actions, thoughts, and opinions. When students faced difficulties, they were provided with minimal hints and guidance to assist them in proceeding with the task.

All 12 thinking aloud sessions (18 min in total) were video recorded and transcribed. The video transcription was analysed using a qualitative inductive content analysis [77]. The data were open coded and then grouped into sub-categories, which were further grouped into categories and the main categories related to the research focus.

4.3.4. Student Questionnaires and Interviews

All the X-is students ($n = 12$) individually completed a questionnaire and participated in a face-to-face interview to evaluate the perceived usefulness and usability of X-is. A 4-point Likert-type scale (ranging from 1 = fully disagree to 4 = fully agree) was adapted from the Usefulness, Satisfaction, and Ease of Use (USE) questionnaire [79], as it is concise and contains the usability dimensions that we investigated. We only used one item from each sub-scale of the USE questionnaire to build our questionnaire, since the selected items best describe each studied dimension. When completing the face-to-face questionnaire, we also verbally other listed items that belonged to the same usability dimension for the participants. The scale was comprised of four items assessing four factors: 'X-is is easy to use' (ease of use), 'It is pleasant to solve equations with X-is' (enjoyment), 'X-is helps me to understand how to solve equations' (usefulness), and 'I would like to solve equations with X-is' (intention for future use). Cronbach's alpha of the scale was 0.69. The interview was conducted at the same time as the questionnaire. The students were asked to give the reason for their response to each questionnaire item. They were also asked for suggestions regarding how X-is could be improved.

All 12 interviews (102 min in total) were video recorded and transcribed. Frequencies of the questionnaire responses were used to determine students' perceptions of X-is. The video transcription was analysed using a qualitative inductive content analysis method. The qualitative findings were used to complement the quantitative findings.

4.3.5. Teacher Questionnaires and Interviews

All teachers ($n = 6$) completed a questionnaire (Appendix B) and a face-to-face interview to assess how teachers perceived the utility of X-is compared to paper-and-pencil instruction (i.e., using worksheets or textbooks). Three 4-point Likert-type scales (ranging from 1 = not at all to 4 = very well) were designed specifically for evaluation of how X-is compares to paper-and-pencil instruction regarding support for:

- Students' understanding of equation-solving concepts (three items; X-is: $\alpha = 0.90$, paper-and-pencil instruction: $\alpha = 0.90$),
- Students' languaging (five items; X-is: $\alpha = 0.71$, paper-and-pencil instruction: $\alpha = 0.67$), and
- Students' learning through discovery and social interaction (three items; X-is: $\alpha = 0.53$, paper-and-pencil instruction: $\alpha = 0.81$).

Although, the Cronbach's alpha value ($\alpha = 0.53$) for the responses to students' learning through discovery and social interaction for X-is was below the generally acceptable level [80]. It is noteworthy that the low alpha value could be due to the small number of items ($N = 3$) in the scale and the small number of the participants ($N = 6$) [80]. Moreover, the Cronbach's alpha value ($\alpha = 0.81$) for the responses to the same scale for the paper-and-pencil instruction was acceptable [80]. Thus, the responses to the scale for X-is were used and interpreted with caution.

During the session, the teachers were required to respond to the questionnaire items as well as provide an explanation for each response. In addition, they were also asked about appropriate grade levels and learning-attaining levels in regard to working with X-is, suggestions for improving X-is, and differences between physical (TUI) and digital (GUI) block manipulation. All six interviews (175 min in total) were audio recorded and transcribed. Descriptive statistics (e.g., frequencies, cumulative sums, and cumulative means) of the questionnaire responses were used to present teachers' perceptions of X-is. A Wilcoxon matched-pairs signed rank test was conducted to determine the difference between X-is and paper-and-pencil instruction according to the teachers. The audio transcription was analysed using a qualitative inductive content analysis. The qualitative findings were used to complement the quantitative findings.

5. Results and Discussion

We next report and discuss the findings in three sections according to our research questions regarding how well X-is promotes learning achievement, learning support, and usability. Then, we reflect on the limitations of the research and give recommendations for future research.

5.1. Learning Achievement

5.1.1. Students' Paper-Based Tests

The Mann-Whitney U test was performed to determine the differences between the total test scores for the languaging-based instructional intervention (with paper-and-pencil or with X-is) groups and the comparison groups. Figure 8 illustrates that there was no significant difference in the test scores for the fourth and fifth graders who received the languaging-based instructional intervention ($M = 11.09$, $SD = 3.69$) compared to the seventh graders in the comparison group ($M = 12.29$, $SD = 5.28$; $U = 314$, $p = 0.10$). However, there was a statistically significant difference between the total test scores for the intervention groups compared to the eighth and ninth graders ($M = 15.57$, $SD = 2.40$; $U = 95$, $p < 0.001$) and for the seventh graders compared to the eighth and ninth graders ($U = 332.5$, $p = 0.01$). In addition, we examined students' low achievement on the test. Passing the test required a student to earn 50% of the maximum score (cut-off score = 9/18). A similar portion of the students in the intervention groups and the seventh graders failed the test (25% and 26%, respectively), whereas none of the eighth or ninth graders did.

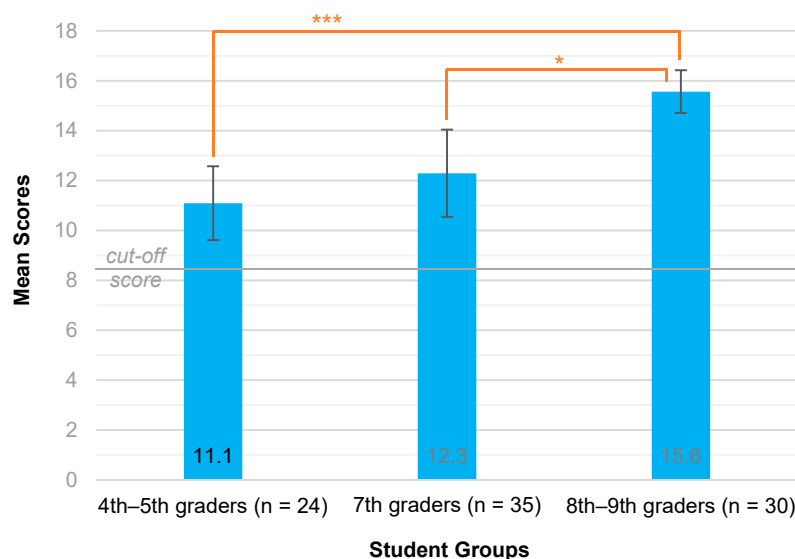


Figure 8. Mean total scores (out of 18) for the intervention and comparison groups. Error bars show 95% CI = Mean \pm (1.96 SE). * $p < 0.05$, *** $p < 0.001$.

To determine the impact of each languaging-based instructional condition on students' learning achievement, we examined the average total test scores of the students in both intervention groups. Overall, the paper-and-pencil group ($M = 11.29$, $SD = 4.08$) and the X-is group ($M = 10.90$, $SD = 3.44$) presented a rather similar test performance. We also investigated the strategies that each intervention group used on the test when solving equations correctly to discover whether the instructional conditions influenced students' use of taught strategies (i.e., reasoning for the unknown in the fourth-grade intervention and doing the same operation on both sides in the fifth-grade intervention) vs other strategies. The analysis did not include any situations in which students arrived at the correct answer without providing an explanation or their steps for equation solving or in which their used strategies could not be identified. Figure 9 shows that the X-is group (31/41) was more likely to use the strategies

taught during the interventions for solving equations correctly than the paper-and-pencil group (24/42). Moreover, two-thirds (8/13) of the X-is group used the taught strategies to solve equations correctly at least once, whereas only half (6/12) of the paper-and-pencil group did.

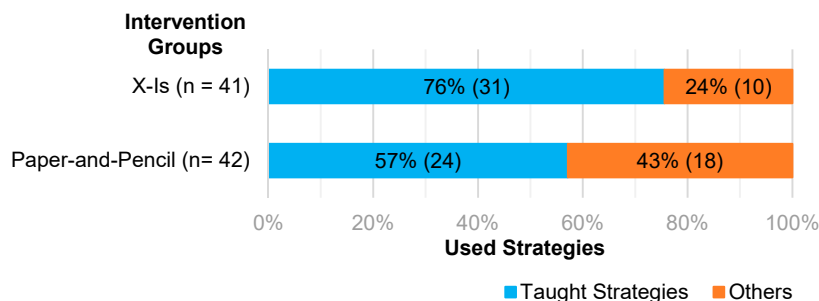


Figure 9. Percentage (number of cases in parentheses) of different strategies used to solve equations correctly on the test by instructional condition.

5.1.2. Students' Thinking Aloud Sessions

In addition to the test analysis, we analysed the thinking aloud transcriptions to examine learning achievement of the X-is group. First, we investigated the students' representational fluency (i.e., how well they could model the math symbolic equations with X-is physical objects), which is an indicator of their equation concept understanding. All students were able to use physical objects to model the given equations successfully. Ten students completed the equation modelling on their own in under 15 s, whereas two students required more time and some guidance from X-is or us. We further examined how well the students could solve the given equations with X-is. All of them were able to solve the equations correctly. Four of them also provided clear argumentation to support their equation-solving process, which indicated their understanding of the equivalent concept and equation-solving principles. For example, after almost 2 min of equation solving, a low-attaining fourth grader was able to explain how she solved the equation by reasoning for the unknown (Figure 10).

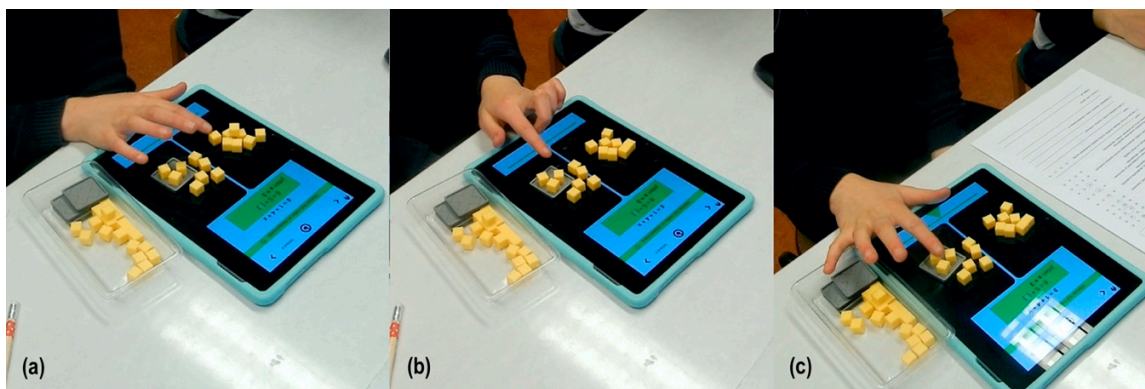


Figure 10. Student solving an equation ($8 = 1 + 4 + x$) at Level 1 (reasoning for the unknown). (a) 'Here are eight' (pointing at eight base-10 units on the left side of the scale); (b) 'And here has to weigh the same' (pointing at the right side of the scale); (c) 'You have to add [base-10 units] here, so that they [units on the right side] are altogether eight' (pointing at three base-10 units inside the X-Box on the right).

Figure 11 shows how a medium-attaining fifth grader verbalised how he solved the equation by doing the same operation on both sides of an equation step by step. The student provided reasons supporting his actions.

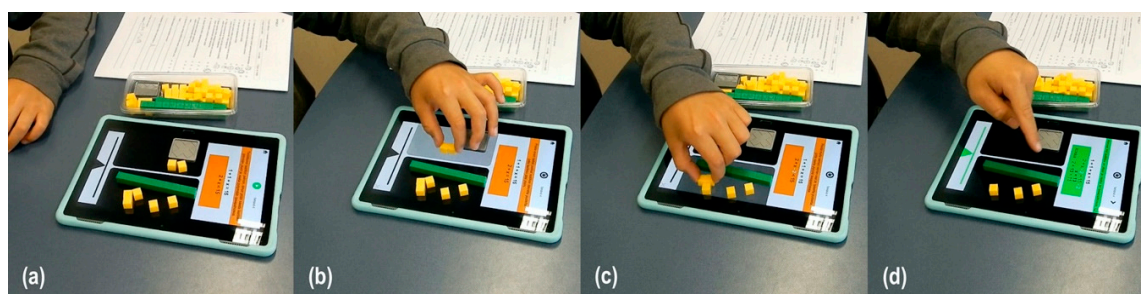


Figure 11. Student solving an equation ($1 + 1 + x = 15$) at Level 2 (doing the same operation on both sides of an equation). (a) Correct modelling of the equation; (b) ‘You have to take two away from here, so that x will stay alone’ (taking two base-10 units from the left side of the scale); (c) ‘Then you have to take also two from here’ (taking two base-10 units from the right side of the scale). ‘Because if you take [blocks] from one side, then you have to take [blocks] also from the other side’; (d) ‘So, here is 13, which is equal to x ’ (pointing at the rest of the blocks on the right).

Hight-attaining fourth and fifth graders used X-is to model and solve an equation correctly within 30 s. It is worth mentioning that the fourth grader demonstrated that he could reason for the unknown by himself without X-is. During the thinking aloud, he calculated the value of the unknown mentally before manipulating X-is.

5.1.3. Discussion of Learning Achievement

In summary, the paper-test result analysis demonstrate that languaging-based instruction (with paper and pencil or with X-is) had a significantly positive impact on equation learning achievement of the fourth and the fifth graders, who had received one intervention lesson, compared to the seven-grade comparison group, who had received approximately 10 normal equation-solving lessons. These quantitative findings suggest that the experimental groups benefited from the intervention instruction, thereby encouraging further development of languaging-based instruction and X-is to support learning equation solving.

The test performance of the X-is students illustrates students’ embodied cognition in the absence of X-is, thereby indicating their independence from the manipulative [9]. A favourable impact of X-is on students’ learning performance and understanding of equation solving was evidenced by (1) students’ likelihood to use the strategies taught during the interventions to solve the equations correctly, (2) their ability to model and solve equations correctly during the thinking aloud sessions, and (3) their clear argumentation to support their equation-solving process. Particularly, their argumentation indicated that they had a good understanding of mathematical equivalence (e.g., the equal sign as a relational symbol) and the equation-solving process. The evidence that X-is is advantageous for students’ learning achievement corroborates the findings of earlier work [14,45,81] in tangible technology-enhanced mathematics learning. Moreover, the findings from the thinking aloud data suggest that X-is facilitated the achievement of equation learning (i.e., modelling and solving equations successfully) among diverse students, particularly low and medium achievers. These findings are consistent with those of Pires et al. [14], who found that children with no proficiency in number combinations benefited more from their TM compared to their virtual manipulative.

5.2. Learning Support

5.2.1. Class Intervention

The video data reveal that, during the pair work sessions, the students mostly concentrated on completing their tasks. Off-task activities were rarely observed. Generally, fourth graders in both conditions could work out the problems on their own. The teacher’s assistance was needed mainly

for translation of the equations in word problems into pictures and mathematical symbols. For the fifth-grade interventions, paper-and-pencil students required more help from the teacher than their X-is peers. The teacher assisted paper-and-pencil students in solving equations by doing the same operation on both sides and translating the equations presented as word problems. X-is students mainly needed the teacher's support for the translation of the equations presented as word problems. Overall, paper-and-pencil students worked silently and separately on their own worksheets at different paces. Thinking aloud while writing, discussing, giving or asking advice, and looking at what the groupmate doing were occasionally observed. In contrast, the X-is group usually manipulated X-is together or took turns (i.e., alternately manipulated X-is and watched when their groupmate manipulated the manipulative). Thinking aloud, discussion, and giving and taking advice were usually observed. After manipulating X-is, each student silently recorded their equation-solving processes on their own worksheet. Most of them also looked at the math sentences on the tablet while recording their work.

Table 3 presents the frequencies and percentages of peer communication episodes regarding directions and types of communication observed during the pair work. It should be noted that all of the students participating in the interventions were unfamiliar with languaging, particularly verbal and written. From the total of 287 observed episodes, most of the peer communication happened during the X-is group's pair work (70%). The average numbers of peer communication episodes were 14 episodes/pair ($SD = 7.7$) for the paper-and-pencil group and 34 episodes/pair ($SD = 13.9$) for the X-is group. Regarding communication directions, most of the paper-and-pencil group communication (60%) was one-way, while most of the X-is group communication (70%) was two-way, indicating increased peer interaction among the latter. In terms of communication types, most of the paper-and-pencil group communication (73%) was verbal followed by verbal and physical (19%), while the verbal communication (48%) and verbal and physical communication (44%) occurred at similar rates for the X-is group. We further examined the types of communication used for each communication direction. Both instructional conditions used mainly verbal communication, particularly, thinking aloud during one-way communication (78% of paper-and-pencil group and 92% of X-is group). For the paper-and-pencil group, most of the two-way communication was verbal (65%) followed by verbal and physical (35%). In contrast, for the X-is group, most of the two-way communication was verbal and physical (63%) followed by verbal (30%). It should be noted that two-way physical communication (7%) was only observed in the fourth-grade X-is intervention group. This kind of communication happened when students used X-is to model or solve equations together without talking to each other. The Pearson's chi-squared test revealed that there were statistically significant associations for instructional conditions with students' peer communication directions [$\chi^2 (1, N = 287) = 23.15, p < 0.001$] and types [$\chi^2 (2, N = 287) = 17.13, p < 0.001$].

Table 3. Observed frequencies and percentages of peer communication episodes regarding directions and types by instructional condition.

Communication	Paper-and-Pencil	X-is	Total
	<i>n</i> (%)	<i>n</i> (%)	<i>n</i> (%)
Directions (<i>N</i> = 287)			
One-way	51 (17.8)	60 (20.9)	111 (38.7)
Two-way	34 (11.8)	142 (49.5)	176 (61.3)
Types (<i>N</i> = 287)			
Verbalisation	62 (21.6)	97 (33.8)	159 (55.4)
Physical actions	7 (2.4)	16 (5.6)	23 (8.0)
Verbalisation and physical actions	16 (5.6)	89 (31.0)	105 (36.6)

5.2.2. Student Questionnaires, Interviews, and Thinking Aloud Sessions

Most of the students (10/12) in the X-is intervention group felt that X-is assisted them in understanding equation solving because of its TUI (i.e., allowing physical input and providing

multimodal output), scaffolding (e.g., the tilting scale and changing background colours of the math expression window), and the balance model used. Pointing at the math expression window, one medium-attaining fifth grader stated, 'From here, you can see what has been done. You can see also from the scale', (pointing at the unbalanced scale), 'whether there is too much or too little. So it helps'. According to a low-attaining fifth grader, 'Being able to put and move these blocks with my hands is better for me and somehow I understand better'.

It is noteworthy that only one high-attaining and one medium-attaining fourth grader disagreed on the supportiveness of X-is because they were able to solve equations without X-is. Moreover, the fifth graders had a higher level of agreement (five strongly agreed, one agreed, and none disagreed or strongly disagreed) than the fourth graders (one strongly agreed, three agreed, and two disagreed or strongly disagreed) regarding the learning support provided by X-is. A possible reason for this might be that the reasoning for the unknown strategy (Level 1) was rather easy for fourth graders, while the strategy for doing the same operation on both sides (Level 2) was appropriately challenging for fifth graders.

Some students found new ways to use X-is in addition to the original intention. Students who were able solve equations by themselves pointed out that they first solved an equation and recorded their solution on the worksheet, and after that, they checked their answer using math sentences in the math expression window on the tablet screen. Thus, the math sentences worked as their answer key instead of as step-by-step scaffolding. During the thinking aloud session, a medium-attaining fourth grader presented her own strategy for how to use X-is by first separating base-10 blocks into spatial groups to find the value of the unknown (Figure 12).

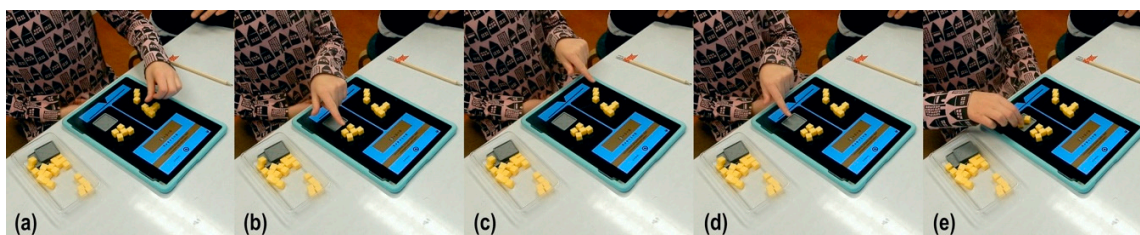


Figure 12. How one student solved an equation ($8 = 1 + 4 + x$) using her own invented strategy. (a) 'First, I move them here' (moving five base-10 units on the left side of the scale further away from three base-10 units on the same side); (b) 'So that they' (pointing at the five separate base-10 units on the left) 'are equal to what is here' (pointing at five base-10 units on the right); (c) 'Then I look at here' (pointing at three base-10 units on the bottom of the left.); (d) 'They are the same [number of units] that I have to put here' (pointing at the empty X-Box on the right); (e) The student then added three base-10 units to the X-Box.

5.2.3. Teacher Questionnaires and Interviews

Based on the questionnaire, the teachers clearly rated X-is as being better at supporting students' equation concept understanding, languaging, and learning through discovery and social interaction than paper-and-pencil instruction (Table 4). A Wilcoxon matched-pairs signed rank test also indicated that these differences were statistically significant ($Z = 2.10, p = 0.04$).

Similar to the students, all six teachers agreed that X-is promoted students' overall understanding of equation-solving concepts better than paper-and-pencil instruction. They pointed out that X-is concretises the equation concepts by allowing students to learn by doing and by providing scaffolding through the dynamic tilting scale and math expression window. For example, one lower secondary school mathematics teacher said, 'I think the manipulative supports students' understanding of equation concepts very well. It is very concrete. Students can try it with their own hands and see it with their own eyes. It is active learning'. The fifth grade class teacher also observed that during the interventions, students working with X-is concentrated more on the process of isolating x by doing

the same operation on both sides of the scale instead of just calculating the value of the unknown in their heads.

Table 4. Teachers' ratings of the learning support of X-is compared to paper-and-pencil instruction.

Scales	Cumulative Sum Mean			Z
	Scale [min, max]	X-is	Paper-and-Pencil Instruction	
Supports students' understanding of equation-solving concepts	[3, 12]	11.0	7.7	2.10 *
Supports students' languaging	[5, 20]	17.3	12.8	2.10 *
Supports students' learning through discovery and social interaction	[3, 12]	10.5	6.5	2.10 *

Note: * $p < 0.05$.

All the teachers believed that X-is better encouraged students to express their mathematical thinking compared to paper-and-pencil instruction, particularly through physical and mathematic symbolic representations. They highlighted that physical block manipulation and math sentences in the math expression window contributed to these physical and symbol representations, respectively.

"The manipulative is very action based and visual. I would say that these help students to explain [the concept] to peers. When students have solved it with their hands, it is easier for them to talk about [the process]. Textbooks and e-textbooks are also very visual, full of pictures and videos. However, textbook exercises usually urge students to move forward too fast instead of talking about the current exercise."(Lower secondary school mathematics teacher).

The fifth-grade class teacher also noticed that during the class interventions, paper-and-pencil students were particularly silent and worked separately, even though he had encouraged them to work together and discuss (Figure 13a). In contrast, the X-is students were more active to discuss and required less encouragement for verbalisation (Figure 13b). The teacher felt that this was because the X-is students had to think about how to manipulate the physical blocks, thereby encouraging thinking aloud and discussion. He also observed that the X-is students learned mathematical symbolic representation from the math expression window. Therefore, they were able to write math sentences explaining their step-by-step solutions more clearly than the paper-and-pencil students.

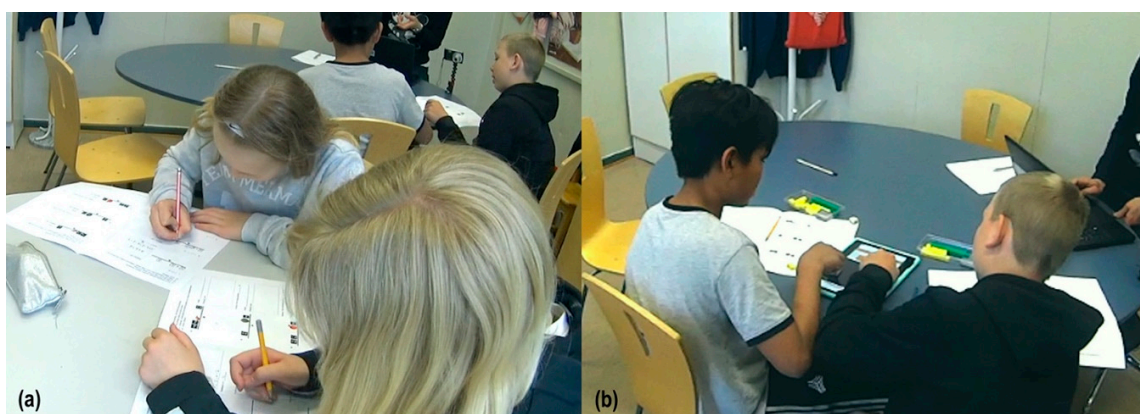


Figure 13. Fifth-graders' communication during the intervention. (a) Paper-and-pencil students solving an equation silently and separately; (b) X-is students discussing while solving an equation.

All the teachers agreed that X-is provided better encouragement for students' learning through discovery than paper-and-pencil instruction. In their opinion, because X-is works automatically and

provides real-time feedback, it allowed students to experiment by themselves and learn independently at their own pace. The fifth-grade teacher noted that X-is students needed less assistance from him during the interventions compared to the paper-and-pencil students. The fourth-grade teacher had a similar finding:

“When students got stuck, they could not get through on their own with the worksheet. Whatever weights students added on the scale [image on the worksheet], the scale wouldn’t move. So students might proceed with the wrong solution. But with the manipulative, students could add and remove blocks and got feedback from the manipulative.”

The teachers had mixed perceptions of how X-is supports students’ learning through social interaction compared to paper-and-pencil instruction. Most of the teachers thought that X-is better encouraged students’ social interaction by allowing them to share the same tablet, thus enhancing peer interaction naturally:

“When working with manipulatives, there are steps that the students can easily talk about. One student can tell another one, for example, ‘First, put it there. Do you notice how the scale moves?’ But when working with a textbook where there is, for example, $x + 2 = 6$, there is not much to discuss, only $x = 6 - 2$. It is difficult for students to invent what to talk about.” (Special education teacher).

However, two mathematics teachers had different views from the majority. They rated both X-is and paper-and-pencil instruction equally. They thought that the teacher and group dynamic played a more important role in peer interaction than the learning materials. Moreover, they felt that some students might try to take control of the manipulative without sharing it with others. Nevertheless, a few incidents of unbalanced participation occurred in the X-is interventions.

According to the teachers, X-is could be used to promote equation concept understanding among pre-primary through ninth-grade students. They believed that Level 1 would be suitable for pre-primary through fourth-grade students, while Level 2 would be suitable for fifth graders onwards. All teachers indicated that they would use X-is with students of all attainment levels in a small group to introduce equation concepts at the beginning of their class. Later on, X-is could be used for differentiation. High achievers could do more challenging exercises with X-is on their own, whereas students who had difficulty with equation solving could use X-is as a recap of what had been taught.

Four teachers were satisfied with X-is as it was. Two teachers provided suggestions for improving the pedagogical benefits of X-is. Instead of the letter x , the unknown could be represented with symbols, pictures, or various letters to emphasise that anything could be used to represent the unknown. Furthermore, after students have learnt how to write math sentences from the math expression window, there could be exercises in which they write math sentences on their own.

When asked about the difference between TUI and GUI—manipulation of physical and digital blocks, respectively—all teachers preferred physical block manipulation. They argued that manipulating physical blocks supports students’ learning by linking their actions with their thinking:

“When everything is digital, students may perceive it as a game. So, they will act like [they do when] playing games [and] just rush to do everything. A good example is when I asked students to learn how to draw points in GeoGebra [an interactive mathematics application]. Some students just kept on clicking [their mouse], so that their screen was full of points. Their minds were in a racing track. I think physical blocks could slow them down to think.” (Lower secondary school mathematics teacher).

Moreover, physical block manipulation is likely to interest students more because the digital world is too familiar for them:

“I think physical blocks are definitely better than digital blocks. They are more interesting for students because nowadays, they have been doing things all the time with the digital world. I noticed that

last semester, [lower secondary school] students were very enthusiastic about playing board games during a math class. I don't see working with physical objects as too childish for lower secondary school students."(Lower secondary school mathematics teacher).

5.2.4. Discussion of Learning Support

With respect to understanding equation-solving concepts, our findings are in line with those of other studies [14,43,45] that found that TUIs are likely to assist students in developing their understanding of mathematical concepts. We found that X-is could benefit diverse-attaining pre-primary through ninth-grade students in understanding equation concepts. Our observations suggest that the unique attributes of X-is, which combines the benefits of physical and virtual manipulatives, might contribute to students' conceptual understanding. First, similar to our initial research [36], physical interactions of grasping and moving base-10 blocks and X-Boxes helped students to concretely develop their understanding of the different terms in an equation and equation-solving concepts. Second, according to Moyer-Packenham and Westenskow [12], the focused constraint of virtual environments promotes students' learning in mathematics by constraining and focusing their attention on certain mathematical objects and processes. In the current study, the scale's tilt and the math expression window's changing colours drew students' attention to the mathematical equivalence concept. Moreover, the GUI provided simultaneous linking [12], that is, linking the visual and mathematical symbolic representations of an equation. Third, tangible technologies concurrently connect multimodal representations (i.e., physical, visual, and symbolic representations) of the same concept. The TUI explicitly bridged students' actions and the effects of their actions in different forms, which not only helped students to perceive the relationship between concrete and abstract representations of equation equivalence but also to present their equation-solving processes with mathematical symbols. One anticipated finding was that some students discovered their own way to utilise X-is. This finding corresponds to Clements' [82] definition of a good manipulative, which is one that allows the learner to control and use it flexibly. Our observation of creating spatial groups of base-10 blocks to find the value of the unknown supports the findings from our initial research [36] and that of others [13,14,83] that physical objects enable students to flexibly move them, thereby exploring more various solutions compared to static pictorial representations on a paper. These findings can possibly be explained using the theory of DPL [7].

Regarding languaging, during the interventions, X-is students not only communicated with their pairs more frequently but also in more varied ways—mainly through speech or gesture and partly through a combination of both—compared to the paper-and-pencil group. The association between instructional conditions and students' peer communication types was also statistically significant. Moreover, all the teachers stated that X-is promoted students' expression of their mathematical thinking through physical and mathematical symbolic representations better than paper-and-pencil instruction. The integration of the physical and digital worlds of X-is contributed to students' various modes of meaning making. Similar to what we found in our initial research [36], we discovered that the manipulation of physical objects promoted physical and verbal representations, whereas the math sentences provided on the screen facilitated students' mathematical symbolic representation. Taken together, these observations suggest that X-is is likely to encourage students to multimodally make meaning from equation-solving concepts and procedures, thereby encouraging their languaging [30], which leads to better conceptual understanding. Additionally, students' languaging made it easier for not only the teachers but also the researchers to evaluate their thinking.

In terms of learning through social interaction, the findings indicate that X-is evidently encouraged peer interactions more than paper-and-pencil instruction. During the intervention, the paper-and-pencil students mainly worked silently and separately. In contrast, the X-is students consistently manipulated X-is together while simultaneously discussing or giving and taking advice from each other, thereby indicating social interaction in learning [31]. This study confirms prior research [43,45] that tangible technologies enhance peer interaction in mathematics classrooms. Most of the teachers believed that X-is would promote social interaction because it allowed students to share and

manipulate simultaneously. The teachers' views support evidence from previous observations [84,85] that shared interfaces (i.e., in our case, physical objects and a tablet) encourage equal participation, which indicates that TUI may promote active collaboration. According to Pontual Falcão and Price [85], the active collaboration observed in our study was probably influenced by three main factors. First, multiple physical objects provided both students in the pair with the opportunity to gain access to them. Second, X-is allowed multiple inputs concurrently, thereby enabling both students to manipulate X-is parallelly. Third, as proposed by Price [16], the co-located design of X-is could resolve the concern of the two mathematics teachers about single-user constraints—one student taking control of the manipulative—which occurred during some of the X-is interventions. Because the input and output occurred at the same point of interaction, the students' attention was drawn to the tablet. Therefore, students could easily see each other's actions through the GUI on the shared screen, which contributes to their learning through social interaction.

Regarding learning through discovery, our results are in line with earlier research [45] that found that TUIs encouraged students to independently experiment and discover to-be-learned mathematics content. According to the findings from our initial research [36] and those of others [11], both physical and digital properties of X-is could contribute to students' self-discovery. When manipulating physical objects, students tended to speak aloud what they were doing or thinking. Therefore, their languaging might not only help them to organise their own mathematical thinking but also allow their peers to listen to and reflect on their thinking [31], thereby building knowledge together. Further, GUIs provide students with guidance and real-time feedback. This scaffolding allows them to explore equation modelling and solving independently.

5.3. Usability

5.3.1. Class Intervention

Both groups of students completed all eight equations within a similar time frame (paper-and-pencil group: $M = 21.1$ min/pair, $SD = 4.8$; X-is group: $M = 23.1$ min/pair, $SD = 2.8$). Originally, the class teachers were supposed to guide the students on how to use X-is at the beginning of the intervention in the same way that they would in normal classes when working with new manipulatives. However, while the teachers were occupied with the paper-and-pencil students, the X-is students started to learn how to use X-is by themselves. After watching a 1 min introductory animation on the tablet, two pairs out of six were able to use X-is to solve all equations on their own. Four other pairs needed our minimal guidance for the first equation and after that were able to solve the rest by themselves. Excluding the time needed to model and solve the first equation, in which students had to learn how to use X-is, the task completion time of both conditions was almost the same (paper-and-pencil group: $M = 2.6$ min/equation, $SD = 0.6$; X-is group: $M = 2.7$ min/equation, $SD = 0.3$). For the X-is group, their task completion time included X-is manipulation and recording the solution on the worksheet.

5.3.2. Student Questionnaires, Interviews, and Thinking Aloud Sessions

All 12 students in the X-is group responded positively regarding the usability of X-is. They perceived X-is as easy and enjoyable to use. Moreover, all of them expressed their intention to use X-is for solving equations in the future. According to the students, the straightforward UI and guidance as well as their familiarity with tablets contributed to its ease of use. For example, one high-attaining fourth grader stated that 'It is not that difficult to use. You just have to put blocks and press an arrow'. Another student stated, 'It is easy to use because I have used a lot of computers and things like. I think this [X-is] is nicer than [solving equations on] paper'. (Medium-attaining fifth grader).

All students (eight who strongly agreed and four who agreed) expressed their enjoyment using X-is for various reasons. Some enjoyed its ease of use, while others appreciated its pedagogical benefits. Some mentioned that it was pleasant that X-is allowed them to work in pairs. A low-attaining fifth grader stated 'I am able more or less to solve equations [using X-is] and start to be interested in solving

equations. That's why it is nice . . . But if it isn't nice, and I can't do it, then I usually give up'. Likewise, a medium-attaining fourth grader stated, 'It was quite fun, because you could do it with your friend'.

Moreover, some students emphasised that the differences between X-is and typical learning materials (e.g., paper and pencil) and its technological aspects made X-is enjoyable. For example, 'It is quite fun using this because you can move these blocks. Writing on the paper is quite boring' (Low-attaining fourth grader). According to a medium-attaining fourth grader, 'Nowadays, it is a must for [those of] us who are at my age to spend time using digital devices. It is nice to have a chance to work with digital devices. I enjoy using computers also at home'.

When asked whether they would like to use X-is again when solving equations, all the students—including the two fourth graders who disagreed with the statement about its effectiveness in terms of learning support—affirmed their intention for future use of X-is. Different pedagogical benefits (e.g., support for understanding and learning through discovery, scaffolding, and guidance) of X-is were the main reasons for the students' positive responses. Other reasons—such as enjoyability, ease of use, and technological aspects—were also mentioned:

"It would be nice if you could have this kind of app for other maths, even from the first grade when learning, for example, addition, subtraction, and multiplication. It's good to have these blocks compared to just digital [elements]." (Medium-attaining fifth grader).

During the thinking aloud sessions, most of the students (9/12) were able to use X-is to model the given equation with physical objects and then solve it within 35–40 s without any difficulty. Only two fourth graders and one fifth grader struggled with how to use X-is. It took them about 2–5 min to finish the same task. This result may be explained by the fact that during the class interventions, these students were somewhat dominated by their pairs, thereby not having a chance to properly learn how X-is functioned. However, after receiving guidance from us or looking at instructions and prompts provided by X-is, they were eventually able to finish the task. Moreover, they could complete another similar task within one minute. During the thinking aloud sessions, some students also benefited from the guidance and scaffolding provided by X-is. For example, textual instruction guided students on how to proceed, while the status of the scale and the blinking working zone prompted students to take correct actions.

Nevertheless, problematic usability issues were identified. The thinking aloud sessions revealed a minor hurdle with the X-is navigation and operation menu. After modelling an equation, about half of the students (5/12) attempted to proceed to the equation solving part without pressing the play button. However, most of the students (8/12) expressed their satisfaction with the current usability of X-is and did not provide any suggestion for improvement. Some students made suggestions for how the usability could be developed, for example, by providing clearer, step-by-step instructions; adding an info button for detailed instructions; and offering an on-off math expression display for hiding or displaying the math expression when needed.

5.3.3. Discussion of Usability

To sum up, the findings demonstrate that the usability of X-is was clearly acceptable. Because the X-is students did not require much time or effort to learn to work with X-is during the interventions, learnability [86] (i.e., ease of learning [79]) of X-is was satisfactory. The efficiency [86] of X-is usability was rather high as students could quickly model and solve equations during the interventions and thinking aloud sessions. There was no evidence that they required substantial additional time to complete the intervention tasks (i.e., manipulating X-is and recording their work on the worksheet) compared to the paper-and-pencil condition. This observation supports the work of Martin et al. [83] but contrasts that of Uttal et al. [10]. The students also rated X-is as easy to use. Manches and O'Malley [15] proposed that one benefit of TMs appears to be that children do not need to learn how to manipulate physical objects. In our case, base-10 blocks were likely already familiar to the students, which may have also contributed to the ease of use of X-is. This finding partly supports the work of Sapounidis

and Demetriadis [84], who reported that children ages 5–8 found interaction with TUI to be easier than with GUI, while children ages 11–12 felt the opposite.

Despite its high learnability, efficiency, and ease of use, some minor usability problems were found. The improvement of these problems would increase the usability of X-is. Regarding satisfaction [86], all students perceived X-is as enjoyable because of its pedagogical and social benefits as well as integration of digital technology. Additionally, all students stated that they intended to use X-is again. Similar findings were also reported in previous studies [44,84] on tangible technologies for learning.

5.4. Limitations and Future Research and Development

The limitations of the current study should be highlighted. First, only the post-test was conducted after the class intervention, and the comparison group was from the higher grade levels. Research employing pre-post test design and using the comparison and experimental groups from the same grade level would increase the precision of the students' learning achievement analysis and decrease the ambiguity of the result interpretation.

Second, the quantitative results must be interpreted with caution due to the small convenience sample size. Future research using larger random samples would contribute to external validity. Moreover, the interventions were conducted in a short period of time, which might not have been long enough to influence students' learning achievement but was still short enough to create a novelty effect. This shortcoming calls for longitudinal studies, which would demonstrate the long-term benefits of the manipulative for students and ensure that positive findings towards the manipulative are not a result of its novelty.

Third, there were some issues regarding research reliability. The internal consistency (i.e., Cronbach's alpha) of some of the questionnaire scales was relatively low. More items within the scales would increase the instrument reliability. Additionally, the qualitative analyses were conducted mainly by the first author. Future research should employ a triangulation of researchers. Despite the lack of researcher triangulation, the quality of the current mixed-methods research was promoted by the triangulation and concurrence of the research methods, data source, and data analysis as well as the integration of quantitative and qualitative findings.

Fourth, our study focused only on an overview of the potentials of tangible technologies for learning linear equations. Future research could investigate specific aspects examined in the current study in greater depth. Research focusing on students with a specific degree of achievement, other educational levels, different mathematics contents, and distance learning could contribute to our better understanding of how tangible technologies facilitate learning.

Fifth, it should be noted that off-shelf technology was employed for the proposed manipulative because our research was aimed at deployment in present classroom contexts. Thus, future studies on TMs employing purposely designed technologies might yield different results. Moreover, the limitations of the Wizard of Oz prototype, such as its delayed output, might have influenced the research results. A reliably working, fully interactive prototype with all functions and features would not only prevent this possibility but also expand the to-be-evaluated attributes of the manipulative. Evidently, our prototype was limited by the available off-shelf technology. The main challenge is of the unreliability of the object tracking using a webcam, especially when the colour contrast or brightness is too low or physical objects are too close to each other. Further software and hardware development and testing in a real classroom environment could enhance the system's reliability. It is possible that future advancements in tangible technology may enable the system to directly detect objects without requiring external devices and connections, thereby increasing product usability, practicality, and reliability. Additional features (e.g., free experiments with the balance scale, students' own equation set-up, equations generated based on students' performance, scoreboard, and analytics data) could enhance the user experience for X-is.

6. Conclusions

This study explored the potentials of tangible technologies for learning linear equations in real classrooms. Taken together, the findings add to the growing body of research indicating that there are advantages to utilising tangible technologies in mathematics classrooms. The initial TM is likely to benefit pre-primary through ninth-grade students of different attainment levels. It not only supported their conceptual understanding, languaging, and learning through discovery and social interaction but also enhanced their learning achievement. Moreover, the usability evaluation results demonstrated that the manipulative was learnable, easy to use, useful, and engaging, which would ensure its successful adoption in real educational contexts. The empirical evidences suggest that the integration of physical and virtual attributes of manipulatives is likely to contribute to these positive findings.

Our research results have important implications for mathematical classroom pedagogy and the development of TMs. Regarding classroom pedagogy, the study demonstrated that TMs should be adopted into mathematics classrooms—which are usually dominated by paper-and-pencil instruction [87,88]—to better assist diverse learners. However, it is worth noting here that TMs alone cannot contribute to mathematics learning [13,46,47]. To be beneficial, they should be used in cooperation with appropriate pedagogy, in our case, languaging. Moreover, the tangible attributes (i.e., physical–digital interactions) of the proposed manipulative, which were perceived as useful and engaging, may encourage the use of manipulatives in the upper-grade classrooms, in which use of manipulatives normally declines [61,89,90]. TM design and development could also benefit from the current work by taking advantage of the unique features of TUIs to facilitate mathematics learning. Additionally, continued efforts are needed to make tangible technologies accessible for all classrooms.

Supplementary Materials: The following are available online at <http://www.mdpi.com/2414-4088/4/4/77/s1>. Video S1: Level 1 equation solving during one fourth-grade thinking aloud session. Video S2: Level 2 equation solving during one fifth-grade thinking aloud session.

Author Contributions: Conceptualisation, D.L.; software, L.M.; methodology, investigation, and formal analysis, D.L.; writing—original draft preparation, D.L. and L.M.; writing—review and editing, D.L., L.M., J.J., and P.P.; visualisation, D.L. and L.M.; supervision, J.J. and P.P.; project administration, D.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

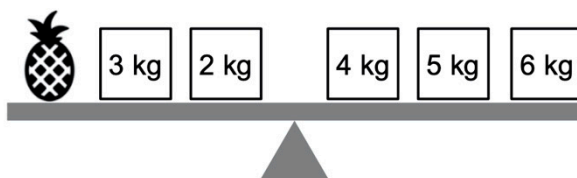
Acknowledgments: The authors would like to acknowledge the contributions of Fouzia Khan, Juho Korkala, Roni Perälä, Niko Sainio, Krishna Bagale, and Pekka Mäkiäho for software development; Tapio Lehtonen for the UI; and Kari Kouhian for the prototyping. The authors appreciate the teachers and the students who participated in the research.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

The paper-based test contains six open-response items. The students were required to (1) translate the six equations presented through natural, pictorial, or mathematical symbolic language into two other representations, (2) verbally explain or show the mathematical steps they used to solve the equations, and (3) provide the value of unknowns. Figure A1 demonstrates the three types of the test items.

1. (a) Explain in your own words what the given picture means. (b) Form an equation from the given picture. (c) Show the steps you follow to determine how much one fruit weighs when the scale is balanced. (d) Provide the correct answer.



2. (a) Draw pictures on the balance scale according to the given equation. (b) Explain in your own words what the given equation means. (c) Show the steps you follow to determine the values of the unknown x that make the equation true. (d) Provide the correct answer.

$$x + 1 + 3 = 11$$



3. (a) Draw pictures on the balance scale according to the given word question. (b) Form an equation from the given word question. (c) Show the steps you follow to determine the values of the unknown x that make the equation true. (d) Provide the correct answer.

When mother weighs the cake ingredients, she notices that 5 g of butter and 12 g of flour are as heavy as one egg and 9 g of sugar. How much does one egg weigh?



Figure A1. The three types of the paper-based test items.

Appendix B

The teacher questionnaire contains the following scales:

1. In your opinion, how well did/would the X-is compared to the paper-and-pencil working method help students with understanding the following equation-solving concepts?
 - 1.1 Both sides of an equation are equal
 - 1.2 An unknown and solving for its value
 - 1.3 An equation stays equivalent when the same operation is performed on both sides
2. In your opinion, how well did/would X-is compared to the paper-and-pencil working method help students with expressing their mathematical thinking by using the following mediums?
 - 2.1 Tactile language
 - 2.2 Pictorial language
 - 2.3 Verbal natural language
 - 2.4 Written natural language

- 2.5 Mathematical symbolic language
3. In your opinion, how well did/would X-is compared to the paper-and-pencil working method support the following aspects for the students?
 - 3.1 Learning through first-hand experience and exploration
 - 3.2 Learning through collaboration with peers
 - 3.3 Active learning

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