

## Article

# Passenger Flow Management in Front of Ticket Booths in Urban Railway Stations

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**Abstract:** In railway stations, queues often form in front of the ticketing booths that provide ticketing services. Proper design of service systems is key to effectively managing these queues, as waiting time is a critical factor affecting customer satisfaction. This research focuses on optimising the location and configuration of queues in front of ticket booths to minimise waiting times and increase service efficiency. Passenger flow management at the station can be understood as the planning and implementation of the orderly movement of the crowd through the infrastructure. Using operational Markov chain analysis, we evaluate different queue configurations and the number of service providers in urban railway stations. The study specifically focuses on the case of the Poprad-Tatry railway station in Slovakia, where we propose the introduction of a common queue for all ticket booths. We propose the distribution of lines and their schedule, based on mathematical analyses, by creating designated service zones with a common queue in front of the ticket booths. The results show that this approach significantly reduces waiting times and improves overall system efficiency. This research focusses on solving the shortcomings in the design of queues in railway stations, specifically on the use of a common queue, thereby contributing to the improvement of passenger movement management.

**Keywords:** passenger flows; urban railway station; queuing system; queues; ticket booths; mathematical analysis



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## 1. Introduction

In everyday life, companies provide various types of services for consumers, where customers subsequently satisfy their needs. The first activity of every customer in any type of public passenger transport facility is to issue a ticket. There are several options for obtaining a travel ticket. The customer can choose to purchase it physically at the ticket booths at the train station or purchase it from the comfort of their home online [1,2]. This need for services is related to the formation of queues in front of service lines.

Many railway stations still use traditional separate queues for each ticket booths, which can lead to an uneven distribution of customers and increased waiting times. The number of workers at sales points often does not change according to current needs or the number of customers. Although many studies focus on individual aspects of queue management, such as the number of service providers or the configuration of queues, there are significant gaps in an integrated approach that would combine multiple factors and technological innovations such as automatic vending machines, online reservations, and others [3,4]. Many queue management systems are not adaptable enough to handle unexpected events, such as peaks during special events or system failures.

A removal of the queue (it means minimising or eliminating customer queues before entering the system) is related to the entry of customers into the system, who enter it with a certain random periodicity and are served with a certain average service time [5].

Customers should receive service almost immediately after entering the system, reducing or eliminating the need for a waiting line. The effort is to capture recurring time intervals of customer entry and prepare for increased demand, for example, by increasing the number of service lines at times of greatest intensity or by introducing service zones for certain customers, thus reducing waiting time, increasing system utilisation, and reducing the number of customers standing in line.

The subject of queuing theory is a system of mathematical analysis (service) that ensures the functioning of the queue. These systems can consist of one or more service points. A typical example of queuing theory in railway transport is the sale of tickets for passengers in urban railway stations. The input flow of passengers to the system is stochastic, and the intensity of the input flow changes during the day. During peak hours, the influx of customers is so high that queues form in front of each service line [6]. The waiting time on each line varies depending on the passenger services provided. If there are several ticket booths at the railway station, passengers can choose the relatively shortest queues, but this does not guarantee the shortest service time [7].

Railway stations, especially in urban areas, must have efficient railway stations as well as quality ticketing facilities for passengers. The increase in the number of rail journeys and the risks of overcrowding at a railway station can lead to adverse effects on passenger satisfaction and, more importantly, on the safety of passengers and other station users. Passenger flow management at the station can be seen as planning and managing the organised movement of passengers through the infrastructure. Therefore, there are opportunities to improve crowd movement, along with changes to ticket booths to help reduce queues.

In the case of the Poprad-Tatry railway station, we used an analytical model to examine and compare the basic characteristics of the current and proposed systems. For the purposes of this research, we collected data from the Poprad-Tatry railway station in the period from 12 January to 25 January 2024. This timeframe included both weekdays and weekends to capture diversity in passenger arrivals and analyse a typical week in station operation. The data were obtained by personal measurement on the spot, while the measurements were carried out manually using a stopwatch at each ticket booth. This method of data collection, although difficult, provided a detailed view of the real state and dynamics of the system.

The aim of this paper is to optimise the ticketing process at railway stations using Markov chains in queuing systems. This approach involves aligning service processes with the intensity of customer arrivals and speed of service, thereby minimising waiting time and determining the optimal configuration of service points based on estimated passenger arrivals. As part of the research, we decided to use analytical models based on the theory of Markov chains and systems with an infinite queue, which effectively describe and predict the behaviour of the system in different configurations and intensities.

It is assumed that the queue capacity is not limited, which means that customers can wait in an infinitely long line without being rejected. This approach enables a detailed analysis of probabilistic service models to optimise the configuration of service lines and minimise waiting times. Based on a mathematical analysis of the current state and proposed solutions, the research evaluates the possibilities of improving ticket sales, including shortening waiting times and optimising the space and distribution of service points.

The introduction of a common queuing system is proposed as a key strategy to increase efficiency, reduce congestion, and improve customer experience. The research also considers the wider implications for urban rail stations, where efficient handling of large numbers of passengers plays a critical role in successful operations.

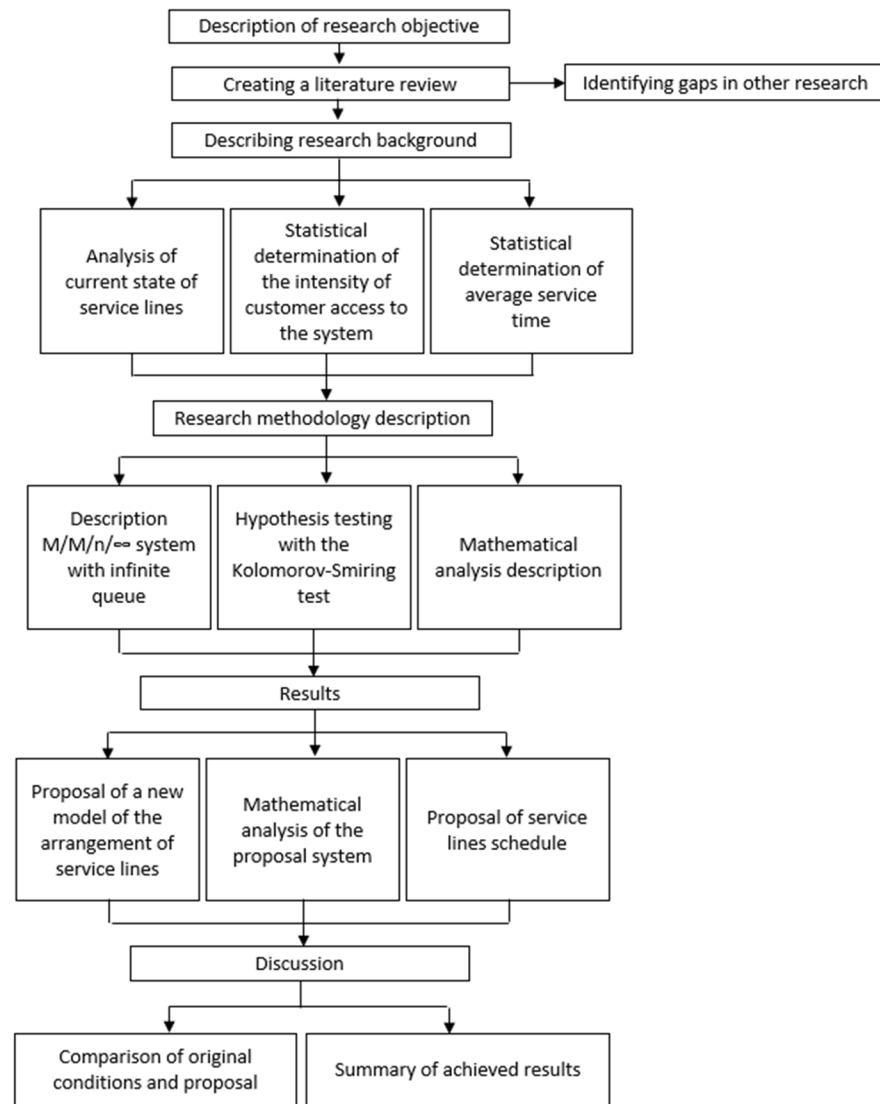
When optimising the process of issuing tickets at railway stations, it is important to consider not only the number of customers and their arrival patterns but also the distribution of employees at the sales windows. This factor plays a key role in line management and system efficiency. The distribution of staff has a direct impact on how quickly customers are served and how evenly distributed waiting times are. In the case of the Poprad-Tatry station, we found that the distribution of employees' working time at individual sales windows was not optimal, which led to excessive overtime and inefficient use of the work-

force. Some employees were overloaded, while others had unused capacity. Therefore, it was necessary to optimise the distribution of employees' working hours so as not only to reduce overtime but also to effectively cover periods with a high intensity of customer arrivals. This approach leads to a better use of resources and reduces overall waiting times for passengers.

Simulations, although useful, were considered unnecessarily complex in this case given the nature of the problem, where analytical models can provide sufficiently accurate and verifiable results without the need for additional complications. This proves particularly practical in passenger flow management applications at railway stations, where Markov chains and infinite queue models can provide fast and verifiable solutions, which is crucial for operations. For example, studies [8,9] in railway station modelling show that the use of mathematical models can effectively identify problems and propose solutions to improve passenger flow. Moreover, their application to real data from the Poprad-Tatry station demonstrated that the results are reliable and practically usable without the need for additional verification steps. The chosen analytical models are therefore considered sufficient not only because of their accuracy but also because of their ability to provide immediate results, which represents a significant contribution to the field of streamlining queue systems in transport. The significant contribution of the selected analytical models lies in their ability to provide immediate and accurate results, which significantly improves the efficiency of the management of waiting systems in transport. By enabling the rapid analysis and optimisation of service processes, the models contribute to the reduction of waiting times and better management of passenger flows in real-time. This ability to quickly apply mathematical results to practice represents a key benefit compared to other, more time-consuming approaches. These models are based on solid mathematical principles and were applied using real data from the Poprad-Tatry railway station, which brings practical knowledge about improving operations.

The main gaps are the lack of an integrated approach to queue management, such as the number of service providers or the configuration of queues, as well as the insufficient distribution of employees (respectively, their working hours). Research also often underestimates the impact that queue design has on customers' perception of waiting time, which can have a significant impact on their satisfaction and tolerance of waiting. Many queue management systems are not flexible enough to handle unexpected events, such as peaks during special events or system failures. The way the approach solves these shortcomings is by using Markov chains, which allows the queue management system to be dynamically adapted to different configurations and customer flow intensities. Markov chains offer flexibility in modelling multiple system states and contribute to queue and service optimisation. As part of the research, the effective distribution of employees is considered, which solves the problem of overtime and increases the effective use of the workforce. This approach leads to an even workload for staff and better handling of high numbers of customers during peak times. Research considers the introduction of ticket machines and online sales as a means of reducing physical queues, thereby improving customer experience and system efficiency.

At the same time, the study acknowledges the limitations of the analytical models, such as assumptions about customer behaviour and arrival patterns, and suggests areas for further research. This article contributes to the ongoing debate on service optimisation in transport hubs, offering both theoretical and practical guidelines for improving efficiency in high-traffic environments. Figure 1 shows the methodological procedure for our research.



**Figure 1.** Methodical procedure of our research solution.

## 2. Literature Review

Queue management and optimisation of passenger flows in railway stations have become important topics in the field of transport, especially in the context of increasing urbanisation and passenger growth. Effective management of these flows is key to improving passenger comfort and safety, which requires detailed studies and the application of advanced methods for analysing and optimising operations.

One area that requires attention is the optimisation of passenger routing in a parallel queue environment, where technologies are used to improve decision-making and reduce waiting times. Most of the literature in this area focuses on routing problems where information about the state of the system is centralised, either perfect or imperfect [10]. These studies are important for understanding how to optimise queues in systems where information is centralised. However, few studies address decentralised routing in parallel queues, which represents a significant gap in the literature. The authors' research [11] proposes a heuristic approach to solving this problem and as part of their work on decentralised control of stochastic systems, they apply approximate dynamic programming, which enables the solution of complex problems of decentralised control without the need for centralised information. This study is key in that it creates a framework for solving stochastic systems, which can also be applied to various traffic management models, while another study [12] focuses on the  $H_\infty$  control strategy for the design of a robust dynamic routing

algorithm in traffic networks, where emphasis is placed on the stability and robustness of the system when handling unpredictable changes in traffic flows. Their approach uses centralised control, which is different from decentralised approaches, and the study [13] introduces the concept of decentralised queue balancing and differentiated services within cooperative control, where the approach focuses on traffic queue management and service differentiation in decentralised systems. This model is more adaptable and does not require centralised control, creating a more flexible way to optimise in real-time. These studies extend the concept of decentralised control and optimisation by providing different approaches to improve the control of traffic flows in systems with centralised and decentralised control mechanisms.

The COVID-19 pandemic has significantly affected the way passenger flows are managed in railway stations, which has required new approaches to optimising these processes. For example, in the study [14], the authors investigated passenger flow at Birmingham New Street railway station and analysed the impact of pandemic measures on passenger behaviour. The use of the SIMUL8 simulation tool made it possible to model different scenarios and assess the impact of these measures on the overall efficiency of station management. The results showed that the pandemic caused significant changes in the behaviour of passengers, especially in using electronic tickets, which reduced the need for physical contact and service. Similarly, the study [15], which dealt with the simulation of buying tickets at the railway station in Bratislava, demonstrated that the introduction of new technologies, such as vending machines, can significantly contribute to reducing the spread of the COVID-19 disease. The following research [16] expanded these findings and proposed procedures to minimise waiting times and the spread of disease in the railway stations of Trnava and Košice.

In the context of growing urbanisation and increasing passenger frequency, it is important to pay attention to crowd management, which includes planning and implementing the efficient movement of passengers through the station infrastructure. The study [17] focuses on the centralised and dynamic management of pedestrians in railway stations. The authors propose a framework that combines centralised control and dynamic approaches, considering the real-time change in pedestrian density. This model focuses on optimising the movement of pedestrians through the control of entrances and exits at the station to reduce congestion and improve passenger comfort. The study [18] investigates crowd dynamics and their management in subway stations with high passenger volumes. In this study, a crowd dynamics model is used, which analyses the behaviour of large groups of people in a limited space, especially during peak hours. The model identifies strategies to control crowd movement and minimises are aimed at streamlining passenger movement and preventing congestion in busy public spaces such as railway stations and metro stations. the risk of accidents through various measures such as infrastructure modifications and service improvements. Both models are aimed at streamlining passenger movement and preventing congestion in busy public spaces such as railway stations and metro stations.

Some studies dealing with queue management without the use of simulation focus on analytical models and probabilistic methods that enable accurate queue analysis and optimisation. For example, study [19] examines the behaviour of queues at airports using analytical approaches that provide sufficiently accurate results without the need for simulation. These models are designed to estimate the length of waiting time and the number of passengers in line, helping to optimise the workforce at airports. Similarly, the study of [20] focuses on M/M/c queuing systems that analyse multi-channel service systems with set-up times, thereby solving efficiency problems without simulation. These analytical approaches allow for accurate analysis without the need for complicated simulations, which is advantageous when managing traffic passenger flows.

The use of simulation tools to describe the theory of queues and the operation of service lines in railway transport is another important topic in the literature. Research [21,22] focuses on the quantification of the occupancy of working lines (ticket sales) and the modelling of queues at main railway stations using simulations. Based on the analysis of

the volume of passengers and the structure of the flow, it is possible to obtain regularities in passenger arrivals, waiting times, and ticket booth capacity. For example, the authors [23] proposed a multi-purpose model of the operational layout of waiting rooms at large railway stations using an optimisation algorithm.

Further research [24,25] shows that strengthening the concept of “accessibility” to online sales services and introducing intelligent ticketing systems can significantly improve sales efficiency and reduce waiting times. These studies also point to the need to consider different approaches and their application to specific conditions and stations.

Overall, the literature review shows that, despite advances in the field of queue and passenger flow management, there are gaps in research, especially in the field of applicability and adaptation of methodologies for specific railway stations. Therefore, this study focuses on the application of queuing theory in the context of the Poprad-Tatry railway station, where the high frequency of passengers, especially due to tourism, requires efficient and customised solutions. The mathematical analysis used enables an accurate representation of the relationships between variables and is applicable to various situations without the need for extensive adjustments. Based on this analysis, it is possible to propose optimal solutions for various problems that are more time-efficient compared to traditional simulation approaches. There are several similar studies dealing with the analysis of the queue from different points of view, for example, some studies that deal with row management without the use of simulations focus on analytical models and probabilistic methods that allow accurate row analysis and optimisation [19,20]. These analytical approaches allow for accurate analysis without the need for complicated simulations, which is advantageous when managing traffic passenger flows. The authors in the study [26] apply multiphase queuing systems to the terminals of international airports in Kerala, analysing the service capacity and the behaviour of queues within the individual phases of the service process. This approach helps to optimise the entire service, from passenger arrival to equipment. The use of a multiphase model allows for a more accurate identification of bottlenecks and a more efficient distribution of the workforce. The study [27] focuses on multi-server queues with setup times, which is important for systems where services are provided at multiple sites and require preparation. The model describes the impact of setup times on the overall efficiency of the system while providing solutions for minimising downtime and optimising personnel. The authors of [28] analyse the capacity of metro stations using queuing theories, evaluating how various factors such as the number of service channels and inter-arrival intervals affect station capacity. This analysis helps optimise passenger movement and increases station efficiency, especially during peak periods. A study [29] analyses the optimal allocation of staff at ticket sales points, using queuing models to predict service demands during different time periods. This approach makes it possible to distribute the workforce effectively, which minimises staff costs and reduces waiting times. The authors of the study [30] propose an optimisation model for ticket distribution in railway networks with a high number of passengers. The study focuses on assigning tickets according to regional requirements, which increases service efficiency and minimises waiting times for passengers.

As an example of the use of different mathematical approaches and methods, we can cite a study [31], which is focused on the use of selected methods from applied mathematics in railway transport. These methods are mainly used to improve the operational efficiency of the transport company, which tends to reduce the overall operating costs. The applied mathematical analysis is based on an extensive mathematical apparatus (detailed statistics carried out in the form of personal measurements). With the use of the theory of probability and random processes, it can examine and compare different states of the investigated models and mathematically analyse the behaviour of the given system under various introduced changes, with changing input values to the model. Mathematical analysis enables the accurate and unambiguous representation of relationships between variables. The performed mathematical analysis can be applied to various situations and conditions without the need for extensive modifications. This means that the results and conclusions from this

study are applicable in different contexts and are not limited to specific conditions. Unlike studies that rely on checkout simulations and are often designed for specific situations, our analytical approaches provide broader flexibility. Simulations can only be tailored to certain conditions, limiting their applicability outside of these settings, while our analytical models can be more easily adapted to different scenarios and environments, increasing their versatility and practical use.

The performed mathematical analysis directly searches for optimal solutions for various problems and can also predict the behaviour of the system based on the specified parameters. On the contrary, simulations are time-consuming, especially if thousands or millions of iterations are needed to obtain reliable results, and often do not directly allow finding optimal solutions but rather provide approximate or indicative results. As an example, to verify our procedures and the advantages of using these models, we can cite a study [32] that provides an overview of simulation optimisations and describes their disadvantages, including the need for many iterations to achieve reliable results, which is time-consuming. Another study [33] analyses different optimisation techniques, comparing the effectiveness of mathematical models and simulation approaches. The results show that analytical models often provide more accurate and faster results.

The literature review also implies that efforts have been made to gradually reduce personal contact sales channels and increase the share of modern sales channels. However, owing to several factors, it is impossible to terminate the sale of travel documents at ticket booths in the long term. Therefore, despite their gradual decline, sales services must be enhanced.

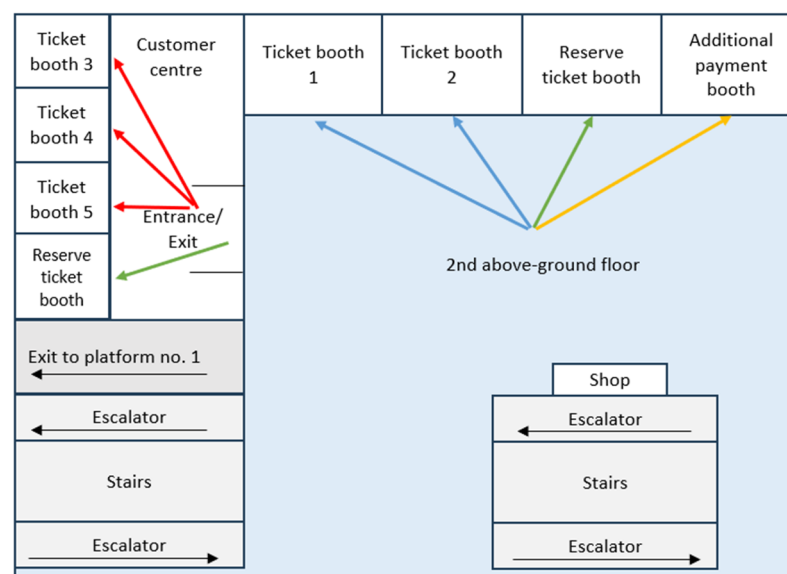
### 3. Research Background

For a railway station, the main function is to provide enough space for passengers to pass and wait inside the station building. The station lobby should also provide a clear layout to ensure that passengers can pass through the station easily and comfortably, which includes clearer route markings, removal of unnecessary obstacles and a well-maintained station floor to prevent slip hazards. In connection with improving the quality of services in urban railway stations, the problem of queues forming in front of ticket booths, which can cause rapid filling of railway station premises, should also be addressed.

The Poprad-Tatry railway station, located in the north of the Slovak Republic, is one of the largest railway stations on the Slovak infrastructure manager network and forms an important transport hub. It is referred to as the gateway to the High Tatras and as one of the oldest stations in Slovakia. The Poprad-Tatry station was chosen as a suitable transfer point for conducting and collecting data for the field research mainly due to increased tourism. Compared to other railway stations such as Bratislava, Košice, or Trnava, this railway station is in a location that experiences the greatest tourist rush throughout the year (not only seasonally), as we consider this railway station not only the gateway to the High Tatras but also the transfer point for all the travelling public. During the whole year, the Poprad-Tatry railway station is dedicated to a high frequency of passengers, especially tourists, so it is not only about seasonal fluctuations. Even though it is not as big in construction and layout as, for example, the Bratislava or Košice railway stations, the onslaught of passengers is approximately the same throughout the year. Unlike the Bratislava and Košice railway stations, the Poprad-Tatry railway station does not have as many service stations as the railway stations. The field research method in this case is the observation of the crowd movement pattern within the station when accessing the platforms from the entrance to the station. It is therefore necessary to examine the ticket booths and their arrangement, i.e., ticket booths, ticket self-service ticket machines, ticket gates, platforms and lobby, when moving the crowd. Since self-service ticket machines are not used in Slovakian railway stations (except for Tatra electric railways and rack railways within the High Tatras region), we focused mainly on ticket booths, which are exposed to a large rush of passengers, especially in the summer months. Examining entry and exit

points, the station has six main entrances/exits, including an entrance/exit that connects the station building to all platforms.

The railway operator Železničná spoločnosť Slovensko (in short ZSSK) sales travel tickets at the Poprad-Tatry railway station in 24 h mode. At the Poprad-Tatry railway station, there are seven service lines for issuing travel tickets, of which five lines are active and two are reserve. Passengers approach the service points randomly and mostly choose the line with the least number of people waiting. Twenty-four-hour continuous operation is valid for the issuance of travel tickets, while at least two service points for the issuance of travel tickets are always open for safety reasons at night. At night, when the system receives the least number of requests, employees work in the ticket booths for emergency work, which is not included in the total working time. This time spent at work is evaluated according to special conditions. The customer centre is open throughout the week from 6.00 a.m. to 10.00 p.m., or according to the estimate of customer access to the system. On Monday, the customer centre is open from 5.00 a.m. due to the rush of passengers to buy travel tickets. There are currently 19 employees working in ticket booths at the Poprad-Tatry station. Eight employees work in the quick sale, which works non-stop. Two service lines are always available in 24 h intervals. Six employees of the operator ZSSK take turns in the customer centre. Spaces for the sale of travel tickets are located on the second floor, which passengers can reach from the underpass by stairs or escalators. From the second floor, passengers can reach platform 1 and stairs and escalators to the 3rd above-ground floor to platforms 2 and 3, where they also have a waiting area. Figure 2 shows the current distribution of ticket booths on the premises of the Poprad-Tatry railway station.



**Figure 2.** Current distribution of ticket booths.

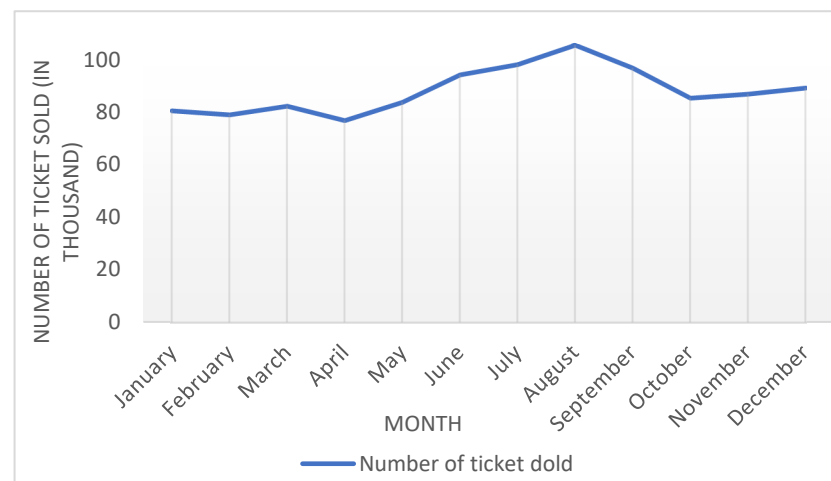
Service lines 1 and 2 (blue line) are internally defined as quick sales, which implies their focus on the quick purchase of a travel ticket for a train that leaves in a few minutes. After interviews with the head of the sales centre and the employees in the ticket booths at the Poprad-Tatry station, it appears that this is not always the case, and both domestic and foreign customers normally approach these lines with different requirements. From a historical point of view and from the habit of customers, it is obvious that in the initial moment after exiting the escalators from the subway, they approach precisely these ticket booths. Service points 3, 4 and 5 (red line) are intended for issuing any travel tickets and are also used to provide information for the travelling public. They are also defined as customer centres. One of the ticket booths is also intended for priority service for passengers travelling by IC trains (we can mark them as a green line). There are also self-service ticket machines at the Poprad-Tatry station, intended for buying travel tickets for the Tatra electric

railway (hereinafter TEŽ) and the rack railway (hereinafter OZ). However, these self-service machines do not work for selling classic train tickets. The surcharge booth is marked as a yellow line. Table 1 shows the number of passengers from the Poprad-Tatry station in 2023, who bought a ticket. The statistics from this year were selected based on the most up-to-date data and the condition of the Poprad-Tatry railway station.

**Table 1.** Number of passengers from the Poprad-Tatry station in 2023; authors, according to [34].

| Year 2023 | Number of Passengers |
|-----------|----------------------|
| January   | 111,740              |
| February  | 110,587              |
| March     | 115,919              |
| April     | 109,868              |
| May       | 116,283              |
| June      | 127,859              |
| July      | 141,577              |
| August    | 148,320              |
| September | 134,383              |
| October   | 130,688              |
| November  | 134,480              |
| December  | 108,823              |
| Total     | 1,490,527            |

The total number of tickets issued for passengers departing from the Poprad-Tatry station in 2023 was 1,490,527. From the total number of issued tickets, 478,749 passengers purchased a ticket in 2023 in a form other than purchasing a travel ticket at the railway station. This means that 1,011,778 tickets were purchased at the ticket booths in the Poprad-Tatry railway station, as shown in Figure 3. The fact that 32.12% of passengers from the Poprad-Tatry station obtained their travel ticket, for example, by purchasing a travel ticket via the Internet or via the ZSSK company’s mobile application “Ideme vlakom” or through the website of the railway operator ZSSK, [www.slovakrail.sk](http://www.slovakrail.sk). Figure 3 shows the number of issued travel tickets in individual months at the Poprad-Tatry station in 2023 out of a total of 1,011,778 travel tickets.

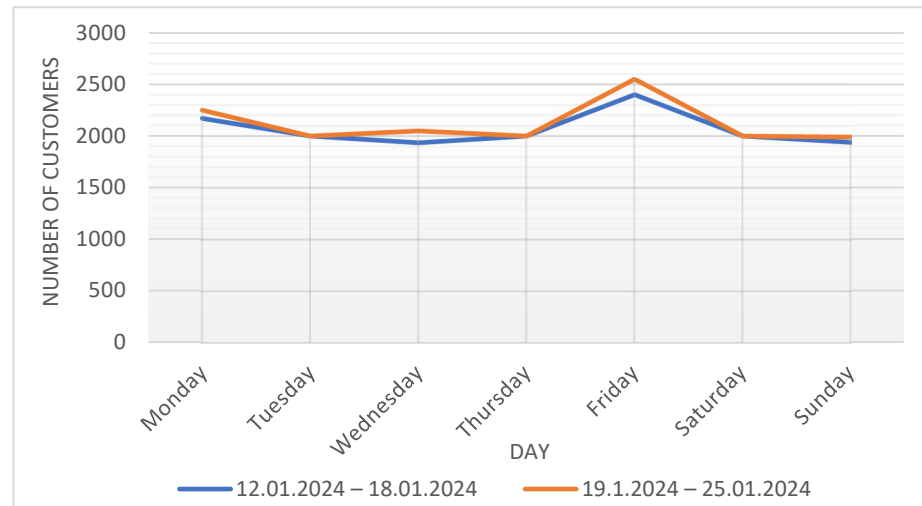


**Figure 3.** Number of issued travel tickets for individual months in 2023; authors, according to [34].

The highest rush of passengers to issue travel tickets is recorded in the months of June to September. During this interval, most tourists heading to the High Tatras for hiking leave or change trains at the Poprad-Tatry station.

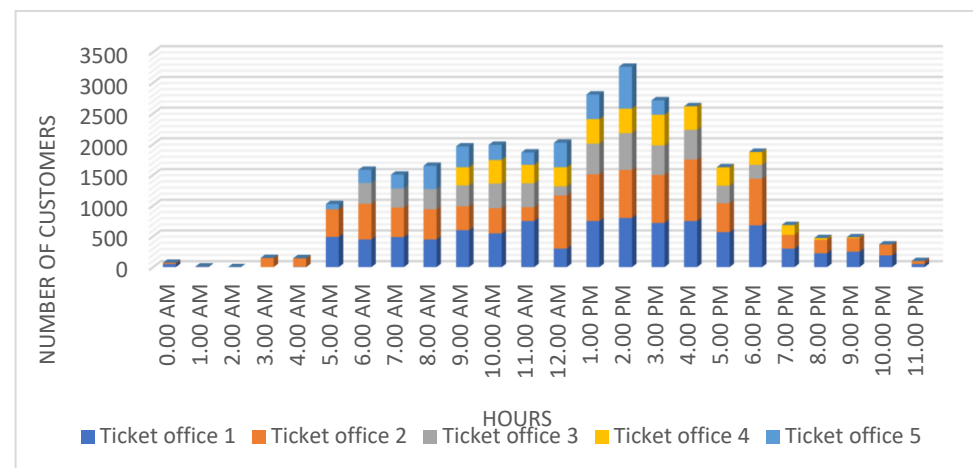
### 3.1. Statistical Determination of the Intensity of Customer Access to the System

There was a total of 29,264 customers who entered the system between 12 January 2024 and 25 January 2024 with a request to issue tickets. Figure 4 shows the intensity of customer entry depending on the date of the request.



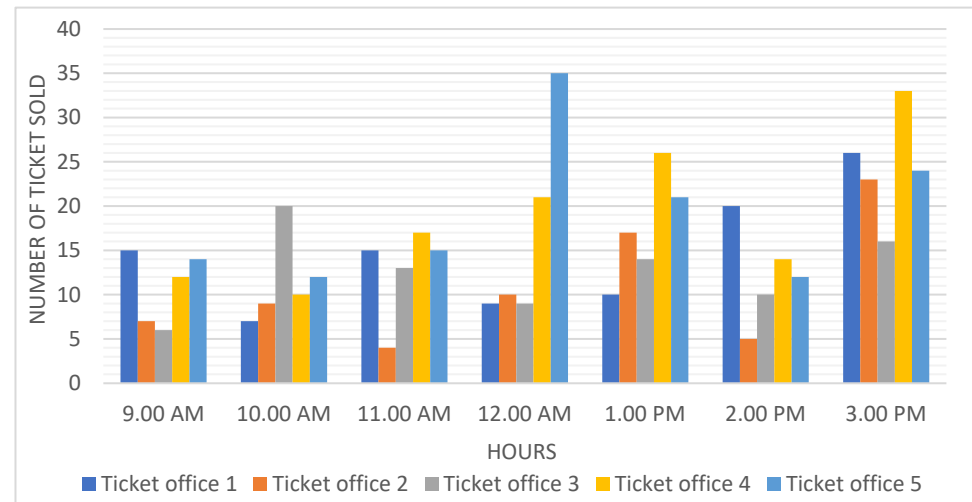
**Figure 4.** Daily intensity of the incoming flow of customers at the Poprad-Tatry station; authors, according to [34].

Figure 4 shows the daily disparity in the number of ticketing requests on Fridays compared to other days of the week. Such an increase of approximately 500 tickets is due to the geographical location of the station, where customers buy a ticket to the High Tatras and customers who study or work in the city of Poprad and go home for the weekend. Passengers who are equipped with a pass for free transportation can travel by TEŽ and OZ trains to the High Tatras, with a zero ticket. On the other hand, passengers who do not own this card are obliged to purchase a ticket for TEŽ and OZ at the ticket booths of ZSSK, respectively, by using a self-service machine. The introduction of free transportation for students and pensioners reduced the intensity of customer entry into the system. There is a 75% increase in the sale of tickets via the Internet, mainly to students who avoid long queues at the station in this way. However, this positive effect is still not sufficient, as evidenced by the fact that approximately 80% of tickets are still issued at ticket booths. The total input flow of customers who enter the system with a request to issue a ticket, with a breakdown by service points, is shown in Figure 5.



**Figure 5.** Number of customers entering the system; authors, according to [34].

Figure 5 expresses the unevenness of the intensity of the incoming stream of passengers to the station Poprad-Tatry. The system registers the most requests between 1.00 p.m. and 4.00 p.m. when many passengers are students and gainfully employed persons returning home from school and work. From 2.00 a.m. to 3.00 a.m. maintenance is being carried out on the service lines, when no tickets are issued and the employees at the ticket booths are working on standby. Figure 6 expresses the requirement for comprehensive service for a customer buying an international ticket. All service lines work from 9.00 a.m. to 3.00 p.m.



**Figure 6.** Difference in the number of issued international tickets according to individual service lines; authors, according to [34].

### 3.2. Statistical Determination of Average Customer Service Time

To apply the proposals for dividing the lines of the queuing system and introducing a common queue, it was also necessary to determine the number of serviced units per unit of time. Customer service time was obtained by personal measurements with a stopwatch at the Poprad-Tatry station in the number of 120 customers. Statistical investigation took place from 12 January 2024 (Friday) to 25 January 2024 (Thursday). The examination of this period was always carried out for two hours at different time intervals in the number of 120 customers. The measurement was carried out by individual lines to capture the work of all employees in the ticket booths as best as possible. Table 2 shows the statistical determination of customer service time by personal measurement at the Poprad-Tatry railway station.

**Table 2.** Statistical determination of customer service time.

| No. | Services             | Time (s) | No. | Services             | Time (s) | No. | Services             | Time (s) |
|-----|----------------------|----------|-----|----------------------|----------|-----|----------------------|----------|
| 1.  | Ticket               | 25.50    | 41. | Information          | 11.40    | 81. | Information          | 34.40    |
| 2.  | Ticket               | 30.40    | 42. | Ticket               | 25.70    | 82. | Information          | 25.70    |
| 3.  | Information          | 20.00    | 43. | International ticket | 200.00   | 83. | International ticket | 312.70   |
| 4.  | Ticket               | 87.45    | 44. | Information          | 20.00    | 84. | Ticket               | 26.70    |
| 5.  | International ticket | 254.40   | 45. | Ticket               | 20.50    | 85. | Information          | 30.70    |
| 6.  | Information          | 45.40    | 46. | Ticket               | 35.50    | 86. | Information          | 100.00   |
| 7.  | Information          | 50.00    | 47. | Ticket               | 40.400   | 87. | International ticket | 190.70   |
| 8.  | Ticket               | 19.80    | 48. | Ticket               | 30.50    | 88. | Ticket               | 50.00    |
| 9.  | Ticket               | 30.80    | 49. | Information          | 7.00     | 89. | International ticket | 119.50   |
| 10. | International ticket | 250.00   | 50. | Ticket               | 60.70    | 90. | Information          | 36.40    |
| 11. | Ticket               | 12.90    | 51. | International ticket | 240.40   | 91. | Information          | 60.70    |
| 12. | Ticket               | 26.70    | 52. | Ticket               | 25.50    | 92. | Ticket               | 30.00    |

Table 2. Cont.

| No. | Services             | Time (s) | No. | Services             | Time (s) | No.  | Services             | Time (s) |
|-----|----------------------|----------|-----|----------------------|----------|------|----------------------|----------|
| 13. | Ticket               | 35.40    | 53. | Ticket               | 43.00    | 93.  | International ticket | 350.00   |
| 14. | Information          | 39.50    | 54. | Information          | 12.40    | 94.  | Ticket               | 40.00    |
| 15. | Ticket               | 13.07    | 55. | Information          | 30.00    | 95.  | International ticket | 504.70   |
| 16. | International ticket | 200.00   | 56. | Ticket               | 45.80    | 96.  | Ticket               | 70.00    |
| 17. | Ticket               | 20.68    | 57. | Ticket               | 66.00    | 97.  | International ticket | 400.10   |
| 18. | Ticket               | 17.54    | 58. | Ticket               | 90.80    | 98.  | Information          | 25.50    |
| 19. | Information          | 9.40     | 59. | Ticket               | 21.40    | 99.  | Information          | 60.50    |
| 20. | Information          | 11.40    | 60. | Information          | 15.60    | 100. | International ticket | 341.70   |
| 21. | Ticket               | 45.40    | 61. | Ticket               | 24.50    | 101. | Ticket               | 20.50    |
| 22. | Information          | 24.30    | 62. | Ticket               | 20.40    | 102. | International ticket | 398.70   |
| 23. | Ticket               | 25.80    | 63. | International ticket | 260.40   | 103. | Ticket               | 25.80    |
| 24. | Information          | 68.40    | 64. | Ticket               | 25.80    | 104. | Ticket               | 80.00    |
| 25. | Information          | 20.40    | 65. | International ticket | 250.40   | 105. | Information          | 87.50    |
| 26. | International ticket | 412.50   | 66. | Ticket               | 20.00    | 106. | Information          | 15.40    |
| 27. | Ticket               | 14.80    | 67. | Information          | 14.50    | 107. | Information          | 36.40    |
| 28. | Ticket               | 29.80    | 68. | International ticket | 29.80    | 108. | International ticket | 397.40   |
| 29. | Ticket               | 30.80    | 69. | Ticket               | 21.40    | 109. | Ticket               | 25.70    |
| 30. | Ticket               | 40.50    | 70. | Ticket               | 15.00    | 110. | Information          | 64.70    |
| 31. | Information          | 20.40    | 71. | International ticket | 250.40   | 111. | International ticket | 250.00   |
| 32. | Ticket               | 39.80    | 72. | Ticket               | 20.50    | 112. | Ticket               | 30.80    |
| 33. | International ticket | 300.00   | 73. | International ticket | 150.70   | 113. | Information          | 12.00    |
| 34. | International ticket | 450.40   | 74. | Ticket               | 31.70    | 114. | Ticket               | 60.00    |
| 35. | Information          | 68.40    | 75. | Information          | 13.70    | 115. | Information          | 15.50    |
| 36. | Information          | 17.50    | 76. | Information          | 12.00    | 116. | Ticket               | 60.50    |
| 37. | Information          | 25.40    | 77. | Ticket               | 24.70    | 117. | Information          | 60.80    |
| 38. | International ticket | 213.60   | 78. | International ticket | 39.40    | 118. | Ticket               | 100.00   |
| 39. | International ticket | 212.40   | 79. | Information          | 13.00    | 119. | International ticket | 300.00   |
| 40. | Ticket               | 40.70    | 80. | Ticket               | 12.70    | 120. | Ticket               | 50.00    |

The above-mentioned measurements were carried out by personal measurement as part of this research from 12 January to 25 January 2024, always in two-hour intervals (this depended on the arrival and departure of trains). The length of service time does not depend on the age of the traveller, as the speed of work (service or ticket issuance) of the employee at the service line (ticket booths) was considered, i.e., how quickly he is able to fulfil the customer's request. Since customers come with different requests, the primary focus was on how quickly the employee on the service line can fulfil the customer's request, which means that the passenger's age does not affect the developed model.

### 3.3. The Current Schedule of Service Lines

A total of 19 employees of the operator ZSSK provide the service of issuing tickets at the Poprad-Tatry railway station. The two service lines are defined as quick sales and are on call 24 h a day. In the customer centre, requests are handled by three lines, whose working hours are changed depending on the estimated number of customers entering the system, or on other factors. The schedule for service line 1, which works non-stop, requires at least four employees to rotate and occupy the entire week. In the night hours, due to the limitation of working hours, a working emergency is introduced, when the employee is not physically present at the service line, but in case of requests for the issuance of a ticket, he is ready to arrange passengers for the purchase of tickets. Preparatory work, which directly serves work performance, must be included in working time. The minimum length of preparatory work resulting from the directive is set at 10 min. Work breaks, when the employee is not obliged to work performance and the employer is not authorised to demand work performance from the employee, are displayed and scheduled in the schedule. Figure 7 shows the current schedule for service line 1. The working mode of line 1 is in a two-week work cycle for two employees.

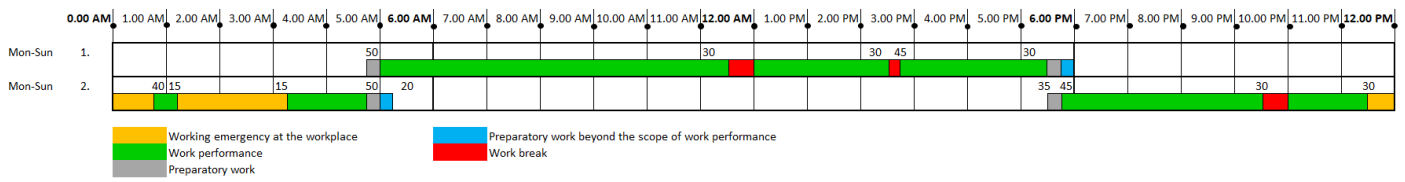


Figure 7. Current schedule of service line 1.

Service line 2 operates in a continuous working mode and issues tickets according to the schedule shown in Figure 8.

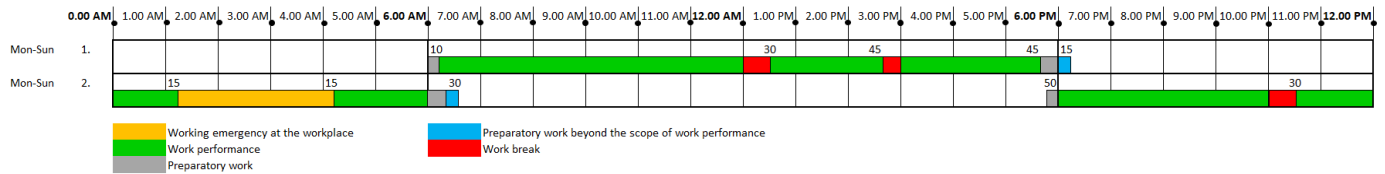


Figure 8. Current schedule of service line 2.

The schedule for line 2, which operates non-stop, is identical to the schedule of line 1. On Monday, line 3 starts providing the ticket service from 5.00 a.m., due to the increased intensity of customers entering the station. From Friday to Sunday, this line is open until 6.00 p.m., also due to the increased intensity of customer entry. Figure 9 shows the current schedule of service line 3.

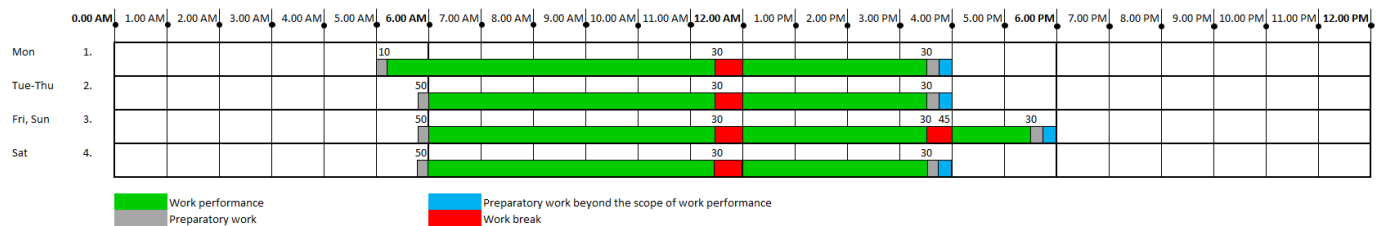


Figure 9. Current schedule of service line 3.

In the customer centre, ticket sales are also provided by line 4, whose schedule is shown in Figure 10.

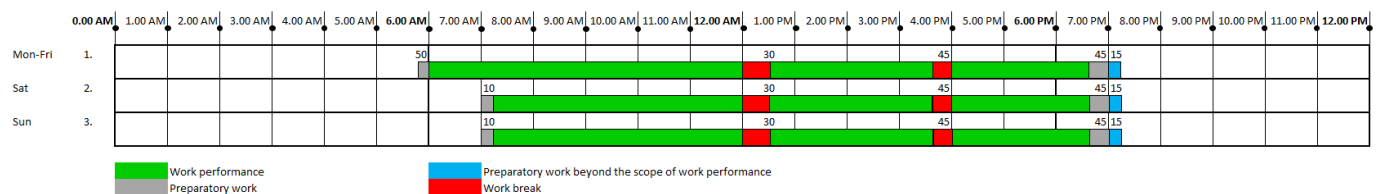


Figure 10. Current schedule of service line 4.

Service line 5 operates in the customer centre and starts issuing tickets at 9.00 a.m., continuing until 10.00 p.m., when the customer centre closes for passengers. Such a schedule of service line 5 is shown in Figure 11.

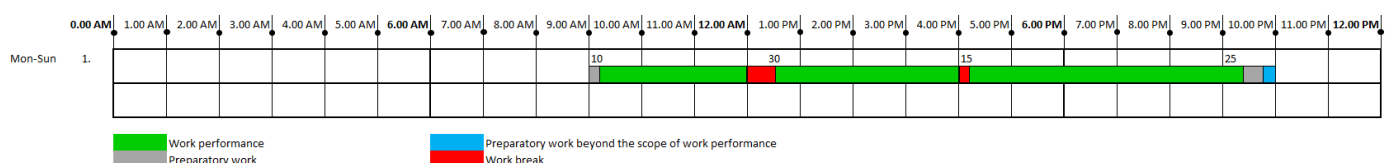


Figure 11. Current schedule of service line 5.

Table 3 shows the calculation of the simultaneously required number of employees for a 13 h schedule. Employees start at 5.50 a.m. work performance is 12 h and 10 min. The night work, when working on call, has a work length of 12 h and 45 min, while the calculated work performance is at the level of 9 h and 50 min. The calculated working time standard refers to a month with 28 days. The maximum monthly performance of an employee in the ticket booths is set at 149.8 h per month based on the directive on the adjustment of the employee's working time.

**Table 3.** Current required number of employees of lines 1 to 5.

| Indicator | Work Performance (h)<br>t | Days<br>d | Total Work Performance (h)<br>d × t | Work Performance per Employee (h)<br>(d × t)/4 | Norm<br>((d × t)/4)/149.8 | The Need for Employees<br>Norm × 4 |
|-----------|---------------------------|-----------|-------------------------------------|------------------------------------------------|---------------------------|------------------------------------|
| Line 1    | 9.83                      | 28        | 275.24                              | 68.81                                          | 0.46                      | 1.84                               |
|           | 12.16                     | 28        | 340.48                              | 85.12                                          | 0.57                      | 2.27                               |
| Total     | 21.99                     | 28        | 615.72                              | 153.93                                         | 1.03                      | 4.11                               |
| Line 2    | 9.91                      | 28        | 277.48                              | 69.37                                          | 0.46                      | 1.85                               |
|           | 12.25                     | 28        | 343                                 | 85.75                                          | 0.57                      | 2.29                               |
| Total     | 22.16                     | 28        | 620.48                              | 155.12                                         | 1.03                      | 4.14                               |
| Line3     | 11                        | 4         | 44                                  | 22.00                                          | 0.15                      | 0.29                               |
|           | 10.16                     | 12        | 121.92                              | 60.96                                          | 0.41                      | 0.81                               |
|           | 12.16                     | 8         | 97.28                               | 48.64                                          | 0.32                      | 0.65                               |
|           | 10.16                     | 4         | 40.64                               | 20.32                                          | 0.14                      | 0.27                               |
| Total     | 22.16                     | 28        | 303.84                              | 151.92                                         | 1.01                      | 2.03                               |
| Line 4    | 13.42                     | 20        | 268.4                               | 134.20                                         | 0.90                      | 1.79                               |
|           | 12.25                     | 4         | 49                                  | 24.50                                          | 0.16                      | 0.33                               |
|           | 12.25                     | 4         | 49                                  | 24.50                                          | 0.16                      | 0.33                               |
| Total     | 37.92                     | 28        | 366.4                               | 183.20                                         | 1.22                      | 2.45                               |
| Line 5    | 13                        | 28        | 364                                 | 182.00                                         | 1.21                      | 2.43                               |

For line 1, it turned out that the work is overcapacity by 4.20 h and therefore it is necessary to replace one work team in two months. Service line 2 works in a continuous working mode. The length of the night work is 12 h and 40 min, while the work performance is 9 h and 55 min. Work performance in the morning team is 12 h and 15 min. The length of work of the night team is 12 h and 40 min, while the work performance is 9 h and 55 min. The work performance of the morning team is 12 h and 15 min. The work of line 2 is overcapacity by 5.36 h. It is necessary to replace one team every two months. At least two employees per week are needed to change the work team. According to the directive on adjustment of working hours, the working time standard for one employee is set at 149.8 h per month in a 28-day month. For line 3, there is overcapacity in the amount of 2.192 h. This line is at least overcapacity, it can work without replacement. A total of 2.45 employees are needed for line 4, while this line is overcapacity by 33.37 h. The need to replace three teams per month. Line 5 is overcapacity by 32.20 h, which results in the need to replace 3 teams per month.

The calculations showed a need for 15.159 employees. For the specified number of employees, a 25% deposit is required to cover the overcapacity work of the lines, or when taking vacations of individual employees. The calculation shows the need for 19 employees in the current system.

#### 4. Research Methodology

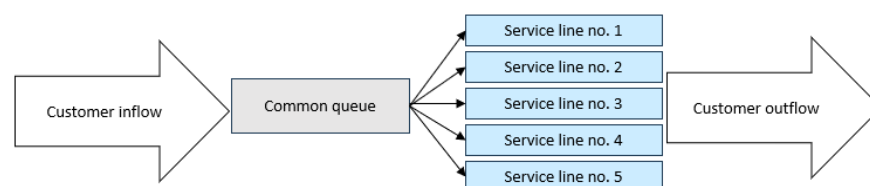
Analytical or simulation models of mass handling can be used to test and investigate system characteristics. Based on known input parameters and the use of suitable software, simulation models examine and test the behaviour of the system over time. The analytical model, which uses operational analysis based on the extensive mathematical theory of probability and random processes, allows for the examination and comparison of different

states of the models and the mathematical analysis of the behaviour of the system under various changes and variations of input values.

As part of our research, we focused on more accurate testing using analytical models. Analytical models are based on mathematical expressions that precisely define the relationships between variables. This means that if the input data are correctly set, it will provide unambiguous results that are mathematically verifiable. This accuracy is especially important for systems where detailed quantitative results are needed, for example, when solving this problem in urban railway stations, which do not require lengthy simulations. Unlike simulation methods, which often use random inputs and generate results based on many iterations, analytical models provide deterministic results. This means that with the same input parameters, the results will always be the same, which reduces uncertainty and increases the reliability of the outputs. This is also proven by the study [35], which examines the difference between deterministic and stochastic models, emphasising the accuracy and predictability of deterministic analytical models. The study [36] proposes a model that shows how analytical approaches can replace simulations and provide clear results in less time.

In railway stations, we encounter ticket booths providing the service of issuing travel tickets, in front of which queues of customers form or requests arise that require a certain form of service. We also call such queuing systems. The situation when the customer waits in the queue is inefficient for the customer as well as for the system. It cannot be completely avoided, but we can influence it. When solving the problem with the waiting queue, two conflicting requirements collide. First, customers demand the shortest waiting time, which can be achieved by building a larger number of service points. On the other hand, companies want to reduce and streamline the construction of service points to reduce maintenance costs. The customer's waiting time in the queue can be reduced either by speeding up the service, which, however, has a certain limit, or by adding service points regarding the operation, or by setting up the process of issuing tickets at railway stations more effectively.

When a customer enters the system and all lines and ticket booths are occupied, he waits in line until a free line or ticket booth becomes available. Customers are often modelled as arriving at certain intervals, as this allows the creation of mathematical models that describe their behaviour within the service line. It is important to distinguish between the "service line" as an overall system and the "queues" in front of individual service points (e.g., ticket booths). In this case, "service line" does not necessarily mean one common row but may include several parallel queues in front of different service boxes. Each of these fronts can be analysed separately, or they can all be included in one common model, depending on the goals of the analysis. Figure 12 shows the basic structure of the queuing system. The queue grows proportionally with the arrival of more customers, and after serving the customers at the service line, it gradually decreases.



**Figure 12.** The basic structure of the queuing system.

Figure 11 represents a queuing system with five service lines into which customers who request service enter in random order. Modelling the queuing system with simulation models consists in the fact that the real system is replaced by a simulation model on which the experiment is performed. This research does not require simulations, or it is not its goal. The simulations can be used as part of further research or in the case of solving this problem for a larger railway station, which experiences an incomparably greater rush of passengers every day, as is the case in the Poprad-Tatry railway station.

#### 4.1. M/M/n/∞ System with Infinite Queue

Queuing theory is a mathematical technique used to calculate performance measures that can be used to propose a decision for various problems in a system. The queuing theory study was intended to manage the queuing phenomenon using relevant performance measures such as the average queue length  $E(L)$ , the average queue waiting time  $E(W)$ , and the utilisation factor  $\rho$  [37].

If events occur randomly and independently at an average rate, then the number of customers and service points per unit time will correspond to a Poisson distribution, and the form of occurrence is described as a Poisson process. The interarrival time of a Poisson process with intensity  $\lambda$  follows an exponential distribution with mean value  $1/\lambda$ . The Poisson probabilities are calculated using Equation (1) as follows:

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \forall x \geq 0, \lambda \geq 0 \quad (1)$$

where:

$X$ —the number of customers arriving per unit of time.

$\lambda$ —the average rate of customer arrival.

The expected number of events per time unit ( $\lambda$ ) should be a positive number. In mathematical terms,  $\lambda > 0$  should apply. In fact,  $\lambda$  can be any non-negative number ( $\lambda \geq 0$ ). At  $\lambda = 0$ , the Poisson distribution would mean that no event will occur with certainty ( $X = 0$  with probability 1). In the Poisson distribution,  $x$  represents the number of events, so it must be an integer and cannot be negative. In mathematical terms,  $x \geq 0$ . The number of events cannot be a fractional number or a negative number;  $x$  must be a non-negative integer (0, 1, 2, ...).

In this system,  $n$  identical lines work, which are independent of each other and are connected in parallel, while the system does not reject customers. If the line is busy, the customer joins the queue. This system perfectly replicates the situation at railway stations such as the Poprad-Tatry station. The formulas for calculating the probabilities of occurrence of individual situations are based on the Chapman–Kolmogor equation [38], which is an application of the absolute probability theorem. Formula (2) expresses the probability that the system will be in state  $j$  at time  $t + \Delta t$ . This probability is calculated as the sum of the probabilities that the system was in state  $k$  at time  $t$  multiplied by the probability that the system will transition from state  $k$  to state  $j$  during the interval  $\Delta t$ . This sum is taken over all possible states  $k$  in the set  $S$ .

$$p_j(t + \Delta t) = \sum_{k \in S} (p_k(t) \times p_{k,j}(\Delta t)) \quad (2)$$

where:

$S$ —the set of all possible states in which the system can be.

$p_j(t + \Delta t)$ —probability that the system will be in state  $j$  at time  $t + \Delta t$ .

$p_k(t)$ —probability that the system is in state  $k$  at time  $t$ .

$p_{k,j}(\Delta t)$ —probability that the system will go from state  $k$  to state  $j$  during the time interval  $\Delta t$ .  
 $k$  and  $j$ —elements of this set.

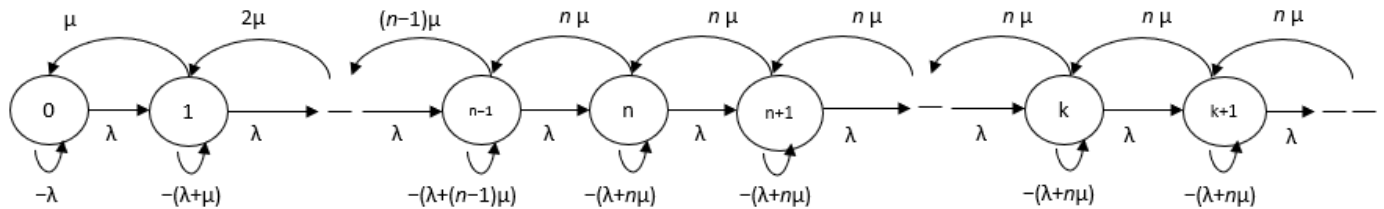
The Chapman–Kolmogor equation describe that for the string to be in state  $j$  at time  $t + \Delta t$ , it could be in any state at time  $t$ , and it had to pass from this state-to-state  $j$  in time  $\Delta t$ . Differential equations describe the change in the probabilities that the system is in state  $j$  at time  $t$  as a function of time  $t$ . After a sufficiently long time, the distribution of states in the Markov chain can stabilise, making the model independent of time and achieving a stationary distribution [39]:

$$\forall k \in S, \lim_{t \rightarrow \infty} p_k(t) = p_k \quad (3)$$

Using Markov's theorem, we obtain a system of homogeneous linear equations, to which we add a norming condition [40,41]:

$$\sum_{k=0}^{\infty} p_k = 1 \quad (4)$$

Figure 13 shows a general transition graph for an M/M/n/∞ system with an infinite queue:



**Figure 13.** Transition graph of a system with infinite waiting queues.

To calculate the system defined in this way, we need to obtain input data, the average service time of one line  $1/\mu$ , the intensity of customer input  $\lambda$  and the number of service lines. From the input values defined in this way, we proceed to the calculation:

$$\alpha = \frac{\lambda}{\mu} \quad (5)$$

Subsequently, we proceed to the condition where we find out whether the system is feasible with a given number of lines, average service time, intensity of customer entry:

$$\rho = \frac{\alpha}{n} < 1 \quad (6)$$

For a simpler calculation, we will use modified Erlang formulas:

$$p_{m+1-\infty} = p_0 \sum_{k=n+1}^{\infty} q_k \quad (7)$$

where:

$p_{m+1-\infty}$ —the probability that the system is in a state that is higher than or equal to a certain threshold point  $m + 1$ , after considering all possible states above this threshold. This threshold point can be interpreted in the context of an infinite number of states as “all states above a certain point”, and the formula expresses the cumulative probability for all these states.

$$p_k = q_k \times p_0 \quad (8)$$

$$p_0 = \frac{1}{\sum_{k=0}^{\infty} q_k} \quad (9)$$

and introduced variables  $q_k$ , where:

$$q_k = \frac{\alpha^k}{k!} = \frac{\alpha}{k} q_{k-1}, \quad k = 1, 2, \dots, n \quad (10)$$

To calculate  $q_k$  with an infinite queue, we use for the state when all lines are:

$$\sum_{k=n+1}^{\infty} q_k = \frac{n^n \rho^{n+1}}{n! (1 - \rho)} \quad (11)$$

From the formulas derived in this way, we can proceed to the calculation of the values of the system characteristics. We calculate the system performance according to the Formula (12):

$$E(S) = \alpha \quad (12)$$

The system performance expresses the average number of serving lines, respectively, ticket booths. This equation is logical, as the system does not reject any customers and thus  $\lambda$  customers enter the system per unit of time and the average service time is  $1/\mu$ . From this we obtain the system performance:

$$E(S) = \lambda \times \frac{1}{\mu} = \alpha \quad (13)$$

We calculate the use of the system according to Formula (14):

$$R = \rho \quad (14)$$

$$R = \frac{\alpha}{n} [\%] \quad (15)$$

The average length of the queue expresses the average number of customers waiting in line for service and is calculated according to the Formula (16):

$$E(L) = \frac{n^n}{n!} \frac{\rho^{n+1}}{(1-\rho)^2} p_0 = \frac{p_{n+1-\infty}}{1-\rho} \quad (16)$$

The average waiting time in the queue is then calculated according to the Formula (17):

$$E(W) = \frac{E(L)}{\lambda} \text{ [s]} \quad (17)$$

Little's formula, which proves the fact of the preservation of the waiting time after the introduction of a common waiting queue, considers the system to be stabilised if, from a certain moment, the further development of the system is independent of time. The average number of customers in the system is calculated as the ratio of the average number of customers in the system and the average time spent by the customer in the system:

$$\lambda = \frac{E(N)}{E(T)} \text{ [customers/h]} \quad (18)$$

The number of customers in the system is thus modelled by a random variable at time  $t \in T$ :

$$N(t) = A(t) - B(t) \text{ [customers]} \quad (19)$$

where:

$A(t)$ —the total number of arrivals to the system until time  $t$ .

$B(t)$ —the number of customers who have already left the system or are being served.

This relationship is advantageously used in queuing system simulations, when we can calculate the basic characteristics of the system only numerically and not analytically. To estimate the average customer service time, the arithmetic means of the statistical file created from the measured values  $x_1, x_2, x_3, \dots, x_n$  is used, where  $n$  is the total number of elements of the statistical file [42].

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\sum_{i=1}^n x_i}{n} \quad (20)$$

The average service time is calculated according to the Formula (20):

$$\bar{x} = \frac{10,497}{120} \doteq 87.47 \text{ [s]}$$

For a better overview of the statistical set, the standard deviation was calculated for this set according to the relationship [42]:

$$s_x = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}} = 133.33 \text{ [s]}$$

The input flow of requests will consist of the number of issued tickets on individual days, which will then be divided into time intervals according to the number of service lines. The average total intensity of customer access to the system from 6.00 a.m. to 6.00 p.m. in the period from 12 January 2024 to 25 January 2024 for all lines and for all types of travel tickets is as follows:

$$\lambda = \frac{12,653 + 13,229}{13 \text{ h} * 14 \text{ days}} = 142.24 \text{ [customers/h]}$$

The formula shows the fact that an average of 142.24 customers per hour entered the system with a request to issue a travel ticket in the interval chosen by us. Considering the distribution of service lines, we calculate the intensity of customer entry for the issuance of a national travel ticket and for the issuance of an international travel ticket. The intensity of the entry of customers with the request to issue a national travel ticket ( $\lambda_{nt}$ ) in the same interval:

$$\lambda_{nt} = \frac{12,313 + 12,834}{13 \text{ h} * 14 \text{ days}} = 138.17 \text{ [customers/h]}$$

Intensity of customers entering with a request to issue an international travel ticket ( $\lambda_{it}$ ) between 9.00 a.m. and 3.00 p.m.:

$$\lambda_{it} = \frac{240 + 291}{7 \text{ h} * 14 \text{ days}} = 5.42 \text{ [customers/h]}$$

The standard deviation is large, which is due to the large difference between the service time for issuing national travel tickets and the service time for issuing international travel tickets. Given that we are considering the division of service lines into lines defined only for the issuance of national travel tickets and lines only for the issuance of international travel tickets, we will calculate the average service times for individual specialised lines based on Table 3. We also calculate the average customer service time for lines specialised in national travel tickets according to Formula (20). The average customer service time for lines specialised in national travel tickets is 39.53 s, for international travel tickets it is 274.67 s. The average customer service time for providing information is 42.29 s, and the average customer service time with the request to provide information together with the sale of travel tickets is 40.70 s. These average service times are considered random variables that are influenced by a series of random events; therefore, they are the mean values of this random variable.

#### 4.1.1. Hypothesis Testing

The exponential probability distribution is commonly used in modelling random phenomena that relate to the time between events in Poisson processes. In the context of railway stations, the exponential distribution can be used to model the time between customer arrivals at the sales windows. We considered the following hypotheses:

**H<sub>0</sub>.** The data follows an exponential distribution.

**H<sub>1</sub>.** The data do not follow an exponential distribution.

The Kolmogorov–Smirnov test (in short K-S test) was chosen for hypothesis testing. This test is used to compare data with a specific distribution, in our case with the exponential distribution. This test provides a  $p$ -value that determines whether the null hypothesis can be rejected.

#### 4.1.2. Calculation of the Empirical Distribution Function (EDF)

The EDF for  $n$  observations  $x_1, x_2, \dots, x_n$  is calculated as the proportion of the number of observations less than or equal to a specific value according to the formula:

$$F_n(x) = \frac{\text{number of data less than or equal to } x}{n} \quad (21)$$

The estimate for the parameter  $\lambda$ , which is the inverse of the average of the data, is calculated according to the formula:

$$\lambda = \frac{1}{\text{average data value}} \quad (22)$$

$D$  is the maximum difference between the empirical distribution function  $F_n(x)$  and the theoretical distribution function  $F(x)$  and is calculated according to the formula:

$$D = \max |F_n(x) - F(x)| \quad (23)$$

This difference is calculated for each value of  $x$  in our data.

The Kolmogorov–Smirnov test for data against an exponential distribution gives the following results (chosen significance level 0.05):

Test statistic  $D = 0.0647$

$p = 0.6727$

Since the  $p$ -value is quite large (greater than 0.05), we failed to reject the null hypothesis. This means that the data follows an exponential distribution.

#### 4.2. Mathematical Analysis

Since the Markov chain usually describes a discrete random (stochastic or probabilistic) process for which the probabilities of the transition to the next state depend only on the current state and not on previous states, we performed a detailed mathematical analysis based on this.

The goal of mathematical analysis is not only to capture the current behaviour of the system during a 24 h period but also to predict the behaviour of the system after the implementation of the proposed changes. This 24 h period is divided into time intervals to more accurately model and analyse the dynamics of the system, which changes depending on several factors, such as the number of tickets issued, the number of available service lines, and the intensity of passenger arrivals. Each time interval represents different operating conditions, which allows a more detailed understanding of how the system reacts to changes in load and capacity. Mathematical analysis focusses on identifying patterns and trends that can help optimise the management and distribution of resources within the station. In addition, this analysis makes it possible to model potential scenarios after the introduction of proposed changes, such as new technologies for ticket sales (self-service kiosks), or changes in the distribution of sales points (change in the layout of ticket booths).

The following calculation of the characteristics of the queuing system is focused on the interval from 2.00 p.m. to 5.00 p.m. on Friday, due to the highest intensity of customer entry into the system,  $\lambda = 203$  customers/h. The total average service time resulting from Table 2 is  $\mu = 41.16$  customers/h, after  $1/\mu = 0.0243$  h/customers (after recalculation to seconds is 87.47 s/customer). The service facilities consist of five lines.

The probabilities of the system states do not depend directly on the intensities  $\lambda$  and  $\mu$ , but on their ratio, which we denote by  $\alpha$ . Therefore, we calculate the load of the system for

the current state in the selected time interval according to the Formula (5), and it reaches  $\alpha = 4.93$ . The stability testing of the current queuing system is tested using the constant  $\rho$ . The condition  $\rho < 1$  expresses the ratio of the number of incoming customers to the system and the ability of the operator. The system stabilisation condition is met if, on average, fewer customers enter the system than the operator can serve at full capacity. According to Formula (6) and reaches  $\rho = 0.98$ .

For the system to be stable,  $\rho$  must be less than 1 ( $\rho < 1$ ). At the intensity of operation  $\rho < 1$ , in our case, we can declare that the system is stable, and we can proceed to the calculation of the transition table between individual states. We will perform the calculation in Table 4, where the first column  $k$  contains the states in which the system can be (no customer, one line is working, etc., or customers are queuing). In the second column, there are variables  $q_k$  calculated according to (10) and (11), while for  $k = 0$  the condition  $q_0 = 1$  and at the end we add the variables. Finally, the last third column contains the probabilities of the individual states of the system  $p_k$ , where  $p_0$  is equal to the reciprocal of the sum of the column  $q_k$ . Table 4 shows the transitivity between the individual states of the system on Friday.

**Table 4.** Transition between individual system states from 2.00 p.m. to 5.00 p.m. on Friday.

| $k$         | $q_k$    | $p_k$ |
|-------------|----------|-------|
| 0           | 1.000    | 0.001 |
| 1           | 4.932    | 0.003 |
| 2           | 12.162   | 0.007 |
| 3           | 19.995   | 0.011 |
| 4           | 24.653   | 0.013 |
| 5           | 24.318   | 0.013 |
| 6– $\infty$ | 1763.032 | 0.953 |
| Total       | 1850.091 | 1     |

Using  $p_k$  probabilities, the characteristics of the current tested system are derived, according to which, in the next step, it is possible to derive its economic and operational evaluations:

- System performance  $E(S) = 4.93$ .
- System utilisation  $R = 0.986 \times 100\% = 98.6\%$ .
- Average queue length  $E(L) = 70.04$  customers.
- Average waiting time in the queue  $E(W) = 1242.112$  s.

In the current system, at the time of the highest intensity of customer entry into the system, when service lines are equal and customers access any of them randomly, the system utilisation is 98.6%. On average, 70.04 customers are waiting to buy a travel ticket in this time interval. Passengers wait an average of 20.7 min for service. This increased waiting time is related to the difference in the waiting time for the issuance of an international travel ticket and a national travel ticket. Due to the uneven number of sales of travel tickets, it is necessary to analyse the intensity of customer entry by day of the week. Intervals of the day are divided according to the number of working service lines. The average service time for the current service system is 41.16 h per customer. On Monday, during the two weeks of measurement, from 12 January 2024 to 25 January 2024, a total of 4295 pieces of all tickets were issued at the Poprad-Tatry railway station. The mathematical analysis of the current state on Monday is shown in Table 5.

The highest rush for the sale of travel tickets, which is based on Figure 5, is in the interval from 1.00 p.m. to 3.00 p.m. when an average of 50.72 customers are waiting in line for issuance. At this point in time, passengers are waiting 15.06 min for the issuance of a travel ticket. Due to the even distribution of the number of working service lines on Tuesday, Wednesday, and Thursday, Table S1, which is part of the Supplementary Materials, analyses the characteristics of the system on these three working days. A total of 12,168 tickets were issued. The highest system utilisation resulting from Table S1 is between 2.00 p.m. and 3.00 p.m. when the system is utilised at 98.15%. In this interval, the

system receives the most requests for the issuance of travel tickets, which are mainly from students and working people who leave the city of Poprad for the surrounding villages. A total of 5048 tickets were issued on Friday. This number of travel tickets is the highest among all monitored days. Table S2, which is part of the Supplementary Materials, analyses this interval during two January days (two Fridays) as part of personal observation. The biggest rush is between 2.00 p.m. and 5.00 p.m. when customers wait in line for up to 20.7 min. The calculated average waiting time is the highest among all monitored waiting times. On Saturday, a total of 3926 customers entered the system with a request to issue a travel ticket; they buy travel tickets mainly for tourist purposes or tickets for TEŽ and OZ. Table S3, which is part of the Supplementary Materials, shows the mathematical analysis of the current system on Saturday. We record the highest rush for the sale of travel tickets in the interval from 1.00 p.m. to 3.00 p.m. when the system utilisation is up to 77.26%, which is the highest number among all intervals. A total of 3827 passengers entered the system on Sunday during the two-week monitoring period to request a travel ticket purchase. A more detailed analysis is shown in Table S4 (in Supplementary Materials).

**Table 5.** Mathematical analysis of the current state on Monday.

| Interval                | 5.00 a.m.—<br>6.00 a.m. | 6.00 a.m.—<br>8.00 a.m. | 9.00 a.m.—<br>12.00 a.m. | 1.00 p.m.—<br>3.00 p.m. | 4.00 p.m.—<br>6.00 p.m. | 7.00 p.m.—<br>9.00 p.m. | 10.00 p.m.—<br>4.00 a.m. |
|-------------------------|-------------------------|-------------------------|--------------------------|-------------------------|-------------------------|-------------------------|--------------------------|
| Number of service lines | 3                       | 4                       | 5                        | 5                       | 4                       | 3                       | 2                        |
| $\Sigma$ tickets        | 163                     | 733                     | 1076                     | 1212                    | 785                     | 198                     | 128                      |
| $\lambda$ (customer/h)  | 82                      | 122                     | 135                      | 202                     | 131                     | 33                      | 9                        |
| $\mu$ (customer/h)      | 41.16                   | 41.16                   | 41.16                    | 41.16                   | 41.16                   | 41.16                   | 41.16                    |
| $E(S)$ (-)              | 1.99                    | 2.96                    | 3.28                     | 4.91                    | 3.06                    | 0.80                    | 0.22                     |
| $R$ (%)                 | 66.41                   | 74.10                   | 65.60                    | 98.15                   | 76.53                   | 26.72                   | 10.93                    |
| $E(L)$ (-)              | 0.87                    | 0.49                    | 0.59                     | 50.72                   | 0.54                    | 0.02                    | 0.00                     |
| $E(W)$ (s)              | 38.26                   | 14.59                   | 15.82                    | 903.88                  | 16.18                   | 2.08                    | 1.06                     |

From 9.00 a.m. to 5.00 p.m., all five service lines work on issuing travel tickets when, based on past observations, the highest number of passengers entering the system is estimated. The unused system is mainly at 6.00 a.m., when three service lines are connected, while in total, during the two Sundays in January, only 55 requests for the issuance of travel tickets came into the system at this time. For this reason, we observe a low utilisation of the system at the level of 22%, and the average occupancy of a working line in this case is 0.668 lines out of 3. We record the highest rush for travel ticket sales at 6.00 p.m. when many customers are students who go to school or to employment outside the city of Poprad. In total, in this two-week monitoring period, customers waited in the queue to issue a ticket for an average of 105.508 s.

## 5. Results

To increase the efficiency and quality of ticket-issuing services at railway stations in urban areas, it is necessary to introduce measures that ensure optimal utilisation of the system. The goal is to minimise or eliminate the waiting times of customers who enter the system at certain random intervals. This means maximising the use of available resources, such as manpower and technology, to ensure smooth and efficient processing of ticketing requests without unnecessary overloading or undersizing. With the queuing system, it is possible to mathematically analyse the given system, which will provide us with information about how the given queuing system will behave in the future when various changes are introduced. Based on mathematical analysis, it is possible to find the optimal configuration of individual service lines depending on the input values that randomly enter this system.

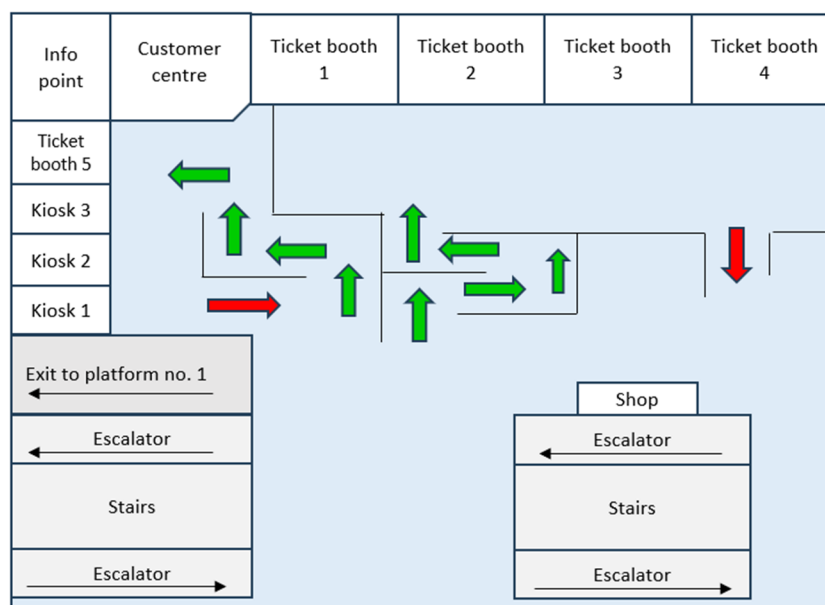
One of the measures that contributes to more efficient customer service at railway stations in urban areas is the introduction of a common waiting queue, which eliminates the customer's waiting time for service. When entering the system, customers line up in a queue that determines the exact order of access to service lines. However, this measure does not affect the reduction of the waiting time in the queue when customers approach any

service line with their requests. The shortening of the waiting time in the queue associated with the increase in system utilisation consists of a more efficient distribution of the tasks of the individual service lines, which are described in the next section.

Another possibility is the introduction of self-service machines for buying tickets, which will allow for a shorter waiting time in the queue. The customer chooses a travel ticket according to his own needs, and there is no prolongation of the waiting time in the queue in front of the ticket booths. This will achieve a fluid flow of customers in the station, as customers will be evenly distributed in front of the ticket booths and in front of the self-service ticket machines.

Among the basic quantities necessary for the application of the proposal for the introduction of a common queue are the average service time and the intensity of the incoming flow of customers at the Poprad-Tatry railway station. We define service time as the time required to service one request entered into the system, which is affected by a series of random events and factors, and therefore, we also refer to it as a random variable. The time expression of the average customer service time at the Poprad-Tatry station and its statistical investigation took place from 12 January 2024 (Friday) to 25 January 2024 (Thursday). The examination of this period was always carried out for two hours at different time intervals with a number of 120 customers. The second quantity required for the application of the proposal of the queuing system was the intensity of the incoming flow of customers, which consists of groups of people who enter the system with a service request.

One of the ways to reduce the waiting time of customers in queues is to introduce specific service zones, which are intended to serve specific customers. As part of improving the quality of services provided to customers, the station has introduced a first-in–first out mode, where customers enter the system in the order in which they arrive, which prevents overtaking and achieves fairness. service order. The goal of the service line configuration in the design is to reduce the number of service facilities to minimise the total cost of waiting time, which can discourage the customer from buying a ticket. To ensure fair access to the service line that issues domestic and international tickets, it is proposed to retain the FIFO (first in, first out) queuing system. Figure 14 shows the new state of arrangement of the service lines at the Poprad-Tatry station, together with the introduction of a common queue in front of the ticket booths. Green arrows indicate the entrance to the service lines and red arrows indicate the exit from the system.



**Figure 14.** Proposal of a new model of the arrangement of service lines in the Poprad-Tatry railway station.

In the proposed model, ticket booths number 1, 2 and 5 are proposed primarily for issuing domestic travel tickets. Ticket booth number 3 and 4 issue international tickets. Ticket booth number 5 is situated next to the information booth in the premises of the customer centre and for the subsequent sale of travel tickets after obtaining information about the connections. The customer centre is an open space with one ticket booth and, at the same time, an info point. The customer centre also includes three self-service payment kiosks located next to ticket booth number 5. As part of the proposal, a mathematical analysis of the proposed system was carried out for all service lines, together with the proposed schedule of the service staff of individual service lines.

### 5.1. Service Lines Specialised in National Travel Tickets Associated with the Submission of Information

In the analysed queuing system, the average service time will increase due to the inclusion of the average duration of information provision. In this case, according to Formula (9), the average service time is determined to be 40.70 s, of which  $1/\mu = 88.25$  h/customer. The intensity of customer entry at the time of the highest demand for the issuance of national travel tickets is on Friday from 2.00 p.m. to 3.00 p.m., which is  $\lambda = 255.75$  customers/h. Due to this intensity of entry, this part will be analysed in more detail in the following section. The system load, which is calculated as the product of the intensity of customer arrival  $\lambda$  and the average service time  $1/\mu$ , expresses the work required from the system according to Formula (5) and reaches the value of 2.90. The condition of stabilisation is tested by the share of work required from the proposed system to the lines that provide their service for this work according to Formula (6), which reaches a value of 0.96. The intensity of operation is less than 1, which results in the stabilisation of the proposed queuing system, where the probabilities cease to be time-dependent after a certain interval. From the calculated values, the probabilities of the system states are calculated according to Formulas (7), (8), (10) and (11). Table 6 shows the passability between service line states for national travel tickets.

**Table 6.** Transition table between states for national service lines.

| $k$         | $q_k$  | $p_k$ |
|-------------|--------|-------|
| 0           | 1      | 0.01  |
| 1           | 2.90   | 0.02  |
| 2           | 4.20   | 0.03  |
| 3           | 4.06   | 0.03  |
| 4– $\infty$ | 115.27 | 0.91  |
| Total       | 127.43 | 1     |

The probability of a situation where customers stand in a queue is 90%, which results in a high load on the system. In this model, with the help of Markov chains, the model was able to stabilise. With a higher number of requests, the system cannot stabilise over time. The characteristics of the proposed system follow from the probability of transition between states:

- System performance  $E(S) = 2.89$ .
- System utilisation  $R = 96.6\%$ .
- Average queue length  $E(L) = 26.61$  customers.
- Average waiting time in the queue  $E(W) = 374.58$  s.

On average, 2.89 lines out of 3 works in the selected time interval from 2.00 p.m. to 3.00 p.m., resulting in an average queue length of 26.61 people for this interval. Proposals for the number of lines for domestic transport in individual parts of the day are analysed in the following section. On Monday, from 15 January 2024 to 25 January 2024, 4155 customers entered with a request to issue a national travel ticket. Table 7 shows the mathematical analysis of the proposed system on Monday. The most loaded system resulting from Table 7 was between 2.00 p.m. and 3.00 p.m. when 821 requests entered the system during these

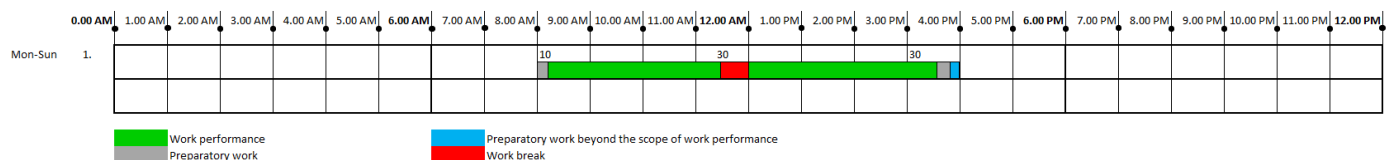
two hours during the two weeks of observation. On average, there were two customers in the queue.

**Table 7.** Mathematical analysis of the proposed system on Monday.

| Interval                | 5.00 a.m.–<br>6.00 a.m. | 6.00 a.m.–<br>7.00 a.m. | 8.00 a.m.–<br>9.00 a.m. | 9.00 a.m.–<br>1.00 p.m. | 2.00 p.m.–<br>3.00 p.m. | 4.00 p.m.–<br>7.00 p.m. | 8.00 p.m.–<br>4.00 a.m. |
|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| Number of service lines | 1                       | 2                       | 3                       | 3                       | 3                       | 2                       | 1                       |
| $\Sigma$ tickets        | 160                     | 489                     | 232                     | 1160                    | 821                     | 791                     | 502                     |
| $\lambda$ (customer/h)  | 80                      | 122.25                  | 116                     | 116                     | 205.25                  | 98.875                  | 25.1                    |
| $\mu$ (customer/h)      | 88.25                   | 88.25                   | 88.25                   | 88.25                   | 88.25                   | 88.25                   | 88.25                   |
| $E(S)$ (-)              | 0.91                    | 1.39                    | 1.31                    | 1.31                    | 2.33                    | 1.12                    | 0.28                    |
| $R$ (%)                 | 90.65                   | 69.26                   | 43.81                   | 43.81                   | 77.53                   | 56.02                   | 28.44                   |
| $E(L)$ (-)              | 8.79                    | 1.28                    | 0.14                    | 0.14                    | 2.10                    | 0.51                    | 0.11                    |
| $E(W)$ (s)              | 395.5                   | 37.62                   | 4.23                    | 4.23                    | 36.75                   | 18.65                   | 16.21                   |

The highest waiting time resulting from Table 7 is on Monday at 5.00 a.m. due to the highest intensity of customer entry into the system. In this interval, customers who go to work or school in the early hours of the morning on Monday instead of travelling on Sunday evening enter the system. Since the service lines work at the same times on Tuesday, Thursday and Friday, Table S5, which is part of the Supplementary Materials, analyses the situation where 11,824 customers entered the system during this interval. After the redistribution of the service lines, we observed a positive effect related to the reduction of the waiting time in the queue for the issuance of a national travel ticket in the monitored days compared to the current situation. The waiting time in the current state was extended by passengers with a request to issue an international travel ticket who accessed any service line. A total of 4930 national travel tickets were issued on Friday, the highest number of all tickets issued. This condition caused increased waiting times during the afternoon peak hour. Table S6, which is part of the Supplementary Materials, shows the mathematical analysis of the proposed system on Friday. The system receives the highest rush to buy travel tickets on Fridays from 2.00 p.m. to 3.00 p.m., when customers wait in line for an average of 6.33 min. A total of 3853 national travel tickets were issued on Saturday, as shown in Table S7 (in Supplementary Materials). Many customers were tourists who went to the High Tatras for relaxation. A total of 3667 national travel tickets were issued on Sunday. Given the intensity of customer entry and the average service time, the system will behave according to Table S8 (in Supplementary Materials). The system records the highest usage on Sundays from 2.00 p.m. to 3.00 p.m., when, on average, two lines out of three work.

Depending on the proposed number of service lines on individual days, schedules of individual service lines were proposed. For lines 1 and 2, the same work performance and the required number of employees are set as in the original case. Service line 5, which will issue national travel tickets during peak hours from 8.00 a.m. to 4.00 p.m., has a schedule shown in Figure 15.



**Figure 15.** Proposal of schedule of service line 5.

The required number of employees for a daily 8 h work interval is 1.5, while the overcapacity of the work interval is 74.20 h.

## 5.2. Service Line Specialised in Issuing International Travel Tickets

Requests for the issuance of an international travel ticket are handled in the customer centre as part of the proposal, where two lines number 3 and 4. The intensity of customer

entry for the analysed model on Sunday during the service time from 4.00 p.m. to 7.00 p.m. is  $\lambda = 8.87$  customer/h; average service time  $1/\mu = 13.10$  customer/h. Based on the input values, the load and utilisation of the proposed system are calculated according to Formulas (5) and (6)  $\alpha = 0.67$ ;  $\rho = 0.67$ . The system stabilisation condition  $\rho < 1$  is met, which means that the need for a reserve to compensate for losses that arise due to the random arrival of customers in the system is met. Table 8 shows the passability between service line states for international travel tickets.

**Table 8.** Transition table between states for international service lines.

| $k$         | $q_k$ | $p_k$ |
|-------------|-------|-------|
| 0           | 1     | 0.32  |
| 1           | 0.68  | 0.22  |
| 3– $\infty$ | 1.42  | 0.46  |
| Total       | 3.10  | 1     |

The service line is free with a 32% probability, which implies a high system utilisation. From the calculated probabilities  $p_k$ , we proceed to calculate the characteristics of the proposed system:

- System performance  $E(S) = 0.67$ .
- System utilisation  $R = 67.7\%$ .
- Average queue length  $E(L) = 1.42$  customers.
- Average waiting time in the queue  $E(W) = 576.25$  s.

The system utilisation represents a high waiting time of 576.25 s. This result is caused by the stochastic arrival of customers in the system and the duration of issuing an international travel ticket, which is at the level of 4.6 min per customer. Table 9 shows the mathematical analysis of the proposed system for international tickets on Monday. On Monday, 140 passengers entered the system with a request to issue an international travel ticket.

**Table 9.** Mathematical analysis of the proposed system for international travel tickets (Monday).

| Interval                | 5.00 a.m.—<br>6.00 a.m. | 6.00 a.m.—<br>8.00 a.m. | 9.00 a.m.—<br>3.00 p.m. | 4.00 p.m.—<br>7.00 p.m. | 8.00 p.m.—<br>4.00 a.m. |
|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| Number of service lines | 1                       | 1                       | 2                       | 1                       | 1                       |
| $\Sigma$ tickets        | 3                       | 15                      | 87                      | 28                      | 7                       |
| $\lambda$ (customer/h)  | 1.50                    | 2.50                    | 6.21                    | 3.50                    | 0.39                    |
| $\mu$ (customer/h)      | 13.1                    | 13.1                    | 13.1                    | 13.1                    | 13.1                    |
| $E(S)$ (-)              | 0.11                    | 0.19                    | 0.47                    | 0.27                    | 0.03                    |
| $R$ (%)                 | 11.45                   | 19.08                   | 23.70                   | 26.72                   | 2.90                    |
| $E(L)$ (-)              | 0.01                    | 0.05                    | 0.03                    | 0.10                    | 0.00                    |
| $E(W)$ (s)              | 35.54                   | 64.81                   | 16.36                   | 100.19                  | 8.21                    |

The highest rush is between 4.00 p.m. and 7.00 p.m. when customers wait in line for 100.19 s, which is caused by the activity of one service line in the customer centre. On Tuesday, Wednesday, and Thursday, 344 customers entered the system with a request to issue an international travel ticket, as shown in Table S9 (in Supplementary Materials). In this interval, the proposed number of service lines is sufficient, and it is not necessary to optimise it. The model contains a reserve in case of accidental arrival of a larger number of customers in the system. On Friday, 118 customers entered the system with a request to issue an international travel ticket. A mathematical analysis of the proposed system for international Friday tickets is shown in Table S10, which is part of the Supplementary Materials.

A total of 73 customers entered the system on Saturday. Customers with this request were handled in the customer centre with the following characteristics. Table S11 (in Supplementary Materials) shows the mathematical analysis of the proposed system for

international Saturday tickets. The most requests are estimated in the interval from 9.00 a.m. to 3.00 p.m. when a total of 51 customers entered the system during the two Saturdays as part of the observation. The use of the system also includes a possible reserve for a higher intensity of customer entry. A total of 160 customers entered the system on Sunday (Table S12 in Supplementary Materials). In the proposed model, a reserve for the random arrival of customers in the system is also preserved.

Between 9.00 a.m. and 7.00 p.m., the highest number of customers, mainly tourists returning from the High Tatras, entered the system. Service line 3 operates according to the schedule shown in Figure 16 and will provide an international ticketing service. The service line starts working from 9.00 a.m. due to the estimated arrival of international passengers in the model. Figure 16 shows the proposal for a schedule of service line 3.

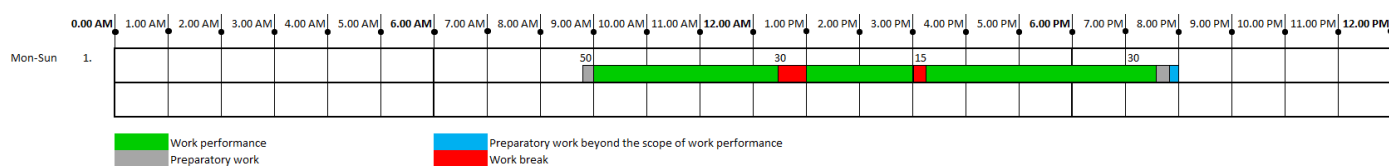


Figure 16. Proposal of schedule of service line 3.

Service line 3 workplace mode is identical to the current system. A total of 312.66 h is needed to complete the given work performance at service line 3. It follows that the work team is slightly overworked by 6.53 h, while there is a need to replace one work team in two months. The proposal of the schedule for line 4 is shown in Figure 17. From Monday to Friday, it starts working as early as 6.00 a.m. and on weekends, when we expect a lower system load, only from 8.00 a.m.

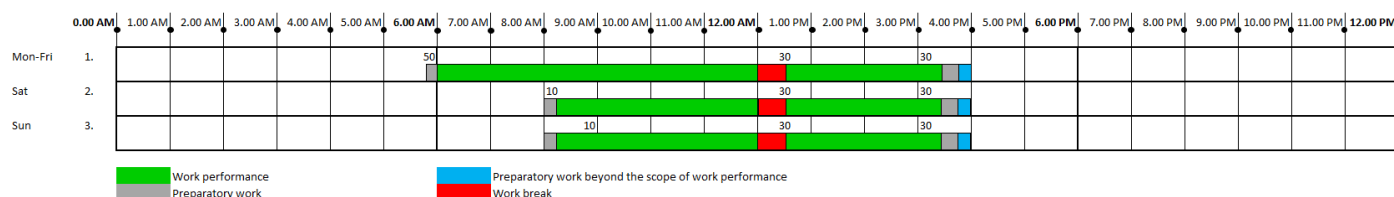


Figure 17. Proposal of schedule of service line 4.

Service line 5 in the customer centre will be alternated with line number 4. This alternation between lines serves to reveal the highest intensity of customer access to the system and the more efficient use of individual employees.

Due to the constantly changing length of the queue in front of the ticket booths, the proposal is to introduce stretchable guide belts. Guide belts will be placed in front of ticket booths, which will lead to four service lines with a notification screen about the issuance of national travel tickets and international travel tickets, as well as in the customer centre, where upon entering, passengers will be directed to service line number 5 with notification screen in Slovak and English language for issuing international travel tickets.

The model set in this way can be verified at any time in the real conditions of the Poprad-Tatry railway station. The primary goal was to set up a model that, based on mathematical analysis, would meet the given conditions. As part of this research, consultations were held to support such a solution, where it was not necessary to create complex simulations. On the contrary, we wanted to emphasise and highlight the applicability and functionality of mathematical analysis approaches as a suitable method for solving this problem. Therefore, it is important to perceive the goal and results of this research from this perspective as well. After the model is set up in this way, it is possible and necessary to introduce this model.

## 6. Discussion

Queueing theory is the mathematical study of waiting lines or queues. Servicing the passenger at the railway station is an example of queueing theory in praxis. Input flow is the high number of customers and travel tickets; ticket booths are waiting lines—queues. During peak hours, the input flow of customers is so high that each service line is forming the queues. When entering complex customer requests into the queue, long queues are formed, and thus, the total waiting time in the queue is extended. The new proposal for the work of individual lines will achieve an even load on all operated service lines.

Markov chains in our formulations are used to model transitions between different states of the system (e.g., the number of customers in a queue) if the future behaviour of the system depends only on the current state and not on the previous one. This model captures dynamic changes in the queue based on state transition probabilities, which allows us to better manage systems with variable arrival and service intensity.

Unlike classical queueing theory, which often works with fixed parameters (e.g., constant service and arrival rates), the Markov model enables dynamic adaptation to change in the system. Classical queueing theory models can be limited by the fact that they do not consider random events that can affect the entire system (e.g., the sudden arrival of many customers). The Markov chain is, therefore, more flexible in handling such changes. Classical models provide results based on average values, while our approach using Markov chains provides more dynamic and accurate predictions because it considers changes in the system in real-time. The optimisation comes from the fact that this model can more effectively redistribute resources (e.g., manpower) based on the current state of the system, thus minimising waiting times and increasing efficiency.

For research purposes, data were collected from the Poprad-Tatry railway station during the period from 12–25 January. This timeframe included both weekdays and weekends to capture diversity in passenger arrivals and analyse a typical week in station operation. The data were obtained by personal measurement on the spot, while the measurements were carried out manually using a stopwatch at each ticket booth. This method of data collection, although difficult, provided a detailed view of the real state and dynamics of the system. Although we did not include advanced simulation tools such as MassMotion, we opted for an analytical approach based on empirical data and Markov chain modelling. The reason was mainly the effort for simplicity and the immediate applicability of the results. Advanced simulation tools were not considered necessary as the goal was to identify practical optimisation options based on directly acquired data. Passenger flow management, which also includes changes in staff allocation, is a key factor in the effective management of a railway station. In this proposal, the configuration of the station and its possibilities to optimise the movement of passengers were considered. Allocation of employees is important because flexible redistribution of labour forces according to the current intensity of passenger arrivals enables more effective processing of queues and shorter waiting times. This practice directly contributes to improving the overall design of the system and increases passenger satisfaction.

As part of the methodology, we first collected and analysed real data from the Poprad-Tatry railway station to identify the patterns of customer arrivals and the time distribution of service at sales points. Subsequently, we applied this data to the infinite queue model, where we used analytical calculations to estimate the probability of individual system states as well as to determine the expected queue length and waiting time. The methodology combines real-world data inputs with sophisticated mathematical models and provides a robust basis for service optimisation in a high-volume customer environment such as the Poprad-Tatry railway station.

Statistical data were used to determine the parameters of the model, which include time intervals between passenger arrivals, queue lengths, service speed, and the total number of passengers during different time intervals. These data were used to calibrate the model and determine parameters such as arrival intensity ( $\lambda$ ) and average service time. To evaluate the accuracy of the model, statistical data that were not used in the

model setup (e.g., data from a different time or from other working days) were used to compare the model's predictions with the real behaviour of the system to verify its accuracy and robustness.

Mathematical analysis allowed us to calculate the effectiveness of the proposed changes in the configuration of service lines while we focused on achieving an optimal balance between the number of service points and the intensity of customer arrivals. Based on the obtained results, we proposed specific measures to improve the functioning of the queue system, including the introduction of a common queue, which could contribute to a more even distribution of customers and shorten the waiting time. The results of the analysis are of practical use not only for the Poprad-Tatry railway station but also for other transport hubs where it is necessary to effectively manage the arrivals and departures of many passengers.

Despite the constant need for ticket booths, and with the use of new technologies, there is a proposal for the future removal of ticket booths to provide additional capacity in the station to improve the customer experience, as well as reduce operating costs in the station [43]. With the growth and advancement of technology, self-service machines have become more dominant for most travellers than quick ticket purchases. The Office of Rail and Road Transport [44] conducted a survey to examine the ability of passengers to make the most appropriate purchase of rail tickets using ticket machines. This was completed through a mystery shopping exercise involving anonymous customers who bought different types of tickets from ticket machines. Their results showed that out of a total sample size of 739, 91% of shoppers chose the most appropriate ticket, and the majority (75%) were overall satisfied with using ticket machines. However, the report notably revealed that 89% of shoppers who use ticket machines did not provide information to show other types of tickets that may have been displayed at the till.

This research, by statistically determining the intensity of customer entry into the system and the average service time at the Poprad-Tatry station and subsequent mathematical analysis, proved the congestion of the service lines (ticket booths) and the long waiting of customers in the queues in front of the ticket booths. After the redistribution of service lines as part of the proposal, a positive effect was achieved related to the reduction of waiting time in the queue for the issuance of a national travel ticket in the observed days compared to the current situation. The waiting time in the current state was extended by passengers with a request to issue an international travel ticket who accessed any service line. Depending on the proposed number of lines on individual days, the schedules of individual service lines were proposed. Lines 1 and 2 will have the same work performance as the required number of employees, as in the original case. Service line 5, located in the customer centre, will issue national travel tickets from 8.00 a.m. to 4.00 p.m. during peak hours. The schedule of line 3 will be from 9.00 a.m. and will provide the service of issuing international travel tickets. Service line 5 in the customer centre will be alternated with service line 4. This alternation between lines will serve to reveal the highest intensity of customer entry into the system and more efficient use of individual employees. Table 10 shows a comparison of the required number of employees before and after the introduction of the proposal.

**Table 10.** Comparison of the number of employees on lines 1–5.

|                                                       | Service Line 1<br>(Employees) | Service Line 2<br>(Employees) | Service Line 3<br>(Employees) | Service Line 4<br>(Employees) | Service Line 5<br>(Employees) |
|-------------------------------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| Original condition                                    | 2                             | 2                             | 4                             | 3                             | 1                             |
| Proposal                                              | 2                             | 2                             | 1                             | 3                             | 1                             |
| The required number of employees according to Table 3 | 4.11                          | 4.14                          | 2.03                          | 2.45                          | 2.43                          |

Referring to Table 3, which shows the calculation of the required number of employees, measures were proposed that made the work of service lines and the number of employees

working on these lines more efficient. The working hours of some service lines have been adjusted, and adjustments have been proposed, for example, in the form of alternating the work of lines 4 and 5. It follows that some lines have maintained the number of employees as in the original state, and vice versa; some have reduced the number of employees.

The proposal includes the introduction of stretchable guide belts due to the constantly changing length of the queue in the station. The conveyor belts will lead to four service lines with a notification screen about the issuance of national travel tickets and international travel tickets, as well as in the customer centre, where upon entry, passengers will be directed to service line 5 with a notification screen in Slovak and English language for issuing international travel tickets. Using queuing theory, the waiting time in queues and the scheduling of individual service lines were improved, with separate arrangements made for purchasing national and international travel tickets.

The mathematical analysis performed within the proposal proves that the service lines will work more efficiently. Considering the operational evaluations of the waiting time, the benefits of the time when the customers stand in an inefficient way in the queue were calculated. In the current system that operates at the Poprad-Tatry station, the customer stood in the queue for an average of 105.507 s between 12 January 2024 and 25 January 2024. This value was calculated as an average of the waiting times of the analysed current system. Since 29,264 customers entered the system during this interval, customers spent an average of 857,662 h in total. Table 11 shows the evaluation of waiting times when buying international (I) or national (N) travel tickets for the original state and the proposed state.

**Table 11.** Evaluation of waiting times in queues for the original and proposed state.

|                | Average Waiting Time (s) |       | $\lambda$ Input Intensity (Customers) |     | Total Waiting Time (Hours) |       | The Difference in Waiting Times (Hours) |       |
|----------------|--------------------------|-------|---------------------------------------|-----|----------------------------|-------|-----------------------------------------|-------|
|                | N                        | I     | N                                     | I   | N                          | I     | N                                       | I     |
| Current system | 105.5                    | 105.5 | 28,429                                | 835 | 829.82                     | 24.47 | -                                       | -     |
| New system     | 46.94                    | 59.15 | 28,429                                | 835 | 370.68                     | 13.72 | 459.13                                  | 10.75 |

The greatest positive effect of the redistribution of service lines will be experienced by passengers with the request to purchase a national travel ticket. Based on the proposed variant, these customers stand in the queue for an average of 46.93 s.

The introduction of self-service Kiosks for the purchase of travel tickets will reduce the waiting time in the queue; the customer chooses a ticket according to his own requirements, and there is no prolongation of the waiting time in line in front of the ticket booths. This will achieve a fluid flow of customers in the station, as customers will be evenly distributed in front of the ticket booths and in front of the self-service machines (kiosks). In addition to the introduction of a common queue in front of the ticket booths, the overall proposal was to change the characteristics of the ticket booths, such as changing the arrangement of individual service lines and dividing the service lines for national travel tickets and especially for international travel tickets. Reorganisation of the customer centre along with the placement of self-service kiosks (ticket machines) to speed up travel ticket sales. These propositions are supported by the mathematical analysis we performed based on the available data. The proposal also contains the schedule of individual lines whose work has been optimised. The smoothness of the flow of passengers in the station will also be ensured by the increased use of ticket purchases using the mobile application Ideme vlakom. In this way, the passenger just passes smoothly through the station directly to the platform. Therefore, it is necessary to pay attention to online forms of ticket purchase. Table 12 shows a comparison of the original conditions with the proposal and describes the advantages of introducing the proposal.

**Table 12.** Comparison of the original conditions and the proposal.

| Original Conditions                             | Change/Proposal                                   | Proposal Advantages                                                                                                                                                                                                                                                                                                                                                                                                                            |
|-------------------------------------------------|---------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Separate queues (1 queue for each service line) | Common queue                                      | <ul style="list-style-type: none"> <li>- Eliminates inefficiencies and injustices associated with multiple ranks.</li> <li>- Minimises the waiting time.</li> <li>- Speeds up service time.</li> </ul>                                                                                                                                                                                                                                         |
| Overloading of employees in ticket booths       | Proposal for assigning employees to ticket booths | <ul style="list-style-type: none"> <li>- It leads to a more balanced load at individual workplaces and reduces line congestion.</li> <li>- Better use of employees' working time and resources, which leads to more efficient management of cash registers and a reduction of overall costs.</li> <li>- The system evenly distributes the workload among all cash registers, thereby improving the working conditions of employees.</li> </ul> |
| Overcrowding of ticket booths (long queues)     | Reducing customer stress                          | <ul style="list-style-type: none"> <li>- Customers do not have to decide which ticket booth to choose, which reduces uncertainty and potential stress.</li> <li>- Customers appreciate faster and more predictable service, which increases their satisfaction with the service.</li> </ul>                                                                                                                                                    |

The proposed model can be verified directly in practice, but the financial and technical resources of the Poprad-Tatry railway station prevent this. Since consultations were carried out for this research, based on which a quick setting of the new model was required, analytical models were chosen, respectively, mathematical analysis, which is accurate, fast, and sufficient for the requirements of the Poprad-Tatry railway station. Therefore, there is no need to create lengthy simulation models, which are also financially demanding. Because of this, we wanted to highlight the constant need to use analytical models, which, even in advanced times, prove their accuracy and usability.

## 7. Conclusions

The creation of common customer waiting queues is closely related to the overall arrangements of service lines. If we consider several service lines with one common waiting queue, we will reach a state where customers who come to the system with random intensity are served exactly in the order in which they arrived. Since passengers cannot estimate the need to issue a ticket to the customer in front of them, by introducing a common line, we will achieve a more efficient and fair service system, which will reduce the variability of the waiting time and distribute the waiting time more evenly among all passengers.

The paper presents the application of the basic principles of Markov chains in the implementation of one common queue at the Poprad-Tatry railway station while serving passengers at the ticket booth. Ticket booths at stations in urban areas are considered essential for some passengers to purchase the correct travel ticket with the help of employees in ticket booths. Based on the analysis of the current system of issuing travel tickets at the Poprad-Tatry station in the two weeks of January, the fact was found that customers requesting the issuance of international travel tickets routinely accessed service lines defined as quick sales for the issuance of national travel tickets and thus on the basis of an extended issuance time of an international travel tickets extended the waiting time of customers behind him. Based on the determination of the average service time of individual customers and the intensity of passenger entry into the system, which was provided by operator ZSSK, it was possible to analyse the current situation using Markov chains of queuing system and to determine the characteristics of the system of average waiting time, the percentage use of individual lines in different intervals, the average by the length of the service queue. The characteristics of the current system were also supplemented by the calculation of the current number of employees with the overall schedule of the operators of individual lines. The aim of this was to propose a common waiting queue for customers,

which, based on the previous analysis, was realised by creating designated service zones for national and international passengers with common queues in the station as well as in the customer centre. After the mathematical characterisation of service lines divided in this way with common queues, the goal of reducing customers' time for service was achieved.

As part of further research, it would be appropriate to focus on streamlining the number of ticket booths, where the optimal number of open ticket booths would be determined for different times of the day, the implementation of dynamic queue management systems that would respond to changes in the number of customers, the design of models for predicting the formation of queues based on historical data and current conditions, researching customer behaviour in various situations (for example, stress, time pressure, comfort), aspects that affect customer satisfaction, the impact of self-service technologies in case of cancellation of ticket booths, analyse the costs associated with inefficient queue management and their impact on the environment, to deal with human resources and working conditions in ticket booths or to examine the design of ticket booth areas. All the mentioned ideas for further research, if implemented, would help to improve the quality of services, increase efficiency, and reduce costs for railway stations.

**Supplementary Materials:** The following supporting information can be downloaded at: <https://www.mdpi.com/article/10.3390/infrastructures9100175/s1>, Table S1: Mathematical analysis of the current system on Tuesday, Wednesday, Thursday; Table S2: Mathematical analysis of the current system on Friday; Table S3: Mathematical analysis of the current system on Saturday; Table S4: Mathematical analysis of the current system on Sunday; Table S5: Mathematical analysis of the proposed system on Tuesday, Wednesday and Thursday; Table S6: Mathematical analysis of the proposed system on Friday; Table S7: Mathematical analysis of the proposed system (Saturday); Table S8: Mathematical analysis of the proposed system (Sunday); Table S9: Mathematical analysis of the proposed system for international travel tickets (Tuesday, Wednesday, Thursday); Table S10: Mathematical analysis of the proposed system for international travel tickets (Friday); Table S11: Mathematical analysis of the proposed system for international travel tickets (Saturday); Table S12: Mathematical analysis of the proposed system for international travel tickets (Sunday).

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