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A Spacetime Metric for the 4 + 1 Formalism

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Abstract

In his foundational work on classical and quantum electrodynamics, Stueckelberg introduced an external evolution parameter, τ, in order to overcome difficulties associated with the problem of time in relativity. Stueckelberg particle trajectories are described by the evolution of spacetime events under the monotonic advance of τ, the basis for the Feynman–Stueckelberg interpretation of particle–antiparticle interactions. An event is a solution to τ-parameterized equations of motion, which, under simple conditions, including the elimination of pair processes, can be reparameterized by the proper time of motion. The $4+1$ formalism in general relativity (GR) extends this framework to provide field equations for a τ-dependent local metric $\gamma_{\mu\nu}(x,\tau)$ induced by these Stueckelberg trajectories, leading to τ-parameterized geodesic equations in an evolving spacetime. As in standard GR, the linearized theory for weak fields leads to a wave equation for the local metric induced by a given matter source. While previous attempts to solve the wave equation have produced a metric with the expected features, the resulting geodesic equations for a test particle lead to unreasonable trajectories. In this paper, we discuss the difficulties associated with the wave equation and set up the more general ADM-like $4+1$ evolution equations, providing an initial value problem for the metric induced by a given source. As in the familiar $3+1$ formalism, the metric can be found as a perturbation to an exact solution for the metric induced by a known source. Here, we propose a metric, ansatz, with certain expected properties; obtain the source that induces this metric; and use them as the initial conditions in an initial value problem for a general metric posed as a perturbation to the ansatz. We show that the ansatz metric, its associated source, and the geodesic equations for a test particle behave as required for such a model, recovering Newtonian gravitation in the nonrelativistic limit. We then pose the initial value problem to obtain more general solutions as perturbations of the ansatz.

Keywords: general relativity; the problem of time; Stueckelberg–Horwitz–Piron theory; parameterized relativistic mechanics

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1. Introduction

A significant obstacle in constructing a quantum theory of gravity centers on a variety of issues known collectively as the problem of time [1–3]. Although sometimes presented as a conceptual conflict between general relativity (GR) and quantum theory (see, for example, [4]), the essential difficulty is already present in the context of classical GR. In relativistic physics, as well as in everyday language, the concept of time play two distinct roles [5]. On the one hand time, time is one of four coordinates describing a location in spacetime, and in GR, these coordinates are dynamic quantities to be determined by equations of motion. But on the other hand, time is the chronological parameter required for posing and solving these equations. While this system is arguably well-posed in classical

GR, its consistency in the presence of quantum fluctuations is a fundamental difficulty facing quantum gravity (see [1] for a discussion of this question).

The distinction between these two aspects of time was already noted by Stueckelberg [6,7] in his foundational work on classical and quantum electrodynamics. In describing antiparticles as particles "going backward in time", he found that neither the coordinate time t nor the proper time of the motion $ds = \sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu}$ could faithfully parameterize chronological evolution in flat spacetime with $\eta_{\mu\nu} = (-1, 1, 1, 1)$. Instead, he introduced a parameter τ, external to the spacetime manifold and independent of the phase space coordinates, whose monotonic increase provides the "arrow of time" that determines chronology. In formulating the Klein–Gordon path integral, Feynman [8] adopted a similar approach, now known as the Feynman–Stueckelberg interpretation.

In the Stueckelberg framework, a particle worldline is the trajectory of a classical event $x^\mu(\tau)$ or quantum event $\psi(x, \tau)$ generated by a Lorentz scalar Hamiltonian K in canonical mechanics whose structure is familiar to nonrelativistic physics. Piron and Horwitz [9] developed Stueckelberg's work into a relativistic canonical many-body theory [10–14] of interacting spacetime events. The resulting Stueckelberg–Horwitz–Piron (SHP) formalism admits τ-dependent gauge freedom, leading to an electrodynamic theory in flat spacetime (see [15–17] and references contained therein) with five gauge potentials $a_\mu(x, \tau)$ and $a_5(x, \tau)$. These fields are designated $a_\alpha(x, \tau)$ with the index convention

$$\alpha, \beta, \gamma, \delta = 0, 1, 2, 3, 5 \qquad \mu, \nu, \lambda, \rho \ldots = 0, 1, 2, 3\,. \tag{1}$$

Although the free fields enjoy 5D gauge and spacetime symmetries, the description of particles must remain Lorentz-covariant, and so the symmetry of the fields breaks to O(3,1) when coupled with matter. The resulting theory of interacting events is integrable and recovers Maxwell electrodynamics in τ-equilibrium.

Horwitz extended the SHP framework to curved spacetime [18,19] with a background metric on 4D spacetime $g_{\mu\nu}(x)$, which reduces to $\eta_{\mu\nu}$ in flat spacetime. The equations of motion are the familiar geodesic equations of GR:

$$\frac{D\dot{x}^\mu}{D\tau} = \ddot{x}^\mu + \Gamma^\mu_{\rho\lambda}\dot{x}^\rho\dot{x}^\lambda = 0 \qquad \Gamma^\rho_{\lambda\mu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial g_{\sigma\lambda}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\sigma}\right) \tag{2}$$

but are parameterized here by the external chronological time τ, where

$$\dot{x}^\mu(\tau) = \frac{dx^\mu(\tau)}{d\tau} \qquad \ddot{x}^\mu = \frac{d^2x^\mu(\tau)}{d\tau^2} \tag{3}$$

describe the Stueckelberg trajectory for the event $x^\mu(\tau)$. For simple trajectories, it is often possible to invert $x^0 = x^0(\tau)$ and reparameterize the system using t, although the resulting description may be significantly more complicated. In the presence of pair creation and annihilation processes, the x^0 trajectory will not be invertible [17].

To develop this system into a fully dynamical theory framework, we must consider that events $x^\mu(\tau)$ will be associated with an event density $\rho(x, \tau)$ and energy–momentum tensor $T_{\mu\nu}(x, \tau)$ that also depend explicitly on τ. We must therefore find τ-dependent field equations to be solved for a local metric $\gamma_{\mu\nu}(x, \tau)$, reflecting Wheeler's characterization of geometrodynamics [20] as "spacetime tells matter how to move; matter tells spacetime how to curve". In a recent series of papers (see [21] and references contained therein), we proposed such field equations by constructing a $4 + 1$ formalism that builds on the theory of embedded surfaces [22–24], as applied in the $3 + 1$ Arnowitt–Deser–Misner (ADM) formalism [25].

In the $3+1$ framework, 4D spacetime $\mathcal{M}$ is decomposed into a timelike direction, defining a time parameter t, and a 3D spacelike hypersurface whose metric $\gamma_{ij}(x)$ evolves with t for $i,j=1,2,3$. The well-known Einstein field equations can then be decomposed into an initial value problem for $\gamma_{ij}(x)$ and the extrinsic curvature $K_{ij}(x)$ that expresses geometrical information about the t-direction.

To construct the $4+1$ formalism, we define a 5D pseudo-spacetime $\mathcal{M}_5$ with coordinates $X^\alpha=(x^\mu,\tau)$ and a metric $g_{\alpha\beta}(x,\tau)$ to be discovered. Mimicking the ADM procedure, we take advantage of the natural foliation to decompose $\mathcal{M}_5$ into the τ-like direction and the 4D hypersurface $\mathcal{M}$, whose metric $\gamma_{\mu\nu}(x,\tau)$ evolves with τ for $\mu,\nu=0,1,2,3$. The Einstein field equations are extended to 5D in such a way as to ensure that the geometrical structures (the free fields of the theory) retain 5D gauge and spacetime symmetry, while the description of matter possesses O(3,1) spacetime symmetry at most. These field equations can then be decomposed into an initial value problem for $\gamma_{\mu\nu}(x,\tau)$ and the extrinsic curvature $K_{\mu\nu}(x,\tau)$ that expresses geometrical information about the τ-direction. Because τ is independent of spacetime in this formalism, the diffeomorphism invariance of general relativity does not raise questions about the evolution of the metric, as might occur under t-evolution. Moreover, the matter dynamics are determined by a Lorentz scalar Hamiltonian and are similarly unaffected by coordinate transformations. These features should also apply to quantized canonical gravity, sidestepping concerns that quantum fluctuations may affect the evolution equations. For simple spacetime configurations, it may be possible to reparameterize the system by x^0 and eliminate τ from the solution for $\gamma_{\mu\nu}$. As Fock observed when first proposing the parameterization of electrodynamics in proper time [26], the resulting solutions will generally be significantly more complicated, so that the $4+1$ formalism is expected to provide more stable numerical solutions for complex energy–momentum configurations.

As in standard GR, the 5D field equations can be linearized for weak fields, leading to a wave equation for small perturbations to the flat space metric. This equation possesses a Green's function [17] that was found in the context of SHP electrodynamics and has been applied to find solutions in GR for sources of various types. For a 'static particle'—a single event evenly distributed along the t-axis in its rest frame, with no detailed information about its location in time—we obtain a Schwarzschild-like metric [21].

However, a free Stueckelberg event in its rest frame is described by $x^\mu(\tau)=(ct(\tau),\mathbf{0})$, and we naturally expect that the time coordinate $t(\tau)$ will advance (or retreat) as τ advances monotonically. A free Stueckelberg particle may be modeled as an ensemble of such events [27] located at some point $\boldsymbol{x}$ in space, but narrowly distributed along the time axis according to some probability distribution $\varphi(t-t(\tau))$, where $t(\tau)$ is the nominal time coordinate. In SHP electrodynamics, the 5D wave equation then produces the Coulomb potential in the form

$$a_0(x,\tau)=-\frac{\varphi(t-R/c-\tau)}{R}+o\left(\frac{1}{R^2}\right), \tag{4}$$

where $\varphi(s)$ has its maximum at $\varphi(0)$. A test event at some spacetime point $x=(ct,R\hat{\boldsymbol{r}})$ will experience a potential whose support is centered around the chronological time $\tau=t-R/c$, the retarded time of the source. An observer located at this static point x will see the potential rise and fall in strength as the chronological time τ approaches and then retreats from the retarded time. But an observer similarly advancing along its worldline with $t=\tau+R/c$ will see the potential with constant strength determined by $\varphi(0)$.

In [21], we showed that this approach is not successful in finding an appropriate metric induced by an event localized on the t-axis in the $4+1$ formalism. A source ensemble of

the type described above produces an energy–momentum tensor narrowly distributed in τ given by

$$T_{\alpha\beta}(x,\tau) = \widehat{T}_{\alpha\beta}(x)\,\varphi(t - R/c - \tau) \tag{5}$$

where $\widehat{T}$ is the static energy–momentum tensor found in standard GR. Solving the wave equation in the linearized $4+1$ formalism thus leads to a Schwarzschild-like metric in which the usual spacetime factor $1-\Phi$ is modified as

$$\Phi = \frac{2GM}{c^2 R} \;\longrightarrow\; \frac{2GM}{c^2 R}\varphi(t - R/c - \tau) \tag{6}$$

describing a localized gravitational field for a test event with a maximum on the lightcone of the source. In this picture, spacetime is flat and empty, except at the chronological moments τ, for which the matter sources and the metric fields they induce have support in the neighborhood of a given spacetime point x^μ. As for the Coulomb potential, an observer advancing along its worldline with $t = \tau + R/c$ will experience the metric with constant strength determined by $\varphi(0)$.

Nevertheless, it was also shown in [21] that the geodesic equations for a test event in any metric of the separable type (6) lead to unreasonable equations of motion, including a dynamic reversal of the sign of the gravitational acceleration. This problem arises from the structure of the 5D Green's function, which is itself subject to a number of open questions. As a result, the wave equation in linearized $4+1$ gravitation does not provide an effective approach to the metric, and we must apply the full 4 + 1 evolution equations. For weak fields, one may pose the initial value problem by choosing an ansatz for the metric and seeking solutions for the desired metric as a perturbation of the ansatz under the influence of the perturbing source.

In this paper, we propose an ansatz with the desired properties, derive the associated source $T_{\mu\nu}$ from the wave equation, and set up the $4+1$ metric evolution equations in the weak field approximation. In Section 2, we briefly review the SHP formalism and $4+1$ general relativity with invariant evolution. Section 3 summarizes solutions to the wave equation using the Green's function, which informs our choice of the ansatz metric. In Section 4, we present an ansatz metric, examine the resulting geodesic equations, and derive the source using the wave equation. In Section 5, we set up the $4+1$ evolution equations by calculating the projected Ricci tensor and extrinsic curvature. Section 6 presents a discussion of these results.

2. Review of General Relativity with Invariant Evolution

2.1. Gauge and Spacetime Symmetries

Using the flat spacetime metric $\eta_{\mu\nu}$ with index convention (1), the electrodynamic event action

$$S_{\text{SHP}} = \int d\tau\, \frac{1}{2}M\dot{x}^\mu\dot{x}_\mu + \frac{e}{c}\left[\dot{x}^\mu a_\mu(x,\tau) + c_5 a_5(x,\tau)\right] \tag{7}$$

$$= \int d\tau\, \frac{1}{2}M\dot{x}^\mu\dot{x}_\mu + \frac{e}{c}\dot{x}^\beta a_\beta(x,\tau) \tag{8}$$

is invariant under the 5D gauge transformation $a_\alpha(x,\tau) \longrightarrow a_\alpha(x,\tau) + \partial_\alpha\Lambda(x,\tau)$, where we introduce $x^5 = c_5\tau$ as a dimensional factor, in analogy to $x^0 = ct$. The superficial 5D spacetime symmetry in the term $\dot{x}^\beta a_\beta$, is broken to O(3,1) by the terms $\dot{x}^\mu\dot{x}_\mu$, $\dot{x}^\mu a_\mu$, and a_5, which are Lorentz scalars, restricting the spacetime symmetry of the action to 4D. Beginning from another point of view, we may treat (8) as a standard 5D action in which we have broken the 5D symmetry of the first term by constraining $\dot{x}^5 \equiv c_5$ and eliminating the constant term so that $\dot{x}^\alpha\dot{x}_\alpha \longrightarrow \dot{x}^\mu\dot{x}_\mu$. This latter observation informs our approach to

τ-dependent GR. For consistency with electrodynamic phenomenology, we take $c_5 < c$ (see [17] for more on the interpretation of c_5).

The action for the electromagnetic field is

$$S_{\text{field}} = \int d\tau\, d^4x f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) \tag{9}$$

where $f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$ appear in the Lorentz force found by varying (8) with respect to x^μ. Since the 5-index signifies an O(3,1) scalar quantity, $f_{\mu\nu}$ is a second-rank tensor, while $f_{5\mu}$ is a vector field strength. Raising the 5-index in (9) seems to require a 5D flat space metric

$$\eta_{\alpha\beta} = \text{diag}(-1,1,1,1,\sigma) \qquad\qquad \sigma = \pm 1 \tag{10}$$

suggesting an O(3,2) or O(4,1) symmetry for the free fields. But by writing

$$f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) = f^{\mu\nu}(x,\tau) f_{\mu\nu}(x,\tau) + 2\sigma f^{\mu}_{\ 5}(x,\tau) f_{\mu 5}(x,\tau) \tag{11}$$

we may regard σ as simply the choice of sign for the vector–vector interaction relative to the tensor–tensor interaction, with no inherent geometrical significance.

Just as we may regard (8) as a symmetry-broken 5D action, we approach GR by embedding 4D spacetime $\mathcal{M}$ in a 5D pseudo-spacetime $\mathcal{M}_5 = \mathcal{M} \times R$ with coordinates $X^\alpha = (x^\mu, c_5\tau)$. We associate with $\mathcal{M}_5$ a metric $g_{\alpha\beta}(x,\tau)$ to be determined using 5D field equations. Exploiting the natural foliation of $\mathcal{M}_5$ into 4D equal-τ spacetimes homeomorphic to $\mathcal{M}$, we construct a constant quintrad frame [28] for the pseudo-spacetime with basis vectors $\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab}$ related to the coordinate frame $\mathbf{g}_\alpha \cdot \mathbf{g}_\beta = g_{\alpha\beta}$ by the vielbein field

$$\mathbf{e}_a = e^{\alpha}_{\ a}(x,\tau)\mathbf{g}_\alpha \tag{12}$$

that "absorbs" the local dependence of the $\mathbf{g}_\alpha(\mathcal{P})$ on the point $\mathcal{P} \in \mathcal{M}_5$. The quintrad indices run $a = 0,1,2,3,5$. In the quintrad frame, the field equations can be immediately written as

$$R_{ab} - \frac{1}{2}\eta_{ab}R = k_G T_{ab} \tag{13}$$

where we initially specify the constant flat metric as (10). As in electrodynamics, we break the 5D spacetime symmetry to O(3,1) at the source by replacing

$$\eta_{ab} = \text{diag}(-1,1,1,1,\sigma) \;\longrightarrow\; \widehat{\eta}_{ab} = \text{diag}(-1,1,1,1,0) = \delta^k_a \delta^l_b\, \eta_{kl} \tag{14}$$

in the matter terms. That is, we break the symmetry of (13) by posing field new equations in the trace reversed form

$$R_{ab} = k_G\left(T_{ab} - \frac{1}{2}\widehat{\eta}_{ab}\widehat{T}\right) \qquad\qquad \widehat{T} = \widehat{\eta}^{ab}T_{ab} = \eta^{kl}T_{kl} \tag{15}$$

where the purely geometrical structure on the LHS enjoys 5D symmetry in the absence of the O(3,1) covariant source terms on the RHS. Transforming this expression back to the coordinate frame provides the O(3,1) symmetric field equations

$$R_{\alpha\beta} = k_G\left(T_{\alpha\beta} - \frac{1}{2}P_{\alpha\beta}\widehat{T}\right) \qquad\qquad P_{\alpha\beta} = g_{\alpha\beta} - \sigma n_\alpha n_\beta \tag{16}$$

where n_α is the unit normal that points in the τ-direction orthogonal to $\mathcal{M}$, and $P_{\alpha\beta}$ is the projection operator from $\mathcal{M}_5$ onto the 4D spacetime hypersurface $\mathcal{M}$. The matter terms in (16) restrict the symmetry of the field equations to O(3,1).

2.2. Event Dynamics in Curved Spacetime

Variation of the scalar event Lagrangian

$$L = \frac{1}{2} M g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{17}$$

with respect to x^γ leads to 5D geodesic equations

$$\frac{D\dot{x}^\gamma}{D\tau} = \ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \qquad \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right) \tag{18}$$

with the Christoffel connection in the standard form. As in electrodynamics, the apparent 5D symmetry is broken to O(3,1) by constraining $\dot{x}^5 = c_5 \longrightarrow \ddot{x}^5 = 0$, leaving the dynamical system in the form

$$\frac{D\dot{x}^\mu}{D\tau} = \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = \ddot{x}^\mu + \Gamma^\mu_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma + 2c_5 \Gamma^\mu_{5\nu} \dot{x}^\nu + c_5^2 \Gamma^\mu_{55} = 0 \tag{19}$$

$$\frac{D\dot{x}^5}{D\tau} = \ddot{x}^5 \equiv 0 \tag{20}$$

which recovers standard 4D GR in the case $g_{5\alpha} = 0$ and $\partial_\tau g_{\mu\nu} = 0$. With canonical momentum

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = M\left(g_{\mu\nu}\dot{x}^\nu + c_5 g_{\mu 5}\right) \tag{21}$$

Legendre transformation of L leads to the Hamiltonian

$$K = \frac{1}{2M} p^2 + \frac{1}{2} c_5 g_{55} g^{5\mu} p_\mu - \frac{1}{2} c_5 g_{5\mu} g^{\mu\lambda} p_\lambda + \frac{1}{2} M c_5^2 g_{5\mu} g^{\mu\lambda} g_{\lambda 5} + \frac{1}{2} M c_5^2 g_{55} \tag{22}$$

representing the total 5D mass of the motion. When $g^{5\mu} = 0$, this assumes the simple form

$$K = \frac{1}{2M} p^\mu p_\mu + \frac{1}{2} M c_5^2 \, g_{55} \tag{23}$$

in which $g_{55}(x, \tau)$ plays the role of a τ-dependent potential on 4D spacetime. In general, the Hamilton equations

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = \frac{dp_\mu}{d\tau} = -\frac{\partial K}{\partial x^\mu} \tag{24}$$

lead to the geodesic equations in a first-order form. As in nonrelativistic mechanics, the Poisson bracket is defined as

$$\{F, G\} = \frac{\partial F}{\partial x^\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial x^\alpha} = \frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial G}{\partial x^\mu} \tag{25}$$

where the second equality follows from $p_5 \equiv 0$. For a scalar function $F(x, p, \tau)$ on phase space, we have

$$\frac{dF}{d\tau} = \{F, K\} + \frac{\partial F}{\partial \tau} \,, \tag{26}$$

generalizing the nonrelativistic result. The Hamiltonian is thus conserved unless K depends explicitly on τ through $g_{\alpha\beta}(x, \tau)$. Nevertheless, the 4D mass $m = -p^\mu p_\mu / 2M$ may vary under $g_{5\alpha}$.

Putting $\partial_0 g_{\alpha\beta} = 0$ in the geodesic Equation (19) and neglecting $\dot{x}^i/c \ll 1$ for $i = 1, 2, 3$, the equations of motion reduce to

$$\frac{d^2t}{d\tau^2} = \frac{dt}{d\tau}\partial_\tau g_{00} \qquad \ddot{\mathbf{x}} = \frac{1}{2}c^2\left(\frac{dt}{d\tau}\right)^2 \nabla g_{00} + \frac{1}{2}c_5^2 \nabla g_{55} \tag{27}$$

which differ from nonrelativistic mechanics when $\partial_\tau g_{00} \neq 0 \longrightarrow \ddot{t} \neq 0$. To compare with Newtonian gravity, we borrow $g_{00} = -1 + 2GM/c^2R$ from the Schwarzschild solution, put $\nabla g_{55} = 0$, and introduce a perturbed source mass $M = M_0 + \delta M(\tau)$. The t equation admits the solution

$$\frac{dt}{d\tau} = \exp\left(\frac{2G}{c^2R}\delta M\right) \xrightarrow[\delta M \longrightarrow 0]{} 1 \tag{28}$$

showing that a mass perturbation in the source leads to acceleration of the t coordinate of the test particle. Introducing the conserved angular momentum $L = Mr^2\dot{\phi}$, we find the radial equation in spherical coordinates

$$\ddot{r} - \frac{L^2}{M^2R^3} + \exp\left(\frac{4G}{c^2R}\delta M\right)\frac{GM_0}{R^2} = 0\,, \tag{29}$$

which recovers Newtonian gravitation in the absence of the perturbation δM. The Hamiltonian for the test event in this coordinate system is

$$K = \frac{1}{2}Mg_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\frac{1}{2}Mc^2\left(1 - \frac{2GM_0}{c^2R}\right)\exp\left(\frac{4G}{c^2R}\delta M\right) + \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\frac{L^2}{MR^2} \tag{30}$$

with the time derivative

$$\frac{d}{d\tau}K = \exp\left(\frac{4G}{c^2R}\delta M\right)\left(-\frac{GM}{r} + \frac{4G^2MM_0}{c^2R^2}\right)\frac{d}{d\tau}\delta M\,. \tag{31}$$

We thus see that the Hamiltonian for the motion of this test particle is not conserved in the presence of a variable mass gravitational source. We may interpret this as a transfer of mass across spacetime mediated by the metric.

Similarly, allowing the time velocity $\dot{x}^0(\tau)$ of the source event to vary in the rest frame, equivalent to variable mass through $p^2 = M^2\dot{x}^2 = M^2(\dot{x}^0)^2$, produces an energy–momentum tensor in the 5D space that exhibits radiation of mass through spacetime. Solving the wave equation for this source produces a post-Newtonian metric similar to Schwarzschild but depending explicitly on τ. Solving the geodesic equations for a test event in this metric leads to motion with variable $\dot{t}(\tau)$ and a radial equation for R characterized by variable dynamic mass p^2. As such, this model also describes a variable mass source event radiating mass across spacetime to a test event whose mass varies in response [21].

For a distribution of geodesically evolving events without mutual interaction ('dust'), the event density $\rho(x, \tau)$ is the number of events per spacetime volume, with dimensions of the length^{-4}. The event current is

$$j^\alpha(x, \tau) = M\rho(x, \tau)\dot{x}^\alpha(\tau) \tag{32}$$

with the continuity equation

$$\nabla_\alpha j^\alpha = \frac{\partial j^\alpha}{\partial x^\alpha} + j^\gamma\Gamma^\alpha_{\gamma\alpha} = \frac{\partial\rho}{\partial\tau} + \nabla_\mu j^\mu = 0 \tag{33}$$

where, again, $j^5 = Mc_5\rho(x, \tau)$ is an O(3, 1) scalar and not the 5-component of a vector

with 5D symmetry. The mass–energy–momentum tensor

$$T^{\alpha\beta} = M\rho\dot{x}^{\alpha}\dot{x}^{\beta} \longrightarrow \begin{cases} T^{\mu\nu} = M\rho\dot{x}^{\mu}\dot{x}^{\nu}, \\ T^{5\beta} = \dot{x}^{5}\dot{x}^{\beta}M\rho = c_5 j^{\beta}, \end{cases} \tag{34}$$

is conserved as

$$\nabla_{\alpha} T^{\alpha\beta} = 0, \tag{35}$$

by virtue of the continuity and geodesic equations. The tensor $T^{\alpha\beta}$ is thus a suitable O(3,1) covariant source for the 5D field equation.

2.3. Weak Field Approximation

In the weak field approximation [17], we write the local metric as a small perturbation $h_{\alpha\beta}$ of the flat metric (10) so that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \longrightarrow \partial_{\gamma} g_{\alpha\beta} = \partial_{\gamma} h_{\alpha\beta} \qquad \left(h_{\alpha\beta}\right)^2 \simeq 0 \tag{36}$$

leading to the Ricci tensor in the form

$$R_{\alpha\beta} \simeq \frac{1}{2}\left(\partial_{\beta}\partial_{\gamma}h^{\gamma}_{\alpha} + \partial_{\alpha}\partial_{\gamma}h^{\gamma}_{\beta} - \partial^{\gamma}\partial_{\gamma}h_{\alpha\beta} - \partial_{\alpha}\partial_{\beta}h\right) \tag{37}$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$. Imposing the Lorenz gauge condition

$$\partial^{\beta}\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h\right) = 0\,, \tag{38}$$

permitted by invariance of the metric under a 5D translation $x^{\alpha} \longrightarrow x^{\alpha} + \Lambda^{\alpha}(x,\tau)$ simplifies the Ricci tensor to

$$R_{\alpha\beta} \simeq -\frac{1}{2}\partial^{\gamma}\partial_{\gamma}h_{\alpha\beta}\,. \tag{39}$$

Using this expression on the LHS of (16), the SHP field equation takes the form of the wave equation

$$-\partial^{\gamma}\partial_{\gamma}h_{\alpha\beta} = -\left(\partial^{\mu}\partial_{\mu} + \sigma\frac{1}{c_5^2}\partial_{\tau}^2\right)h_{\alpha\beta} = 2k_G\left(T_{\alpha\beta} - \frac{1}{2}P_{\alpha\beta}\widehat{T}\right) \tag{40}$$

for which the principal part Green's function [17] is

$$G(x,\tau) = \frac{1}{2\pi}\delta(x^2)\delta(\tau) + \frac{c_5}{2\pi^2}\frac{\partial}{\partial x^2}\theta(-\sigma g_{\alpha\beta}x^{\alpha}x^{\beta})\frac{1}{\sqrt{-\sigma g_{\alpha\beta}x^{\alpha}x^{\beta}}}\,. \tag{41}$$

The leading term, which is denoted G_{Maxwell}, has lightlike support at equal-τ and is dominant at long distances. The second term, denoted $G_{\text{correlation}}$, drops off as $1/\text{distance}^2$ with spacelike support for $\sigma = -1$ and timelike support for $\sigma = +1$. As discussed in Section 1, the leading term was used in SHP electrodynamics to obtain the Coulomb force. Nevertheless, significant complexities arise in determining the appropriate Green's function for a 5D wave equation [17], related to the treatment of multiple poles in the Fourier domain and the associated order of integration. In Section 3, we discuss the difficulties that arise when applying this Green's function in GR.

For a matter source in GR, we consider a distribution of events moving in tandem in the neighborhood of a point with 5D coordinates and normalized velocity

$$X^\alpha(\tau) = (X^\mu(\tau), c_5\tau) \qquad \xi^\alpha(\tau) = \frac{1}{c}\dot{X}^\alpha(\tau) = \frac{1}{c}\frac{dX^\alpha}{d\tau}\,. \tag{42}$$

Writing the spacetime event density in the form $\rho(x,\tau) = \rho(x - X(\tau))$ leads to the mass-energy–momentum tensor

$$T^{\alpha\beta} = M\rho(x,\tau)\dot{X}^\alpha\dot{X}^\beta = M\rho(x,\tau)u^\alpha u^\beta = Mc^2\rho(x,\tau)\xi^\alpha(\tau)\xi^\beta(\tau) \tag{43}$$

which is seen to be conserved by noting that $\partial_\tau\rho(x,\tau) = -\xi^\mu\partial_\mu\rho(x,\tau)$. For this source, a generic solution for the metric perturbation is then

$$h^{\alpha\beta}(x,\tau) = 2k_G\int d^4x'd\tau' G(x - x', \tau - \tau')\left(\xi^\alpha\xi^\beta - \frac{1}{2}\widehat{\eta}^{\alpha\beta}\widehat{\xi}^{\,2}\right)\rho(x',\tau') \tag{44}$$

where $\xi^\alpha = \xi^\alpha(\tau')$ and $\widehat{\xi}^{\,2} = \widehat{\eta}^{\mu\nu}\xi_\mu\xi_\nu$.

2.4. Evolution of the Local Metric

The $3+1$ ADM formalism [25] in GR defines a foliation of 4D spacetime that permits the Einstein field equations to be split into a pair of first-order evolution equations and a pair of propagating but non-evolving constraints. After finding initial conditions that satisfy the constraints, the evolution equations provide an initial value problem for the metric γ_{ij} of space as a function of t. The extraction of the evolution equations and constraints from Einstein's equations relies heavily on the theory of embedded surfaces [22–24], treating 3D space as an equal-t spacelike hypersurface embedded in 4D spacetime. For SHP GR, we generalized the ADM methods to a $4+1$ formalism [29], exploiting the natural foliation of $\mathcal{M}_5$ into equal-τ spacetime hypersurfaces to construct an initial value problem for the spacetime metric $\gamma_{\mu\nu}$ as a function of τ. This section briefly summarizes the main results.

Although we constructed the 5D pseudo-spacetime described in Section 2.1 as an embedding of equal-τ 4D spacetimes $\mathcal{M}$ in $\mathcal{M}_5$, we may take advantage of the ADM techniques by treating $\mathcal{M}_5$ as a starting point, for which we have the known field Equation (16). To facilitate the natural foliation of $\mathcal{M}_5$ into spacetime hypersurfaces we extend the previous partition of coordinate indices to the quintrad indices as

$$a, b, c, d, = 0, 1, 2, 3, 5 \qquad k, l, m, n, \ldots = 0, 1, 2, 3 \tag{45}$$

expressing the vielbein transformation and its inverse as

$$\mathbf{e}_k = e^\mu{}_k\mathbf{g}_\mu + e^5{}_k\mathbf{g}_5 \qquad \mathbf{g}_\mu = E_\mu{}^k\mathbf{e}_k + E_\mu{}^{\bar{5}}\mathbf{e}_5 \tag{46}$$

where the spacetime hypersurface (quatrad) is spanned by $\mathbf{e}_k$, while $\mathbf{e}_5$ points in the direction of τ-evolution normal to $\mathcal{M}$. We introduce the ADM parameterization

$$\mathbf{g}_5 = N^\mu\mathbf{g}_\mu + N\mathbf{e}_5, \tag{47}$$

where N^μ generalizes the shift 3-vector, and N is the lapse function with respect to τ. The vielbein field is now

$$e^\alpha{}_a = \delta^k_a\delta^\alpha_\mu e^\mu{}_k - \delta^5_a\delta^\alpha_\mu\frac{1}{N}N^\mu + \delta^5_a\delta^\alpha_5\frac{1}{N} \qquad E_\alpha{}^a = \delta^\mu_\alpha\delta^a_k E_\mu{}^k + \delta^5_\alpha\left(E_\mu{}^k N^\mu\delta^a_k + N\delta^a_5\right) \tag{48}$$

leading to the coordinate metric on $\mathcal{M}_5$.

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix}, \tag{49}$$

which generalizes the ADM decomposition. The induced metric $\gamma_{\mu\nu}$ is the evolving metric on 4D spacetime $\mathcal{M}$. As in the $3+1$ formalism, the arbitrary functions N and N^μ act as Lagrange multipliers whose choice is comparable to gauge freedom [25], and so the dynamical content in the vielbein field $e^\alpha{}_a$ is contained entirely in the spacetime vierbein field $e^\mu{}_k$.

The foliation of $\mathcal{M}_5$ into equal-τ spacetimes $\mathcal{M}$ provides $P_{\alpha\beta}$, the projection operator used to find (16). Following the ADM procedure, the remaining geometrical structures are then projected from $\mathcal{M}_5$ onto $\mathcal{M}$ in the following steps (see [29] for detail):

1. The covariant derivative D_α on $\mathcal{M}$ is found using $P_{\alpha\beta}$ to project the covariant derivative ∇_α on $\mathcal{M}_5$.
2. The extrinsic curvature $K_{\alpha\beta}$ is defined by projecting the covariant derivative of the unit normal n_α.
3. The projected curvature $\bar{R}^\delta_{\gamma\alpha\beta}$ on $\mathcal{M}$ is defined through the non-commutation of projected covariant derivatives D_α and D_β.
4. The Gauss relation is found by decomposing the 5D curvature $R^\delta_{\gamma\alpha\beta}$ in terms of $\bar{R}^\delta_{\gamma\alpha\beta}$ and $K_{\alpha\beta}$.
5. The energy–momentum tensor is decomposed through the projections

$$\kappa = n_\alpha n_\beta T^{\alpha\beta} \qquad p_\beta = -n_{\alpha'} P_{\beta\beta'} T^{\alpha'\beta'} \qquad S_{\alpha\beta} = P_{\alpha\alpha'} P_{\beta\beta'} T^{\alpha'\beta'} \qquad S = P^{\alpha\beta} S_{\alpha\beta},$$

   .
6. Projecting the 5D curvature $R^\delta_{\gamma\alpha\beta}$ on the unit normal n_α leads to the Codazzi relation, providing a relationship between $K_{\alpha\beta}$ and p_α.
7. Lie derivatives of $P_{\alpha\beta}$ and $K_{\alpha\beta}$ along the direction of τ evolution, given by the unit normal n_α in the coordinate frame, are combined with these ingredients, along with the O(3,1)-symmetric field Equation (16), to obtain τ-evolution equations for $\gamma_{\mu\nu}$ and $K_{\mu\nu}$, as well as a pair of constraints on the initial conditions.

Finally, the evolution equations are found to be

$$\frac{1}{c_5}\partial_\tau \gamma_{\mu\nu} = \mathcal{L}_{\mathbf{N}}\, \gamma_{\mu\nu} - 2NK_{\mu\nu} \tag{50}$$

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -D_\mu D_\nu N + \mathcal{L}_{\mathbf{N}} K_{\mu\nu} + N\left\{ -\sigma \bar{R}_{\mu\nu} + K K_{\mu\nu} - 2K^\lambda_\mu K_{\nu\lambda} + \sigma k_G \left(S_{\mu\nu} - \frac{1}{2} P_{\mu\nu} S \right) \right\} \tag{51}$$

where $\mathcal{L}_{\mathbf{N}}$ is the Lie derivative along N^μ. Their solutions must satisfy the Hamiltonian and momentum constraints

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu} K_{\mu\nu} \right) = -k_G (S + \sigma\kappa) \qquad D_\mu K^\mu_\nu - D_\nu K = k_G p_\nu \,. \tag{52}$$

3. The Metric as Solution to a 5D Wave Equation

We now obtain the metric for a source event of constant velocity using Green's function (41) and discuss the difficulties with this approach. For such a source event, expression (44) for the metric perturbation reduces to

$$h_{\alpha\beta}(x,\tau) = 2k_G Mc^2\, Z_{\alpha\beta}\mathcal{G}[\rho(x,\tau)] \tag{53}$$

where we denote

$$Z_{\alpha\beta} = \xi_\alpha\xi_\beta - \frac{1}{2}\widehat{\eta}_{\alpha\beta}\xi^\mu\xi_\mu \tag{54}$$

$$\mathcal{G}[\rho(x,\tau)] = \int d^4x'\, d\tau' G(x-x',\tau-\tau')\rho(x',\tau') \tag{55}$$

as the kinematic and dynamical factors. For a source evenly distributed along the t-axis in its rest frame, we may write

$$\rho(t-\tau) = 1 \tag{56}$$

so that the event density becomes

$$\rho(x,\tau) = \rho(x - X(\tau)) = \rho(ct - c\tau)\delta^{(3)}(\mathbf{x}) = \delta^{(3)}(\mathbf{x}) \tag{57}$$

as is typically written for a "static" particle in standard relativity. Integration of the event density (57) with Green's function (41) leads to

$$\mathcal{G}_{\text{Maxwell}}[\rho(x,\tau)] = \int d^4x' d\tau' \frac{1}{2\pi}\delta((x-x')^2)\delta(\tau-\tau')\delta^{(3)}(\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}|} \tag{58}$$

$$\mathcal{G}_{\text{correlation}}[\rho(x,\tau)] = \frac{c_5}{2\pi^2}\int d^4x' d\tau' \frac{\partial}{\partial x^2}\frac{\theta(-\sigma(x-x')^\alpha(x-x')_\alpha)}{\sqrt{-\sigma(x-x')^\alpha(x-x')_\alpha}}\delta^{(3)}(\mathbf{x}') = 0 \tag{59}$$

and so taking

$$k_G = \frac{8\pi G}{c^4}\,, \tag{60}$$

the spacetime part of the metric becomes

$$g_{\mu\nu} = \text{diag}\left(-1 + \frac{2GM}{c^2 r}, \left(1 + \frac{2GM}{c^2 r}\right)\delta_{ij}\right) \simeq \text{diag}\left(-U, U^{-1}\delta_{ij}\right)\,, \tag{61}$$

where

$$U = \left(1 - \frac{2GM}{c^2 r}\right) \tag{62}$$

and $r = |\mathbf{x}|$. Naturally, this metric is spatially isotropic and t-independent because the event density is spread evenly along the t-axis. Transforming to spherical coordinates, (61) becomes

$$g_{\mu\nu} = \text{diag}\left(-U, U^{-1}, U^{-1}r^2, U^{-1}r^2\sin^2\theta\right), \tag{63}$$

which, for weak fields, is recognized as the Schwarzschild metric

$$g_{\mu\nu} = \text{diag}\left(-U, U^{-1}, R^2, R^2\sin^2\theta\right) \tag{64}$$

when expressed in the isotropic coordinates [30] defined through

$$R = r\left(1 + \frac{k}{2r}\right)^2. \tag{65}$$

This metric is well-known to be Ricci-flat, $R_{\mu\nu} = 0$, a consequence of t-independence.

In [21], we attempted to modify the calculation for the "static" event by specifying a trajectory narrowly distributed along the t-axis at the spatial origin $\mathbf{x} = 0$. We wrote the event density

$$\rho(x,\tau) = \varphi(t-\tau)\delta^{(3)}(\mathbf{x}) \qquad \varphi_{\max} = \varphi(0) \tag{66}$$

with support in a neighborhood around $t = \tau$. The kinematic factors are

$$\begin{aligned} Z_{00} &= \frac{1}{2} & Z_{05} = Z_{50} &= -\sigma\frac{c_5}{c} = -\sigma\xi^5 \\ Z_{ij} &= \frac{1}{2}\delta_{ij} & Z_{55} &= \frac{c_5^2}{c^2} = \xi_5^2 \end{aligned} \tag{67}$$

where $i, j = 1, 2, 3$, and the dynamic factors are

$$\mathcal{G}_{\text{Maxwell}} = \int d^4x' d\tau' \frac{1}{2\pi}\delta((x-x')^2)\delta(\tau-\tau')\varphi(t'-\tau')\delta^{(3)}(\mathbf{x}') \tag{68}$$

$$\mathcal{G}_{\text{correlation}} = \frac{c_5}{2\pi^2}\int d^4x' d\tau' \frac{\partial}{\partial x^2}\frac{\theta(-\sigma(x-x')^\alpha(x-x')_\alpha)}{\sqrt{-\sigma(x-x')^\alpha(x-x')_\alpha}}\varphi(t'-\tau')\delta^{(3)}(\mathbf{x}')\,. \tag{69}$$

The leading term is easily found as

$$\mathcal{G}_{\text{Maxwell}} = \frac{\varphi(t-R/c-\tau)}{4\pi R} \tag{70}$$

producing a gravitational field with a maximum at $\tau = t - R/c$ for a given $R > 0$. Thus, for a source located at $t_{\text{source}} = \tau$, a test event will feel the strongest gravitational force when moving on the lightcone of the source, accounting for the propagation time of the gravitational field. The similarity to the Coulomb force in (4) in electrodynamics is clear. Detailed analysis of $\mathcal{G}_{\text{correlation}}$ shows [21] that this term vanishes for $\sigma = -1$, while for $\sigma = +1$

$$\mathcal{G}_{\text{Correlation}} = \frac{c_5}{2\pi^2}\frac{1}{\mathbf{x}^2 - c_5^2\tau(2t-\tau)} \tag{71}$$

which drops off as $1/R^2$, leaving the contribution from $\mathcal{G}_{\text{Maxwell}}$ dominant at a long distance. Neglecting the subdominant term, the metric becomes

$$g_{00} = -U = -1 + \frac{k_G M c^2}{4\pi R}\varphi(t-R/c-\tau) \tag{72}$$

$$g_{ij} = V\delta_{ij} = \left[1 + \frac{k_G M c^2}{4\pi r}\varphi(t-R/c-\tau)\right]\delta_{ij} \qquad i,j = 1,2,3 \tag{73}$$

$$g_{05} = g_{50} = -2\sigma\xi^5\frac{k_G M c^2}{4\pi R}\varphi(t-R/c-\tau) \tag{74}$$

$$g_{55} = 2\xi_5^2\frac{k_G M c^2}{4\pi R}\varphi(t-R/c-\tau) \tag{75}$$

which seems reasonable as it modifies the Schwarzschild-like metric (61) by restricting the support of its influence to $\tau \simeq t_{\text{retarded}}$. At some chosen point $x = (ct, \boldsymbol{x})$ in spacetime, the metric is flat at $\tau \to -\infty$, rises to a maximum at $\tau = t - R/c = t_{\text{retarded}}$, and again returns to the flat metric at $\tau \to \infty$. Thus, the effect of the metric in $4+1$ GR depends on the trajectories of the source and test events moving through spacetime. In particular, for a test event evolving as $x = (c(\tau + R/c), \boldsymbol{x})$, the factor $\varphi(t - R/c - \tau) \longrightarrow 1$ and the gravitational field will be determined by the time-independent Schwarzschild metric.

The equations of motion for a nonrelativistic test event are found by expanding (19) as

$$0 = \ddot{x}^\mu + c^2\left(\Gamma^\mu_{00}\dot{t}^2 + 2\Gamma^\mu_{i0}\frac{\dot{x}^i}{c}\dot{t} + \Gamma^\mu_{ij}\frac{\dot{x}^i}{c}\frac{\dot{x}^j}{c} + 2\frac{c_5}{c}\Gamma^\mu_{50}\dot{t} + 2\frac{c_5}{c}\Gamma^\mu_{5i}\frac{\dot{x}^i}{c} + \frac{c_5^2}{c^2}\Gamma^\mu_{55}\right) \tag{76}$$

and neglecting terms containing $\dot{x}^i/c \ll 1$. The nonzero Christoffel symbols are

$$\Gamma^\mu_{00} = -\frac{1}{2c}\delta^{\mu 0}\frac{\partial h_{00}}{\partial t} - \frac{1}{2}\delta^{\mu k}\frac{\partial h_{00}}{\partial x^k} \qquad \Gamma^\mu_{i0} = \frac{1}{2c}\delta^{\mu j}\frac{\partial h_{ji}}{\partial t} - \frac{1}{2}\delta^{\mu 0}\frac{\partial h_{00}}{\partial x^i} \tag{77}$$

$$\Gamma^\mu_{ij} = \frac{1}{2}\delta^{\mu k}\left(\frac{\partial h_{ki}}{\partial x^j} + \frac{\partial h_{kj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^k}\right) + \frac{1}{2c}\delta^{\mu 0}\frac{\partial h_{ij}}{\partial t} \tag{78}$$

$$\Gamma^\mu_{50} = -\frac{1}{2c_5}\delta^{\mu 0}\frac{\partial h_{00}}{\partial \tau} \qquad \Gamma^\mu_{5i} = \frac{1}{2c_5}\delta^{\mu k}\frac{\partial h_{ki}}{\partial \tau} \tag{79}$$

where we used $h_{0i} = 0$, $i = 1, 2, 3$ and dropped $c_5^2/c^2 \ll 1$. The equations of motion split into time and space components

$$0 = \ddot{t} - \frac{1}{2}\frac{\partial h_{00}}{\partial t}\dot{t}^2 - \left(\frac{\partial h_{00}}{\partial \tau} + \dot{\mathbf{x}}\cdot\nabla h_{00}\right)\dot{t} \qquad 0 = \ddot{\mathbf{x}} - \frac{1}{2}c^2\dot{t}^2\,\nabla h_{00}, \tag{80}$$

which differ from (27) because, in this case, h_{00} is explicitly t-dependent. In spherical coordinates, the equations of motion are

$$\ddot{t} = \frac{1}{2}(\partial_t h_{00})\dot{t}^2 + (\partial_\tau h_{00} + \dot{R}\partial_R h_{00})\dot{t} \qquad \ddot{R} = \frac{1}{2}c^2(\partial_R h_{00})\dot{t}^2 + \frac{L^2}{M^2R^3} \tag{81}$$

where again, the conserved angular momentum is $L = MR^2\dot{\phi}$. For a nonrelativistic test event, we can neglect $\ddot{t} \approx 0$ and take $\dot{t} = 1$. Then

$$\partial_r h_{00} = \frac{k_G Mc^2}{4\pi}\partial_r\left(\frac{1}{R}\varphi\right) = \frac{k_G Mc^2}{4\pi}\left(-\frac{1}{R^2}\varphi + \frac{1}{R}\partial_R\varphi\right) \tag{82}$$

and again, using (60) for k_G, we find

$$\ddot{R} = -\frac{GM}{R^2}\varphi\left(1 - \frac{R}{\varphi}\partial_R\varphi\right) + \frac{L^2}{M^2R^3} \tag{83}$$

for the radial equation. Any normalized distribution with maximum $\varphi(0) = 1$ will satisfy $\partial_R\varphi = 0$ when the test event is precisely on the source event's lightcone and so the radial equation will recover Newtonian gravitation. However, for a narrow distribution, $\partial_R\varphi$ will become large and negative just slightly away from the lightcone, and so the terms in parentheses in (83) will likely become very large and possibly negative.

For example, if we consider the Gaussian distribution $\varphi(s) = \exp(-s^2/\lambda_0^2)$ where λ_0 is a time scale representing the width of the event distribution along the t-axis, then for an event precisely on the lightcone of the source

$$1 - \frac{R}{\varphi}\partial_R\varphi = 1 + \frac{R}{\lambda_0 c}\frac{2(t - R/c - \tau)}{\lambda_0} \longrightarrow 1\,. \tag{84}$$

However, if gravity accelerates the test event away from the lightcone and toward the source, so that $R \longrightarrow R - c\lambda_0$, then

$$1 - \frac{R}{\varphi}\partial_R\varphi \longrightarrow 1 + \frac{R - c\lambda_0}{\lambda_0 c}\frac{2(c\lambda_0/c)}{\lambda_0} = -1 + 2\frac{R}{\lambda_0 c} \tag{85}$$

which depends on the ratio $R/\lambda_0 c$. Since we expect the radial distance R to be large, while the width of the event distribution $\lambda_0 c$ is small by assumption, this term magnifies the gravitational field by a large factor. Worse still, if the initial conditions are such that $t - R/c - \tau \sim -\lambda_0$, then the gravitational force will become repulsive. To eliminate these problems from this model would require taking $\lambda_0 \longrightarrow \infty$, which puts $\varphi = 1$. But with this restriction, the metric components (72) and (73) recover the τ-independent metric (61) and the t-localization is lost, thus defeating the goal of the model.

We conclude that these problems will be present whenever the metric takes the separable form φ/R, which must apply to any solution derived from the truncated Green function. As seen in (4), the equal-τ leading term in (41) provides an approximate solution to the 5D wave function adequate for electrodynamics. However, it also provides an exact solution to the 4D wave equation on spacetime, as appears in the linearized form of standard 4D Einstein GR

$$-\frac{1}{2}\partial^\lambda \partial_\lambda h_{\mu\nu} = R^{(4)}_{\mu\nu} = -k_G T_{\mu\nu}(x,\tau) \tag{86}$$

where $R^{(4)}_{\mu\nu}$ is the 4D Ricci tensor. The 4D wave equation thus fails to capture the full geometrical content of $4+1$ weak field GR, where the 5D wave equation is obtained from the linearized 5D Ricci tensor (37). By separating the spacetime terms as

$$\begin{aligned} R_{\mu\nu} &\simeq \frac{1}{2}\left(\partial_\nu \partial_\gamma h^\gamma_\mu + \partial_\mu \partial_\gamma h^\gamma_\nu - \partial^\gamma \partial_\gamma h_{\mu\nu} - \partial_\mu \partial_\nu h\right) && (87)\\ &= R^{(4)}_{\mu\nu} + \frac{1}{2}\left(\partial_\nu \partial_5 h^5_\mu + \partial_\mu \partial_5 h^5_\nu - \partial^5 \partial_5 h_{\mu\nu}\right) && (88)\\ &= R^{(4)}_{\mu\nu} + \partial_5 \Gamma^5_{\mu\nu} && (89) \end{aligned}$$

we immediately notice the reappearance of the Christoffel symbol $\Gamma^5_{\mu\nu}$ that is absent from the geodesic Equation (20) because of the constraint $\ddot{x}^5 \equiv 0$. Since $\Gamma^5_{\mu\nu}$ represents contributions to the 5D curvature from τ-evolution (extrinsic curvature), we see that (86) must fail to take proper account of the τ-evolution of the 4D geometry, regardless of the τ-dependence of the source. So while G_{Maxwell} provides approximate solutions that are adequate for electrodynamics, it misrepresents the evolving spacetime geometry we seek to describe. For this reason, it becomes necessary to approach the metric through the 4 + 1 evolution equations described in Section 2.4.

4. Ansatz Metric

We now switch directions, and instead of specifying the expected source for a gravitational field, we propose an ansatz for the metric that possesses certain expected properties. Using the 5D wave equation in the weak field approximation, we can then derive the source for the ansatz metric, evaluate the Ricci tensor and extrinsic curvature, and set up the $4+1$ evolution equations for general mass–energy configurations. While the ansatz metric for the derived source satisfies the evolution equations exactly, the procedure leads to alternative metrics found as perturbations under general sources. The construction of general expressions characterizing perturbed metrics is discussed in standard texts on numerical relativity, such as [22].

We propose an ansatz metric in the form

$$g^{\alpha\beta} = H^{\alpha\beta}\Phi(t,r,\tau) \tag{90}$$

where $H^{\alpha\beta}$ is a constant kinematic term, $\Phi(t, r, \tau)$ is a spherically symmetric function containing the dependence on t, $r = |\boldsymbol{x}|$, and τ. We specify the component structure of $H^{\alpha\beta}$ to be

$$\begin{gathered} g^{\mu\nu} = \operatorname{diag}\left(-1 + H^{00}\Phi, \left(1 + H^{00}\Phi\right)\delta^{ij}\right) \\ g^{05} = 2\xi^5 H^{00}\Phi \qquad g^{i5} = 0 \qquad g^{55} = 2\xi_5^2 H^{00}\Phi \end{gathered} \tag{91}$$

as found from (67) for a source event evolving on the t-axis.

We are interested in a metric that recovers the $1/r$-dependence of Newtonian gravitation but whose support is restricted to a neighborhood of τ for a given test particle. We consider a source event in its rest frame evolving along its time axis as $x^0(\tau) = x^5$, and a test event similarly evolving along its time axis at a spatial distance r. This suggests a possible functional form of the type

$$\Phi(t, r, \tau) = \frac{1}{\sqrt{r^2 + \left(x^0 - x^5\right)^2}} = \frac{1}{\sqrt{r^2 + c^2(t - \xi^5\tau)^2}} \tag{92}$$

where we denote

$$\rho = \sqrt{r^2 + c^2(t - \xi^5\tau)^2} \tag{93}$$

for convenience. Evaluating the connection for this metric, we study the trajectory of a test event determined by the geodesic Equation (19) with the initial conditions

$$r(0) = R \qquad \dot{r}(0) = 0 \qquad t(0) = 0 \qquad \dot{t}(0) = \xi^5 \tag{94}$$

for which $\rho \longrightarrow R$, and Φ takes on its maximum value. If the test event deviates from the trajectory $\left(x^0, \boldsymbol{x}\right) = \left(\xi^5\tau, R\hat{\boldsymbol{r}}\right)$, then the strength of the metric will diminish. Because $\Phi(t, r, \tau)$ has a maximum at $t = \xi^5\tau$ with respect to t but not with respect to r, this functional form does not suffer from the difficulties associated with the geodesic Equation (83), as discussed in Section 3.

To write the geodesic equations, we use the derivatives

$$\frac{\partial\Phi}{\partial t} = -\frac{c^2\left(t - \xi^5\tau\right)}{\rho^3} \qquad \frac{\partial\Phi}{\partial\tau} = \frac{\xi^5 c^2\left(t - \xi^5\tau\right)}{\rho^3} \qquad \frac{\partial\Phi}{\partial x^i} = -\frac{x_i}{\rho^3} \tag{95}$$

leading to the connection in the explicit form

$$\Gamma^\mu_{00} = \frac{1}{2}H^{00}\frac{1}{\rho^3}\left(\delta^{\mu 0}c\left(t - \xi^5\tau\right) + \delta^{\mu k}x_k\right) \tag{96}$$

$$\Gamma^\mu_{i0} = H^{00}\frac{1}{2}\frac{1}{\rho^3}\left[-\delta^\mu_i c\left(t - \xi^5\tau\right) + \delta^{\mu 0}x_i\right] \tag{97}$$

$$\Gamma^\mu_{ij} = -\frac{1}{2}H^{00}\frac{1}{\rho^3}\left[\delta^\mu_i x_j + \delta^\mu_j x_i - \delta^{\mu k}\delta_{ij}x_k + \delta^{\mu 0}\delta_{ij}c\left(t - \xi^5\tau\right)\right] \tag{98}$$

$$\Gamma^\mu_{50} = -\frac{1}{2}H^{00}\frac{1}{\rho^3}\left[2\sigma\xi^5\delta^{\mu k}x_k + \delta^{\mu 0}c\left(t - \xi^5\tau\right)\right] \tag{99}$$

$$\Gamma^\mu_{5i} = -\frac{1}{2}H^{00}\frac{1}{\rho^3}\left(-\delta^\mu_i c\left(t - \xi^5\tau\right) + \xi^5\delta^{\mu 0}2\sigma x_i\right) \tag{100}$$

$$\Gamma^\mu_{55} = H^{00}\frac{1}{\rho^3}\left[\delta^{\mu k}\xi_5^2 x_k + \delta^{\mu 0}c_5\left(t - \xi^5\tau\right)\left(2\sigma - \xi^5\right)\right] \tag{101}$$

so that splitting the equations of motion (76) into time and space components, we arrive at

$$0 = \ddot{t} + H^{00}\frac{1}{\rho^3}\left\{\frac{1}{2}c^2\left(t-\xi^5\tau\right)\dot{t}^2 + x_i\dot{x}^i\dot{t} - \frac{1}{2}\left(t-\xi^5\tau\right)\delta_{ij}\dot{x}^i\dot{x}^j \right.$$
$$\left. -cc_5\left(t-\xi^5\tau\right)\dot{t} - 2\sigma\xi_5^2 x_i\dot{x}^i + cc_5\xi_5^2\left(t-\xi^5\tau\right)\left(2\sigma-\xi_5^2\right)\right\} \tag{102}$$

$$0 = \ddot{x}^k + H^{00}\frac{1}{\rho^3}c^2\left\{\frac{1}{2}x^k\dot{t}^2 - c\left(t-\xi^5\tau\right)\frac{\dot{x}^k}{c}\dot{t} - x_j\frac{\dot{x}^k}{c}\frac{\dot{x}^j}{c} + \frac{1}{2}x^k\frac{\dot{x}_j}{c}\frac{\dot{x}^j}{c}\right.$$
$$\left. -2\sigma\xi_5^2 x^k\dot{t} + c_5\left(t-\xi^5\tau\right)\frac{\dot{x}^k}{c} + \xi_5^4 x^k\right\}. \tag{103}$$

In the neighborhood of the initial conditions, the equations of motion reduce to

$$0 = \ddot{t} + H^{00}\frac{x_i}{r^3}\frac{\dot{x}^i}{c}c\left(\dot{t} - 2\sigma\xi_5^2\right) \tag{104}$$

$$0 = \ddot{x}^k + H^{00}\frac{1}{r^3}c^2\left\{\frac{1}{2}x^k\dot{t}^2 - x_j\frac{\dot{x}^k}{c}\frac{\dot{x}^j}{c} + \frac{1}{2}x^k\frac{\dot{x}_j}{c}\frac{\dot{x}^j}{c} - 2\sigma\xi_5^2 x^k\dot{t} + \xi_5^4 x^k\right\} \tag{105}$$

so that neglecting $\dot{x}/c \ll 1$ in the nonrelativistic regime the time equation reduces to

$$0 = \ddot{t} \;\longrightarrow\; \dot{t}(\tau) = \xi^5 \tag{106}$$

and the space equation becomes

$$0 = \ddot{x}^k + H^{00}\frac{1}{r^3}c^2\xi_5^2\left\{\frac{1}{2}x^k - 2\sigma\xi^5 x^k + \xi_5^2 x^k\right\} \simeq \ddot{x}^k + \frac{1}{2}c^2\xi_5^2 H^{00}\frac{1}{r^2}\hat{x}^k \tag{107}$$

which recovers Newtonian gravitation if we take $H^{00} = 2GM/\xi_5^2 c^2 = 2GM/c_5^2$.

Now, considering a relativistic test event with the initial condition $t = \xi^5\tau$, and again keeping only lower order terms in ξ^5, the equations of motion become

$$0 = \ddot{t} + H^{00}\frac{1}{r^2}\hat{x}_i\dot{x}^i\dot{t} \;\longrightarrow\; c_5\ddot{t} \simeq -\frac{2GM}{r^2}\hat{x}_i\frac{\dot{x}^i}{c} \tag{108}$$

$$0 = \ddot{x}^k + \frac{GM}{r^2}\left\{\hat{x}^k - \left(\frac{2\hat{x}_j\dot{x}^k - \hat{x}^k\dot{x}_j}{c}\right)\frac{\dot{x}^j}{c}\frac{c^2}{c_5^2}\right\}. \tag{109}$$

Since $c/c_5 > 1$ by assumption, while for non-tachyonic particles $\dot{x}^k < c$, we see here that the event trajectory may differ from the post-Newtonian relativistic form seen in (80).

In linearized GR, we may attribute the ansatz metric to a source by writing the field Equation (16) in the linearized form (40)

$$k_G\left(T_{\alpha\beta} - \frac{1}{2}\hat{\eta}_{\alpha\beta}\widehat{T}\right) = -\frac{1}{2}\partial^\gamma\partial_\gamma h_{\alpha\beta} \tag{110}$$

with spacetime components

$$k_G\left(T_{\mu\nu} - \frac{1}{2}\hat{\eta}_{\mu\nu}\widehat{T}\right) = -\frac{1}{2}H^{00}\delta_{\mu\nu}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \sigma\frac{1}{c_5^2}\frac{\partial^2}{\partial\tau^2}\right)\Phi \tag{111}$$

for any spacetime functional Φ. Combining the derivatives of (92)

$$\nabla^2\Phi = -\frac{3c^2\left(t-\xi^5\tau\right)^2}{\rho^5} \tag{112}$$

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = \frac{r^2-2c^2\left(t-\xi^5\tau\right)^2}{\rho^5} \tag{113}$$

and

$$\sigma\frac{1}{c_5^2}\frac{\partial^2}{\partial \tau^2}\Phi = -\sigma\frac{r^2-2c^2\left(t-\xi^5\tau\right)^2}{\rho^5} \tag{114}$$

we are led to

$$k_G\left(T_{\mu\nu}-\frac{1}{2}\hat{\eta}_{\mu\nu}\widehat{T}\right) = -\frac{1}{2}\frac{H^{00}}{\rho^5}\left[(\sigma-1)r^2+(5-2\sigma)c^2\left(t-\xi^5\tau\right)^2\right]\delta_{\mu\nu} \tag{115}$$

and if we again take $H^{00} = 2GM/c_5^2$ along with $k_G = 8\pi G/c^2c_5^2$, then

$$T_{\mu\nu} = \frac{Mc^2}{4\pi\rho^5}\left[(\sigma-1)r^2+(5-2\sigma)c^2\left(t-\xi^5\tau\right)^2\right]\mathrm{diag}(-1,0,0,0) \tag{116}$$

where only the energy component T_{00} is non-vanishing. The structure of this source is easiest to see for $\sigma = 1$, in which case

$$T_{\mu\nu} = 3\frac{Mc^2}{4\pi\rho^5}\left[c\left(t-\xi^5\tau\right)\right]^2\mathrm{diag}(-1,0,0,0) \tag{117}$$

where $c^2\left(t-\xi^5\tau\right)^2/\rho^5$ has units of length^{-3}, as expected for a particle density in space. Although $T_{\mu\nu}$ appears to vanish at $t = \xi^5\tau$, we recall that under this condition, $\rho \longrightarrow r$ so that Φ becomes independent of t and τ. In this case, the wave Equation (111) from which we derive the source reduces to

$$\partial^\gamma\partial_\gamma\left(\frac{1}{\rho}\right) \longrightarrow \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^{(3)}(\boldsymbol{x}) \tag{118}$$

which describes a point source evenly spread along the t-axis. In contrast, at a small spatial distance $r \ll c(t-\xi^5\tau)$, we have $c^2(t-\xi^5\tau)^2/\rho^5 \longrightarrow [c^2(t-\xi^5\tau)]^{-3}$, which describes a narrow particle density along the t-axis centered at $t = \xi^5\tau$. This localization at a coordinate time t determined by the chronological time τ expresses the desired τ-evolution. As expected, the source (117) describes a matter distribution evolving with τ, leading to the metric (91), which similarly evolves with τ, and the geodesic Equations (102) and (103), whose coefficients evolve with τ.

5. $4+1$ Evolution Equations

In $4+1$ numerical relativity, we obtain the metric induced by a given source $T_{\mu\nu}$ by writing the evolution Equations (50) and (51) for a metric $\gamma_{\mu\nu}$ expressed as a perturbation of a known metric $\tilde{\gamma}_{\mu\nu}$. In $3+1$ relativity, it is common to use the York–Lichnerowicz [31,32] conformal decomposition $\gamma_{\mu\nu} = \phi^4\tilde{\gamma}_{\mu\nu}$, for which the evolution equations become an initial value problem for the factor ϕ, with initial conditions for the projected Ricci tensor $\bar{R}_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$ derived from the unperturbed metric.

In weak field approximation, the $4+1$ evolution equations are simplified [29] by discarding terms of the order $\left(h_{\alpha\beta}\right)^2 \approx 0$. Combining Equations (49) and (91), we express the 5D metric as

$$\|g_{\alpha\beta}\| = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} = \begin{bmatrix} \eta_{\mu\nu} + h_{\mu\nu} & h_{\mu 5} \\ h_{\mu 5} & \eta_{55} + h_{55} \end{bmatrix} \tag{119}$$

from which

$$N \approx 1 + \sigma h_{55}/2 \qquad N_\mu = h_{\mu 5}\,, \tag{120}$$

the Lie derivative of the metric reduces to

$$\mathcal{L}_{\mathbf{N}} \gamma_{\mu\nu} \approx \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu}\,, \tag{121}$$

and the metric evolution Equation (50) becomes

$$\frac{1}{c_5}\partial_\tau \gamma_{\mu\nu} = \frac{1}{c_5}\partial_\tau h_{\mu\nu} \approx \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} - 2K_{\mu\nu}\,, \tag{122}$$

confirming that $K_{\mu\nu}$ is of the order $h_{\alpha\beta}$. As a result, we may neglect the Lie derivative in the second evolution Equation (51)

$$\mathcal{L}_{\mathbf{N}} K_{\mu\nu} = N^\lambda \partial_\lambda K_{\mu\nu} + K_{\lambda\nu}\partial_\mu N^\lambda + K_{\mu\lambda}\partial_\nu N^\lambda \propto \left(h_{\alpha\beta}\right)^2 \approx 0 \tag{123}$$

as well as quadratic terms in $K_{\mu\nu}$. Equation (51) for the extrinsic curvature now reduces to

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -\frac{1}{2}\sigma\partial_\mu\partial_\nu h_{55} - \sigma\bar{R}_{\mu\nu} + \sigma\frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\bar{\eta}_{\mu\nu}\bar{S}\right) \tag{124}$$

and the constraints become

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) \approx \bar{R} = -\sigma\frac{16\pi G}{c^4}\kappa \tag{125}$$

$$D_\mu K^\mu_\nu - D_\nu K \approx \partial_\mu K^\mu_\nu - \partial_\nu K = \frac{8\pi G}{c^4} p_\nu \tag{126}$$

with source terms

$$S_{\mu\nu} \approx T_{\mu\nu} \qquad p_\nu \approx -T_{5\nu} \qquad \kappa \approx T_{55}\,. \tag{127}$$

As an aside, we note that the 4 + 1 initial value problem for weak fields can be simply related to the 5D wave equation found from the Ricci tensor [29]. To see this, we write the five components of Equations (16) and (39) as

$$R_{5\mu} = -\frac{1}{2}\partial^\beta\partial_\beta h_{5\mu} = k_G T_{\mu 5} = -k_G p_\mu \tag{128}$$

$$R_{55} = -\frac{1}{2}\partial^\beta\partial_\beta h_{55} = k_G T_{55} = k_G \kappa \tag{129}$$

from which it follows that any weak field solution with the component structure (91) will satisfy the constraints. Before imposing the Lorenz gauge for the spacetime terms, we may split (37) as

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\eta^{\lambda\rho}\left(\partial_\rho\partial_\mu h_{\lambda\nu} - \partial_\nu\partial_\mu h_{\lambda\rho} + \partial_\nu\partial_\lambda h_{\mu\rho} - \partial_\rho\partial_\lambda h_{\mu\nu}\right) \\ &\quad + \frac{1}{2}\eta^{55}\left(\partial_5\partial_\mu h_{5\nu} - \partial_\nu\partial_\mu h_{55} + \partial_\nu\partial_5 h_{\mu 5} - \partial_5^2 h_{\mu\nu}\right) \\ &= \frac{1}{2}\left(\partial_\mu\partial^\lambda h_{\lambda\nu} + \partial_\nu\partial^\rho h_{\mu\rho} - \partial^\lambda\partial_\lambda h_{\mu\nu} - \partial_\nu\partial_\mu h\right) + \frac{1}{2}\sigma\partial_5\left(\partial_\mu h_{5\nu} + \partial_\nu h_{\mu 5} - \partial_5 h_{\mu\nu}\right) \\ &= \bar{R}_{\mu\nu} + \sigma\partial_5 K_{\mu\nu} \end{aligned} \tag{130}$$

in which the first term is the projected Ricci tensor $\bar{R}_{\mu\nu}$ and the second term contains the form of the extrinsic curvature $K_{\mu\nu}$ in the weak field approximation. Using the terms in parentheses for $K_{\mu\nu}$, this expression can be rearranged as

$$\frac{1}{c_5}\partial_\tau h_{\mu\nu} = \partial_\mu h_{5\nu} + \partial_\nu h_{\mu 5} - 2K_{\mu\nu} \tag{131}$$

which recovers the first evolution Equation (122). Similarly, combining (130) with (16) and absorbing the negligible term $-\sigma\partial_\mu\partial_\nu h_{55}/2$ into $\bar{R}_{\mu\nu}$, we obtain

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -\sigma\bar{R}_{\mu\nu} + \sigma k_G\left(T_{\mu\nu} - \frac{1}{2}\bar{\eta}_{\mu\nu}\hat{T}\right) \tag{132}$$

which recovers (124). We thus see that the first-order evolution equations can be found by introducing the auxiliary variable $K_{\mu\nu}$ (almost, but not quite conjugate to $\gamma_{\mu\nu}$) and rearranging terms in the linearized field equations.

The initial value problem requires the Ricci tensor and extrinsic curvature computed from the unperturbed metric as initial conditions. Splitting the Lorenz gauge condition into spacetime and five parts

$$\partial^\lambda h_{\mu\lambda} = -\partial^5 h_{\mu 5} + \frac{1}{2}\partial_\mu h \tag{133}$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$ is the 5D trace, the projected Ricci tensor may be written as

$$\bar{R}_{\mu\nu} = -\frac{1}{2}\partial^\lambda\partial_\lambda h_{\mu\nu} - \frac{1}{2}\sigma\partial_5\left(\partial_\mu h_{\nu 5} + \partial_\nu h_{\mu 5}\right) \tag{134}$$

providing a relatively simple means to calculate $\bar{R}_{\mu\nu}$. As an aside, we note that combining (132) and (134) to eliminate $K_{\mu\nu}$, we recover the wave Equation (40)

$$\begin{aligned} R_{\mu\nu} &= \bar{R}_{\mu\nu} + \sigma\partial_5 K_{\mu\nu} \\ &= -\frac{1}{2}\partial^\lambda\partial_\lambda h_{\mu\nu} - \frac{1}{2}\sigma\partial_5\left(\partial_\mu h_{\nu 5} + \partial_\nu h_{\mu 5}\right) + \frac{1}{2}\sigma\partial_5\left(\partial_\mu h_{5\nu} + \partial_\nu h_{\mu 5} - \partial_5 h_{\mu\nu}\right) \\ &= -\frac{1}{2}\left(\partial^\lambda\partial_\lambda + \sigma\partial_5^2\right)h_{\mu\nu} = k_G\left(T_{\mu\nu} - \frac{1}{2}\bar{\eta}_{\mu\nu}\hat{T}\right), \end{aligned} \tag{135}$$

much as the elimination of momentum in favor of velocity in the Hamiltonian equations leads to second-order differential equations in particle mechanics.

For an ansatz metric with the component structure (91), the source can be found directly from (111) as we achieved in expression (115). The projected Ricci tensor is found using (134)

$$\bar{R}_{\mu\nu} = H^{00}\left[-\frac{1}{2}\delta_{\mu\nu}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) + \xi^5\frac{1}{c_5}\frac{\partial}{\partial\tau}\left(\delta_{\nu 0}\frac{\partial}{\partial x^\mu} + \delta_{\mu 0}\frac{\partial}{\partial x^\nu}\right)\right]\Phi \tag{136}$$

and the extrinsic curvature is

$$K_{\mu\nu} = -H^{00}\left[\sigma\xi^5\left(\delta_{\nu 0}\frac{\partial}{\partial x^\mu} + \delta_{\mu 0}\frac{\partial}{\partial x^\nu}\right) + \frac{1}{2}\delta_{\mu\nu}\frac{1}{c_5}\frac{\partial}{\partial\tau}\right]\Phi \tag{137}$$

from which

$$\sigma\partial_5 K_{\mu\nu} = H^{00}\left[-\xi^5\frac{1}{c_5}\frac{\partial}{\partial\tau}\left(\delta_{\nu 0}\frac{\partial}{\partial x^\mu} + \delta_{\mu 0}\frac{\partial}{\partial x^\nu}\right) - \sigma\frac{1}{2}\delta_{\mu\nu}\frac{1}{c_5^2}\frac{\partial^2}{\partial\tau^2}\right]\Phi\,. \tag{138}$$

We see that the off-diagonal terms in $\bar{R}_{\mu\nu}$ and $\sigma\partial_5 K_{\mu\nu}$ mutually cancel, leaving the source diagonal as required. Inserting the explicit functional form (92) for $\Phi(t, r, \tau)$ we obtain

$$\bar{R}_{\mu\nu} = \frac{H^{00}}{\rho^5}\left\{-\frac{1}{2}\left[r^2 - 5c^2(t-\tau\xi)^2\right]\delta_{\mu\nu} + 2\xi^5\left(r^2 - 2c^2\left(t-\tau\xi^5\right)^2\right)\delta_{\mu 0}\delta_{\nu 0} \right.$$
$$\left. -3c\left(t-\xi^5\tau\right)\left(\delta^0_\mu\delta^i_\nu + \delta^i_\mu\delta^0_\nu\right)x_i\right\} \tag{139}$$

and

$$K_{\mu\nu} = \sigma\frac{H^{00}}{\rho^5}\left\{-\frac{1}{2}c\left(t-\xi^5\tau\right)\delta_{\mu\nu} + 2\xi^5 c\left(t-\xi^5\tau\right)\delta_{\mu 0}\delta_{\nu 0} \right.$$
$$\left. +\xi^5\left(\delta^0_\mu\delta^i_\nu + \delta^i_\mu\delta^0_\nu\right)x_i\right\}. \tag{140}$$

Using any suitable functional $\Phi(t, r, \tau)$ to derive an ansatz metric (91), its unperturbed source (111), along with initial conditions (136) and (137), the weak field evolution Equations (122) and (124) are satisfied exactly. Writing the unperturbed source as $T^{(0)}_{\mu\nu}$, the perturbed source $T_{\mu\nu}$ can be found using standard approximation methods. For example, we may write a source parameterized by λ as

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \lambda T^{(1)}_{\mu\nu} + \lambda^2 T^{(2)}_{\mu\nu} + \cdots \tag{141}$$

where $\lambda = 0 \longrightarrow T_{\mu\nu} = T^{(0)}_{\mu\nu}$, and we seek a perturbed metric

$$h_{\alpha\beta} = h^{(0)}_{\alpha\beta} + \lambda h^{(1)}_{\alpha\beta} + \lambda^2 h^{(2)}_{\alpha\beta} + \cdots \tag{142}$$

up to the desired order in λ. For example, writing

$$\Phi = \frac{1}{\sqrt{r^2 + c^2(t-\xi^5\tau)^2}} \longrightarrow \frac{1}{\sqrt{r^2 + c^2 A(t,\tau)}} \tag{143}$$

where

$$A(t,\tau) = \left(t-\xi^5\tau\right)^2 + \lambda\alpha(t,\tau) \tag{144}$$

leads to

$$\Phi \simeq \frac{1}{\sqrt{r^2 + c^2(t-\xi^5\tau)^2}} - \frac{1}{2}\frac{\lambda\alpha(t,\tau)}{r^2 + c^2(t-\xi^5\tau)^2} \tag{145}$$

to first order. In the linearized theory, we also have the first-order

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + \lambda R^{(1)}_{\mu\nu} \qquad K_{\mu\nu} = K^{(0)}_{\mu\nu} + \lambda K^{(1)}_{\mu\nu} \tag{146}$$

where $R^{(0)}_{\mu\nu}$ and $K^{(0)}_{\mu\nu}$ are derived from $T^{(0)}_{\mu\nu}$. And since the evolution equations are linear and solved exactly by $h^{(0)}_{\mu\nu}$ for $T^{(0)}_{\mu\nu}$, the evolution equations for the perturbed metric reduce to the evolution equations for the perturbation itself.

$$\partial_5 h^{(1)}_{\mu\nu} = -2K^{(1)}_{\mu\nu} + \partial_\mu h^{(1)}_{5\nu} + \partial_\nu h^{(1)}_{5\mu} \tag{147}$$

$$\partial_5 K^{(1)}_{\mu\nu} = -\sigma\left[R^{(1)}_{\mu\nu} - \frac{8\pi G}{c^4}\left(S^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S^{(1)}\right)\right] \tag{148}$$

where we may use (136) and (137) to express $R^{(1)}_{\mu\nu}$ and $K^{(1)}_{\mu\nu}$ in terms of the perturbation $\alpha(t,\tau)$. We note that in order to preserve the structure of the evolution equations as an initial value problem, we must preserve $\partial_5\alpha(t,\tau)$ as an independent dynamical quantity and choose the initial values for $\alpha(t,0)$ and $\partial_5\alpha(t,0)$.

6. Discussion

In this paper, we considered the relationship between the evolving 4D metric $\gamma_{\mu\nu}(x,\tau)$ and the evolving matter source $T_{\mu\nu}(x,\tau)$ in weak-field $4+1$ general relativity and discussed an approach to determining the metric for a given source. We demonstrated the difficulties associated with applying the principal part Green's function (41) for the 5D wave equation to a source localized and evolving uniformly in its rest frame. We saw that the geodesic equations for a test event moving in the spacetime described by the resulting metric are highly sensitive to initial conditions and become unstable under general choices for τ-synchronization with the source. This issue is attributed to the product structure $\phi(t,r,\tau)/r$ for general solutions found from the leading term $G_{\text{Maxwell}}(x,\tau)$ of (41), where ϕ is a narrow distribution that localizes the source along the t-axis at a given τ. This product structure was seen to result from the fact that in addition to approximate solutions to the 5D wave equations, G_{Maxwell} provides exact solutions to the 4D wave function. Consequently, these solutions lead to metric solutions to the 4D Einstein equations, not correctly accounting for the evolution of spacetime under τ. Thus, while solving the wave equation for the linearized field equations is normally the simplest way to obtain the metric, we concluded that the shortcomings of the available Green's function for the 5D wave equation make this method impractical. As for the case of general fields in 4D GR, the most suitable approach to the metric is to solve the ADM-like first-order evolution equations.

In order to apply the 4 + 1 formalism, we must begin with an unperturbed metric ansatz. Thus, we proposed an ansatz with many of the expected properties for the metric and found the associated matter source from the linearized field equations. In general, the source was seen to describe an event density evolving at a constant velocity along the t-axis in its rest frame, within a small volume $[c(t-\xi^5\tau)]^3$. At a given point $(t,\mathbf{0})$ in the rest frame of the source, $T_{\mu\nu}(x,\tau)$ only has support in a narrow region of chronological time around $\tau = t/\xi^5$. Consequently, the induced metric at a given point $(t,\boldsymbol{x})$ drops off as $1/|\boldsymbol{x}|$, but its support is similarly restricted to this narrow region around τ. As a test event evolves along some trajectory $x^\mu(\tau)$, it experiences the spacetime geometry as a function of τ, as well as its coordinate location. The geodesic equations for this metric were seen to recover Newtonian gravitation in the nonrelativistic limit, with possible deviations from standard models at relativistic energies.

We next proceeded to characterize the $4+1$ evolution equations in the weak field limit and demonstrate their relationship with the 5D Ricci tensor. We showed that the first-order initial value problem emerges from the linearized 5D field equations by introducing the extrinsic curvature $K_{\mu\nu}$ as an auxiliary variable, much in the way that Hamiltonian mechanics poses first-order equations of motion for position and its conjugate momentum, treated as an independent quantity. The 5D wave equation is easily found by combining expressions for the projected Ricci tensor $\bar{R}_{\mu\nu}$ and $K_{\mu\nu}$, in such a way that off-diagonal terms cancel. Given the linearized evolution equations and the ansatz metric that solves them exactly for a given source, the initial value problem for a perturbed metric is just the linearized evolution equations for the perturbation terms themselves. We showed that with a presumed general structure for the perturbed metric, the evolution equations become a set of first-order (in τ) partial differential equations in the parameters of the perturbation.

While the ansatz chosen for study in this paper contains the general features expected in an evolving metric, these methods can be applied to other forms. For example, writing the functional dependence of the metric as

$$\Phi(r,t,\tau) = \frac{1}{r} e^{-\left(t-\xi^5\tau\right)^2/\tau_0^2} \tag{149}$$

for some short time scale τ_0, we easily find the source to be

$$T_{\mu\nu} = \frac{Mc^2}{4\pi\rho^5}\mathrm{diag}(-1,0,0,0)\left[-\frac{1}{4\pi}\delta^3(\mathbf{x}) + (1-\sigma)\frac{2}{rc_5^2\tau_0^2}\left(1-\frac{2(t-\tau)^2}{\tau_0^2}\right)\right]e^{-(t-\tau)^2/\tau_0^2}\,. \tag{150}$$

In the neighborhood of $t = \xi^5\tau$, the geodesic equations become

$$\ddot{t} = 0 \;\longrightarrow\; \dot{t} = \xi^5 \tag{151}$$

and

$$0 = \dot{x}^k + \frac{GM}{r^2}\left\{\hat{x}^k - \left(\frac{2\hat{x}_j\dot{x}^k - \hat{x}^k\dot{x}_j}{c}\right)\frac{\dot{x}^j}{c}\frac{c^2}{c_5^2}\right\} \tag{152}$$

which is identical to the equation of motion (109) found using the functional form for Φ in (92). These results suggest that other forms for Φ may lead to phenomenological behavior of a similar type. As we showed, perturbative modifications to (149) may be considered by again replacing $(t-\xi^5\tau)^2 \;\longrightarrow\; A(t,\tau)$, as we performed in (144). Numerical solutions for perturbed metrics will be presented in a subsequent paper.

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