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Mathematics and Poetry · Yang–Baxter Equations, Boolean Algebras, and BCK-Algebras

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Abstract: The current paper explores the potential of the areas between mathematics and poetry. We will first recall some definitions and results that are needed to construct solutions of the Yang–Baxter equation. A new duality principle is presented and Boolean coalgebras are introduced. A section on poetry dedicated to the Yang–Baxter equation is presented, and a discussion on a poem related to a mathematical formula follows. The final section presents our conclusions and further information on these topics.

Keywords: Yang–Baxter equation; Boolean (co)algebra; BCK-algebra; poetry

MSC: 16T25; 06F35; 06E20; 00B10; 00B15

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1. Introduction

Sometimes, mathematicians desire to express their enthusiasm and original ideas to friends (who may not be highly-trained mathematicians). So, they write literary works. The current paper explores the potential of some areas situated between mathematics and poetry. It contains sufficient mathematical content to attract the attention of professional mathematicians, yet some sections could be read by poets as well. The impact of this kind of approach is impressive; for example, one could look at the successful AMS sessions dedicated to poetry and mathematics, the articles on this matter from some Romanian publications (Apostrof, Caiete Critice, Convorbiri Literare, Tribuna Educationala, etc.), as well as several web sites and online journals.

As poets search for new ways to express their thoughts and feelings, they create poetical experiments, which sometimes lead to valuable contributions to literature.

The Yang–Baxter equation was first discovered by Nobel laureate C.N. Yang in theoretical physics [1] and by R.J. Baxter in statistical mechanics [2,3]. It is one of the main equations used in mathematical physics, integrable systems, quantum algebraic systems, the theory of quantum groups, quantum computing, knot theory, braided categories, etc. (see [4]). Yang initially considered the matrix equation $F(x)G(x+y)F(y) = G(y)F(x+y)G(x)$, and found an explicit solution where $F(x)$ and $G(x)$ are rational functions. Many scientists have used the axioms of various algebraic structures in order to obtain solutions for these versions of the Yang–Baxter equation [5]. F.F. Nichita et al. obtained results on Jordan algebras and Jordan coalgebras, and related them to the Yang–Baxter equations (see, for example, [6,7] and the references therein). Constructions of quantum gates and link invariants from solutions of the Yang–Baxter equation were described in [8,9]. Some solutions for the Yang–Baxter equation in MV algebras, Wajsberg algebras, MTL-algebras, weak implication algebras, and lattice effect algebras were investigated in [10–13]. BCK-algebras are concepts introduced by Y. Imai and K. Iseki [14]. BCK-algebras involve generalizations of the notion of algebraic sets with subtraction and the notion of implication algebra [14,15]. In the next section, we will recall some fundamental definitions, lemmas, and theorems that are needed to construct solutions of the Yang–Baxter equation in BCK-algebras. We will also define Boolean coalgebras. In Section 5, we will present explicit set-theoretical solutions. Our propositions, lemmas, and theorems will hopefully provide new perspectives on the Yang–Baxter equation (in BCK-algebras). We also recall a braid-quantum Yang–Baxter equation, whose solutions include both solutions of the braid equation and solutions to the quantum Yang–Baxter equation. There is a new duality principle about solutions to the braid condition (in Boolean algebras).

2. Rudiments of *BCK*-Algebras

Throughout this section, we provide fundamental definitions, lemmas, and theorems about the structures of BCK-algebras. These notions are taken from [16].

Definition 1. *An algebra $\mathcal{A} = (A; \to, 1)$ of type $(2, 0)$ is said to be a $BCK-$algebra if it verifies the following identities*

(i) $(x \to y) \to ((y \to z) \to (x \to z)) = 1,$

(ii) $x \to ((x \to y) \to y) = 1,$

(iii) $x \to x = 1,$

(iv) $x \to 1 = 1,$

(v) $x \to y = 1$ *and* $y \to x = 1$ *imply* $x = y.$

for each $x, y \in A$.

Lemma 1. *The binary relation $\leq$ on A given by*

$$x \leq y \Leftrightarrow x \to y = 1$$

is a partial order on A with 1 as the biggest element.

As opposed to Lemma 1, the poset $(A; \leq)$ has no particular property because any poset $(P; \leq)$ with 1 can be made a $BCK-$algebra by setting $a \to b := 1$ for $a \leq b$, and $a \to b := b$ otherwise for any $a, b \in P$.

Definition 2. *An algebra $\mathcal{A} = (A; \to, 1)$ is said to be a bounded $BCK-algebra$, where $(A; \to, 1)$ is a BCK-algebra with the least element 0, such that $0 \to x = x$.*

Lemma 2. *Let $(A; \to, 1)$ be a BCK-algebra. Then*

(a) $x \leq y \Rightarrow y \to z \leq x \to z,$
(b) $x \leq y \Rightarrow z \to x \leq z \to y,$
(c) $x \to (y \to z) = y \to (x \to z),$
(d) $y \leq x \to y,$
(e) $1 \to x = x,$
(f) $x \to y \leq (z \to x) \to (z \to y),$
(g) $((x \to y) \to y) \to y = x \to y,$

are satisfied for all $x, y, z \in A$.

The commutative $BCK-$algebras can be characterized as join-semilattices by defining the $\vee-$operation as follows:

$$x \vee y := (x \to y) \to y. \tag{1}$$

Definition 3. *Let $\mathcal{A} = (A; \rightarrow, 0, 1)$ be a bounded BCK−algebra. The mapping N is defined on A as*

$$N(x) := x \rightarrow 0$$

for each $x \in A$.

Lemma 3. *Let $\mathcal{A} = (A; \rightarrow, 0, 1)$ be a bounded BCK−algebra. The mapping N is an antitone involution on A.*

Proof. Let $a, b \in A$ and $a \leq b$. By substituting $[c := 0]$ in Lemma 2 (a), we obtain $b \rightarrow 0 \leq a \rightarrow 0$. By the definition of the mapping N, we obtain $N(b) \leq N(a)$. Then N is an antitone mapping.

Let $a \in A$. From the Equation (1), we have $N(N(x)) = (x \rightarrow 0) \rightarrow 0 = x \vee 0 = x$. Then $N(N(x)) = x$. So, N is an involution mapping. □

Definition 4. *Let $\mathcal{A} = (A; \rightarrow, 0, 1)$ be a bounded BCK−algebra. The binary operations $\sqcup$ and $\sqcap$ are defined as*

$$x \sqcup y := (x \rightarrow y) \rightarrow y,$$

$$x \sqcap y := N(N(x) \sqcup N(y)),$$

for each $x, y \in A$.

Definition 5. *A commutative BCK-algebra is a BCK-algebra that satisfies the identity*

$$x \sqcup y = y \sqcup x$$

for each $x, y \in A$.

Theorem 1. *Let $\mathcal{A} = (A; \rightarrow, 1)$ be a BCK−algebra. Define a unary operation N_k on the section $[k, 1] = \{x \in A : k \leq x\}$ for each $a \in A$ by*

$$N_k(x) = x \rightarrow k.$$

Then the structure $\Omega(\mathcal{A}) = (A; \sqcup, N_k, 1)$ satisfies the following quasi-identities:

(i) $a \sqcup a = a$,
(ii) $a \sqcup b = b$ *and* $b \sqcup a = a$ *imply* $a = b$,
(iii) $a \sqcup b = (a \sqcup b) \sqcup b = a \sqcup (a \sqcup b) = b \sqcup (a \sqcup b)$,
(iv) $(a \sqcup c) \sqcup ((a \sqcup b) \sqcup c) = (a \sqcup b) \sqcup c$,
(v) $a \sqcup 1 = 1$,
(vi) $N_a(a) = 1, \ \ N_a(1) = a$,
(vii) $a \sqcup b = N_b(N_b(a \sqcup b)) = N_b(N_b(a \sqcup b) \sqcup b)$,
(viii) $N_b(a \sqcup b) \sqcup N_{b \sqcup c}((a \sqcup c) \sqcup (b \sqcup c)) = N_{b \sqcup c}((a \sqcup c) \sqcup (b \sqcup c))$,
(ix) $N_{b \sqcup c}(N_c(a \sqcup c) \sqcup (b \sqcup c)) = N_{a \sqcup c}(N_c(b \sqcup c) \sqcup (a \sqcup c))$,
(ix) $N_a((a \sqcup b) \sqcup a) = N_a(a \sqcup b)$.

for each $a, b, c \in A$.

Lemma 4. *Let $(A; \sqcup)$ be defined as Theorem 1. The binary relation $\leq$ is defined by*

$$x \leq y \quad \text{if and only if} \quad x \sqcup y = y.$$

Then, the binary relation $\leq$ is a partial order on A. Moreover, $x \sqcup y$ is the least upper bound of x and y. Dually, $x \sqcap y$ is the greatest lower bound of x and y.

Lemma 5. *Let $\mathcal{A} = (A; \rightarrow, 1)$ be a BCK−algebra. The binary operation $\sqcup$ is defined as Theorem 1. Then the following statements are equivalent to each other:*

(i) $\mathcal{A}$ *is commutative;*
(ii) $(A;\sqcup)$ *is a directoid;*
(iii) $(A;\sqcup)$ *is a a join-semilattice.*

Theorem 2. *Let* $\Omega = (S;\sqcup, N_k, 1)$ *be a structure on* S*. The binary operation* $\to$ *is defined on* S *as*

$$a \to b := N_b(a \sqcup b).$$

Then, $\mathcal{A}(\Omega) = (S;\to,1)$ *is a* $BCK-$*algebra.*

Definition 6. *Let* $\mathcal{A} = (A;\to,1)$ *be a* $BCK-$*algebra.*
- *If it verifies*

$$x \to (y \to z) = (x \to y) \to (x \to z)$$

for each $x, y, z \in A$ *then, it is called a positive implicative* $BCK-$*algebra.*
- *If it verifies*

$$(x \to y) \to z = (x \to z) \to (y \to z)$$

for each $x, y, z \in A$ *then, it is called a negative implicative* $BCK-$*algebra.*

At the end of this preliminary section, let us define a new structure that will be used in our search for solutions to the Yang–Baxter equation. Further investigations in the framework of BCK-algebras will continue in the future.

Definition 7. *A Boolean coalgebra is defined as a 6-tuple* $\mathcal{C} = (C,\ \vee,\ \Delta,\ N,\ 0,\ 1)$*, where* $\vee,\ N,\ 0$ *and* 1 *have the usual properties. (So,* $\vee$ *is an associative and commutative operation,* N *is an involution,* $x \vee 0 = x,\ x \vee 1 = 1$*, etc.)*

The new structure is $\Delta : C \to C \times C,\ \Delta(a) = (a_1,\ a_2)$*, and we require:*

(i) Δ *is coassociative (i.e.,* $(\Delta \times I) \circ \Delta = (I \times \Delta) \circ \Delta$*),*
(ii) $a_1 \vee a_2 = a\ \ \forall a \in C$*,*
(iii) $\Delta(0) = (0,\ 0)$ *and*
(iv) $\Delta(a \vee b) = \Delta(a) \vee \Delta(b)\ \ \forall a, b \in C$ *(Notice that this equality takes place in* $C \times C$*).*

For an arbitrary Boolean algebra, we can associate a Boolean coalgebra with $\Delta(a) = (a \wedge c, a \wedge N(c))$. Moreover, if we recall that $a \to b = N(a) \vee b$, we obtain a BCK-algebra with the following property: $\Delta(a \to b) = \Delta(a \to 0) \to \Delta(b)\ \ \forall a, b \in C$.

3. Perspectives on the Yang–Baxter Equation in BCK-Algebras

In this section, we present some set-theoretical solutions of the Yang–Baxter equation in BCK-algebras. Moreover, we define new operators on BCK-algebras, then we obtain new solutions by using these operators.

Let V be a vector space over the field k. The tensor products are defined over k. We also use the (set-theoretical) twist map $\mho : V \otimes V \to V \otimes V,\ \ \mho(p \otimes q) = q \otimes p$.

The identity map of this vector space is defined $I : V \to V$. For a $k-$linear map, $R : V \otimes V \to V \otimes V$ we define $R^{12} = R \otimes I$, $R^{23} = I \otimes R$ and $R^{13} = (I \otimes \mho)(R \otimes I)(\mho \otimes I)$.

Definition 8 ([17]). *A Yang–Baxter operator is an invertible* $k-$*linear map* $R : V \otimes V \to V \otimes V$ *and it verifies the braid condition (known as the "Yang–Baxter equation" or the "braid condition")*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{2}$$

If R *verifies the Equation (2), then* $\mho \circ R$ *and* $R \circ \mho$ *supply the quantum Yang–Baxter equation (known as* $QYBE$*):*

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \tag{3}$$

Lemma 6 ([17]). *The Equations (2) and (3) are equivalent.*

Lemma 7 ([18]). *The Equations (2) and (3) lead each to solutions for the following "braid-quantum Yang–Baxter equation":*

$$R^{12} \circ X^{13} \circ R^{23} \circ Y^{12} = R^{23} \circ X^{13} \circ R^{12} \circ Y^{23}, \tag{4}$$

where $R = XY$.

Obviously, finding all solutions for the *braid-quantum Yang–Baxter equation* is an open problem. The first step to solve this would be to construct solutions for it, and to make a small analysis of those solutions, which are neither solutions for the braid condition nor for the quantum Yang–Baxter equation.

Back to $BCK-$algebras, we recall the following definition.

Definition 9 ([17]). *Let P be any set. The mapping $S(p,q) = (p', q')$ is defined from $P \times P$ to $P \times P$. The mapping S satisfies the Yang–Baxter equation (or equivalently, "S is a set-theoretical solution of the Yang–Baxter equation") if it holds the following equation*

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}, \tag{5}$$

which is also equivalent to

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}, \tag{6}$$

where

$$S^{12} : P \times P \times P \to P \times P \times P, \;\; S^{12}(p,q,r) = (p', q', r),$$
$$S^{23} : P \times P \times P \to P \times P \times P, \;\; S^{23}(p,q,r) = (p, q', r'),$$
$$S^{13} : P \times P \times P \to P \times P \times P, \;\; S^{13}(p,q,r) = (p', q, r').$$

Now, we may handle verifying the Yang–Baxter equation in BCK-algebras. First of all, we provide the following lemma, which is needed for further processing of this work.

Proposition 1 ([16]). *Let $(A; \to, 1)$ be a BCK-algebra. Then*

(1) $(0 \to 0) \to x = x$
(2) $(x \to 0) \to 0 = x$
(3) $(z \to 0) \to y = (y \to 0) \to z$
(4) $x \to (y \to x) = 1.$

hold for all $x, y, z \in A$.

Lemma 8. *Let $(A, \to, 0, 1)$ be a bounded BCK-algebra. Then, the mapping $S(x,y) = (y \to 0, x \to 0)$ verifies the braid condition on this structure.*

Lemma 9. *Let $(A, \to, 1)$ be a BCK-algebra. Then, the mapping $S(x,y) = (1 \to x, 1 \to y)$ verifies the braid condition on this structure.*

Lemma 10. *Let $(A; \to, 0, 1)$ be a bounded commutative BCK-algebra. Then, the mapping $S(x,y) = ((x \to 0) \to y, 0)$ verifies the braid condition on this structure. As a conclusion, the Yang–Baxter equation has a set-theoretical solution in BCK-algebra.*

Proof. We define S^{12} and S^{23} as follows:

$$\begin{aligned} S^{12}(x,y,z) &= ((x \to 0) \to y, 0, z), \\ S^{23}(x,y,z) &= (x, (y \to 0) \to z, 0). \end{aligned}$$

We show that the equilibrium $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ are satisfied for each $(x, y, z) \in A \times A \times A$. By the help of Definition 1, Lemma 2 (c) and (e), and Proposition 1 (2) and (3), we have

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= S^{12}(S^{23}(S^{12}(x, y, z))) \\
&= S^{12}(S^{23}((x \to 0) \to y, 0, z)) \\
&= S^{12}((x \to 0) \to y, (0 \to 0) \to z, 0) \\
&= S^{12}((x \to 0) \to y, z, 0) \\
&= ((((x \to 0) \to y) \to 0) \to z, 0, 0) \\
&= ((((x \to 0) \to y) \to 0) \to ((z \to 0) \to 0), 0, 0) \\
&= ((z \to 0) \to ((((x \to 0) \to b) \to 0) \to 0), 0, 0) \\
&= ((z \to 0) \to ((x \to 0) \to y), 0, 0) \\
&= ((x \to 0) \to ((z \to 0) \to y), 0, 0) \\
&= ((x \to 0) \to ((y \to 0) \to z), 0, 0)
\end{aligned}$$

and

$$\begin{aligned}
(S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= S^{23}(S^{12}(S^{23}(x, y, z))) \\
&= S^{23}(S^{12}(x, (y \to 0) \to z, 0)) \\
&= S^{23}((x \to 0) \to ((y \to 0) \to z), 0, 0) \\
&= ((x \to 0) \to ((y \to 0) \to z), (0 \to 0) \to 0, 0) \\
&= ((x \to 0) \to ((y \to 0) \to z), 0, 0).
\end{aligned}$$

Thus, the Yang–Baxter equation is satisfied in $BCK-$algebras. The mapping $S(x, y) = ((x \to 0) \to y, 0)$ is a set-theoretical solution of it on these structures. □

Lemma 11. *Let $(A, \to, 1)$ be a commutative BCK-algebra. Then, the mapping $S(x, y) = ((x \to y) \to y, y)$ verifies the braid condition on this structure. Therefore, the Yang–Baxter equation has a set-theoretical solution in BCK-algebras.*

Proof. We define S^{12} and S^{23} as follows:

$$\begin{aligned}
S^{12}(x, y, z) &= ((x \to y) \to y, y, z), \\
S^{23}(x, y, z) &= (x, (y \to z) \to z, z).
\end{aligned}$$

We show that the equality $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ is satisfied for each $(x, y, z) \in A \times A \times A$. By the Definition 5, Lemma 2 (c) and (e) and Proposition 1 (4), we obtain

$$
\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\
&= S^{12}(S^{23}((x \to y) \to y, y, z)) \\
&= S^{12}((x \to y) \to y, (y \to z) \to z, z) \\
&= ((((x \to y) \to y) \to ((y \to z) \to z))) \to ((y \to z) \to z), \\
&\quad (y \to z) \to z, z) \\
&= ((((x \to y) \to y) \to ((z \to y) \to y))) \to ((z \to y) \to y), \\
&\quad (y \to z) \to z, z) \\
&= (((z \to y) \to (((x \to y) \to y) \to y)) \to ((z \to y) \to y), \\
&\quad (y \to z) \to z, z) \\
&= (((z \to y) \to ((y \to (x \to y)) \to (x \to y))) \to ((z \to y) \to y), \\
&\quad (y \to z) \to z, z) \\
&= (((z \to y) \to (x \to y)) \to ((z \to y) \to y), (y \to z) \to z, z) \\
&= ((x \to ((y \to z) \to z)) \to ((y \to z) \to z), (y \to z) \to z, z)
\end{aligned}
$$

and

$$
\begin{aligned}
(S^{23} \circ S^{12} \circ S^{23})(x,y,z) &= S^{23}(S^{12}(S^{23}(x,y,z))) \\
&= S^{23}(S^{12}(x, (y \to z) \to z, z)) \\
&= S^{23}((x \to ((y \to z) \to z)) \to ((y \to z) \to z), (y \to z) \to z, z) \\
&= ((x \to ((y \to z) \to z)) \to ((y \to z) \to z), \\
&\quad (((y \to z) \to z) \to z) \to z, z) \\
&= (((x \to ((y \to z) \to z)) \to ((y \to z) \to z), \\
&\quad ((z \to (y \to z)) \to (y \to z)) \to z, z) \\
&= ((x \to ((y \to z) \to z)) \to ((y \to z) \to z), (y \to z) \to z, z).
\end{aligned}
$$

Then, the Yang–Baxter equation has a one-set theoretical solution $S(x,y) = ((x \to y) \to y, y)$ in $BCK-$algebra. □

Lemma 12. *Let $(A, \to, 0, 1)$ be a bounded BCK-algebra. Then, the mapping $S(x,y) = ((y \to (x \to 0)) \to 0, 1)$ verifies the braid condition on this structure. Therefore, the Yang–Baxter equation has a set-theoretical solution in BCK-algebra.*

Proof. Let S^{12} and S^{23} be defined as follows:

$$
\begin{aligned}
S^{12}(x,y,z) &= ((y \to (x \to 0)) \to 0, 1, z), \\
S^{23}(a,b,c) &= (x, (z \to (y \to 0)) \to 0, 1).
\end{aligned}
$$

With the help of Definition 1, Lemma 2 (e), and Proposition 1 (2), we have

$$
\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\
&= S^{12}(S^{23}((y \to (x \to 0)) \to 0, 1, z) \\
&= S^{12}((y \to (x \to 0)) \to 0, (z \to (1 \to 0)) \to 0, 1) \\
&= S^{12}((y \to (x \to 0)) \to 0, z, 1) \\
&= ((z \to (((y \to (x \to 0)) \to 0) \to 0)) \to 0, 1, 1) \\
&= ((z \to (((y \to 0) \to 0) \to (x \to 0))) \to 0, 1, 1)
\end{aligned}
$$

and

$$\begin{aligned}
(S^{23} \circ S^{12} \circ S^{23})(x,y,z) &= S^{23}(S^{12}(S^{23}(x,y,z))) \\
&= S^{23}(S^{12}(x,(z \to (y \to 0)) \to 0,1) \\
&= S^{23}((((z \to (y \to 0)) \to 0) \to (x \to 0)) \to 0,1,1) \\
&= ((((z \to (y \to 0)) \to 0) \to (x \to 0)) \to 0, \\
&\quad (1 \to (1 \to 0)) \to 0,1) \\
&= ((((z \to (y \to 0)) \to 0) \to (x \to 0)) \to 0,1,1).
\end{aligned}$$

Then, the Yang–Baxter equation is satisfied in $BCK-$algebras. The mapping $S(x,y) = ((x \to y) \to y, y)$ is a set-theoretical solution of this equation on $BCK-$algebras. □

Example 1. *Let $X = \{0, x, 1\}$. The operation $\to$ is defined as the following table:*

$\to$	0	x	1
0	1	1	1
x	x	1	1
1	0	x	1

Then, $(X, \to, 0, 1)$ is a bounded BCK-algebra. Moreover, the mapping $S(a,b) = (((b \to 0) \to (a \to 0)) \to 0, 0)$ verifies the braid condition on this structure. Therefore, the Yang–Baxter equation has a set-theoretical solution in this structure.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned}
S^{12}(a,b,c) &= (((b \to 0) \to (a \to 0)) \to 0, 0, c), \\
S^{23}(a,b,c) &= (a, ((c \to 0) \to (b \to 0)) \to 0, 0).
\end{aligned}$$

By using Definition 1, Lemma 2 (e), Proposition 1 (2), we obtain

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(a,b,c) &= S^{12}(S^{23}(S^{12}(a,b,c))) \\
&= S^{12}(S^{23}(((b \to 0) \to (a \to 0)) \to 0, 0, c) \\
&= S^{12}(((b \to 0) \to (a \to 0)) \to 0, ((c \to 0) \to (0 \to 0)) \\
&\quad \to 0, 0) \\
&= S^{12}(((b \to 0) \to (a \to 0)) \to 0, ((c \to 0) \to 1) \to 0, 0) \\
&= S^{12}(((b \to 0) \to (a \to 0)) \to 0, 0, 0) \\
&= (((((0 \to 0) \to ((((b \to 0) \to (a \to 0)) \to 0) \to 0) \\
&\quad \to 0), 0, 0) \\
&= (((((b \to 0) \to (a \to 0)) \to 0) \to 0) \to 0, 0, 0) \\
&= (((b \to 0) \to (a \to 0)) \to 0, 0, 0), \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
(S^{23} \circ S^{12} \circ S^{23})(a,b,c) &= S^{23}(S^{12}(S^{23}(a,b,c))) \\
&= S^{23}(S^{12}(a, ((c \to 0) \to (b \to 0)) \to 0, 0) \\
&= S^{23}((((((c \to 0) \to (b \to 0)) \to 0) \to 0) \to (a \to 0)) \\
&\quad \to 0, 0, 0) \\
&= ((((((c \to 0) \to (b \to 0)) \to 0) \to 0) \to (a \to 0)) \to 0, \\
&\quad ((0 \to 0) \to (0 \to 0)) \to 0, 0) \\
&= ((((((c \to 0) \to (b \to 0)) \to 0) \to 0) \to (a \to 0)) \qquad (8) \\
&\quad \to 0, 0, 0).
\end{aligned}$$

Since the Equation (7) is equal to the Equation (8) for all $a, b, c \in X$, we see that the Yang–Baxter equation is satisfied in this structure. The mapping $S(a,b) = (((b \to 0) \to (a \to 0)) \to 0, 0)$ is a set-theoretical solution of this equation on this structure, whereas it is not a set-theoretical solution of the Yang–Baxter equation in bounded BCK-algebras. □

Example 2. *The mapping $S(x,y) = (x \to y, x)$ is a set-theoretical solution of the Yang–Baxter equation in Boolean algebras and implicative BCK-algebras (see [19]), while it is not a set-theoretical solution of the Yang–Baxter equation in MV-algebras:*

$$\begin{aligned}(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\ &= S^{12}(S^{23}(x \to y, x, z)) \\ &= S^{12}(x \to y, x \to z, x) \\ &= ((x \to y) \to (x \to z), x \to y, x)\end{aligned}$$

and

$$\begin{aligned}(S^{23} \circ S^{12} \circ S^{23})(x,y,z) &= S^{23}(S^{12}(S^{23}(x,y,z))) \\ &= S^{23}(S^{12}(x, y \to z, y)) \\ &= S^{23}(x \to (y \to z), x, y) \\ &= (x \to (y \to z), x \to y, x).\end{aligned}$$

Then, the Yang–Baxter equation has a set-theoretical solution $S(x,y) = (x \to y, x)$ in positive implicative BCK-algebras. Since $x \to y$ corresponds to $\neg x \oplus y$ in $MV-$algebras, we obtain

$$\begin{aligned}(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\ &= S^{12}(S^{23}(\neg x \oplus y, x, z)) \\ &= S^{12}(\neg x \oplus y, \neg x \oplus z, x) \\ &= (\neg(\neg x \oplus y) \oplus (\neg x \oplus z), \neg x \oplus y, x)\end{aligned}$$

and

$$\begin{aligned}(S^{23} \circ S^{12} \circ S^{23})(x,y,z) &= S^{23}(S^{12}(S^{23}(x,y,z))) \\ &= S^{23}(S^{12}(x, \neg y \oplus z, y) \\ &= S^{23}(\neg x \oplus (\neg y \oplus z), x, y) \\ &= (\neg x \oplus (\neg y \oplus z), \neg x \oplus y, x)\end{aligned}$$

is not a set-theoretical solution of the Yang–Baxter equation as $\neg(\neg x \oplus y) \oplus (\neg x \oplus z) \neq \neg x \oplus (\neg y \oplus z)$.

Lemma 13. *Let $(A, \to, 0, 1)$ be a positive implicative BCK-algebra. Then, the mapping $S(x,y) = (x \to y, 1 \to x)$ verifies the braid condition on this structure, i.e., the Yang–Baxter equation has a set-theoretical solution in BCK-algebras.*

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned}S^{12}(x,y,z) &= (x \to y, 1 \to x, z), \\ S^{23}(x,y,z) &= (x, y \to z, 1 \to z).\end{aligned}$$

From Lemma 2 (e), we have

$$\begin{aligned}(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\ &= S^{12}(S^{23}(x \to y, 1 \to x, z) \\ &= S^{12}(x \to y, (1 \to x) \to z, 1 \to (1 \to x)) \\ &= ((x \to y) \to ((1 \to x) \to z), 1 \to (x \to y), 1 \to (1 \to x))\end{aligned}$$

and

$$\begin{aligned}(S^{23} \circ S^{12} \circ S^{23})(x,y,z) &= S^{23}(S^{12}(S^{23}(x,y,z))) \\ &= S^{23}(S^{12}(x, y \to z, 1 \to y) \\ &= S^{23} x \to (y \to z), 1 \to x, 1 \to y) \\ &= (x \to (y \to z), (1 \to x) \to (1 \to y), 1 \to (1 \to x)).\end{aligned}$$

Therefore, the Yang–Baxter equation has a one set-theoretical solution $S(x,y) = (x \to y, 1 \to x)$ in positive implicative BCK-algebras. □

Lemma 14. *Let $(A, \to, 0, 1)$ be a bounded negative implicative BCK-algebra. The mapping $S(x,y) = (((y \to 0) \to (x \to 0)) \to 0, x)$ verifies the braid condition on this structure. Therefore, the Yang–Baxter equation has a set-theoretical solution in BCK-algebras.*

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned}S^{12}(x,y,z) &= (((y \to 0) \to (x \to 0)) \to 0, x, z), \\ S^{23}(x,y,z) &= (x, ((z \to 0) \to (y \to 0)) \to 0, y).\end{aligned}$$

We have

$$\begin{aligned}(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\ &= S^{12}(S^{23}(((y \to 0) \to (x \to 0)) \to 0, x, z) \\ &= S^{12}(((y \to 0) \to (x \to 0)) \to 0, ((z \to 0) \to \\ &\quad (x \to 0)) \to 0, x) \\ &= (((((z \to 0) \to (x \to 0)) \to 0) \to 0) \to ((((y \to 0) \to \\ &\quad (x \to 0)) \to 0) \to 0) \to 0, ((y \to 0) \to (x \to 0)) \to 0, x)\end{aligned}$$

and

$$\begin{aligned}(S^{23} \circ S^{12} \circ S^{23})(x,y,z) &= S^{23}(S^{12}(S^{23}(x,y,z))) \\ &= S^{23}(S^{12}(x, ((z \to 0) \to (y \to 0)) \to 0, y) \\ &= S^{23}(((((z \to 0) \to (y \to 0)) \to 0) \to 0) \to (x \to 0))) \\ &\quad \to 0, x, y) \\ &= (((((z \to 0) \to (y \to 0)) \to 0) \to 0) \to (x \to 0))) \to 0, \\ &\quad ((y \to 0) \to (x \to 0)) \to 0, x).\end{aligned}$$

Then, the Yang–Baxter equation is satisfied in negative implicative BCK-algebras. The mapping $S(x,y) = (((y \to 0) \to (x \to 0)) \to 0, x)$ is a set-theoretical solution of it. □

Lemma 15. *Let $(A, \to, 0, 1)$ be a bounded BCK-algebra. The mapping $S(x,y) = (N(y), N(x))$ is a set-theoretical solution of the Yang–Baxter equation in BCK$-$ algebras. Moreover, for each $k \in A$ and for every $x, y \in [k, 1]$, $S(x,y) = (N_k(y), N_k(x))$ also verifies the braid condition on this structure. Therefore, the Yang–Baxter equation has a set-theoretical solution in bounded BCK-algebras.*

Proof. It follows from the Definition 3, Lemma 6, and Theorem 1. □

Lemma 16. *Let $\mathcal{A} = (A; \to, 0, 1)$ be a bounded BCK−algebra. Then the following identity*

$$x \to y = N(y) \to N(x)$$

holds for each $x, y \in A$.

Proof. Assume that $x, y \in A$. By using the Definition 3 and Lemma 2 (c), we obtain

$$\begin{aligned} N(y) \to N(x) &= (y \to 0) \to (x \to 0) \\ &= x \to ((y \to 0) \to 0) \\ &= x \to y. \end{aligned}$$

□

Lemma 17. *Let $\mathcal{A} = (A; \to, 0, 1)$ be a bounded commutative BCK−algebra. Then the identity*

$$N(x \to N(x \to y)) = N((y \to x) \to N(y))$$

holds for each $x, y \in A$.

Proof. By using commutativity and the Lemma 16, we obtain

$$\begin{aligned} N((y \to x) \to N(y)) &= N((N(x) \to N(y)) \to N(y)) \\ &= N((N(y) \to N(x)) \to N(x)) \\ &= N((x \to y) \to N(x)) \\ &= N(x \to N(x \to y)) \end{aligned}$$

for each $x, y \in A$. □

Lemma 18. *Let $(A, \to, 0, 1)$ be a bounded commutative BCK-algebra. The mapping $S(x, y) = (x \sqcup y, x \sqcap y)$ verifies the braid condition on this structure. Therefore, the Yang–Baxter equation has a set-theoretical solution in bounded BCK-algebras.*

Proof. It follows from Definitions 4, 9 and Lemma 17. □

Lemma 19. *Let $\mathcal{C} = (C,\ \vee,\ \Delta,\ N,\ 0,\ 1)$, be a Boolean coalgebra, then the mapping $S(x, y) = (x_1, x_2)$ verifies the braid condition.*

Proof. It follows from the (co)associativity of Δ. □

One can ask about the relationship between the maps from Example 2 and Lemma 19, but we will leave our proposed problems for the future.

Theorem 3. *([19]—Solutions to the Yang–Baxter equation from the material implication.)*
For a Boolean algebra, the map $R(a,\ b) = (a \to b,\ a)$ is a solution for the braid condition.

Remark 1. *The equality $R^{12} \circ R^{23} \circ R^{12}(a,\ b,\ c) = R^{23} \circ R^{12} \circ R^{23}(a,\ b,\ c)$ implies some kind of left self-distributivity: $a \to (b \to c) = (a \to b) \to (a \to c)$.*

Theorem 4. *(Solutions to the Yang–Baxter equation from Boolean subtractions.)*

For a Boolean algebra, the map $R(a,\ b) = (b,\ a \wedge N(b)) = (b,\ a - b)$ *is a solution for the braid condition:*

$$R^{12} \circ R^{23} \circ R^{12} \ = R^{23} \circ R^{12} \circ R^{23}\,. \tag{9}$$

Proof. The equality $R^{12} \circ R^{23} \circ R^{12}(a,\ b,\ c) = R^{23} \circ R^{12} \circ R^{23}(a,\ b,\ c)$ implies a right self-distributivity: $(a - b) - c = (a - c) - (b - c)$. It is easy to check the right self-distributivity of the Boolean subtraction using Venn diagrams. □

Remark 2. *Duality principle. "If a Boolean map* $f(x, y)$ *is a solution for the braid condition, then its dual,* $g(x, y) = \overline{f(\overline{x}, \overline{y})}$*, is also a solution for the same equation".*

Remark 3. *Notice that the maps* $(a,\ b) \mapsto (a \to b,\ a)$ *and* $(a,\ b) \mapsto (b,\ a - b)$ *could be considered dual to each other, and that the left self-distributivity is dual to the right self-distributivity. More precisely, if* $R(a, b) = (a \to b,\ a)$*, then* $\mho \circ \overline{R(\overline{a}, \overline{b})} \circ \mho = (b,\ a - b)$*.*

The next sections are about poetry and mathematics.

4. Poetry and the Yang–Baxter Equation

In this section, we will present poetry related to the Yang–Baxter equation.

When B. Karlgren, a member of the Royal Academy of Sciences, addressed the Nobel laureates, Mr. Lee and Mr. Yang, he recalled some pieces of poetry [20]:

"... your culture of 3000 years is really as it is said in the sacred ancient hymn:

like the Kiang river,
like the Han river,
massive like the mountains,
voluminously flowing like the rivers."

"... in the words of a famous T'ang poet:

how can we for a single day
do without these men"

When Chen Ning Yang, received a birthday present, at the age of 90, lines from the poet Tu Fu were engraved on the top of it:

"A piece of literature
Is meant for the millennium
But its ups and downs are known
Already in the author's heart."

In "Thoughts on my first theorem", the author describes, in a poetic manner, the beginning of the unification theory of algebra and coalgebra structures in the framework of Yang–Baxter equations [21]:

(...)
The small particle
was captured...
The common piece of information...
The two streams
arrived on my table from overseas
were unified...

Returning from a daily walk, the same author finds out that his office is filled with literature works, but there is no contradiction in this mixture:

A POST-MODERN MANIFEST

Once… after a promenade,
I gorged a "pomegranate":

Abstract cocktail of notes,
mixed with books,
flowers and clothes,
resting on my desks,
falling from the shelves,
rolling on the chair,
flying in the air…

From the open volumes,
on inevitable social inequalities,
to the open problems,
on classical means inequalities…

Subtle metaphors,
musical measures,
philosophical concepts,
mathematical models,
entwined structures,
historical phrases…

Amalgamated groups…

Kaleidoscopic traces…

5. Poetry and Mathematics—Other Aspects

The following poem resembles the "double poem" by Nina Casian.

CHANGE OF GUARDS IN THE WINTER

The silver lake,
Cooked as a steak…
Icicles serving,
Drinking and dancing…

"Look at those marvels:
Windows with flowers…"
"Take with a fork
Some fine art work!"

Wine from the steam
Snows on the realm…
The wild flame bites
The icy coulds…

The poem was built around the following formula, where a_i represents the distance from the fire that warms the house:

$$\sum_{1}^{6} a_i - b_i = 0, \quad a_i = 7 - i, \;\; b_i = i.$$

We assigned numbers from 1 to 6 to each verse of the poem that refers to the first team, and negative numbers for for each verse of the poem that refers to the second team:

The silver lake,	6	(first team)
Cooked as a steak...	−1	(second team)
Icicles serving,	5	(first team)
Drinking and dancing...	−2	(second team)
"Look at those marvels:	4	(first team)
Windows with flowers..."	−3	(second team)
"Take with a fork	3	(first team)
Some fine art work!"	−4	(second team)
Wine from the steam	2	(first team)
Snows on the realm...	−5	(second team)
The wild flame bites	1	(first team)
The icy coulds...	−6	(second team)

The sum is equal to zero, and suggests an equilibrium: some people come to the warmed house, and others go out into the cold weather of winter.

6. Conclusions

The current paper is a continuation of the article [21] belonging to the Special Issue "Mathematics and Poetry, with a View towards Machine Learning" https://www.mdpi.com/journal/sci/special_issues/Mathematics_Poetry_View_Machine_Learning (accessed on 10 March 2020). The articles [22,23] from the same special issue present complementary views on mathematics and poetry. We also recommend the book [24] and the article [25].

This paper had two aims that were intertwined. The first aim involved a discussion on the Yang–Baxter equation, leading to a duality principle for Boolean algebras. The second aim of this paper was to present poetry related to mathematics, leading to new forms of literary work. One could conclude that both mathematics and poetry are characterized by lyricism.

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