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An Analysis of the Convergence Problem of a Function in Functional Norms by Applying the Generalized Nörlund-Matrix Product Operator

Hari M. Srivastava [1,2,3,4,5,*], Hare K. Nigam [6] and Swagata Nandy [6]

[1] Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[3] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan
[4] Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy
[5] Center for Converging Humanities, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea
[6] Department of Mathematics, Central University of South Bihar, Gaya 824236, Bihar, India; hknigam@cusb.ac.in (H.K.N.); swagata@cusb.ac.in (S.N.)
* Correspondence: harimsri@math.uvic.ca

Abstract: In this paper, we analyze the convergence problems of function g of Fourier series in Besov and generalized Zygmund norms using generalized Nörlund-Matrix $(N^{p,q}A)$ means of Fourier series. Convergence results are also compared by means of applications.

Keywords: degree of convergence; Besov space (B.S.); generalized Zygmund spaces (G.Z.S.); generalized Nörlund-Matrix $(N^{p,q}A)$ operator; Fourier series (F.S.)

MSC: 41A10; 42A10; 42B05

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1. Introduction

Besov $(B_\alpha^\rho(L^\upsilon))$ spaces describe the smoothness of the functions. In fact, Besov $B_\alpha^\rho(L^\upsilon)$ spaces are a set of functions from fundamental spaces L^υ, where ρ and α denote smoothness and its finer gradation, respectively (see Equation (8), p. 5). These spaces naturally appear in many fields of pure and applied mathematics in general and in the field of mathematical analysis in particular. Presently, Besov spaces are defined by using Fourier transforms and modulus of smoothness of the function. These two definitions of Besov spaces are equivalent unless $\upsilon < 1$ and ρ is small. The Besov spaces defined by the modulus of smoothness appear more naturally in the area of approximation theory [1]. Generalized Zygmund spaces are also powerful spaces which use the definition of modulus of smoothness of the function and appear naturally in the area of approximation theory.

Besides their wide applications in different studies of approximation theory, these spaces are also used for studying convergence problems of a function using summability means.

We note that the investigators [2–19], etc., have studied the degree of approximation of a function of Fourier series in Lipschitz, Hölder and generalized Hölder spaces using summability means.

This paper contains an analysis of the convergence problems of function g of Fourier series in Besov and generalized Zygmund norms using matrix-generalized Nörlund $(AN^{p,q})$ means of Fourier series. We compare our convergence results by means of applications.

The organization of the paper is as follows: In Section 2, we provide important definitions related to our work. In Section 3, we prove the auxiliary results, which are

used in the proofs of our main results. In Section 4, we prove our convergence results. In Section 5, convergence results are compared by means of applications. Section 6 contains a conclusion.

2. Preliminaries

2.1. Fourier Series

Let g be a Lebesgue integrable function with period 2π on the interval $[-\pi, \pi]$. The Fourier transform of a function g is given by

$$g(\omega) = \int_{-\infty}^{\infty} \exp(-i\omega x) g(x) dx =< g, e^{i\omega x} >, \tag{1}$$

where $g(\omega)$ is a function of frequency ω and $< g, e^{i\omega x} >$ is the inner product in a Hilbert space. Thus, the transform of a signal decomposes it into a sine wave of different frequencies and phases.

The Fourier series of function g is given by

$$g(x) \sim \frac{a_0}{2} + \sum_{\mu=1}^{\infty} (a_\mu \cos \mu x + b_\mu \sin \mu x), \tag{2}$$

where a_0, a_μ and b_μ are the Fourier coefficients.

The μth partial sum of (1) is given by ([20])

$$s_\mu(g; x) - g(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(x, y) D_\mu(y) dy, \tag{3}$$

where

$$\phi(x, y) = g(x + y) + g(x - y) - 2g(x)$$

and $D_\mu(y)$ (Dirichlet Kernal) is defined by

$$D_\mu(y) = \frac{\sin\left(\mu + \frac{1}{2}\right) y}{\sin\left(\frac{y}{2}\right)}.$$

2.2. Summability Operators

Let

$$u_0 + u_1 + u_3 + \cdots = \sum_{\kappa=0}^{\infty} u_\kappa \tag{4}$$

be an infinite series with the sequence of its κth partial sum s_κ.

The infinite sequence of κth partial sum $\{s_\kappa\}$ is

$$\{u_0, u_1, u_3, \cdots\} = \{s_\kappa\}_0^{\infty}. \tag{5}$$

Generalized Nörlund Matrix $(N^{p,q}A)$ Operator

Let $\{p_\kappa\}$ and $\{q_\kappa\}$ be the sequence of constants, real or complex such that

$$P_\kappa = p_0 + p_1 + \cdots + p_\kappa = \sum_{v=0}^{\kappa} p_v \to \infty, \text{ as } \kappa \to \infty,$$

$$Q_\kappa = q_0 + q_1 + \cdots + q_\kappa = \sum_{v=0}^{\kappa} q_v \to \infty, \text{ as } \kappa \to \infty$$

and

$$R_\kappa = p_0 q_\kappa + p_1 q_{\kappa-1} + \cdots + p_\kappa q_0 = \sum_{v=0}^{\kappa} p_v q_{\kappa-v} \to \infty, \text{ as } \kappa \to \infty.$$

Given two sequences $\{p_\kappa\}$ and $\{q_\kappa\}$, convolution $(p*q)_\kappa$ is defined as

$$R_\kappa = (p*q)_\kappa = \sum_{\mu=0}^{\kappa} p_{\kappa-\mu} q_\mu.$$

We write

$$\gamma_\kappa^{N^{p,q}} = \frac{1}{R_\kappa} \sum_{\mu=0}^{\kappa} p_{\kappa-\mu} q_\mu s_\mu.$$

If $R_\kappa \neq 0 \; \forall \; \kappa$, the generalized Nörlund transform $(N^{p,q})$ of the sequence $\{s_\kappa\}$ is the sequence $\{\gamma_\kappa^{N^{p,q}}\}$.

If $\{\gamma_\kappa^{N^{p,q}}\} \to s$, as $\kappa \to \infty$, the infinite series given by (4) or a sequence given by (5) is summable to s by the generalized Nörlund $(N^{p,q})$ method and is denoted by $s_\kappa \to s(N^{p,q})$.

The necessary and sufficient conditions for $(N^{p,q})$ method to be regular are

$$\sum_{\mu=0}^{\kappa} |p_{\kappa-\mu} q_\mu| = O(|R_\kappa|) \text{ and } p_{\kappa-\mu} = O(|R_\kappa|) \text{ as } \kappa \to \infty$$

for every fixed $\mu \geq 0$ for which $q_\mu \neq 0$.

Let $A = (a_{\kappa,\mu})$, $\kappa, \mu = 0, 1, 2, \cdots$ be an infinite triangular matrix satisfying the Silverman–Toeplitz conditions of regularity, i.e.,

$$\begin{cases} \sum_{\mu=0}^{\kappa} a_{\kappa,\mu} & = 1 \text{ as } \kappa \to \infty, \\ a_{\kappa,\mu} & = 0, \text{ for } \mu > \kappa, \\ \lim_{\kappa\to\infty} a_{\kappa,\mu} & = 0 \text{ for each } \mu, \\ \sum_{\mu=0}^{\kappa} |a_{\kappa,\mu}| & \leq N, \text{ a finite constant.} \end{cases} \tag{6}$$

Now, we superimpose the generalized Nörlund $(N^{p,q})$ method on the matrix (A) method and obtain a new product summability method $(N^{p,q}A)$ which is defined as

$$\begin{aligned} \gamma_\kappa^{N^{p,q}A} &= \frac{1}{R_\kappa} \sum_{\mu=0}^{\kappa} p_{\kappa-\mu} q_\mu \gamma_\mu^A \\ &= \frac{1}{R_\kappa} \sum_{\mu=0}^{\kappa} p_{\kappa-\mu} q_\mu \sum_{j=0}^{\mu} a_{\mu,j} s_j. \end{aligned}$$

If $\gamma_\kappa^{N^{p,q}A} \to s$ as $\kappa \to \infty$, then (4) is summable to a finite value s by the $N^{p,q}A$ method. The regularity of $N^{p,q}$ and A methods implies the regularity of the product $N^{p,q}A$ method.

Remark 1. *$N^{p,q}A$ product operator is reduced to*

(i) $N^{p,q}C^1$ *operator if* $a_{\kappa,\mu} = \frac{1}{\kappa+1}$.

(ii) $N^{p,q}H$ *operator if* $a_{\kappa,\mu} = \frac{1}{(\kappa-\mu+1)\log\kappa}$.

(iii) $N^{p,q}C^\beta$ *operator if* $a_{\kappa,\mu} = \frac{\binom{\kappa-\mu+\beta-1}{\beta-1}}{\binom{\kappa+\beta}{\beta}}$.

(iv) $N^{p,q}H^p$ *operator if* $a_{\kappa,\mu} = \frac{1}{\log^{p-1}(\kappa+1)\prod_{m=0}^{p-1}\log^m(\mu+1)}$.

(v) $N^{p,q}N^p$ *operator if* $a_{\kappa,\mu} = \frac{p_{\kappa-\mu}}{P_\kappa}$, *where* $P_\kappa = \sum_{\mu=0}^{\infty} p_\mu$.

(vi) $N^{p,q}\overline{N}^p$ *operator if* $a_{\kappa,\mu} = \frac{p_\mu}{P_\kappa}$, *where* $P_\kappa = \sum_{\mu=0}^{\infty} p_\mu$.

(vii) $N^p A$ *operator if* $q_\kappa = 1$, *for all* κ.

(viii) $\overline{N}^p A$ *operator if* $p_\kappa = 1$, *for all* κ.

(ix) $C^\beta A$ *operator if* $p_\kappa = \binom{\kappa+\beta-1}{\beta-1}$, $\beta > 0$ *and* $q_\kappa = 1$, *for all* κ.

2.3. Lipschitz Spaces

The norm of $g \in L^v[0, 2\pi]$ is given by

$$||g||_v := \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(y)|^v dy \right\}^{\frac{1}{v}}, & \text{for } 1 \leq v < \infty; \\ \operatorname{ess\,sup}_{0<y<2\pi} |g(y)|, & \text{for } v = \infty. \end{cases}$$

Lip ρ class of function:
The function $g \in \text{Lip } \rho$ if

$$|g(y+t) - g(y)| = O(t^\rho), \text{ for } 0 < \rho \leq 1.$$

Lip (ρ, v) class of a function:
The function $g \in \text{Lip}(\rho, v)$ if

$$\left(\int_0^{2\pi} |g(y+t) - g(y)|^v dy \right)^{\frac{1}{v}} = O(t^\rho), \ 0 < \rho \leq 1, v \geq 1.$$

Lip$^*(\rho, v)$ class of a function:
For $\rho > 0, \mu > \rho$, i.e., $\mu = [\rho] + 1$, the smallest integer is larger than ρ; function $g \in \text{Lip}^*(\rho, v)$ if

$$\omega_\mu(g, t)_v = O(t^\rho), \quad t > 0.$$

2.4. Besov Spaces

Let $C_{2\pi} := C[0, 2\pi]$ denote the Banach space of all 2π-periodic continuous functions defined on $[0, 2\pi]$ under the supremum norm and

$$L^v := L^v[0, 2\pi] := \left\{ g : [0, 2\pi] \to \mathbb{R} : \int_0^{2\pi} |g(x)|^v dx < \infty, v \geq 1 \right\}$$

be the space of all 2π-periodic integrable functions.

The L^v-norm of a function g is defined by

$$\|g\|_v = \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(x)|^v dx \right\}^{\frac{1}{v}} & \text{for } 1 \leq v < \infty, \\ \operatorname{ess\,sup}_{x \in [0,2\pi]} |g(x)| & \text{for } v = \infty. \end{cases}$$

We define for function $g \in L^v$ the spaces

$$\omega(g; t) := \sup_{\substack{x, x+h \in [0,2\pi] \\ |h| < t}} |g(x+h) - g(x)|, \tag{7}$$

which is said to be the modulus of continuity.

We define for function $g \in L^v$ the spaces

$$\omega_\mu(g, t)_v := \sup_{0<h\leq t} \|\Delta_h^\mu(g, \cdot)\|_v, \quad t > 0, \tag{8}$$

which is said to be the μth order modulus of smoothness
where

$$\Delta_h^\mu(g, x) := \sum_{\lambda=0}^{\mu} (-1)^{\mu-\lambda} \binom{\mu}{\lambda} g(x + \lambda h), \quad \mu \in \mathbb{N}. \tag{9}$$

Remark 2. *(i)* *If* $v = \infty, \mu = 0, \omega(g, t) \subseteq \omega_\mu(g, t)_v$.
(ii) *If* $0 < v < \infty, \mu = 1, \omega_1(g, t)_v \subseteq \omega_\mu(g, t)_v$.

(iii) *If* $g \in C_{2\pi}$ *and* $\omega(g,t) = O(t^\rho)$, $0 < \rho \leq 1$ *then* $g \in Lip\ \rho$.
(iv) *If* $g \in L^v$ *for* $0 < v < \infty$ *and* $\omega(g,t) = O(t^\rho)$ *for* $0 < \rho \leq 1$, *then* $g \in Lip(\rho, v)$.
(v) *If* $v = \infty$, $Lip\ \rho \subseteq Lip(\rho, v)$.

Let $\rho > 0, \mu > \rho$, i.e., $\mu = [\rho] + 1$, be the smallest integer larger than ρ. For $g \in L^v$, if

$$\omega_\mu(g,t)_v = O(t^\rho), \quad t > 0, \tag{10}$$

then $g \in \mathrm{Lip}^*(\rho, v)$ (generalized Lipschitz class of function g).

The semi-norm of (10) is given by

$$|g|_{\mathrm{Lip}^*(\rho,v)} = \sup_{t>0}\left(\frac{\omega_\mu(g,t)_v}{t^\rho}\right). \tag{11}$$

Thus, $\mathrm{Lip}(\rho, v) \subseteq \mathrm{Lip}^*(\rho, v)$.

Let $\mu = [\rho] + 1$ for $\rho > 0, \mu > \rho$, i.e., $0 < v, \alpha \leq \infty$, the Besov $B^\rho_\alpha(L^v)$ spaces are defined by a collection of all 2π-periodic functions $g \in L^v$ given by

$$\|\omega_\mu(g,\cdot)\|_{\rho,\alpha} = |g|_{B^\rho_\alpha(L^v)} := \begin{cases} \left\{\int_0^\pi \left(\frac{\omega_\mu(g,t)_v}{t^\rho}\right)^\alpha \frac{dt}{t}\right\}^{\frac{1}{\alpha}}, & 0 < \alpha < \infty \\ \sup_{t>0}\left(\frac{\omega_\mu(g,t)_v}{t^\rho}\right), & \alpha = \infty. \end{cases} \tag{12}$$

Note that (12) is finite [21].

Moreover, (12) is a semi-norm for $1 \leq v, \alpha \leq \infty$ and a quasi semi-norm in cases [22].

Thus, the quasi-norm for Besov $B^\rho_\alpha(L^v)$ spaces is given by

$$\|g\|_{B^\rho_\alpha(L^v)} := \|g\|_v + |g|_{B^\rho_\alpha(L^v)} = \|g\|_v + \|\omega_\mu(g,\cdot)\|_{\rho,\alpha}. \tag{13}$$

2.5. Modulus of Continuity

The modulus of continuity is defined by

$$\omega_g(\delta) = \omega(g,\delta) = \sup_{y;|t|<\delta;} |\Delta_t(g,y)|.$$

The first-order modulus of continuity of function $g \in L^v$ is defined by

$$\omega_1(g;t)_v := \sup_{|y|<t, x\in\mathbb{R}} ||g(x+y) - g(x)||_v.$$

This is also said to be the integral modulus of continuity.

The second-order modulus of continuity of function $g \in L^v$ is defined by

$$\omega_2(g,\delta) = \sup_{y;|t|<\delta;} \left|\Delta_t^2(g,y)\right|.$$

2.6. Generalized Zygmund Spaces

Let $g : [0, 2\pi] \to \mathbb{R}$ be an arbitrary function with $\omega(y) > 0$ for $0 < y \leq 2\pi$ and $\lim_{y\to 0^+} \omega(y) = \omega(0) = 0$. We define

$$\chi_v^{(\omega)} = \left\{g \in L^v[0,2\pi] : 1 \leq v < \infty, \sup_{y\neq 0} \frac{||g(\cdot+y) + g(\cdot-y) - 2g(\cdot)||_v}{\omega(y)} < \infty, v \geq 1\right\},$$

and its norm is given by

$$||g||_v^{(\omega)} = ||g||_v + \sup_{y\neq 0} \frac{||g(\cdot+y) + g(\cdot-y) - 2g(\cdot)||_v}{\omega(y)}, v \geq 1.$$

Also,

$$||g||_v^{(\xi)} = ||g||_v + \sup_{y\neq 0} \frac{||g(\cdot+y)+g(\cdot-y)-2g(\cdot)||_v}{\xi(y)}, v \geq 1.$$

We note that the G.Z.S. $(\chi_v^{(\omega)})$ is a Banach space under the norm $||\cdot||_v^{(\omega)}$.
The completeness of spaces $L^v (v \geq 1)$ implies the completeness of spaces $\chi_v^{(\omega)}$.

Remark 3. *Let* $\frac{\omega(y)}{\xi(y)}$ *be positive and non-decreasing in* y,

$$||g||_v^{(\xi)} \leq \max\left(1, \frac{\omega(2\pi)}{\xi(2\pi)}\right)||g||_v^{(\omega)} < \infty,$$

where ω *and* ξ *denote the second-order moduli of continuity.*

Thus, we can write

$$\chi_v^{(\omega)} \subset \chi_v^{(\xi)} \subset L^v, v \geq 1.$$

Remark 4. *(i) If* $\omega(y) = y^\rho$, *then* $\chi^{(\omega)}$ *class reduces to* χ_ρ *class.*
(ii) If $\omega(y) = y^\rho$, *then* $\chi_v^{(\omega)}$ *class reduces to* $\chi_{\rho,v}$ *class.*
(iii) If $v \to \infty$, *then* $\chi_v^{(\omega)}$ *class reduces to* $\chi^{(\omega)}$ *class.*
(iv) If we take $v \to \infty$, *then* $\chi_{\rho,v}$ *class becomes* χ_ρ *class.*
(v) Let $0 \leq \tau < \rho < 1$, *if* $\omega(y) = y^{\rho_2}$ *and* $\xi(y) = y^{\rho_1}$, *then* $\frac{\omega(y)}{\xi(y)}$ *is non-decreasing in* y *and* $\frac{\omega(y)}{y\xi(y)}$ *is non-increasing in* y.

2.7. Degree of Convergence

The degree of convergence of function g is a measure of the speed at which γ_κ converges to g, which is given by ([23])

$$\|g - \gamma_\kappa\| = O\left(\frac{1}{\rho_\kappa}\right),$$

where γ_κ is a trigonometric polynomial of degree κ and $\rho_\kappa \to \infty$ as $\kappa \to \infty$.

2.8. Notations

$$\phi(x,y) = g(x+y) + g(x-y) - 2g(x).$$

For Besov $B_\alpha^\rho(L^v)$ spaces,

$$\Phi(x,t,y) = \begin{cases} \phi(x+t,y) - \phi(x,y), & 0 < \rho < 1 \\ \phi(x+t,y) + \phi(x-t,y) - 2\phi(x,y), & 1 \leq \rho < 2; \end{cases}$$

$$Z_\kappa(x) = \gamma_\kappa^{N^{p,q}A}(g;x) - g(x) = \int_0^\pi \phi(x,y) M_n^{N^{p,q}A}(y)dy. \tag{14}$$

We know that

$$\begin{aligned} \omega_\mu(Z_\kappa,t)_v &= \sup_{0<h\leq t} \|\Delta_h^\kappa(Z_\kappa,\cdot)\|_v, \ \ t > 0 \\ &= \begin{cases} \sup_{0<h\leq t} \|Z_\kappa(\cdot+h) - Z_\kappa(\cdot)\|_v, & 0 < \rho < 1 \\ \sup_{0<h\leq t} \|Z_\kappa(\cdot+h) + Z_\kappa(\cdot-h) - 2Z_\kappa(\cdot)\|_v, & 1 \leq \rho < 2, \end{cases} \\ &= \|Y_\kappa(\cdot,t)\|_v. \end{aligned}$$

$$\begin{aligned}
Y_\kappa(x,t) &= \begin{cases} Z_\kappa(x+t) - Z_\kappa(x), & 0 < \rho < 1 \\ Z_\kappa(x+t) + Z_\kappa(x-t) - 2Z_\kappa(x), & 1 \le \rho < 2, \end{cases} \\
&= \begin{cases} \int_0^\pi \{\phi(x+t,y) - \phi(x,y)\} M_n^{N^{p,q}A}(y)dy, & 0 < \rho < 1 \\ \int_0^\pi \{\phi(x+t,y) + \phi(x-t,y) - 2\phi(x,y)\} M_n^{N^{p,q}A}(y)dy, & 1 \le \rho < 2, \text{using (14)} \end{cases} \\
&= \int_0^\pi \phi(x,t,y) M_n^{N^{p,q}A}(y)dy
\end{aligned}$$

For generalized Zygmund spaces,

$$\Phi(x,t,y) = \phi(x+t,y) + \phi(x-t,y) - 2\phi(x,y);$$
$$\Delta a_{\kappa,\mu} = a_{\kappa,\mu} - a_{\kappa,\mu+1}, \ 0 \le \mu \le \kappa - 1;$$
$$M_\kappa^{N^{p,q}A}(y) = \frac{1}{2\pi}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\frac{\sin\left(j+\frac{1}{2}\right)y}{\sin(\frac{y}{2})}.$$

3. Auxiliary Results

The following auxiliary results are required for the proof of the following Theorems.

Lemma 1. *If* $\{p_\kappa\}$ *and* $\{q_\kappa\}$ *are monotonic decreasing and monotonic increasing sequences, respectively, then*

$$(\kappa+1)p_\kappa q_0 = O(R_\kappa). \tag{15}$$

Proof. The proof of this Lemma is straightforward. □

Lemma 2. $|M_\kappa^{N^{p,q}A}(y)| = O(\kappa+1)$ *for* $0 < y \le \frac{1}{\kappa+1}$.

Proof. For $0 < y \le \frac{1}{\kappa+1}, \sin(\frac{y}{2}) \ge \frac{y}{\pi}, |\sin(\kappa y)| \le \kappa y.$

$$\begin{aligned}
\left|M_\kappa^{N^{p,q}A}(y)\right| &= \frac{1}{2\pi}\frac{1}{R_\kappa}\left|\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\frac{\sin(j+\frac{1}{2})y}{\sin(\frac{y}{2})}\right| \\
&\le \frac{1}{2\pi}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\frac{|\sin(j+\frac{1}{2})y|}{|\sin(\frac{y}{2})|} \\
&\le \frac{1}{2\pi}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\frac{(j+\frac{1}{2})y}{\frac{y}{\pi}} \\
&\le \frac{1}{4}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}(2j+1) \\
&= \frac{1}{4}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \left\{2\sum_{j=0}^{\mu} j a_{\mu,j} + \sum_{j=0}^{\mu} a_{\mu,j}\right\} \\
&= \frac{1}{4}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \{2(a_{\mu,1} + 2a_{\mu,2} + \cdots + \mu a_{\mu,\mu}) + 1\} \\
&= \frac{1}{4}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \{2(\mu a_{\mu,1} + \mu a_{\mu,2} + \cdots + \mu a_{\mu,\mu}) + 1\} \\
&= \frac{1}{4}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \{2\mu(a_{\mu,0} + a_{\mu,1} + a_{\mu,2} + \cdots + a_{\mu,\mu}) - 2\mu a_{\mu,0} + 1\} \\
&= \frac{1}{4}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu 2\mu\{(a_{\mu,0} + a_{\mu,1} + a_{\mu,2} + \cdots + a_{\mu,\mu}) - a_{\mu,0}\} + 1 \\
&= \frac{1}{4}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu 2\mu\{1 - a_{\mu,0}\} + 1 \\
&\le \frac{1}{4}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu (2\mu+1) \\
&\le \frac{(2\kappa+1)}{4}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \\
&= O(\kappa+1).
\end{aligned}$$

□

Lemma 3. $|M_{\kappa}^{N^{p,q}A}(y)| = O\left(\frac{1}{y^2(\kappa+1)}\right)$ *for* $\frac{1}{\kappa+1} < y \leq \pi$.

Proof. For $\frac{1}{\kappa+1} < y \leq \pi, \sin(\frac{y}{2}) \geq \frac{y}{\pi}$.

$$\begin{aligned}
\left|M_{\kappa}^{N^{p,q}A}(y)\right| &= \left|\frac{1}{2\pi}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\frac{\sin\left(j+\frac{1}{2}\right)y}{\sin(\frac{y}{2})}\right| \\
&\leq \frac{1}{2\pi}\frac{1}{R_\kappa}\left|\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\frac{\sin\left(j+\frac{1}{2}\right)y}{|\sin(\frac{y}{2})|}\right| \\
&\leq \frac{1}{2\pi}\frac{1}{R_\kappa}\left|\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\frac{\sin\left(j+\frac{1}{2}\right)y}{\frac{y}{\pi}}\right| \\
&\leq \frac{1}{2y}\frac{1}{R_\kappa}\left|\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\sin\left(j+\frac{1}{2}\right)y\right|.
\end{aligned}$$

By Abel's lemma, we obtain

$$\begin{aligned}
\left|M_{\kappa}^{N^{p,q}A}(y)\right| &\leq \frac{1}{2y}\frac{1}{R_\kappa}\left[\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu\left|\sum_{j=0}^{\mu-1}(a_{\mu,j}-a_{\mu-1,j+1})\sum_{v=0}^{j}\sin\left(v+\frac{1}{2}\right)y\right|\right. \\
&\quad \left. + a_{\mu,\mu}\sum_{j=0}^{\mu}\sin\left(j+\frac{1}{2}\right)y\right] \\
&\leq \frac{1}{2y}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu\left|\sum_{j=0}^{\mu-1}\Delta a_{\mu,j}\sum_{v=0}^{j}\sin\left(v+\frac{1}{2}\right)y\right| + a_{\mu,\mu}\left|\sum_{j=0}^{\mu}\sin\left(j+\frac{1}{2}\right)y\right| \\
&\leq \frac{1}{2y}\frac{1}{R_v}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu\left[\sum_{j=0}^{\mu-1}|\Delta a_{\mu,j}| + a_{\mu,\mu}\right]\max_{0\leq v\leq m}\left|\sum_{v=0}^{m}\sin\left(v+\frac{1}{2}\right)y\right| \\
&\leq \frac{1}{2y}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu\left[O\left(\frac{1}{\mu+1}\right)+O\left(\frac{1}{\mu+1}\right)\right]\cdot\frac{1}{y} \\
&\leq \frac{1}{y^2}\frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu\left(\frac{1}{\mu+1}\right).
\end{aligned}$$

Again, by Abel's lemma, we obtain

$$\begin{aligned}
\left|M_{\kappa}^{N^{p,q}A}(y)\right| &\leq \frac{1}{y^2}\frac{1}{R_\kappa}\left|\sum_{\mu=0}^{\kappa-1}(p_{\kappa-\mu}q_\mu - p_{\kappa-\mu-1}q_{\mu+1})\sum_{v=0}^{\mu}\frac{1}{\mu+1} + p_0q_\kappa\sum_{\mu=0}^{\kappa}\frac{1}{\mu+1}\right| \\
&\leq \frac{1}{y^2}\frac{1}{R_\kappa}\left[\sum_{\mu=0}^{\kappa-1}|(p_{\kappa-\mu}q_\mu - p_{\kappa-\mu-1}q_{\mu+1})| + |p_0q_\kappa|\right]\max_{0\leq v\leq m}\left|\sum_{v=0}^{m}\frac{1}{v+1}\right|
\end{aligned}$$

Lemma 1 provides

$$\begin{aligned}
\left|M_{\kappa}^{N^{p,q}A}(y)\right| &\leq \frac{1}{y^2}\frac{1}{R_\kappa}p_\kappa q_0 \\
&= O\left(\frac{1}{y^2(\kappa+1)}\right).
\end{aligned}$$

□

Lemma 4 ([24]). *For* $1 < v \leq \infty$, $0 < \rho < 2$ *and for* $0 < t,\ y \leq \pi$.

(i) $\|\Phi(\cdot,t,y)\|_v \leq 4\omega_\mu(g,t)_v$,
(ii) $\|\Phi(\cdot,t,y)\|_v \leq 4\omega_\mu(g,y)_v$,
(iii) $\|\Phi(\cdot,y)\|_v \leq 2\omega_\mu(g,y)_v$,
where $g \in L_v$ *and* $\mu = [\rho]+1$.

Lemma 5. *Let* $0 \leq \beta < \rho < 2$. *If* $g \in B^\rho_\alpha(L^v), v \geq 1, 1 < \alpha < \infty$, *then*

(i) $\int_0^\pi |M_\kappa^{N^{p,q}A}(y)|\left(\int_0^\pi \frac{\|\Phi(\cdot,t,y)\|_v^\alpha}{t^{\beta\alpha}}\frac{dt}{t}\right)^{\frac{1}{\alpha}}dy = O\left\{\int_0^\pi (y^{\rho-\beta}|M_\kappa^{N^{p,q}A}(y)|)^{\frac{\alpha}{\alpha-1}}dy\right\}^{1-\frac{1}{\alpha}}$

(ii) $\int_0^\pi |M_\kappa^{N^{p,q}A}(y)|\left(\int_0^\pi \frac{\|\Phi(\cdot,t,y)\|_v^\alpha}{t^{\beta\alpha}}\frac{dt}{t}\right)^{\frac{1}{\alpha}}dy = O\left\{\int_0^\pi (y^{\rho-\beta+\frac{1}{\alpha}}|M_\kappa^{N^{p,q}A}(y)|)^{\frac{\alpha}{\alpha-1}}dy\right\}^{1-\frac{1}{\alpha}}$

Proof. Proof of this Lemma is parallel to the the proof of Lemma 1 of [24]. □

Lemma 6 ([24]). *Let* $0 \leq \beta < \rho < 2$. *If* $g \in B^\rho_\alpha(L^v), v \geq 1, \alpha = \infty$ *then*

$$\sup_{0<t,y\leq\pi}\left(\frac{\|\Phi(\cdot,t,y)\|_v}{t^\beta}\right) = O(y^{\rho-\beta}). \tag{16}$$

Lemma 7 ([25]). *If* $g \in \chi_v^{(\omega)}$ *then for* $0 < y \leq \pi$,
(i) $||\phi(\cdot,y)||_v = O(\omega(y))$.
(ii) $||\phi(\cdot+t,y)+\phi(\cdot-t,y)-2\phi(\cdot,t)||_v = O\left(\xi(|t|)\frac{\omega(y)}{\xi(y)}\right)$.
where $\omega(y)$ *and* $\xi(y)$ *are the moduli of continuity of order* L^v

4. Main Results

4.1. Analysis of Convergence of Function in the Besov Norm

In this subsection, we establish a theorem to study the convergence of function g of Fourier series in Besov $B^\rho_\alpha(L^v)$ spaces $(v \geq 1, 1 < \alpha \leq \infty)$ using the generalized Nörlund Matrix means.

Theorem 1. *Let* g *be a* 2π*-periodic and Lebesgue integrable function. For* $0 \leq \beta < \rho < 2$, *the degree of convergence of function of* g *of Fourier series in the Besov* $B^\rho_\alpha(L^v)$ *space,* $v \geq 1, 1 < \alpha < \infty$ *using* $N^{p,q}A$ *operator, is given by*

$$\|Z_\kappa(\cdot)\|_{B^\beta_\alpha(L^v)} = O\begin{cases}(\kappa+1)^{-1}, & \rho-\beta-\frac{1}{\alpha} > 1,\\ (\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha} < 1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha} = 1.\end{cases} \tag{17}$$

Proof. Using the integral representation [26] of $s_\kappa(g;x)$, we have

$$s_\kappa(g;x) - g(x) = \frac{1}{2\pi}\int_0^\pi \phi(x,y)\frac{\sin(\kappa+\frac{1}{2})y}{\sin(\frac{y}{2})}dy. \tag{18}$$

Denoting the $N^{p,q}A$ summability mean of $s_\kappa(g;x)$ by $\gamma_\kappa^{N^{p,q}A}(g;x)$, we obtain

$$\begin{aligned}\gamma_\kappa^{N^{p,q}A}(g;x) - g(x) &= \frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\{s_j(g;x)-g(x)\}\\ &= \frac{1}{R_\kappa}\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\left\{\frac{1}{2\pi}\int_0^\pi \phi(x,y)\frac{\sin(j+\frac{1}{2})y}{\sin(\frac{y}{2})}dy\right\}\\ &= \frac{1}{2\pi}\frac{1}{R_\kappa}\int_0^\pi \phi(x,y)\sum_{\mu=0}^{\kappa} p_{\kappa-\mu}q_\mu \sum_{j=0}^{\mu} a_{\mu,j}\frac{\sin(j+\frac{1}{2})y}{\sin(\frac{y}{2})}dy\\ &= \int_0^\pi \phi(x,y)M_\kappa^{N^{p,q}A}(y)dy.\end{aligned} \tag{19}$$

In view of (13), Equation (14) can be written as

$$\|Z_\kappa(\cdot)\|_{B_\alpha^\beta(L^\upsilon)} = \|Z_\kappa(\cdot)\|_\upsilon + \|\varpi_\mu(Z_\kappa,\cdot)\|_{\beta,\alpha}. \tag{20}$$

Now, let us consider the first norm of (20).
Using generalized Minkowski's inequality [27] and Lemma 4(iii), we have

$$\begin{aligned} \|Z_\kappa(\cdot)\|_\upsilon &\le \int_0^\pi \|\phi(\cdot,y)\|_\upsilon |M_\kappa^{N^{p,q}A}(y)|dy \\ &\le \int_0^\pi 2\varpi_\mu(g;y)_\upsilon |M_\kappa^{N^{p,q}A}(y)|dy. \end{aligned} \tag{21}$$

Using Hölder's inequality and (12),

$$\begin{aligned} \|Z_\kappa(\cdot)\|_\upsilon &\le 2\left\{\int_0^\pi \left(|M_\kappa^{N^{p,q}A}(y)|y^{\rho+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}} dy\right\}^{1-\frac{1}{\alpha}} \left\{\int_0^\pi \left(\frac{\varpi_\mu(g,y)_\upsilon}{y^{\rho+\alpha^{-1}}}\right)^\alpha dy\right\}^{\frac{1}{\alpha}} \\ &= O\left[\left\{\int_0^\pi \left(|M_\kappa^{N^{p,q}A}(y)|y^{\rho+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}} dy\right\}^{1-\frac{1}{\alpha}}\right] \\ &= O\left[\left\{\left(\int_0^{\frac{1}{\kappa+1}} + \int_{\frac{1}{\kappa+1}}^\pi\right)\int_0^\pi \left(|M_\kappa^{N^{p,q}A}(y)|y^{\rho+\frac{1}{\alpha}}\right)^{\frac{\alpha}{(\alpha-1)}} dy\right\}^{1-\frac{1}{\alpha}}\right] \\ &= O\left[\left\{\int_0^{\frac{1}{\kappa+1}} \left(|M_n^{N^{p,q}A}(y)|y^{\rho+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}} dy\right\}^{1-\frac{1}{\alpha}}\right. \\ &\left. + \left\{\int_{\frac{1}{\kappa+1}}^\pi \left(|M_\kappa^{N^{p,q}A}(y)|y^{\rho+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}} dy\right\}^{1-\frac{1}{\alpha}}\right] \\ &= I_1 + I_2. \end{aligned} \tag{22}$$

Using Lemma 2, we obtain

$$\begin{aligned} I_1 &= O\left\{\int_0^{\frac{1}{\kappa+1}} \left(|M_\kappa^{N^{p,q}A}(y)|y^{\rho+\frac{1}{\alpha}}\right)^{\frac{\alpha}{(\alpha-1)}} dy\right\}^{1-\frac{1}{\alpha}} \\ &= O\left(\int_0^{\frac{1}{\kappa+1}} \left\{(\kappa+1)y^{\rho+\frac{1}{\alpha}}\right\}^{\frac{\alpha}{\alpha-1}} dy\right)^{\frac{\alpha-1}{\alpha}} \\ &= O\left[(\kappa+1)^{\frac{\alpha}{\alpha-1}} \int_0^{\frac{1}{\kappa+1}} \left\{(y)^{\rho+\frac{1}{\alpha}}\right\}^{\frac{\alpha}{\alpha-1}} dy\right]^{\frac{\alpha-1}{\alpha}} \\ &= O\left[(\kappa+1)^{\frac{\alpha}{\alpha-1}} \int_0^{\frac{1}{\kappa+1}} (y)^{\frac{\rho\alpha+1}{\alpha-1}} dy\right]^{\frac{\alpha-1}{\alpha}} \\ &= O\left[(\kappa+1)^{\frac{\alpha}{\alpha-1}} \cdot \left(\frac{\alpha-1}{\rho\alpha+\alpha}\right)\left(\frac{1}{\kappa+1}\right)^{\frac{\alpha(\rho+1)}{\alpha-1}}\right]^{\frac{\alpha-1}{\alpha}} \\ &= O\left[(\kappa+1)\cdot\left(\frac{\alpha-1}{\rho\alpha+\alpha}\right)\frac{1}{(\kappa+1)^{\rho+1}}\right] \\ &= O\left(\frac{1}{(\kappa+1)^\rho}\right). \end{aligned} \tag{23}$$

Using Lemma 3, we obtain

$$
\begin{aligned}
I_2 &= O\left\{\int_{\frac{1}{\kappa+1}}^{\pi}\left(|M_{\kappa}^{N^{p,q}}A(y)|y^{\rho+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}dy\right\}^{1-\frac{1}{\alpha}}\\
&= O\left\{\int_{\frac{1}{\kappa+1}}^{\pi}\left(\frac{1}{(\kappa+1)y^2}\cdot y^{\rho+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}dy\right\}^{1-\frac{1}{\alpha}}\\
&= O\left\{\left(\frac{1}{\kappa+1}\right)^{\frac{\alpha}{\alpha-1}}\int_{\frac{1}{\kappa+1}}^{\pi}\left(y^{\rho+\frac{1}{\alpha}-2}\right)^{\frac{\alpha}{\alpha-1}}dy\right\}^{1-\frac{1}{\alpha}}\\
&= O\left(\frac{1}{\kappa+1}\right)\left\{\int_{\frac{1}{\kappa+1}}^{\pi}\left(y^{\frac{\rho\alpha+1-2\alpha}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}dy\right\}^{\frac{\alpha-1}{\alpha}}\\
&= O\left(\frac{1}{\kappa+1}\right)\left\{\int_{\frac{1}{\kappa+1}}^{\pi}\left(y^{\frac{\rho\alpha+1-2\alpha}{\alpha-1}}\right)dy\right\}^{\frac{\alpha-1}{\alpha}}\\
&= O\left(\frac{1}{\kappa+1}\right)\left\{\left(\frac{y^{\frac{\rho\alpha-\alpha}{\alpha-1}}}{\frac{\rho\alpha-1}{\alpha-1}}\right)_{\frac{1}{\kappa+1}}^{\pi}\right\}^{\frac{\alpha-1}{\alpha}}\\
&= O\left(\frac{1}{\kappa+1}\right)\left\{\frac{\pi^{\frac{\rho\alpha-\alpha}{\alpha-1}}}{\frac{\rho\alpha-1}{\alpha-1}}-\frac{\left(\frac{1}{\kappa+1}\right)^{\frac{\rho\alpha-\alpha}{\alpha-1}}}{\frac{\rho\alpha-1}{\alpha-1}}\right\}^{\frac{\alpha-1}{\alpha}}
\end{aligned}
$$

$$
I_2 = O\begin{cases}(\kappa+1)^{-1}, & \text{for all } \rho>1, 1<\alpha\leq\infty,\\ (\kappa+1)^{-\rho}, & \text{for all } \rho<1, 1<\alpha\leq\infty,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \text{for } \rho=1, 1<\alpha\leq\infty.\end{cases} \tag{24}
$$

Combining (22)–(24), we have

$$
\|Z_{\kappa}(\cdot)\|_{v} = O\begin{cases}(\kappa+1)^{-1}, & \rho>1,\\ (\kappa+1)^{-\rho}, & \rho<1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho=1.\end{cases} \tag{25}
$$

Now, let us consider the second norm of (20).
Using generalized Minkowski's inequality [27], we obtain

$$
\begin{aligned}
\|\varpi_{\mu}(Z_{\kappa},\cdot)\|_{\beta,\alpha} &= \left[\int_0^{\pi}\left(\frac{\varpi_{\mu}(Z_{\kappa},t)}{t^{\beta}}\right)^{\alpha}\frac{dt}{t}\right]^{\frac{1}{\alpha}}\\
&= \left[\int_0^{\pi}\left(\frac{\|\Upsilon_{\kappa}(\cdot,t)\|_{v}}{t^{\beta}}\right)^{\alpha}\frac{dt}{t}\right]^{\frac{1}{\alpha}}\\
&\leq \int_0^{\pi}|M_{\kappa}^{N^{p,q}}A(y)|dy\left(\int_0^{\pi}\|\Phi(\cdot,t,y)\|_{v}^{\alpha}\frac{dt}{t^{\beta\alpha+1}}\right)^{\frac{1}{\alpha}}\\
&\leq \int_0^{\pi}|M_{\kappa}^{N^{p,q}}A(y)|\left[\int_0^{y}\frac{\|\Phi(\cdot,t,y)\|_{v}^{\alpha}}{t^{\beta\alpha}}\cdot\frac{dt}{t}\right]^{\frac{1}{\alpha}}dy\\
&+ \int_0^{\pi}|M_{\kappa}^{N^{p,q}}A(y)|\left[\int_y^{\pi}\frac{\|\Phi(\cdot,t,y)\|_{v}^{\alpha}}{t^{\beta\alpha}}\cdot\frac{dt}{t}\right]^{\frac{1}{\alpha}}dy
\end{aligned}
$$

Using Lemma 5, we obtain

$$
\begin{aligned}
\|\varpi_{\mu}(Z_{\kappa},\cdot)\|_{\beta,\alpha} &= O\left[\int_0^{\pi}\left\{|M_{\kappa}^{N^{p,q}}A(y)|y^{\rho-\beta}\right\}^{\frac{\alpha}{\alpha-1}}dy\right]^{1-\frac{1}{\alpha}}\\
&+ O\left[\int_0^{\pi}\left\{|M_{\kappa}^{N^{p,q}}A(y)|y^{\rho-\beta+\frac{1}{\alpha}}\right\}^{\frac{\alpha}{\alpha-1}}dy\right]^{1-\frac{1}{\alpha}}\\
&= (I_3+I_4).
\end{aligned} \tag{26}
$$

Since $(c+d)^v \leq c^v + d^v$ for positive c, d and $0 < v \leq 1$ for $v = 1 - \frac{1}{\alpha} < 1$, then

$$\begin{aligned} I_3 &= O\left[\int_0^{\pi} \left(|M_{\kappa}^{N^{p,q}A}(y)| y^{\rho-\beta}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left[\int_0^{\frac{1}{\kappa+1}} \left(|M_{\kappa}^{N^{p,q}A}(y)| y^{\rho-\beta}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} + O\left[\int_{\frac{1}{\kappa+1}}^{\pi} \left(|M_{\kappa}^{N^{p,q}A}(y)| y^{\rho-\beta}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= I_{31} + I_{32}. \end{aligned} \tag{27}$$

Using Lemma 2, we have

$$\begin{aligned} I_{31} &= O\left[\int_0^{\frac{1}{\kappa+1}} \left(|M_{\kappa}^{N^{p,q}A}(y)| y^{\rho-\beta}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left[\int_0^{\frac{1}{\kappa+1}} \left\{y^{\rho-\beta}(\kappa+1)\right\}^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left[(\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}\right]. \end{aligned} \tag{28}$$

Using Lemma 3, we have

$$\begin{aligned} I_{32} &= O\left[\int_{\frac{1}{\kappa+1}}^{\pi} \left(|M_{n}^{N^{p,q}A}(y)| y^{\rho-\beta}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left[\int_{\frac{1}{\kappa+1}}^{\pi} \left\{\left(\frac{1}{(\kappa+1)y^2}\right) y^{(\rho-\beta)}\right\}^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left(\frac{1}{\kappa+1}\right) \cdot \left[\int_{\frac{1}{\kappa+1}}^{\pi} y^{(\rho-\beta-2)\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\begin{cases} (\kappa+1)^{-1}, & \rho-\beta-\frac{1}{\alpha} > 1, \\ (\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha} < 1, \\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha} = 1. \end{cases} \end{aligned} \tag{29}$$

Combining (27)–(29), we have

$$I_3 = O\begin{cases} (\kappa+1)^{-1}, & \rho-\beta-\frac{1}{\alpha} > 1, \\ (\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha} < 1, \\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha} = 1. \end{cases} \tag{30}$$

Again, using inequality $(c+d)^v \leq c^v + d^v$ for positive c, d and $0 < v \leq 1$ for $v = 1 - \frac{1}{\alpha} < 1$,

$$\begin{aligned} I_4 &= O\left[\int_0^{\pi} \left\{|M_{\kappa}^{N^{p,q}A}(y)| y^{\rho-\beta+\frac{1}{\alpha}}\right\}^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left[\int_0^{\frac{1}{\kappa+1}} \left(|M_{\kappa}^{N^{p,q}A}(y)| y^{\rho-\beta+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} + O\left[\int_{\frac{1}{\kappa+1}}^{\pi} \left(|M_{\kappa}^{N^{p,q}A}(y)| y^{\rho-\beta+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= I_{41} + I_{42}. \end{aligned} \tag{31}$$

Using Lemma 2, we have

$$\begin{aligned} I_{41} &= O\left[\int_0^{\frac{1}{\kappa+1}} \left(|M_\kappa^{N^{p,q}} A(y)| y^{\rho-\beta+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left[\int_0^{\frac{1}{\kappa+1}} \left(y^{\rho-\beta+\frac{1}{\alpha}}(\kappa+1)\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left(\frac{1}{(\kappa+1)^{\rho-\beta}}\right) \end{aligned} \tag{32}$$

Using Lemma 3, we have

$$\begin{aligned} I_{42} &= O\left[\int_{\frac{1}{\kappa+1}}^{\pi} \left(|M_\kappa^{N^{p,q}} A(y)| y^{\rho-\beta+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left[\int_{\frac{1}{\kappa+1}}^{\pi} \left(\frac{1}{(\kappa+1)y^2} \cdot y^{\rho-\beta+\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\left(\frac{1}{\kappa+1}\right)\left[\int_{\frac{1}{\kappa+1}}^{\pi} y^{(\rho-\beta+\frac{1}{\alpha})\frac{\alpha}{\alpha-1}} dy\right]^{1-\frac{1}{\alpha}} \\ &= O\begin{cases} (\kappa+1)^{-1}, & \rho-\beta-\frac{1}{\alpha}>1, \\ (\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}<1, \\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}=1. \end{cases} \end{aligned} \tag{33}$$

Combining (31)–(33), we have

$$I_4 = O\begin{cases} (\kappa+1)^{-1}, & \rho-\beta-\frac{1}{\alpha}>1, \\ (\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}<1, \\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}=1. \end{cases} \tag{34}$$

Combining (26), (30) and (34), we have

$$\|\omega_\mu(Z_\kappa,\cdot)\|_{\beta,\alpha} = O\begin{cases} (\kappa+1)^{-1}, & \rho-\beta-\frac{1}{\alpha}>1, \\ (\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}<1, \\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}=1. \end{cases} \tag{35}$$

Combining (20), (25) and (35), we have

$$\|Z_\kappa(\cdot)\|_{B_\alpha^\beta(L^\upsilon)} = O\begin{cases} (\kappa+1)^{-1}, & \rho-\beta-\frac{1}{\alpha}>1, \\ (\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}<1, \\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}=1. \end{cases} \tag{36}$$

□

Remark 5. *When $\alpha = \infty$, Besov space $B_\infty^\rho(L^\upsilon), \upsilon \geq 1, \rho \geq 0$ reduces to generalized Lipschitz class $Lip^*(\rho,\upsilon)$ and the corresponding norm $\|\cdot\|_{B_\infty^\rho(L^\upsilon)}$ is given by*

$$\|g\|_{B_\infty^\rho(L^\upsilon)} = \|g\|_{Lip^*(\rho,\upsilon)} = \|g\|_\upsilon + \sup_{t>0} \frac{\omega_\mu(g,t)_\upsilon}{t^\rho}. \tag{37}$$

Thus, in view of Remark 6, we establish the following theorem to obtain convergence for $g \in Lip^*(\rho,\upsilon), \upsilon \geq 1, \alpha = \infty$.

Theorem 2. *If g is a 2π-periodic and Lebesgue integrable function belonging to the generalized Lipschitz class $Lip^*(\rho, v)$, $v \geq 1$ and $\alpha = \infty$, then, for $0 \leq \beta < \rho < 2$, the degree of convergence of a function g of Fourier series using the $N^{p,q}A$ operator is given by*

$$\|Z_\kappa(\cdot)\|_{Lip^*(\beta,L^v)} = O\begin{cases}(\kappa+1)^{-1}, & \rho-\beta>1,\\ (\kappa+1)^{-\rho+\beta}, & \rho-\beta<1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}, & \rho-\beta=1.\end{cases} \tag{38}$$

Proof. Using (37), we write

$$\|Z_\kappa(\cdot)\|_{B^\beta_\infty(L^v)} = \|Z_\kappa(\cdot)\|_v + \|\varpi_\mu(Z_\kappa,\cdot)\|_{\beta,\infty}. \tag{39}$$

Using (10) in (21), we obtain

$$\begin{aligned}\|Z_\kappa(\cdot)\|_v &\leq 2\int_0^\pi \varpi_\mu(g,y)_v|M_\kappa^{N^{p,q}A}(y)|dy\\ &= O\left[\int_0^{\frac{1}{\kappa+1}} y^\rho|M_\kappa^{N^{p,q}A}(y)|dy + \int_{\frac{1}{\kappa+1}}^\pi y^\rho|M_\kappa^{N^{p,q}A}(y)|dy\right]\\ &= O(J_1+J_2).\end{aligned} \tag{40}$$

Using Lemma 2, we obtain

$$\begin{aligned}J_1 &= \int_0^{\frac{1}{\kappa+1}} y^\rho|M_\kappa^{N^{p,q}A}(y)|dy\\ &= O(\kappa+1)\int_0^{\frac{1}{\kappa+1}} y^\rho dy\\ &= O(\kappa+1)^{-\rho}.\end{aligned} \tag{41}$$

Using Lemma 3, we obtain

$$\begin{aligned}J_2 &= \int_{\frac{1}{\kappa+1}}^\pi y^\rho|M_\kappa^{N^{p,q}A}(y)|dy\\ &= \int_{\frac{1}{\kappa+1}}^\pi y^\rho\frac{1}{(\kappa+1)y^2}dy\\ &= \frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^\pi y^{\rho-2}dy\\ &= \begin{cases}(\kappa+1)^{-1}, & \rho>1,\\ (\kappa+1)^{-\rho}, & \rho<1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}, & \rho=1.\end{cases}\end{aligned} \tag{42}$$

Combining (40)–(42), we obtain

$$\|Z_\kappa(\cdot)\|_v = \begin{cases}(\kappa+1)^{-1}, & \rho>1,\\ (\kappa+1)^{-\rho}, & \rho<1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}, & \rho=1.\end{cases} \tag{43}$$

Using generalized Minkowski's inequality [27], we obtain

$$\begin{aligned}
\|\omega_{\mu}(Z_{\kappa},\cdot)\|_{\beta,\infty} &= \sup_{t>0}\left(\frac{\omega_{\mu}(Z_{\kappa},t)_{v}}{t^{\beta}}\right)\\
&= \sup_{t>0}\left(\frac{\|\mathrm{Y}_{\kappa}(\cdot,t)\|_{v}}{t^{\beta}}\right)\\
&= \sup_{t>0}\left[\frac{1}{t^{\beta}}\left(\frac{1}{2\pi}\int_{0}^{2\pi}|\mathrm{Y}_{\kappa}(x,t)|^{v}dx\right)^{\frac{1}{v}}\right]\\
&= \sup_{t>0}\left[\frac{1}{t^{\beta}}\left(\frac{1}{2\pi}\int_{0}^{2\pi}\left|\int_{0}^{\pi}\phi(x,t,y)M_{\kappa}^{N^{p,q}A}(y)dy\right|^{v}dx\right)^{\frac{1}{v}}\right]\\
&\leq \sup_{t>0}\left[\frac{1}{t^{\beta}}\left(\frac{1}{2\pi}\right)^{\frac{1}{v}}\int_{0}^{\pi}\left\{\int_{0}^{2\pi}|\phi(x,t,y)|^{v}\cdot|M_{\kappa}^{N^{p,q}A}(y)|^{v}dx\right\}^{\frac{1}{v}}dy\right]\\
&= \sup_{t>0}\left[\frac{1}{t^{\beta}}\int_{0}^{\pi}\|\phi(\cdot,t,y)\|_{v}|M_{\kappa}^{N^{p,q}A}(y)|dy\right]\\
&\leq \int_{0}^{\pi}\left(\sup_{t>0}\frac{\|\phi(\cdot,t,y)\|_{v}}{t^{\beta}}\right)|M_{\kappa}^{N^{p,q}A}(y)|dy
\end{aligned}$$

Using Lemma 6, we obtain

$$\begin{aligned}
\|\omega_{\mu}(Z_{\kappa},\cdot)\|_{\beta,\infty} &= O\int_{0}^{\pi}y^{\rho-\beta}|M_{\kappa}^{N^{p,q}A}(y)|dy\\
&= O\left[\int_{0}^{\frac{1}{\kappa+1}}y^{\rho-\beta}|M_{\kappa}^{N^{p,q}A}(y)|dy+\int_{\frac{1}{\kappa+1}}^{\pi}y^{\rho-\beta}|M_{\kappa}^{N^{p,q}A}(y)|dy\right]\\
&= O(R_{1}+R_{2}).
\end{aligned}\tag{44}$$

Using Lemma 2, we obtain

$$\begin{aligned}
R_{1} &= \left[\int_{0}^{\frac{1}{\kappa+1}}y^{\rho}|M_{\kappa}^{N^{p,q}A}(y)|dy\right]\\
&= O(\kappa+1)\int_{0}^{\frac{1}{\kappa+1}}y^{\rho-\beta}dy\\
&= O\left(\frac{1}{(\kappa+1)^{\rho-\beta}}\right)
\end{aligned}\tag{45}$$

Using Lemma 3, we obtain

$$\begin{aligned}
R_{2} &= \left[\int_{\frac{1}{\kappa+1}}^{\pi}y^{\rho-\beta}|M_{\kappa}^{N^{p,q}A}(y)|dy\right]\\
&= \left[\int_{\frac{1}{\kappa+1}}^{\pi}y^{\rho-\beta}\cdot\frac{1}{(\kappa+1)y^{2}}dy\right]\\
&= O\left(\frac{1}{\kappa+1}\right)\int_{0}^{\frac{1}{\kappa+1}}y^{\rho-\beta-2}dy\\
&= O\begin{cases}(\kappa+1)^{-1}, & \rho-\beta>1,\\ (\kappa+1)^{-\rho+\beta}, & \rho-\beta<1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}, & \rho-\beta=1.\end{cases}
\end{aligned}\tag{46}$$

Combining (44)–(46), we obtain

$$\|\varpi_{\mu}(Z_{\kappa},\cdot)\|_{\beta,\infty} = O\begin{cases}(\kappa+1)^{-1}, & \rho-\beta>1,\\ (\kappa+1)^{-\rho+\beta}, & \rho-\beta<1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}, & \rho-\beta=1.\end{cases} \tag{47}$$

Combining (39), (43) and (47), we have

$$\|Z_{\kappa}(\cdot)\|_{B^{\beta}_{\infty}(L^{\upsilon})} = O\begin{cases}(\kappa+1)^{-1}, & \rho-\beta>1,\\ (\kappa+1)^{-\rho+\beta}, & \rho-\beta<1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}, & \rho-\beta=1.\end{cases} \tag{48}$$

□

The following corollaries are deduced from Theorems 1.

Corollaries

Corollary 1. *Following Remark 1(i), the convergence of g of Fourier series in the Besov spaces* $B^{\rho}_{\alpha}(L^{\upsilon}), \upsilon \geq 1; 1 < \alpha < \infty$ *using the* $N^{p,q}C^1$ *operator is given by*

$$\| \gamma^{N^{p,q}C^1}_{\kappa}(g;\cdot) - g(\cdot) \|_{B^{\beta}_{\alpha}(L^{\upsilon})} = O\begin{cases}(\kappa+1)^{-1}, & \rho-\beta-\frac{1}{\alpha}>1,\\ (\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}<1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}=1.\end{cases}$$

Corollary 2. *Following Remark 1(i), the convergence of g of Fourier series in the Besov spaces* $B^{\rho}_{\alpha}(L^{\upsilon}), \upsilon \geq 1; 1 < \alpha < \infty$ *using the* $N^{p,q}H$ *operator is given by*

$$\| \gamma^{N^{p,q}H}_{\kappa}(g;\cdot) - g(\cdot) \|_{B^{\beta}_{\alpha}(L^{\upsilon})} = O\begin{cases}(\kappa+1)^{-1}, & \rho-\beta-\frac{1}{\alpha}>1,\\ (\kappa+1)^{-\rho+\beta+\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}<1,\\ (\kappa+1)^{-1}\{\log(\kappa+1)\pi\}^{1-\frac{1}{\alpha}}, & \rho-\beta-\frac{1}{\alpha}=1.\end{cases}$$

Remark 6. *Other Corollaries can be deduced from Theorem 1 in view of Remark 1(iii) to 1(ix).*

4.2. Analysis of the Convergence of Function in the Generalized Zygmund Norm

In this subsection, we establish the following theorem to study the convergence of function g of Fourier series in generalized Zygmund $(\chi^{(\varpi)}_{\upsilon})$ spaces using the generalized Nörlund Matrix $(N^{p,q}A)$ means.

Theorem 3. *Let g be a* 2π *periodic and Lebesgue integrable function. Then, the convergence of function g of Fourier series in the generalized Zygmund* $(\chi^{(\varpi)}_{\upsilon})$ *spaces by* $N^{p,q}A$ *means is given by*

$$||\gamma^{N^{p,q}A}_{\kappa}(g;\cdot) - g(\cdot)||^{(\xi)}_{\upsilon} = O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^2\xi(y)}dy\right),$$

where ϖ *and* ξ *denote the moduli of continuity of order two such that* $\frac{\varpi(y)}{\xi(y)}$ *is non-negative and non-decreasing.*

Proof. Using the integral representation [26] of $s_{\kappa}(g;x)$, we have

$$s_{\kappa}(g;x) - g(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(x,y)\frac{\sin(\kappa+\frac{1}{2})y}{\sin(\frac{y}{2})}dy.$$

Denoting the $N^{p,q}A$ transform of $s_{\kappa}(g;x)$ by $\gamma^{N^{p,q}A}_{\kappa}$, we obtain

$$\begin{aligned}\gamma_{\kappa}^{N^{p,q}A}(g;x)-g(x) &= \frac{1}{R_{\kappa}}\sum_{\mu=0}^{\kappa}p_{\kappa-\mu}q_{\mu}\sum_{j=0}^{\mu}a_{\mu,j}\{s_j(g;x)-g(x)\}\\ &= \frac{1}{R_{\kappa}}\sum_{\mu=0}^{\kappa}p_{\kappa-\mu}q_{\mu}\sum_{j=0}^{\mu}a_{\mu,j}\left\{\frac{1}{2\pi}\int_0^{\pi}\phi(x,y)\frac{\sin(j+\frac{1}{2})y}{\sin(\frac{y}{2})}dy\right\}\\ &= \frac{1}{2\pi}\frac{1}{R_{\kappa}}\int_0^{\pi}\phi(x,y)\sum_{\mu=0}^{\kappa}p_{\kappa-\mu}q_{\mu}\sum_{j=0}^{\mu}a_{\mu,j}\frac{\sin(j+\frac{1}{2})y}{\sin(\frac{y}{2})}dy\\ &= \int_0^{\pi}\phi(x,y)M_{\kappa}^{N^{p,q}A}(y)dy.\end{aligned}$$

We define

$$T_{\kappa}(x)=\gamma_{\kappa}^{N^{p,q}A}(g;x)-g(x)=\int_0^{\pi}\phi(y)M_{\kappa}^{N^{p,q}A}(y)dy.$$

Then,

$$\begin{aligned}&T_{\kappa}(x+t)+T_{\kappa}(x-t)-2T_{\kappa}(x)\\ &=\int_0^{\pi}\Big(\phi(x+t,y)+\phi(x-t,y)-2\phi(x,t)\Big)M_{\kappa}^{N^{p,q}A}(y)dy.\end{aligned}$$

Using generalized Minkowski's inequality, we write

$$\begin{aligned}&||T_{\kappa}(\cdot+t)+T_{\kappa}(\cdot-t)-2T_{\kappa}(\cdot)||_{v}\\ &\leq\int_0^{\pi}||\phi(\cdot+t,y)+\phi(\cdot-t,y)-2\phi(\cdot,t)||_{v}|M_{\kappa}^{N^{p,q}A}(y)|dy\\ &=\int_0^{\frac{1}{\kappa+1}}||\phi(\cdot+t,y)+\phi(\cdot-t,y)-2\phi(\cdot,t)||_{v}|M_{\kappa}^{N^{p,q}A}(y)|dy\\ &+\int_{\frac{1}{\kappa+1}}^{\pi}||\phi(\cdot+t,y)+\phi(\cdot-t,y)-2\phi(\cdot,t)||_{v}|M_{\kappa}^{N^{p,q}A}(y)|dy\\ &=K_1+K_2\end{aligned}\tag{49}$$

Using Lemmas 2 and 7(ii), we have

$$\begin{aligned}K_1&=\int_0^{\frac{1}{\kappa+1}}||\phi(\cdot+t,y)+\phi(\cdot-t,y)-2\phi(\cdot,t)||_{v}|M_{\kappa}^{N^{p,q}A}(y)|dy\\ &=O\left(\int_0^{\frac{1}{\kappa+1}}\xi(|t|)\cdot\frac{\varpi(y)}{\xi(y)}(\kappa+1)dy\right)\\ &=O\left((\kappa+1)\xi(|t|)\int_0^{\frac{1}{\kappa+1}}\frac{\varpi(y)}{\xi(y)}dy\right)\\ &=O\left((\kappa+1)\xi(|t|)\frac{\varpi(\frac{1}{n+1})}{\xi(\frac{1}{\kappa+1})}\int_0^{\frac{1}{\kappa+1}}dy\right)\\ &=O\left(\xi(|t|)\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\right).\end{aligned}\tag{50}$$

Using Lemmas 3 and 7(ii), we have

$$\begin{aligned} K_2 &= \int_{\frac{1}{\kappa+1}}^{\pi} ||\phi(\cdot+t,y)+\phi(\cdot-t,y)-2\phi(\cdot,t)||_v |M_\kappa^{N^{p,q}A}(y)|dy \\ &= O\left(\int_{\frac{1}{\kappa+1}}^{\pi} \xi(|t|)\cdot\frac{\varpi(y)}{\xi(y)}\cdot\frac{1}{y^2(\kappa+1)}dy\right) \\ &= O\left(\frac{1}{\kappa+1}\xi(|t|)\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^2\xi(y)}dy\right). \end{aligned} \tag{51}$$

Combining (49)–(51),

$$\begin{aligned} ||T_\kappa(\cdot+t)+T_\kappa(\cdot-t)-2T_\kappa(\cdot)||_v &= O\left(\xi(|t|)\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\right) \\ &\quad + O\left(\frac{1}{\kappa+1}\xi(|t|)\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^2\xi(y)}dy\right). \end{aligned} \tag{52}$$

Thus,

$$\begin{aligned} \sup_{t\neq 0}\frac{||T_\kappa(\cdot+t)+T_\kappa(\cdot-t)-2T_\kappa(\cdot)||_v}{\xi(|t|)} &= O\left(\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\right) \\ &\quad + O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^2\xi(y)}dy\right). \end{aligned} \tag{53}$$

Using Lemmas 2, 3 and 7(i), we obtain

$$\begin{aligned} ||T_\kappa(\cdot)||_v &= ||\gamma_\kappa^{N^{p,q}A}(g;\cdot)-g(\cdot)||_v \\ &\leq \left(\int_0^{\frac{1}{\kappa+1}}+\int_{\frac{1}{\kappa+1}}^{\pi}\right)||\phi(\cdot,y)||_v|M_\kappa^{N^{p,q}A}(y)|dy \\ &= \int_0^{\frac{1}{\kappa+1}}||\phi(\cdot,y)||_v|M_\kappa^{N^{p,q}A}(y)|dy+\int_{\frac{1}{\kappa+1}}^{\pi}||\phi(\cdot,y)||_v|M_\kappa^{N^{p,q}A}(y)|dy \\ &= O\left((\kappa+1)\int_0^{\frac{1}{\kappa+1}}\varpi(y)dy\right)+O\left(\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^2(\kappa+1)}dy\right) \\ &= O\left(\varpi\left(\frac{1}{\kappa+1}\right)\right)+O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^2}dy\right). \end{aligned} \tag{54}$$

From (53) and (54),

$$\begin{aligned} ||T_\kappa(\cdot)||_v^\xi &= ||T_\kappa(\cdot)||_v+\sup_{t\neq 0}\frac{||T_\kappa(\cdot+t)+T_\kappa(\cdot-t)-2T_\kappa(\cdot)||_v}{\xi(|t|)} \\ &= O\left(\varpi\left(\frac{1}{\kappa+1}\right)\right)+O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^2}dy\right) \\ &\quad + O\left(\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\right)+O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^2\xi(y)}dy\right). \end{aligned}$$

Using the monotonicity of function $\xi(y)$,

$$\varpi(y)=\frac{\varpi(y)}{\xi(y)}\xi(y)\leq\xi(\pi)\frac{\varpi(y)}{\xi(y)}=O\left(\frac{\varpi(y)}{\xi(y)}\right)\ \text{ for } 0<y\leq\pi.$$

We have

$$O\left(\varpi\left(\frac{1}{\kappa+1}\right)\right) = O\left(\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\right) \text{ for } y = \frac{1}{\kappa+1}.$$

Again, the monotonicity of function $\xi(y)$ yields

$$\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}\xi(y)dy \leq \xi(\pi)\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy = O\left(\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy\right).$$

Thus,

$$||T_\kappa(\cdot)||_\upsilon^\xi = O\left(\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\right) + O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy\right). \tag{55}$$

Using Remark 3, we write

$$\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy \geq \frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\int_{\frac{1}{\kappa+1}}^{\pi} \frac{dy}{y^2} \geq (\kappa+1)\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}.$$

Hence,

$$\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})} = O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy\right). \tag{56}$$

From (55) and (56), we obtain

$$\begin{aligned} ||T_\kappa(\cdot)||_\upsilon^\xi &= O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy\right) + \left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy\right) \\ &= O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy\right) \\ ||\gamma_\kappa^{N^{p,q}A}(g;\cdot) - g(\cdot)||_\upsilon^{(\xi)} &= O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy\right). \end{aligned} \tag{57}$$

□

In addition to the conditions of Theorem 3, we establish the following theorem.

Theorem 4. *Let g be a Lebesgue integrable and 2π a periodic function; thenm, the degree of convergence of function g of Fourier series in the generalized Zygmund $(\chi_\upsilon^{(\varpi)})$ spaces by $N^{p,q}A$ means is given by*

$$||\gamma_\kappa^{N^{p,q}A}(g;\cdot) - g(\cdot)||_\upsilon^{(\xi)} = O\left(\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\right)(\log \pi(\kappa+1)),$$

where ϖ and ξ denote the moduli of continuity of order two such that $\frac{\varpi(y)}{y\xi(y)}$ is non-negative and non-increasing.

Proof. Following the proof of Theorem 3,

$$||\gamma_\kappa^{N^{p,q}A}(g;\cdot) - g(\cdot)||_\upsilon^{(\xi)} = O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi} \frac{\varpi(y)}{y^2\xi(y)}dy\right)$$

since $\frac{\varpi(\frac{1}{\kappa+1})}{y\xi(\frac{1}{\kappa+1})}$ is positive and non-increasing; then, by the second mean value theorem of integral calculus,

$$\begin{aligned} ||\gamma_{\kappa}^{N^{p,q}A}(g;\cdot)-g(\cdot)||_{v}^{(\xi)} &= O\left(\frac{1}{\kappa+1}(\kappa+1)\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\int_{\frac{1}{\kappa+1}}^{\pi}\frac{dy}{y}\right) \\ &= O\left(\frac{\varpi(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})}\right)(\log\pi(\kappa+1)). \end{aligned} \tag{58}$$

□

Corollaries

Corollary 3. *Let* $g \in \chi_{v}^{(\xi)}, v \geq 1$, *and* $0 \leq \tau < \rho < 1$. *Then*

$$||\gamma_{\kappa}^{N^{p,q}A}(g;\cdot)-g(\cdot)||_{v}^{(\xi)} = \begin{cases} O\left(\frac{((\kappa+1)\pi)^{\rho-\tau-1}-1}{\kappa+1}\right), & 0 \leq \tau < \rho < 1, \\ O\left(\frac{\log\pi(\kappa+1)}{\kappa+1}\right), & \tau = 0, \rho = 1. \end{cases}$$

Proof. Taking $\varpi(y) = y^{\rho}, \xi(y) = y^{\tau}, 0 \leq \tau < \rho < 1$ in (57), we write

$$||\gamma_{\kappa}^{N^{p,q}A}(g;\cdot)-g(\cdot)||_{v}^{(\xi)} = O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi} y^{\rho-\tau-2}dy\right).$$

Now,

$$||\gamma_{\kappa}^{N^{p,q}A}(g;\cdot)-g(\cdot)||_{v}^{(\xi)} = \begin{cases} O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi} y^{\rho-\tau-2}dy\right), & 0 \leq \tau < \rho < 1, \\ O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi}\frac{dy}{y}\right), & \tau = 0, \rho = 1. \end{cases}$$

$$||\gamma_{\kappa}^{N^{p,q}A}(g;\cdot)-g(\cdot)||_{v}^{(\xi)} = \begin{cases} O\left(\frac{((\kappa+1)\pi)^{\rho-\tau-1}-1}{\kappa+1}\right), & 0 \leq \tau < \rho < 1, \\ O\left(\frac{\log\pi(\kappa+1)}{\kappa+1}\right), & \tau = 0, \rho = 1. \end{cases}$$

□

Corollary 4. *Following Remark 1(i), the convergence of g of Fourier series in the generalized Zygmund spaces* $(\chi_{v}^{(\varpi)})$ *by the* $N^{p,q}C^{1}$ *operator is given by*

$$||\gamma_{\kappa}^{N^{p,q}C^{1}}(g;\cdot)-g(\cdot)||_{v}^{(\xi)} = O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^{2}\xi(y)}dy\right),$$

where ϖ *and* ξ *denote the moduli of continuity of order two such that* $\frac{\varpi(y)}{\xi(y)}$ *is non-negative and non-decreasing.*

Corollary 5. *Following Remark 1 (ii), the convergence of g of Fourier series in the generalized Zygmund spaces* $(\chi_{v}^{(\varpi)})$ *by the* $N^{p,q}H$ *operator is given by*

$$||\gamma_{\kappa}^{N^{p,q}H}(g;\cdot)-g(\cdot)||_{v}^{(\xi)} = O\left(\frac{1}{\kappa+1}\int_{\frac{1}{\kappa+1}}^{\pi}\frac{\varpi(y)}{y^{2}\xi(y)}dy\right),$$

where ϖ *and* ξ *denote the moduli of continuity of order two such that* $\frac{\varpi(y)}{\xi(y)}$ *is non-negative and non-decreasing.*

Remark 7. *Other Corollaries can be obtained from Theorem 3 in view of Remark 1(iii) to 1(ix).*

5. Applications

In this section, we present an application of the convergence of our results.

5.1. Application of the Convergence of Function in the Besov Norm

We have $M_{\kappa}^{N^{p,q}A}(y) = O(\kappa+1)$ for $0 < y \leq \frac{1}{\kappa+1}$ and $M_{\kappa}^{N^{p,q}A}(y) = O\left(\frac{1}{(\kappa+1)y^2}\right)$ for $\frac{1}{\kappa+1} < y \leq \pi$. Taking $\rho = 1, \beta = 0$ and $\alpha = \infty$, we have $\|Z_{\kappa}(\cdot)\|_{v} = O\left(\frac{\log(\kappa+1)\pi}{\kappa+1}\right)$ and $\|\varpi_{\mu}(Z_{\kappa}, \cdot)_{v}\|_{0,\infty} = O\left(\frac{\log(\kappa+1)\pi}{\kappa+1}\right)$.

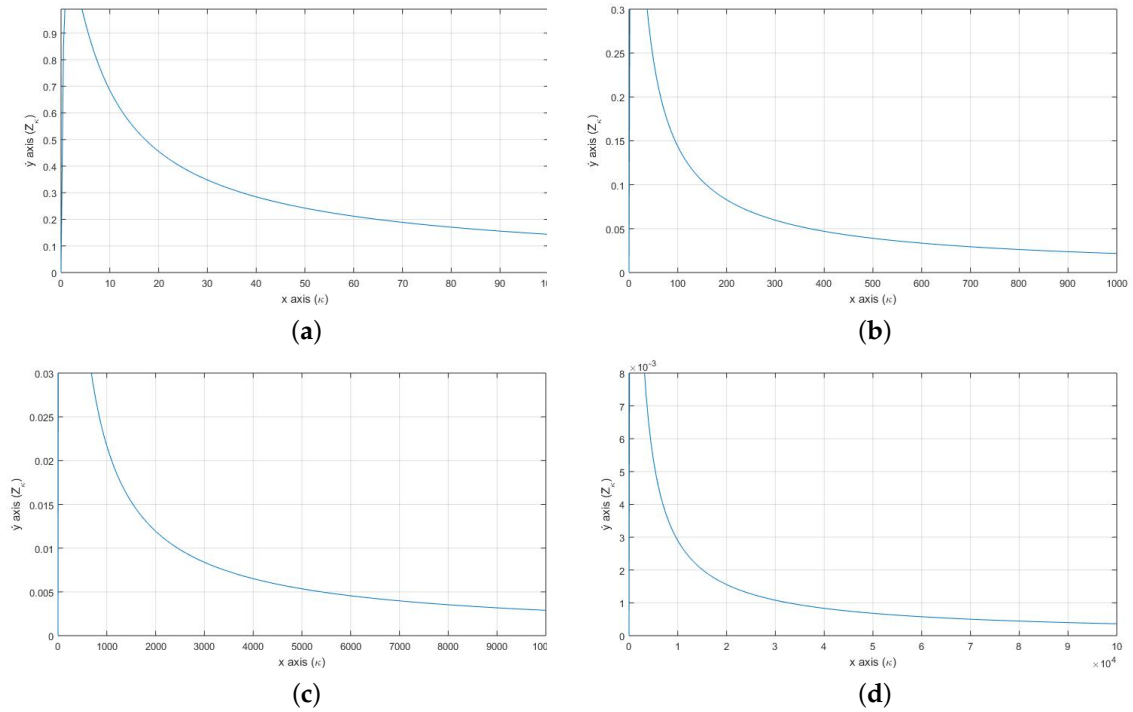
Thus,

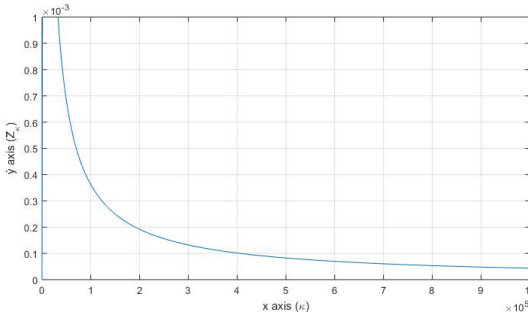
$$\|Z_{\kappa}(\cdot)_{v}\|_{B_{\infty}^{0}(L_2)} = \|Z_{\kappa}(\cdot)\|_{v} + \|\varpi_{\mu}(Z_{\kappa}, \cdot)_{v}\|_{0,\infty} = O\left(\frac{\log(\kappa+1)\pi}{\kappa+1}\right).$$

Now, we construct the Table 1 then plot graphs for $Z_{\kappa}(\cdot)$ for different values of κ (Figure 1).

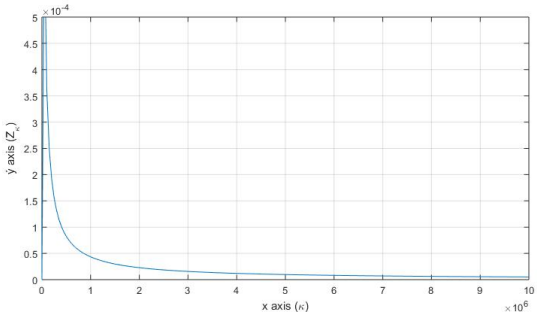
Table 1. Values of $\|Z_{\kappa}(\cdot)\|$ for different values of κ.

κ	$\|Z_{\kappa}(\cdot)\| = \frac{\log(\kappa+1)\pi}{(\kappa+1)}$
100	0.057028
1000	0.008045
10,000	0.001035
100,000	0.000127
1,000,000	0.000015
10,000,000	0.000002
...	...
∞	0





(**e**)



(**f**)

Figure 1. Graphs of $\|Z_\kappa(\cdot)\|$ for different values of κ. (**a**) For $\kappa = 100$; (**b**) For $\kappa = 1000$; (**c**) For $\kappa = 10{,}000$; (**d**) For $\kappa = 100{,}000$; (**e**) For $\kappa = 1{,}000{,}000$; (**f**) For $\kappa = 10{,}000{,}000$.

5.2. Application of the Convergence of Function in the Generalized Zygmund Norm

Consider function $\frac{\omega(\frac{1}{\kappa+1})}{\xi(\frac{1}{\kappa+1})} = \frac{1}{(\kappa+1)^2}$.

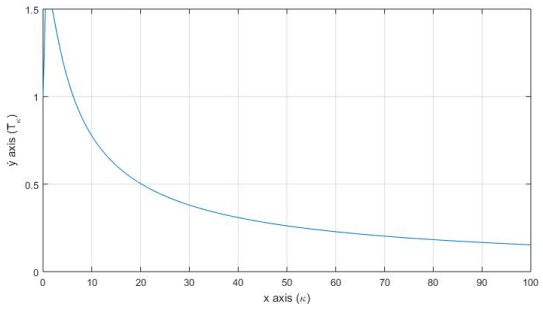
Therefore,

$$||T_\kappa(\cdot)||_v^\xi = O\left(\frac{\log \pi(\kappa+1)}{(\kappa+1)^2}\right).$$

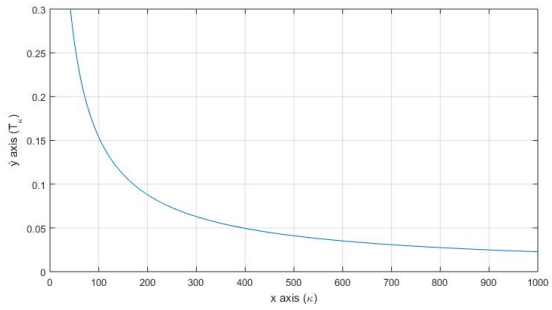
Now, we construct the Table 2 then plot graphs for $T_\kappa(\cdot)$ for different values of κ (Figure 2).

Table 2. Values of $T_\kappa(\cdot)$ for different values of κ.

κ	$\|\|T_\kappa(\cdot)\|\|_v^\xi = \frac{(\log \pi(\kappa+1)}{(\kappa+1)^2}$
100	0.000564636
1000	0.000008037
10,000	0.000000104
100,000	0.000000001
1,000,000	0.000000000
10,000,000	0.000000000
...	...
∞	0



(**a**)



(**b**)

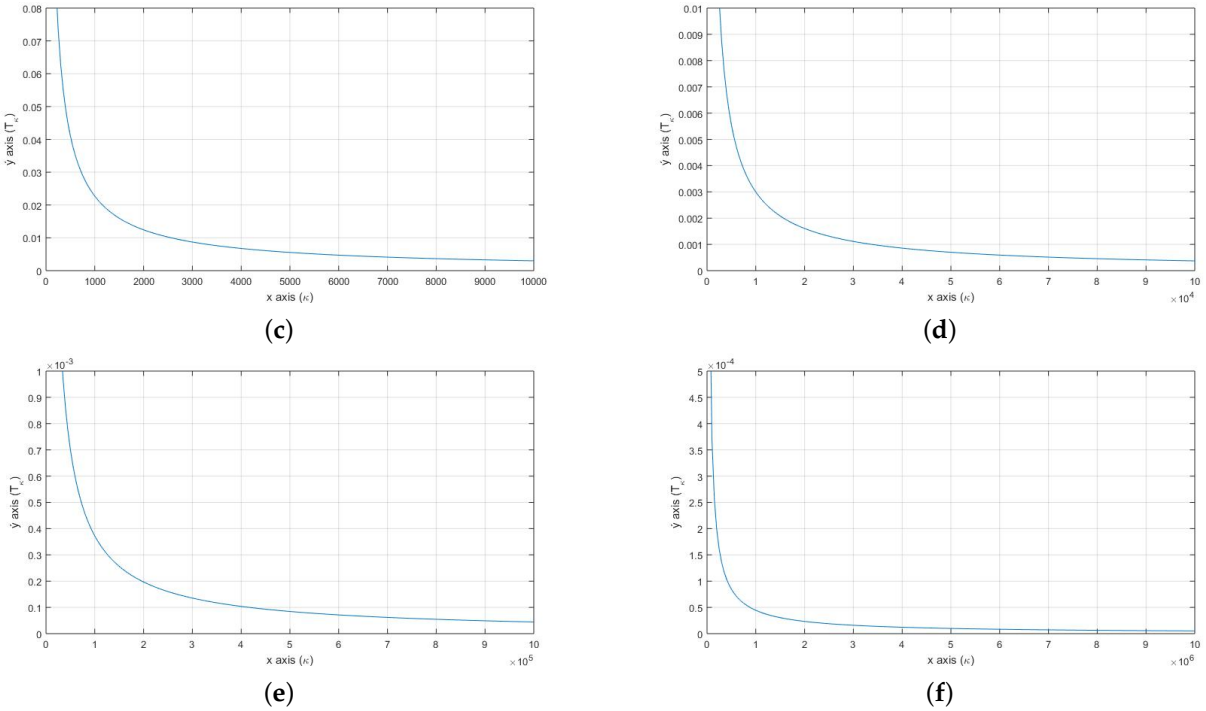


Figure 2. Graphs of $||T_\kappa(\cdot)||$ for different values of κ. (**a**) For $\kappa = 100$; (**b**) For $\kappa = 1000$; (**c**) For $\kappa = 10,000$; (**d**) For κ = 100,000; (**e**) For κ =1,000,000; (**f**) For κ = 10,000,000.

6. Conclusions

In Application 1 (Table 1 and the Figure 1a–f), we observe that the rate of convergence of g becomes faster as κ increases, i.e., $\|Z_\kappa(\cdot)_v\|_{B^0_\infty} \to 0$ as $\kappa \to \infty$.
In Application 2 (Table 2 and the Figure 2a–f), we observe that the rate of convergence of g becomes faster as κ increases, i.e., $||T_\kappa(\cdot)||^\zeta_v \to 0$ as $\kappa \to \infty$.
The analysis of both convergence results shows that both the norms provide the best approximation. However, the generalized Zygmund norm provides rapid convergence compared to the Besov norm.

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