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Algebras of Vector Functions over Normed Fields

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Abstract: This article is devoted to study of vector functions in Banach algebras and Banach spaces over normed fields. A structure of their Banach algebras is investigated. Banach algebras of vector functions with values in $*$-algebras, finely regular algebras, B^*-algebras, and operator algebras are scrutinized. An approximation of vector functions is investigated. The realizations of these algebras by operator algebras are studied.

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1. Introduction

Algebras and operator algebras over the real field and the complex field were intensively studied and have found multi-faceted applications (see, for example, [1–7] and the references therein).

For normed algebras and operator algebras over the real and complex fields, many results were already obtained (see, for example, [3,5,6,8] and the references therein). On the contrary, for algebras over non-Archimedean normed (i.e., ultranormed) fields, comparatively few results are known. This is explained by their specific features and additional difficulties arising from structure of fields [9–18].

Many results in the classical case use strong conditions to which the real field and the complex field satisfy. Among them there are the following: the real field, $\mathbf{R}$, has a linear ordering that is compatible with its additive and multiplicative structure; complex field $\mathbf{C}$ is algebraically closed and norm complete and locally compact, and it is the quadratic extension of $\mathbf{R}$; moreover, there are no other commutative fields with Archimedean multiplicative norms and that are complete relative to their norms besides these two fields.

For comparison, in the non-Archimedean case, the algebraic closure of the field of p-adic numbers is not locally compact [9,18,19]. Each ultranormed field can be embedded into a larger ultranormed field. There is no ordering of an infinite ultranormed field such as $\mathbf{Q_p}$, $\mathbf{C_p}$ or $\mathbf{F_p}(t)$ that is compatible with its algebraic structure.

Non-Archimedean analyses, functional analyses, and the representation theory of groups over non-Archimedean fields have developed quickly in recent years [18,20–24]. This is motivated not only by the needs of mathematics but also their applications in other sciences such as physics, quantum mechanics, quantum field theory, informatics, etc. (see, for example, [25–31] and the references therein). Henceforward, a norm (or normed) will be written shortly instead of an ultranorm or a non-Archimedean norm (or ultranormed correspondingly). That is, it satisfies the strong triangle inequality $|x+y| \leq \max(|x|, |y|)$ for each x and y in a normed space, X.

This article is devoted to the study of vector functions in Banach algebras and Banach spaces over normed fields. A structure of their Banach algebras is investigated. Ideals in them are studied in Theorem 1, Proposition 1, and Corollaries 1 and 2. The Banach algebras of vector functions with values in $*$-algebras, finely regular algebras, and B^*-algebras are scrutinized in Propositions 2 and 3. An approximation of vector functions is investigated in Theorems 2 and 3. The realizations of these algebras by operator algebras are studied. Examples 1–5 are provided. An invariance of such algebras relative to multiplication on bounded continuous functions is given by Theorems 4 and 5. Necessary definitions and notations are recalled in Appendix A.

All main results of this paper are obtained for the first time. Their possible applications are discussed in the Conclusion.

2. Algebras of Vector Functions over Normed Fields

Definition 1. *Let F be an infinite field with a non-Archimedean multiplicative norm $|\cdot|_F = |\cdot|$ such that $\exists v \in F$, $0 < |v| < 1$. Let F be norm complete. Let S be a nonvoid zero-dimensional topological space. Let each $s \in S$ be posed as a Banach algebra A_s over F, where norm $|\cdot|_{A_s}$ is non-Archimedean on A_s. Let $Ba(S, \{A_s\})$ be a family of all vector functions $x = \{x(s) \in A_s : s \in S\}$ possessing the following properties (i), (ii):*

(i) $x(s) \in A_s$ for each $s \in S$;

(ii) $|x(s)|$ is a bounded continuous function on S, where $|x(s)| = |x(s)|_{A_s}$.

By an algebra of vector functions generated by S and the family $\{A_s : s \in S\}$ of Banach algebras, we use a subset $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ of $Ba(S, \{A_s\})$by forming a Banach algebra over F relative to the following operations:

(1) $fx = \{f(s)x(s) : s \in S\}$, $x + y = \{x(s) + y(s) : s \in S\}$,

$xy = \{x(s)y(s) : s \in S\}$

for every x and y belonging to $\mathcal{A}$, $f \in C_b(S, F)$, where $C_b(S, F)$ denotes the space of all continuous bounded functions $f : S \to F$. This Banach algebra is supplied with the following norm.

(2) $|x| = \sup_{s \in S} |x(s)|$.

Shortly, $x = \{x(s) \in A_s : s \in S\}$ is also denoted by $\{x(s) : s \in S\}$ or $\{x(s)\}$ if S is specified. Moreover,

(3) $|f| = \sup_{s \in S} |f(s)| < \infty$ for each $f \in C_b(S, F)$, where $|f(s)| = |f(s)|_F$.

If X_s is a Banach space over F for each $s \in S$, then Banach spaces $Ba(S, \{X_s\})$ and $\mathcal{A}(S, \{X_s\})$ are similarly defined instead of algebras.

A subset K in $\mathcal{A}$ by $cl_{\mathcal{A}}K$ is denoted the closure of K in $\mathcal{A}$. Henceforward, χ_W denotes the characteristic function of a subset W in set S such that $\chi_W(x) = 1$ for each $x \in W$, while $\chi_W(x) = 0$ for each $x \notin W$.

Example 1. *In particular, there may be $A_s = D$ for each $s \in S$, where D is the Banach algebra over F. In this case, $\mathcal{A}(S, \{A_s\})$ will also be denoted by $\mathcal{A}(S, D)$, while $Ba(S, \{A_s\})$ is denoted by $Ba(S, D)$. In particular, the Banach algebra exists as $C_b(S, D)$ of all continuously bounded maps, x, from S into D.*

Example 2. *If $S = \{s_1, \ldots, s_n\}$ is a finite discrete space, then $Ba(S, \{A_s\})$ is isomorphic to the direct sum $A_{s_1} \oplus \ldots \oplus A_{s_n}$.*

Definition 2. *Let $\mathcal{F} \subseteq F^S$. If $fx \in \mathcal{A}$ for each $x \in \mathcal{A}$ and $f \in \mathcal{F}$, then it is said that algebra $\mathcal{A}$ is invariant relative to $\mathcal{F}$. Let $J \subseteq \mathcal{A}$; then, family $\{x(s_0) : x \in J\}$ is called a projection of J onto A_{s_0} and is denoted by J_{s_0}.*

Theorem 1. *Let S be an ultrametric space. Let the algebra $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ (see Definitions 1 and 2) satisfy the following conditions:*

(i) $\mathcal{A}_s = A_s$ for each $s \in S$;

(ii) fx belongs to the closed ideal in $\mathcal{A}$ generated by x for each $f \in C_b(S, F)$ and each $x \in \mathcal{A}$. Let V be the closed left (or right or two-sided) ideal in $\mathcal{A}$ such that

(iii) $\sum_{j \in \lambda} \chi_{W_j} x_j \in cl_{\mathcal{A}}V$ for each $x_j \in V$ with $\sup_j |x_j| < \infty$ and each disjoint clopen covering $\{W_j : j \in \lambda\}$ of S, where λ is a set.

Then, there exists a closed left (or right or two-sided correspondingly) ideal J_s in A_s for each $s \in S$ such that

(1) $V = \{x \in \mathcal{A} : \ \forall s \in S, \ x(s) \in J_s\}$.

Proof. We consider the case of left ideals similarly to other cases. Let $y \in \mathcal{A}$, and et $J_s = cl_{A_s} V_s$ be the closure of V_s in A_s for each $s \in S$. Let $y(s) \in J_s$ for each $s \in S$. The case $y = 0$ is trivial. Let $y \neq 0$. Since $\mathcal{A}$ and V are over the field, F, it is sufficient to consider the case of $|y| = 1$. We chose a monotonously decreasing positive sequence $\{\epsilon_j : \forall j \in \mathbf{N},\ 0 < \epsilon_j\}$ with $\lim_{j\to\infty} \epsilon_j = 0$.

Note that the norm on $\mathcal{A}$ is non-Archimedean by Formula (2) in Definition 1, since the norm is non-Archimedean for each $s \in S$on Banach algebra A_s . On the other hand, from Condition (i) and the definition of J_s and for y as stated above, it follows that for each $q \in S$, there exists $x_{\epsilon_j,q}$ in V such that $|x_{\epsilon_j,q}(q) - y(q)| < \epsilon_j$. By the continuity of $|x_{\epsilon_j,q}(s) - y(s)|$ as the function of s, there exists a clopen ball $B(S, q, r_{j,q})$ containing q in S such that

(2) $|x_{\epsilon_j,q}(s) - y(s)| \leq \epsilon_j$ for each $s \in B(S, q, r_{j,q})$, where $0 < r_{j,q} < \infty$, since S is ultrametrizable and hence zero-dimensional. Notice that $\chi_W \in C_b(S, F)$ for each clopen subset W in S, where χ_W denotes the characteristic function of W; that is, $\chi_W(s) = 1$ for each $s \in W$ and $\chi_W(s) = 0$ for each $s \in S \setminus W$. The condition (ii) of this theorem implies that $cl_{\mathcal{A}} \chi_W V \subseteq V$ for each W clopen in S, since $cl_{\mathcal{A}} V = V$, $C_b(S, F) \subseteq Z(\mathcal{A})$, where $Z(\mathcal{A})$ denotes the center of $\mathcal{A}$. Therefore, $\chi_W y \in \mathcal{A}$, $\chi_W x_{\epsilon_j,q} \in V$.

Thus $\Upsilon_j = \{B(S, q, r_{j,q}) : q \in S\}$ is the clopen covering of S. It is known that in ultrametric space S, as it follows from the strong triangle inequality for the ultrametric space, either each pair of clopen balls do not intersect or one of them is contained in the other [18,22]. Therefore, for S, there exists a disjoint clopen subcovering $\Psi_j = \{W_{j,k} = B(S, q_k, r_{j,q_k}) : k \in \lambda_j\}$, where λ_j is a set. Thus, $\bigcup_{k\in\lambda_j} W_{j,k} = S$, $W_{j,k} \cap W_{j,l} = \emptyset$ for each $k \neq l$ in λ_j, and $W_{j,k}$ is clopen in S for each $j \in \mathbf{N}$, $k \in \lambda_j$. Therefore, $h_j = \sum_{k\in\lambda_j} \chi_{W_{j,k}} x_{\epsilon_j,q_k}$ belongs to V by Condition (iii) of this theorem. By the construction above $|h_j - y| \leq \epsilon_j$, since $\lim_{j\to\infty} \epsilon_j = 0$ and $\mathcal{A}$ comprise the Banach algebra, then $\lim_{j\to\infty} |h_j - y| = 0$; consequently, $y \in V$. $\square$

Example 3. *If J is a closed ideal in D, either $\mathcal{A} = Ba(S, D)$ or $\mathcal{A} = C_b(S, D)$, $V = Ba(S, J)$, or $V = C_b(S, J)$; then, they satisfy the conditions of Theorem 1, where S is the ultrametric space, and D is the Banach algebra over F.*

Corollary 1. *Let S, $\mathcal{A}$, and A_s be the same as in Theorem 1. Let V be a maximal closed left (or right, or two-sided) ideal in $\mathcal{A}$ satisfying Condition (iii) in Theorem 1. Then, for each $s \in S$, V_s is a closed maximal left (or right or two-sided correspondingly) ideal in A_s. If, moreover, the algebra A_s is simple for each $s \in S$, then for each closed two-sided ideal V in $\mathcal{A}$, there exists a closed subset S_V in S such that $V = \{x \in \mathcal{A} :\ \forall s \in S_V,\ x(s) = 0\}$.*

Proposition 1. *Let S be an ultrametric space, and let $\mathcal{A} = \mathcal{A}(S, \{A_s\})$. Let the following also be the case:*

(i) $\mathcal{A}_s = A_s$ for each $s \in S$;

(ii) $\mathcal{A}$ be invariant relative to multiplication on each $f \in C_b(S, F)$;

(iii) $\sum_{j\in\lambda} \chi_{W_j} x_j \in cl_{\mathcal{A}} V$ for each $x_j \in V$ with $\sup_j |x_j| < \infty$ and each disjoint clopen covering $\{W_j : j \in \lambda\}$ of S, where λ is a set;

(iv) $\exists 1 < \delta < \infty$, $\forall 0 < \epsilon < 1$, $\forall s \in S$, $\forall u \in A_s$, $(u \in cl_{A_s}(uA_s)$ & $(\exists v \in A_s$, $|v| < \delta$, $|uv - u| < \epsilon))$.

Then, gx belongs to the closed left ideal J generated by x in $\mathcal{A}$ for each $x \in \mathcal{A}$ and each $g \in C_b(S, F)$.

Proof. Let $x \in \mathcal{A}$, $0 < \epsilon_{j+1} < \epsilon_j < 1$ for each $j \in \mathbf{N}$, $\lim_{j\to\infty} \epsilon_j = 0$. From Conditions (i),and(iv), it follows that for each $q \in S$, there exists $v_{j,q} \in \mathcal{A}$ such that $|x(q)v_{j,q}(q) - x(q)| < \epsilon_j$ and $|v_{j,q}(q)| < \delta$. By the continuity of $|xv_{j,q} - x|$ and $|v_{j,q}|$, there exists a clopen ball $B(S, q, r_{j,q})$ such that $|x(s)v_{j,q}(s) - x(s)| \leq \epsilon_j$ and $|v_{j,q}(s)| \leq \delta$ for each $s \in B(S, q, r_{j,q})$. As in the proof of Theorem 1, the clopen covering $\Upsilon_j = \{B(S, q, r_{j,q}) : q \in S\}$ of S possesses a disjoint clopen subcovering $\Psi_j = \{W_{j,k} = B(S, q_k, r_{j,q_k}) : k \in \lambda_j\}$, where λ_j is a set.

From Conditions (ii), (iii), and (iv), we deduce that $|x\sum_{k\in\lambda_j}\chi_{W_{j,k}}v_{j,q_k}-x|\leq\epsilon_j$ and $h_j=g\sum_{k\in\lambda_j}\chi_{W_{j,k}}v_{j,q_k}$ belong to $\mathcal{A}$, since $\mathcal{A}$ is the Banach algebra over F and $\sup_{k\in\lambda_j}|\chi_{W_{j,k}}v_{j,q}|\leq\delta$. Hence, $|xh_j-gx|\leq\epsilon_j|g|<\infty$, where $|g|=\sup_{s\in S}|g(s)|<\infty$, since $g\in C_b(S,F)$. Note that $xh\in J$, where $J=cl_{\mathcal{A}}(x\mathcal{A})$. Therefore, $gx\in J$, since $C_b(S,F)\subseteq Z(\mathcal{A})$, $\lim_{j\to\infty}\epsilon_j=0$, and $\mathcal{A}$ is the Banach algebra. $\square$

Example 4. *If zero-dimensional space S is locally compact, there exists the Banach algebra*

$B_\infty(S,\{A_s\})=\{y\in Ba(S,\{A_s\}):\ \forall\epsilon>0,\ \exists U$ *compact subset in* $S,\ \forall s\in S\setminus U,\ |y(s)|<\epsilon\}$.

In particular, if $A_s=D$ for each $s\in S$, there exists Banach subalgebra $C_\infty(S,D)=C_b(S,D)\cap B_\infty(S,D)$. For $\mathcal{A}\subseteq B_\infty(S,\{A_s\})$, Condition (iii) of Theorem 1 can evidently be omitted, since for each $\epsilon>0$ and $y\in\mathcal{A}$, the set $\{s\in S:|y(s)|\geq\epsilon\}=:\Omega_{y,\epsilon}$ is compact so that each open (or clopen in particular) covering of $\Omega_{y,\epsilon}$ has a finite subcovering. Moreover, in this case of algebra $\mathcal{A}$, Theorem 1 and Proposition 1 remain valid for the zero-dimensional locally compact space, S, instead of the ultrametric space, S.

Corollary 2. *Let S be an ultrametric space, A be a Banach algebra over F, and let $z\in cl_A(zA)$ for each $z\in A$. Let V be a closed ideal in $\mathcal{A}=C_b(S,A)$ satisfying Condition (iii) in Theorem 1. Then, for each $s\in S$, there exists a closed ideal J_s in A such that*

$V=\{x\in A:\ \forall s\in S,\ x(s)\in J_s\}$.

Remark 1. *In particular, if S is a zero-dimensional compact space, then it implies that Condition (iii) in Theorem 1 and Proposition 1 is satisfied. This condition can also be omitted for the zero-dimensional locally compact space S with $\mathcal{A}\subseteq C_\infty(S,A)$, since S has an Alexandroff compactification $\alpha S=S\cup\{\alpha\}$, for which $C_\infty(S,A)$ is isomorphic with $\mathcal{A}(\alpha S,\{A_s\})$ such that $\mathcal{A}(\alpha S,\{A_s\})\subseteq C_\infty(\alpha S,A)$ with $A_s=A$ for each $s\in S$ and $A_\alpha=\{0\}$ (see also Examples 3, 4).*

Theorem 1, Proposition 1, and Corollaries 1 and 2 remain valid for the zero-dimensional Lindelöf space S instead of the ultrametric space. Indeed, each clopen covering $\Upsilon=\{W_k:k\in\lambda\}$ of S has a countable subcovering $\Psi=\{W_{k_i}:i\in\nu\}$, where λ is a set, $\nu\subseteq\mathbf{N}$ (see also [32]). Then, $U_1=W_{k_1}$ and $U_j=W_{k_j}\setminus(\bigcup_{i\in\nu,i<j}W_{k_i})$ for each $1<j\in\nu$ provide a disjoint clopen covering $\{U_i:i\in\nu\}$ of S.

Definition 3. *Let $y\in Ba(S,\{A_s\})$; for each $q\in S$, there exists $x_q\in\mathcal{A}(S,\{A_s\})$ and an open neighborhood $U(q)$ of q such that $y(s)=x_q(s)$ for each $s\in U(q)$. Then, it is said that y locally belongs to the algebra $\mathcal{A}(S,\{A_s\})$ at q. If y locally belongs to $\mathcal{A}(S,\{A_s\})$ at q for each $q\in S$, then it is said that y locally belongs to $\mathcal{A}(S,\{A_s\})$.*

Vector function $y\in Ba(S,\{A_s\})$ is called continuous relative to the algebra $\mathcal{A}(S,\{A_s\})$, if for each $x\in\mathcal{A}(S,\{A_s\})$, and the function $|y(s)-x(s)|$ is continuous.

Example 5. *If $A_s=D$ for each $s\in S$, $\mathcal{A}\subseteq C_b(S,D)$, $y\in C_b(S,D)$, then y is continuous relative to algebra $\mathcal{A}$.*

Definition 4. *A family $\mathcal{G}\subseteq C(S,F)$ on S is called F completely regular if there exists $x\in\mathcal{G}$ for each closed subset K in S and $q\in S\setminus K$ such that $x(q)\neq0$ and $x(s)=0$ for each $s\in K$, where S is a topological space and where $C(S,F)$ denotes the space of all continuous functions $f:S\to F$.*

The family $\mathcal{G}$ is called normal on S, if there exists $x\in\mathcal{G}$ for each closed subsets K and K_1 in S with $K\cap K_1=\emptyset$ with $x(s)=0$ for each $s\in K$ and $x(s)=1$ for each $s\in K_1$.

Theorem 2. *Let S be an ultrametric space, let Banach algebra $\mathcal{A}=\mathcal{A}(S,\{A_s\})$*

(i) Be invariant relative to the multiplication on each $f\in C_b(S,F)$;

(ii) $\sum_{j\in\lambda}\chi_{W_j}x_j\in\mathcal{A}$ for each $x_j\in\mathcal{A}$ with $\sup_j|x_j|<\infty$ and each disjoint clopen covering $\{W_j:j\in\lambda\}$ of S, where λ is a set.

Let y belong to $\mathcal{A}$ locally. Then, $y\in\mathcal{A}$.

Proof. For each $q \in S$, there exists $x_q \in \mathcal{A}$ and a clopen ball $B(S, q, r_q)$ in S such that $y(s) = x_q(s)$ for each $s \in B(S, q, r_q)$, since y locally belongs to $\mathcal{A}$ and $\chi_{B(S,q,r_q)} \in C_b(S, F)$. The covering, $\Upsilon = \{B(S, q, r_q) : q \in S\}$, has a disjoint subcovering $\Psi = \{W_k = B(S, q_k, r_{q_k}) : k \in \lambda\}$, since S is the ultrametric space where λ is a set. Then, $\chi_{W_k} y = \chi_{W_k} x_{q_k} \in \mathcal{A}$ for each $k \in \lambda$, since $\chi_{W_k} \in C_b(S, F)$ and $\mathcal{A}$ satisfies Condition (i). From $\sum_{k\in\lambda} \chi_{W_k}(s) = 1$ for each $s \in S$, it follows that $\sum_{k\in\lambda} \chi_{W_k} y = y$. Hence, $y \in \mathcal{A}$ by Condition (ii) of this theorem, since $\sup_{k\in\lambda} |x_k| \le |y| < \infty$. □

Theorem 3. *Let S be the ultrametric space. Let the Banach algebra $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ satisfy the following conditions (i)–(iii):*

(i) $\mathcal{A}_s = A_s$ for each $s \in S$;

(ii) $\mathcal{A}$ is invariant relative to multiplication on each $f \in C_b(S, F)$;

(iii) $\sum_{j\in\lambda} \chi_{W_j} x_j \in \mathcal{A}$ for each $x_j \in \mathcal{A}$ with $\sup_j |x_j| < \infty$ and each disjoint clopen covering $\{W_j : j \in \lambda\}$ of S, where λ is a set.

Let y be continuous relative to $\mathcal{A}$. Then, $y \in \mathcal{A}$.

Proof. From Condition (i), it follows that there exists x_q in $\mathcal{A}$ such that $x_q(q) = y(q)$. Let $0 < \epsilon < \infty$. By the continuity of $|y(s) - x_q(s)|$ as a function of $s \in S$, there exists clopen ball $B(S, q, r_q)$ such that $|y(s) - x_q(s)| < \epsilon$ for each $s \in B(S, q, r_q)$. The covering $\Upsilon = \{B(S, q, r_q) : q \in S\}$ has a disjoint subcovering $\Psi = \{W_k = B(S, q_k, r_{q_k}) : k \in \lambda\}$, since S is the ultrametric space, where λ is a set. Analogously to the proof of Theorem 1, we infer that $|y(s) - \sum_{k\in\lambda} \chi_{W_k}(s) x_q(s)| < \epsilon$ for each $s \in S$.

By Conditions (ii), (iii) $g_\epsilon(s) = \sum_{k\in\lambda} \chi_{W_k}(s) x_q(s) \in \mathcal{A}$. Hence, $y \in \mathcal{A}$, since $\lim_{\epsilon\to 0} |y - g_\epsilon| = 0$, and $\mathcal{A}$ is the Banach algebra. □

Corollary 3. *Let S be the ultrametric space. Let $\mathcal{A}$ be a closed subalgebra in $C_b(S, D)$, where D is the Banach algebra over F. Let $\mathcal{A}$ satisfy the following:*

(i) $\mathcal{A}_s = D$ for each $s \in S$

and Conditions (ii) and (iii) of Theorem 3. Then, $\mathcal{A} = C_b(S, D)$.

Definition 5. *Let $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ be the algebra over field F of characteristic $char(F) \neq 2$, where A_s is the subalgebra in $L(X_s, X_s)$, where $X_s = c_0(\alpha_s, F)$ is the Banach space over F, α_s is a set for each $s \in S$. Let $\mathcal{A}$ also be a B_2-bimodule, where $B_2 = B_2(F)$ is the commutative associative algebra with one generator i_1 such that $i_1^2 = -1$ and it possesses involution $(vi_1)^* = -vi_1$ for each $v \in F$. Let $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ be a continuous bijective F-linear operator:*

(1) $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ such that

(2) $\mathcal{I}(ab) = (\mathcal{I}b)(\mathcal{I}a)$ and

(3) $\mathcal{I}(ga) = (\mathcal{I}a)g^*$ and $\mathcal{I}(ag) = g^*(\mathcal{I}a)$;

(4) $\mathcal{I}\mathcal{I}a = a$

(5) $(\theta(y))(ax) = \theta((\mathcal{I}a)y)(x)$

for each a and b in $\mathcal{A}$, $g \in B_2$ and x and y in the Banach space $Ba(S, \{X_s\})$, where $\theta : Ba(S, \{X_s\}) \to Ba(S, \{X'_s\})$ is the canonical embedding such that $(\theta x)(s) = \theta_s x(s)$, $\theta_s : X_s \to X'_s$ is the canonical embedding of X_s into the topological dual space X'_s for each $s \in S$.

Then, $\mathcal{A}$ is called a $*^m$-algebra, and operator $\mathcal{I}$ is called the involution. Briefly, a^* can also be written instead of $\mathcal{I}a$.

Proposition 2. (i). *If A_s is the Banach $*$-algebra over field F for each $s \in S$, $char(F) \neq 2$, then $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ can be supplied with the $*^m$-algebra structure.*

(ii). *If $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ is the $*^m$-algebra and $\mathcal{A}_s = A_s$ for each $s \in S$, then A_s is the $*$-algebra for each $s \in S$.*

Proof. (i). For each $a \in \mathcal{A}$ and $s \in S$, we define $a^*(s) = (a(s))^*$. As the $*$-algebra A_s has an embedding into $L(X_s, X_s)$, where $X_s = c_0(\alpha_s, F)$, α_s is a set. Therefore, there exist Banach spaces $Ba(S, \{X_s\})$ and $Ba(S, \{X'_s\})$. This implies that Conditions (1)–(5) in Definition 5 are satisfied. From (4) in Definition 5, it follows that $\mathcal{I} = \mathcal{I}^{-1}$ and, consequently, $|\mathcal{I}| = 1$, where $(\mathcal{I}a)(s) = a^*(s)$ for each $a \in \mathcal{A}$ and $s \in S$. Hence, $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ is the F-linear continuous bijective operator.

(ii). For each $s \in S$ and $b \in A_s$, there exists $a_{s,b} \in \mathcal{A}$ with $a(s) = b$, since $A_s = \mathcal{A}_s$ for each $s \in S$. Inserting $b^* = a^*(s)$ and using Conditions (1)–(5) in Definition 5, A_s is the $*$-algebra. $\square$

Definition 6. *Let $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ be the Banach algebra over field F of characteristic $char(F) \neq 2$. Let $\mathcal{A}$ satisfy the following conditions:*

(1) *$\mathcal{A}$ is the $*^m$-algebra;*

(2) *There exists a bilinear operator $(\cdot,\cdot) : \mathcal{A}^2 \to C_b(S, F)$ such that $|(a, b)| \leq \gamma|a||b|$ for each a and b in $\mathcal{A}$, where $0 < \gamma < \infty$ is a constant independent of a and b;*

(3) *$(a, b) = (b, a)$ and $(a, b) = (a^*, b^*)$ for each a, b in $\mathcal{A}$;*

(4) *If there exists $s \in S$ such that $(a, b)(s) = 0$ for each $b \in \mathcal{A}$, then $a(s) = 0$;*

(5) *$(ab, c) = (a, cb^*)$ for every a, b, c in $\mathcal{A}$;*

(6) *$a(s)a^*(s) \neq 0$, if $a \in \mathcal{A}$, $s \in S$ and $a(s)$ is a non-zero element in $\mathcal{A}_s$.*

Then, $\mathcal{A}$ will be called a B^{*^m}-algebra.

Proposition 3. (i). *Let A_s be the Banach B^*-algebra over the field, F, for each $s \in S$, and let $char(F) \neq 2$. Let also $0 < \sup_{s\in S} q_s < \infty$, where $0 < q_s < \infty$ is a constant such that $|(c, d)_s| \leq q_s|c||d|$ for each c and d in A_s, and $(\cdot,\cdot)_s$ is the bilinear functional on A_s. Then, $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ can be supplied with the B^{*^m}-algebra structure.*

(ii). *Let $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ be the B^{*^m}-algebra and $\mathcal{A}_s = A_s$ for each $s \in S$. Then, A_s is the B^*-algebra for each $s \in S$ and $\sup_{s\in S} q_s \leq \gamma$.*

Proof. (i). By virtue of Proposition 2, (i) $\mathcal{A}$ can be supplied with the $*^m$-algebra structure. Let $(a, b)(s) = (a(s), b(s))_s$ for each $s \in S$, where $(\cdot,\cdot)_s$ is the bilinear functional on A_s corresponding to its B^*-algebra structure by Definition 4 in [10]. This implies (3)–(6) in Definition 6, since $\mathcal{A}_s \subseteq A_s$ for each $s \in S$. From $0 < \sup_{s\in S} q_s < \infty$, it follows that $0 < \gamma < \infty$.

(ii). Since $A_s = \mathcal{A}_s$ for each $s \in S$, then there exists $a_{s,b} \in \mathcal{A}$ for any given $s \in S$ and $b \in A_s$ with $a(s) = b$. In view of Proposition 2(ii), A_s is the $*$-algebra for each $s \in S$. From Conditions (1)–(6) in Definition 6, it follows that A_s is the B^*-algebra for each $s \in S$ and $\sup_{s\in S} q_s \leq \gamma$, since $q_s \leq \gamma$ for each $s \in S$. $\square$

Theorem 4. *Let $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ be the unital algebra over field F satisfying the following conditions:*

(i) *S is the ultrametric space;*

(ii) *For each $s_1 \neq s_2$ in S, there exists $x \in \mathcal{A}$ such that $x(s_1) = 0$ and $x(s_2) = e$, where $e = e(s_2)$ denotes the unit element in A_{s_2}, $|e(s_2)| = 1$;*

(iii) *$\sum_{j\in\lambda} \chi_{W_j} x_j \in \mathcal{A}$ for each clopen disjoint covering $\{W_j : j \in \lambda\}$ of S and each $x_j \in \mathcal{A}$ such that $\sup_{j\in\lambda} |x_j| < \infty$, where λ is a set.*

Then, $\mathcal{A}$ contains fe for each $f \in C_b(S, F)$, where $e = \{e(s) : s \in S\}$ denotes the unit element in $\mathcal{A}$.

Proof. Let $s_1 \neq s_2$ belong to S. By condition (ii), there exists $x = x_{s_1,s_2} \in \mathcal{A}$ such that $x(s_1) = 0$ and $x(s_2) = e(s_2)$. Let $0 < \epsilon < 1$. By the continuity of $|x|$ there exists a clopen ball $B(S, s_1, r_{s_1})$ such that $|x(s)| < \epsilon$ for each $s \in B(S, s_1, r_{s_1})$.

Let U be a clopen subset in S and $s_2 \in S \setminus U$. The covering $\Upsilon_{U,\epsilon} = \{B(S, s_1, r_{s_1}) \cap U = B(U, s_1, r_{s_1}) : s_1 \in U\}$ of U has a disjoint subcovering $\Psi_{U,\epsilon} = \{W_k = B(U, s_{1,k}, r_{s_{1,k}}) : k \in \lambda_{U,\epsilon}\}$, where $\lambda_{U,\epsilon}$ is a set. From Condition (iii), we deduce that $h_{U,\epsilon,s_2} = \chi_{S\setminus U} x_{s_{1,k_0},s_2} + \sum_{k\in\lambda_{U,\epsilon}} \chi_{W_k} x_{s_{1,k},s_2} \in \mathcal{A}$ and $|h_{U,\epsilon,s_2}(s)| < \epsilon$ for each $s \in U$, while $h_{U,\epsilon,s_2}(s_2) = e(s_2)$, where k_0 is a fixed element in $\lambda_{U,\epsilon}$. Let U and U_2 be two disjoint clopen subsets in S. By the continuity of $|h_{U,\epsilon,s_2}(s) - e(s)|$, there exists a clopen ball $B(S \setminus U, s_2, p_{s_2})$ with

$0 < p_{s_2} < \infty$ such that $|h_{U,\epsilon,s_2}(s) - e(s)| < \epsilon$ for each $s \in B(S \setminus U, s_2, p_{s_2})$. The covering $\Upsilon_{U_2,\epsilon} = \{B(S \setminus U, s_2, p_{s_2}) \cap U_2 = B(U_2, s_2, p_{s_2}) : s_2 \in U_2\}$ has the disjoint subcovering $\Psi_{U_2,\epsilon} = \{E_j = B(U_2, s_{2,j}, p_{s_{2,j}}) : j \in \lambda_{U_2,\epsilon}\}$ by Condition (i) of this theorem, where $\lambda_{U_2,\epsilon}$ is a set. Therefore, $g_{U,U_2,\epsilon} = \chi_{S\setminus U_2} h_{U,\epsilon,s_{2,j_0}} + \sum_{j \in \lambda_{U_2,\epsilon}} \chi_{E_j} h_{U,\epsilon,s_{2,j}}$ belongs to $\mathcal{A}$ by Condition (iii), where j_0 is a fixed element in $\lambda_{U_2,\epsilon}$. Therefore, $|g_{U,U_2,\epsilon}(s) - e(s)| < \epsilon$ for each $s \in U_2$, while $|g_{U,U_2,\epsilon}(s)| < \epsilon$ for each $s \in U$.

Take any fixed $f \in C_b(S, F)$. For each, $q \in S$ there exists a clopen ball $B(S, q, t_q)$ with $0 < t_q < \infty$ such that $|f(q) - f(s)| < \epsilon$ for each $s \in B(S, q, t_q)$. The covering $\Upsilon_{S,\epsilon} = \{B(S, q, t_q) : q \in S\}$ has the disjoint subcovering $\Psi_{S,\epsilon} = \{H_i = B(S, q_i, t_{q_i}) : i \in \lambda_{f,S,\epsilon}\}$, where $\lambda_{f,S,\epsilon}$ is a set. Hence, $f = \sum_{i \in \lambda_{f,S,\epsilon}} \chi_{H_i} f$. Therefore, $v_\epsilon = \sum_{i \in \lambda_{f,S,\epsilon}} \chi_{H_i} g_{S\setminus H_i, H_i, \epsilon}$ belongs to $\mathcal{A}$, since $\mathcal{A}$ is the F-algebra satisfying Condition (iii) of this theorem. From the construction above and the strong triangle inequality, we deduce that $|f(s)e - v_\epsilon(s)| < \epsilon$ for each $s \in S$. Taking $\lim_{\epsilon \to 0}$ and using $\mathcal{A}$ as the Banach algebra, we obtain the assertion of this theorem. □

Theorem 5. *Let $\mathcal{A} = \mathcal{A}(S, \{A_s\})$ be the algebra over the field F such that the following is the case:*

(i) S is the ultrametric space;

(ii) For each q in S, x_q in A, there exists $x \in \mathcal{A}$ such that $x(q) = x_q$;

(iii) $\sum_{j \in \lambda} \chi_{W_j} x_j \in \mathcal{A}$ for each disjoint clopen covering $\{W_j : j \in \lambda\}$ of S and each $x_j \in \mathcal{A}$ with $\sup_{j \in \lambda} |x_j| < \infty$, where λ is a set. Then, $\mathcal{A}$ is invariant relative to multiplication on each f in $C_b(S, F)$.

Proof. If $y = 0$, then evidently $fy = 0 \in \mathcal{A}$ for each $f \in C_b(S, F)$. Let $y \in \mathcal{A}$ be a nonzero element and $f \in C_b(S, F)$, and let $0 < \epsilon < |y|$. Since $y \in \mathcal{A}$ and $f \in C_b(S, F)$, then $fy \in Ba(S, \{A_s\})$ and $|fy| \leq |f||y|$. For each $q \in S$, there exists $z_q \in \mathcal{A}$ such that $z_q(q) = f(q)y(q)$ by Condition (ii), since A_q is the F-algebra. By the continuity of $|z_q(s) - f(s)y(s)|$, there exists the clopen ball $B(S, q, t_q)$ in S such that $|f(s)y(s) - z_q(s)| < \epsilon \max(1, |f(q)||y(q)|)$ for each $s \in B(S, q, t_q)$. The covering $\Upsilon_\epsilon = \{B(S, q, t_q) : q \in S\}$ has the disjoint subcovering $\Psi_\epsilon = \{W_j = B(S, q_j, t_{q_j}) : j \in \lambda_\epsilon\}$, where λ_ϵ is a set. Then, $v_\epsilon = \sum_{j \in \lambda_\epsilon} \chi_{W_j} z_{q_j} \in \mathcal{A}$ by Condition (iii). From the construction above, we deduce that $|v_\epsilon(s) - f(s)y(s)| < \epsilon$ for each $s \in S$. Taking $\lim_{\epsilon \to 0}$ and using $\mathcal{A}$ as the Banach algebra, we obtain $fy \in \mathcal{A}$. □

Corollary 4. *Let $\mathcal{A} = \mathcal{A}(S, A)$ be the algebra over field F such that $\mathcal{A} \subseteq C_b(S, A)$ and the following is the case:*

(i) S is the ultrametric space;

(ii) Either A contains the unit element and $\mathcal{A}$ satisfies Condition (ii) in Theorem 4 or $\mathcal{A}$ satisfies Condition (ii) in Theorem 5;

(iii) $\mathcal{A}$ satisfies Condition (iii) in Theorem 4, where A is the Banach algebra over F. Then, $\mathcal{A} = C_b(S, A)$.

This corollary follows from Theorems 4 and 5.

Remark 2. *Theorems 2–5 and Corollaries 3 and 4 are also accomplished for (i) a zero-dimensional Lindelöf space S or (ii) a locally compact zero-dimensional space S with $B_\infty(S, \{A_s\})$, $C_\infty(S, F)$, $C_\infty(S, A)$ instead of $Ba(S, \{A_s\})$, $C_b(S, F)$, $C_b(S, A)$ correspondingly.*

3. Conclusions

The results of this article can be used for further studies of normed algebras and operator algebras on Banach spaces, the spectral theory of operators, PDEs, integral equations, the representation theory of groups, algebraic geometry, and applications in the sciences, including mathematical physics, gauge theory, quantum field theory, informatics, and mathematical geology, because they are based on normed algebras and vector-valued functions with values in normed algebras [3,5,8,15,18,20,21,25–31,33,34]. Indeed, the spectrum of Banach algebra A over normed field F is generally contained in the extension, G, of the initial normed field by Theorem 2 in [11]. This extension, in its turn, is the normed field. If set S is contained in G, then S is ultrametrizable, since norm on G is non-Archimedean. Therefore algebras $Ba(S, \{A_s\})$ are related with representations of operators using their spectra.

Evidently, the considered above exposition in this article encompasses a particular case of the algebras of compact operators. This also can be useful for some integral operators or for finding subalgebras in algebras appearing in the representation theory of algebras or groups and in the approximation of operator spectra.

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Appendix A

In order to avoid misunderstandings, we recall necessary definitions from previous articles [10,12,13]. If the readers are familiar with them, they can skip this appendix.

Remark A1. *Let F be an infinite field supplied with a multiplicative non-trivial non-Archimedean norm (i.e., ultranorm) $|\cdot|_F$ relative to which it is complete so that F is non-discrete and also $|x+y|_F \le \max(|x|_F, |y|_F)$ and $|xy|_F = |x|_F|y|_F$ for each x and y in F.*

Usually, a commutative field is called a field, while a noncommutative field is called a skew field or a division algebra.

A metric ρ on a metrizable space S is an ultrametric if it satisfies the strong triangle inequality $\rho(x,y) \le \max(\rho(x,z), \rho(y,z))$ for every x, y, and z in S. Notice that each ultrametric space is topologically zero-dimensional [18,32].

Remark A2. *$c_0(\alpha, F)$ is denoted as a Banach space consisting of all vectors $x = (x_j : \forall j \in \alpha\ x_j \in F)$ satisfying condition*

$card\{j \in \alpha : |x_j| > \epsilon\} < \aleph_0$ for each $\epsilon > 0$

and furnished with the norm

$(A1)$ $|x| = \sup_{j\in\alpha} |x_j|$,

where α is a set. For normed spaces X and Y, the linear space $L(X, Y)$ of all linear continuous operators $A : X \to Y$ is supplied with the operator norm

$(A2)$ $|A| := \sup_{x\in X\setminus\{0\}} |Ax|/|x|$.

Speaking about Banach spaces and Banach algebras, we stress that a field over which it is defined is norm complete.

If $X = c_0(\alpha, F)$, then to each $A \in L(X, X)$ an infinite matrix $(A_{i,j} : i \in \alpha, j \in \alpha)$ corresponds in the standard basis $\{e_j : j \in \alpha\}$ of X, where

$(A3)$ $x = \sum_j x_j e_j$

for each $x \in X = c_0(\alpha, F)$.

For a subalgebra V of $L(X, X)$, operation $B \mapsto B^t$ from V into $L(X, X)$ will be called a transposition operation if it is induced by that of its infinite matrix such that $(aA + bB)^t = aA^t + bB^t$ and $(AB)^t = B^tA^t$ and $(A^t)^t = A$ for every A and B in V and a and b in F; that is, $(A^t)_{i,j} = A_{j,i}$ for each i and j in α. Then, $V^t := \{A : A = B^t, B \in V\}$.

An operator A in $L(X, X)$ is called symmetric if $A^t = A$.

$L_0(X, X)$ is denoted as the family of all continuous linear operators $U : X \to X$ matrices $(U_{i,j} : i \in \alpha, j \in \alpha)$, of which all fulfill the conditions

$(A4)$ $\forall i\ \exists\ \lim_j U_{j,i} = 0$ and $\forall j\ \exists\ \lim_i U_{j,i} = 0$.

For an algebra A over F, it is supposed that a norm $|\cdot|_A$ on A satisfies the following conditions:

$|a|_A \ge 0$ for each $a \in A$, also;

$|a|_A = 0$ if and only if $a = 0$ in A;

$|ta|_A = |t|_F|a|_A$ for each $a \in A$ and $t \in F$;

$|a + b|_A \le \max(|a|_A, |b|_A)$ and;

$|ab|_A \leq |a|_A |b|_A$ for each a and b in A.

In short, it also will be written $|\cdot|$ instead of $|\cdot|_F$ or $|\cdot|_A$.

Definition A1. *Suppose that F is an infinite field with a nontrivial non-Archimedean norm such that F is norm complete, of the characteristic $char(F) \neq 2$ and $B_2 = B_2(F)$ is the commutative associative algebra with one generator i_1 such that $i_1^2 = -1$ and with the involution $(vi_1)^* = -vi_1$ for each $v \in F$. Let A be a subalgebra in $L(X, X)$ such that A is also a two-sided B_2-module, where $X = c_0(\alpha, F)$ is the Banach space over F, and α is a set. We say that A is an $*$-algebra if there is a continuous bijective (i.e injective and surjective) F-linear operator $\mathcal{I} : A \to A$ such that*

$(A5)$ $\mathcal{I}(ab) = (\mathcal{I}b)(\mathcal{I}a)$

$(A6)$ $\mathcal{I}(ga) = (\mathcal{I}a)g^*$ *and* $\mathcal{I}(ag) = g^*(\mathcal{I}a)$

$(A7)$ $\mathcal{I}\mathcal{I}a = a$

$(A8)$ $(\theta(y))(ax) = (\theta((\mathcal{I}a)y))(x)$

for every a and b in A and $g \in B_2$ and x and y in X, where $\theta : X \hookrightarrow X'$ is the canonical embedding of X into the topological dual space X' so that $\theta(y)x = \sum_{j \in \alpha} y_j x_j$. In summary, we can write a^* instead of $\mathcal{I}a$. The mapping $\mathcal{I}$ is what we call the involution. An element $a \in A$ is called self-adjoint if $a = a^*$.

Definition A2. *An algebra A is called an annihilator algebra if conditions $(A9)$–$(A11)$ are fulfilled:*

$(A9)$ $A_l(A) = A_r(A) = 0$ *and*

$(A10)$ $A_l(J_r) \neq 0$ *and*

$(A11)$ $A_r(J_l) \neq 0$

for all closed right J_r and left J_l ideals in A.

If for all closed (proper or improper) left J_l and right J_r ideals in A

$(A12)$ $A_l(A_r(J_l)) = J_l$ *and*

$(A13)$ $A_r(A_l(J_r)) = J_r$

then A is called a dual algebra.

If A is an $*$-algebra and for each $x \in A$ elements, $a \in A$ and $a_1 \in A$ exist such that a norm on A for these elements satisfies the following conditions:

$(A14)$ $|axx^*a_1^*| = |x|^2$ *and* $|a||a_1^*| \leq 1$,

then the algebra A is called finely regular.

Definition A3. *Let A be an normed algebra over field F satisfying the following conditions:*

$(A15)$ *A is a Banach $*$-algebra and*

$(A16)$ *There exists a bilinear functional $(\cdot, \cdot) : A^2 \to F$ such that $|(x, y)| \leq q|x||y|$ for all x and y in A, where $0 < q < \infty$ is a constant independent of x and y,*

$(A17)$ *$(x, y) = (y, x)$ and $(x, y) = (x^*, y^*)$ for each x and y in A,*

$(A18)$ *if $(x, y) = 0$ for each $y \in A$, then $x = 0$;*

$(A19)$ *$(xy, z) = (x, zy^*)$ for every x, y and z in A,*

$(A20)$ *$xx^* \neq 0$ for each nonzero element $x \in A \setminus (0)$.*

Then, we call A a B^*-algebra.

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