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Application of Various Price-Discount Policy for Deteriorated Products and Delay-in-Payments in an Advanced Inventory Model

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Abstract: In this proposed research, clear prospects of a real life marketing scenario, by analyzing a price discount policy and variable demand, are derived. The proposed study presents a production model along with time-dependent and selling price related demand for decaying items. Items deteriorate over time, therefore, considering deterioration in this model makes it more acceptable to the present marketing situation. The concept of delay-in-payments is utilized in this inventory system. In this research, a retailer buys some products, enjoys constant credit-period offers which are provided by the supplier. This model depicts a price discount strategy which is based on purchasing cost to attract more consumers in any business industry. By using this strategy, any manufacturer or business may gain more profit in comparison to methods suggested by earlier literature. The average profit function of the inventory system is maximized analytically and also finds the selling-price per unit and duration of the inventory cycle optimally. A numerical example, along with a case study and their graphical representations, are incorporated to verify the optimality of this research very clearly. The findings of this research have maximized the average profit function more than the existing literature.

Keywords: variable demand; credit period; price discount; production; deterioration

1. Introduction

Deterioration of items is one of the major problems to be considered in the manufacturing industry. In a practical life situation, it is too hard to maintain the deterioration of products such as fruits and vegetables. These types of products, which deteriorate over time, are not in good enough condition to fulfill the customers demand. Therefore, the effect of deterioration cannot be disregarded in production lot-size. Dave and Pandya [1] presented two separate inventory models, that is, one was an infinite finite horizon model, the other was a finite horizon model with deteriorated items. Bhunia et al. [2] discovered an inventory system for decayed items from two different warehouses with preserving technologies. Partially backlogged shortages were considered as waiting time related. Different realistic cases, sub cases, and scenarios corresponding problems were considered as non-linear constrained optimization problems. By mentioning the rate of deterioration as controllable variables, an inventory system in a novel freshness-preservation effort (FPE) indicator was demonstrated by Yang et al. [3]. They observed the optimal values of the FPE indicator and with that the optimal order quantity was also examined. After that, Khakzad and Gholamian [4] obtained an inventory system for decaying

products regarding the impact of inspection times. They mentioned the inspection times throughout the replenishment duration. They also assumed that the supplier was going to inflict some payments in an advance policy on the retailer. Their study depicted that the average decaying rate was related to the inspections number during every period.

Another important factor in an inventory system is the time. Whenever any business launches their branded new product, it is observed that the demand graph of that item is not always continuous over time. The demand graph of that product is constant within a couple of times but not always. After that, as time passes, demand for that product may increase or may gradually decrease. Due to the ups and downs nature of that demand graph, it is better to consider demand as time dependent. Hsu and Li [5] surveyed a mathematical programming model which was non-linear for optimizing the delivery process with time related demand. Their results proved that discriminating service policy was a more beneficial technique with respect to uniform strategy. Khanra et al. [6] examined an inventory system with quadratic demand with respect to time. The concept of delay-in-payments was used in their model. Shortages were considered after some variable time. In this direction of research, Wang et al. [7] showed a mixed-integer programming system while the demand was a function of time. Their model assumed the train capacity. They minimized the passenger total waiting duration along with the amount of passengers unable to transfer. Their study considered Genetic Algorithm (GA) and Grey Wolf Optimizer (GWO) for determining the model. Afterwards, Chen et al. [8] represented a decaying inventory system for variable demand in a multi-period finite horizon. They described several demand patterns such as constant, time-dependent, selling price-dependent, and stock-dependent.

Pricing is a major factor in the success of a business. Like time, demand of products may be affected by the selling price. Selling price plays a vital role in every business association. As much as the selling price of an item decreases, the demand graph of that product increases. For the other case, where the selling price of any products gradually increases, the demand graph becomes low. Therefore, selling price has significant value in an inventory system. Krugon and Nagaraju [9] established a two-echelon inventory model in which non-linear price dependent demand was assumed along with the credit period. A sustainable integrated inventory system was formulated by Dey et al. [10] to reduce discrete setup cost for price dependent demand. In their model, poisson distributed lead time demand was considered. Modak and Kelle [11] established a dual-channel inventory model for price and variable demand. Later, a decentralized channel optimization model was developed by Gholami et al. [12]. They modeled their study by taking into account that demand was measured as a function of pricing history. There were two price-adjusting factors in a bi-level (Stackelberg) model in which uncertain demand patterns were described for perishable items. In their stochastic demand model, to describe an uncertain demand pattern, they highlighted price-dependent memory functions. By focusing the demand as selling price related, Torkaman et al. [13] discovered a production-routing model with the classification of various demand functions. Demand was assumed as a continuous diminishing function of selling price. Their study consists of two Outer Approximation (OA) grounded algorithms.

Every e-commerce business industry or any other industrial enterprise is always concerned about their profit level. They often provide some discount policy to attract more consumers as much as they can. For this phenomenon, customers are much more willing to buy products from marketing companies. As a result, those companies are able to achieve higher profit value. Alfares and Ghaithan [14] developed an inventory system by assuming demand rate which was variable and holding cost that was time-dependent. Buratto [15] studied a supply chain in which two mechanisms were discussed with a cooperative advertising policy and another was a price discount technique. Jadidi et al. [16] designed a one period inventory model where a supplier offered some discount policies to the buyer. They also added demand of products which were stochastic and selling price-dependent. Li et al. [17] surveyed two-period models for obtaining a platform's price-discount policy. Their model mainly consists of three several online coupon redemption policies. These policies were—the first one, described as a platform, provides one short-term coupon in period 1 and customers can only redeem that coupon in period 1. For the second coupon policy, customers can only redeem the coupon in

period 2. For the third coupon policy, the platform recommended two coupons, that is, a short-term as well as a long-term coupon in period 1, and customers can redeem only one coupon in every period. Further, Qiu and Lee [18] designed a rail transport pricing model with some quantity discount policy for a dry port system. In their model, they enhanced the dry port's profit by utilizing the concept of a single breakpoint pricing strategy. They also considered the Stackelberg game approach to solve their model under the single breakpoint pricing scheme.

The trade-credit period is generally defined as the constant time-period provided by the supplier to retailers for adjusting their payment. In that time-period, retailers are not charged to pay any type of interest. After that particular time-period, some interest is charged to the retailer along with their due payment. In addition, the retailer allows their consumer a trade-credit duration which is partial. Interest is charged to consumers if they are not able to settle the purchasing amount. For this reason, the retailer can postpone their due payment until the last moment of a given duration. Hence, the retailer can gain more profits. Li et al. [19] generated an inventory model where the retailer's due payments was allowed by the supplier. They incorporated a corresponding inventory game for delay-in-payments, and showed that its core was nonempty. Their model generated a core allocation rule that can be reached through a population monotonic allocation scheme. Zou and Tian [20] formulated a supply chain model for developing a trade-credit matter. They also depicted some payment and ordering techniques. Aljazzar et al. [21] derived a supply chain model by designing the idea of delays-in-payments. Their model presented environmental as well as economic execution of a supply chain model. They also described the transportation cost along with the carbon emissions cost. Optimizing results in their model showed that adopting delay-in-payments policy improved both environmental as well as the economic execution of the supply chain model. Later, Jory et al. [22] examined the connection between trade-credit and its significance value for U.S. public firms with government economic policy uncertainty (EPU). They portrayed the effect of how the trade-credit strategy changed on firm value. They also observed that fastening trade-credit throughout periods of high economic policy uncertainty enhanced shareholder value only to a certain level.

It is observed that each of the above parameters has its own impact in any manufacturing companies for enhancing their profit value. Therefore, it is necessary to incorporate each of those parameters in any advanced inventory model. This model is established with the goal of improving the benefit level of production companies. For this phenomenon, a production model is developed in which the average profit function of supplier-retailer inventory system is to be maximized. This study extended earlier research by inserting some price discount technique, variable demand, consideration of deterioration of products, trade-credit policy, and finite replenishment rate. This model determines a practical approach to raise a higher profit level in comparison to earlier research studies. The obtained result in this proposed study would help to improve the profit level in every business organization.

Contribution of this Research

This research is mainly formulated on the basis of some factors. These factors are as follows—the first factor is to increase the growth of the profitability level of any business organization. To fulfill this factor of this research, some essential policies are inserted that are defined as delay-in-payments technique and a price-discount policy. The thought behind this idea of the delay-in-payments technique is that by using this policy any industry sector will be benefitted by earning some interest. That price-discount policy is primarily included in this research to attract the attention of consumers.

The second factor is to develop this model more acceptable in the present scenario by incorporating the realistic assumptions, that is, deterioration while considering any products with their shelf life. Because it is obvious that none of the products remain in a good condition over the whole time within their shelf life. These products deteriorate during the changes of time interval. For this reason, this research did not neglect the important assumption of deterioration. The consideration of deterioration constructed in this study is even more fruitful in comparison to the other research that does not include deterioration.

The third factor is to extend this research by adding some other realistic parameters which are variable demand, trade-credit strategy, and finite replenishment rate.

By analyzing all these key parameters, this study built a better profit level compared with previous research. With the help of these factors, one by one, this research contributes to any marketing agencies to achieve their goal of enriching the optimum values of the profit function.

2. Literature Review

Products deteriorate over time. Therefore, the quality of deteriorated products decrease automatically. Hence, these products are not in a good condition to sell in the market. For this matter, companies are not able to fulfil customer demand. Thus, the affect of deterioration cannot be discarded in production factories. Heng et al. [23] derived the effect of decay in an inventory model. Skouri and Papachristos [24] depicted an inventory model by highlighting deterioration, variable shortage, and opportunity cost. Further, Sarkar et al. [25] described a study regarding two ways trade-credit policy. They mentioned that deterioration was measured as time dependent in their model. Later, Sarkar et al. [26] minimized the total setup cost and raised the item's quality for deteriorating products. Iqbal and Sarkar [27] incorporated preservation technology for deteriorated items in an integrated inventory model.

In traditional studies, the demand rate was measured as a constant over whole time. That assumption for constant demand is not true as the demand of any product may fluctuate over time. Some notable research models about linear-trend demand were created by many researchers. Goswami and Chaudhuri [28] explained the inventory replenishment technology for a deteriorating item. By introducing the concept of exponentially deteriorating items, Hariga [29] proposed two distinct efficient solution procedures to develop optimal replenishment schedules for decayed items whose lifetime was fixed and variable demand. Li [30] studied the delivery method of a distributor by spotlighting carbon-emissions and retailers' variable demand. Zhao et al. [31] described a multi layered stages inventory system with the fact that was the time-dependent demand.

Most of the products are available in the market, where the rate of demand of products and selling-price are closely related. Whenever the price of any product reduces, customers are more affective to that product, that is, demand of that product increases directly. Avinadav et al. [32] discussed a study related to perishable products in an inventory system for several demand patterns. Sarkar and Saren [33] obtained the ordering and transferring policy with several demand functions for deteriorating items. These items were transferred by using a multi-delivery policy with equal delivery lot-size. Demand was taken to be as time, time-price, price and exponentially time dependent. Sarkar et al. [34] depicted the concept that demand was credit period related and selling price related. They extended their research by adding time dependent deteriorating items. By considering demand as quality dependent and selling price, Dey et al. [35] incorporated the automation policy to classify defective products. Khanna et al. [36] generated the inventory model with defective products and selling price-dependent demand.

In general, several companies provide some price-discount to consumers if they purchase a bulk size of inventories. For this reason, companies can gain more demand of items and increase their profit level. At the same time, for selling a large amount of inventories, companies can minimize the holding cost of stocks. Xu et al. [37] represented the effect of price discount policy and trade-ins. A "low" substitutability level in their model stated that trade-ins were lower. Sheehan et al. [38] highlighted the impact of the price discount's magnitude on customer purchase intentions at the time of online shopping. Hota et al. [39] described a supply chain with the impact of unreliable players, backorder price discounts, and service-dependent demand. They also considered an investment function for reducing the lead time crashing cost (LTCC).

Earlier, it was assumed that retailers pay instantly for the products they purchase from a supplier. Nowadays, the supplier provides a constant time to retailer for adjusting the remaining purchasing cost. This constant period of time is known as the trade-credit period. Nowadays, retailers also offer a

trade-credit procedure to consumers for gaining maximum profit. The view-point of credit-connected demand and trade-credit method was introduced by Jaggi et al. [40] to reflect the present-life situations. Sarkar et al. [41] considered the business policy in which suppliers provide a credit-period to customers. By inserting a cash flow technique, Mishra et al. [42] included a deteriorating inventory system with variable demand. The trade-credit technique on re-manufactured products was depicted along with time value for money. Later, Taleizadeh et al. [43] formulated a manufacturing model with defective quantities and a delayed payment system. They developed this manufacturing system to produce multi-products. See Table 1 for various research knowledge from distinct authors.

Table 1. Contribution Table of distinct Author(s).

Author(s)	Demand (Price and Time Related)	Discount Policy	Trade- Credit Policy	Deterioration
Dey et al. [10]	Price related	-	-	-
Alfares and Ghaithan [14]	Price related	Quantity discount	-	-
Li et al. [19]	-	-	One level	-
Avinadav et al. [32]	Price and time related	-	-	Constant
Xu et al. [37]	-	Discount in price	-	-
Jaggi et al. [40]	-	-	Two level	-
Sarkar et al. [44]	Time related	-	-	-
Sarkar and Sarkar [45]	-	-	-	Variable
Teng and Chang [46]	Price related	-	-	Constant
Wu et al. [47]	-	-	-	Variable
This model	Price and time related	Discount in price	One level	Constant

“-” means this concept does not exist in that study.

In this study, the authors extended the research of Sana and Chaudhuri [48] for deteriorated items with different functions of demand and the rate of replenishment being finite. Demand for commodities is a function of price and time. The supplier allows a constant duration of time for the retailer to pay the buying amount. In addition, the supplier provides a price-discount strategy for the buying amount to the retailer. The profit function of the retailer is maximized for a finite production rate by determining the selling-price per unit and the duration of the inventory cycle. Section 3 elucidates the mathematical model of this study. Section 4 portrays the solution methodology of this study. In Section 5, a numerical example along with a case study are designed to verify the optimality of this research clearly. In Section 6, sensitivity analysis of parameters with proper discussions are given. Section 7 depicts the managerial insights of this research. In Section 8, some concluding remarks as well as future directions are presented.

3. Mathematical Model

In this model, the same assumptions as in Sana and Chaudhuri [48] are considered except for the demand, deterioration, and finite production rate.

1. The proposed inventory system is basically developed for considering a single-type of products (Dey et al. [10], Sarkar and Saren [33]).
2. Deterioration of the product is incorporated in this model. Products deteriorate over time. In view of a real life situation, this model assumes deterioration of the product (Skouri and Papachristos [24], Sarkar and Sarkar [45], Shaw et al. [49], Daryanto et al. [50], Mahapatra et al. [51]).
3. In this study, the demand rate is variable. This model depicts that the demand rate of any product is affected by several parameters, for instance, time and selling price. The demand rate of products

may change regarding time (Hsu and Li [5], Sarkar et al. [45]). In addition, the product's demand rate generally enhances if the selling price of that product diminishes (Krugon and Nagaraju [9], Khanna et al. [36], Sarkar et al. [52]). For this reason, the demand of products is measured as time and price dependent. This model considers that demand increases quadratically with time and decreases linearly with selling-price.

4. Delay-in-payments is highlighted in this model. Instead of allowing a single credit period, supplier permits different credit periods to the retailer for adjusting due payment (Mishra et al. [42]).
5. Several price discount policies on purchasing cost are assumed in this model. As the supplier provides three credit periods to the retailer, the supplier offers a distinct price discount to the retailer in each credit period. In the first credit period, the supplier offers the biggest price discount on the purchasing cost. After that, in the second credit period the supplier offers a lesser price discount to the retailer. In the third and final credit period, no price discount is allotted for the retailer. The supplier generally utilizes this price discount technique to attract more retailers to earn maximum profit (Xu et al. [37], Sheehan et al. [38], Sana and Chaudhuri [48]).
6. The production rate is finite with an infinite time horizon. Shortage and backlogging are not permitted. This model considers no lead time.

The inventory cycle starts from zero. The production rate μ continues until time t_1 and at $t = T$ to adjust the market's demand. For adjusting the due amount, the supplier offers several credit-periods K_i , when, $i = 1, 2, 3$ to the retailer. The buying cost at several credit-durations is

$$C_P = \left\{ \begin{array}{ll} C_M(1 - \delta_1) & \text{while } K = K_1 \\ C_M(1 - \delta_2) & \text{when } K = K_2 \\ C_M(1 - \delta_3) & \text{when } K = K_3 \end{array} \right\},$$

where K_i 's are the i th allowable delay-period at while retailer's discount rate is δ_i .

Considering this technique, there are two cases:

Case 1 $T \geq K$

(The above condition is briefly shown in Figure 1).

Figure 1. Pictorial illustration of inventory system while $T \geq K$.

Case 2 $T \leq K$

(See Figure 2).

Then the carrying cost admitting charges of interest are

$$\begin{aligned}
 &= C_H \left[\int_0^{t_1} I_1(t) dt + \int_{t_1}^T I_2(t) dt \right] \\
 &= C_H \left[X_1 \left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta} \right) - t_1^2 \left(\frac{b_1}{2\theta} - \frac{c_1}{\theta^2} \right) - \frac{c_1 t_1^3}{3\theta} + \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) (X_2 + Y_1 T + Y_2 T^2) \right. \\
 &\quad \left. - X_2(T - t_1) - Y_1 \frac{(T^2 - t_1^2)}{2} - Y_2 \frac{(T^3 - t_1^3)}{3} \right].
 \end{aligned}$$

The profit earns credit-balance during delayed time interval $[0, K]$ is

$$\begin{aligned}
 &= U_c P \int_0^K (K - t) D(P, t) dt \\
 &= U_c P \left((a_1 - hP) \frac{K^2}{2} + b_1 \frac{K^3}{6} + c_1 \frac{K^4}{12} \right).
 \end{aligned}$$

The charge of interest for financing inventory throughout $[K, T]$ is

$$\begin{aligned}
 &= U_f C_P \int_K^T I_2(t) dt \\
 &= U_f C_P \left[(X_2 + Y_1 T + Y_2 T^2) \left(\frac{e^{\theta(T-K)} - 1}{\theta} \right) - X_2(T - K) - Y_1 \frac{(T^2 - K^2)}{2} - Y_2 \frac{(T^3 - K^3)}{3} \right].
 \end{aligned}$$

Therefore, the total profit is

$$\begin{aligned}
 P_{1i} &= [(P - C_P)\mu t_1 + U_c P \{ \int_0^{K_i} (K_i - t) D(P, t) dt \} - C_H \{ \int_0^{t_1} I_1(t) dt \\
 &\quad + \int_{t_1}^T I_2(t) dt \} - U_f C_P \int_{K_i}^T I_2(t) dt - C_1], \text{ while } i \in \{1, 2, 3\}.
 \end{aligned} \tag{1}$$

From Equation (1), the average profit for Case 1 is

$$\begin{aligned}
 AVP_{1i} &= \frac{P_{1i}}{T} \\
 &= \frac{1}{T} [U_c P \{ \int_0^{K_i} (K_i - t) D(P, t) dt \} + (P - C_P)\mu t_1 \\
 &\quad - C_H \{ \int_0^{t_1} I_1(t) dt + \int_{t_1}^T I_2(t) dt \} - U_f C_P \int_{K_i}^T I_2(t) dt - C_1], \text{ for } i \in \{1, 2, 3\}. \\
 &= \frac{(P - C_P)\mu t_1}{T} + \frac{PU_c}{T} \left[(a_1 - hP) \frac{K_i^2}{2} + b_1 \frac{K_i^3}{6} + c_1 \frac{K_i^4}{12} \right] - \frac{C_H}{T} \left(X_1 \left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta} \right) \right. \\
 &\quad \left. - \frac{b_1 t_1^2}{2\theta} - \frac{c_1 t_1^3}{3\theta} + \frac{c_1 t_1^2}{\theta^2} \right) - \frac{C_H}{T} \left[\left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) (X_2 + Y_1 T + Y_2 T^2) - X_2(T - t_1) \right. \\
 &\quad \left. - Y_1 \frac{(T^2 - t_1^2)}{2} - Y_2 \frac{(T^3 - t_1^3)}{3} \right] - \frac{U_f C_P}{T} \left((X_2 + Y_1 T + Y_2 T^2) \left(\frac{e^{\theta(T-K_i)} - 1}{\theta} \right) \right. \\
 &\quad \left. - X_2(T - K_i) - Y_1 \frac{(T^2 - K_i^2)}{2} - Y_2 \frac{(T^3 - K_i^3)}{3} \right) - \frac{C_1}{T}, \text{ where } i \in \{1, 2, 3\}.
 \end{aligned} \tag{2}$$

Case 2 When $(T \leq K)$ The carrying cost omitting interest rates is

$$\begin{aligned}
 &= C_H \left[\int_0^{t_1} I_1(t) dt + \int_{t_1}^T I_2(t) dt \right] \\
 &= C_H \left[X_1 \left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta} \right) - t_1^2 \left(\frac{b_1}{2\theta} - \frac{c_1}{\theta^2} \right) - \frac{c_1 t_1^3}{3\theta} + \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) (X_2 + Y_1 T \right. \\
 &\quad \left. + Y_2 T^2) - X_2(T - t_1) - Y_1 \frac{(T^2 - t_1^2)}{2} - Y_2 \frac{(T^3 - t_1^3)}{3} \right].
 \end{aligned}$$

The profit gains throughout the delay-period $[0, K]$ is

$$\begin{aligned}
 &= U_c P \left[\int_0^T (T - t) D(P, t) dt + \mu t_1 (K - T) \right] \\
 &= U_c P \left[(a_1 - hP) \frac{T^2}{2} + b_1 \frac{T^3}{6} + c_1 \frac{T^4}{12} + \mu t_1 (K_i - T) \right].
 \end{aligned}$$

Thus, the total profit is

$$\begin{aligned}
 P_{2i} &= [(P - C_P) \mu t_1 + U_c P \{ \int_0^T (T - t) D(P, t) dt + \mu t_1 (K - T) \} - C_H \{ \int_0^{t_1} I_1(t) dt \\
 &\quad + \int_{t_1}^T I_2(t) dt \} - C_1], \text{ while } i \in \{1, 2, 3\}.
 \end{aligned} \tag{3}$$

From Equation (3), the average profit for Case 2 is

$$\begin{aligned}
 AVP_{2i} &= \frac{P_{2i}}{T} \\
 &= \frac{1}{T} [U_c P \{ \int_0^T (T - t) D(P, t) dt + \mu t_1 (K_i - T) \} + (P - C_P) \mu t_1 - C_H \{ \int_0^{t_1} I_1(t) dt \\
 &\quad + \int_{t_1}^T I_2(t) dt \} - C_1], \text{ for } i \in \{1, 2, 3\}. \\
 &= \frac{(P - C_P) \mu t_1}{T} + \frac{P U_c}{T} \left[(a_1 - hP) \frac{T^2}{2} + b_1 \frac{T^3}{6} + c_1 \frac{T^4}{12} + \mu t_1 (K_i - T) \right] - \frac{C_H}{T} \left[X_1 \left(t_1 \right. \right. \\
 &\quad \left. \left. + \frac{e^{-\theta t_1} - 1}{\theta} \right) - t_1^2 \left(\frac{b_1}{2\theta} - \frac{c_1}{\theta^2} \right) - \frac{c_1 t_1^3}{3\theta} + (X_2 + Y_1 T + Y_2 T^2) \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) \right. \\
 &\quad \left. - X_2(T - t_1) - Y_1 \frac{(T^2 - t_1^2)}{2} - Y_2 \frac{(T^3 - t_1^3)}{3} \right] - \frac{C_1}{T}, \text{ for } i \in \{1, 2, 3\}.
 \end{aligned} \tag{4}$$

4. Solution Methodology

In this model, a classical optimization technique is applied to obtain the optimal values for the system's average profit AVP_{1i} and AVP_{2i} when $i \in \{1, 2, 3\}$. The optimality of system's average profit AVP_{1i} and AVP_{2i} are determined by the help of Hessian matrix. The concavity of AVP_{1i} and AVP_{2i} are obtained for inventory cycle's duration T and selling price P by assuming second order derivatives and by checking the globally maximum condition.

The Hessian matrix for AVP_{1i} is H as follows

$$H = \begin{bmatrix} \frac{\partial^2 AVP_{1i}}{\partial T^2} & \frac{\partial^2 AVP_{1i}}{\partial T \partial P} \\ \frac{\partial^2 AVP_{1i}}{\partial P \partial T} & \frac{\partial^2 AVP_{1i}}{\partial P^2} \end{bmatrix}.$$

The necessary condition of the optimal solution AVP_{1i} is the first order partial derivatives of AVP_{1i} with respect to the inventory cycle's duration T and selling price P are equating to zero. that is, $\frac{\partial AVP_{1i}}{\partial T} = 0$ and $\frac{\partial AVP_{1i}}{\partial P} = 0$.

Now,

$$\begin{aligned} \frac{\partial AVP_{1i}}{\partial T} &= -\frac{(P - C_P)\mu t_1}{T^2} - \frac{PU_c}{T^2} \left[(a_1 - hP)\frac{K_i^2}{2} + b_1\frac{K_i^3}{6} + c_1\frac{K_i^4}{12} \right] + \frac{C_H}{T^2} \left[X_1 \left(t_1 \right. \right. \\ &+ \left. \left. \frac{e^{-\theta t_1} - 1}{\theta} \right) - \frac{b_1 t_1^2}{2\theta} - \frac{c_1 t_1^3}{3\theta} + \frac{c_1 t_1^2}{\theta^2} \right] - C_H \left[\left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) \left(Y_2 - \frac{X_2}{T^2} \right) \right. \\ &+ \left. e^{\theta(T-t_1)} \left(\frac{X_2}{T} + Y_1 + Y_2 T \right) - \frac{1}{T^2} \left(X_2 t_1 + \frac{Y_1 t_1^2}{2} + \frac{Y_2 t_1^3}{3} \right) - \frac{Y_1}{2} - \frac{2Y_2 T}{3} \right] \\ &- U_f C_p e^{\theta(T-K_i)} \left(\frac{X_2}{T} + Y_1 + Y_2 T \right) - U_f C_p \left(\frac{e^{\theta(T-K_i)} - 1}{\theta} \right) \left(Y_2 - \frac{X_2}{T^2} \right) \\ &- X_2 - Y_1 T - Y_2 T^2 + \frac{c_1}{T^2} \\ &= 0. \end{aligned}$$

Now, T^* will be determined from $\frac{\partial AVP_{1i}}{\partial T} = 0$ that is, from $MT^4 + LT^3 + g(T) + J = 0$. where,

$$\begin{aligned} M &= -Y_2 \\ L &= -\left(Y_2 + \frac{2Y_2 C_H}{3} \right) \\ g(T) &= T^2 [U_f C_p \left[\frac{(e^{\theta(T-K_i)} - 1)}{\theta} \left(\frac{X_2}{T^2} - Y_2 \right) - \left(\frac{X_2}{T} + Y_1 + Y_2 T \right) e^{\theta(T-K_i)} \right] - C_H \left[\frac{(e^{\theta(T-t_1)} - 1)}{\theta} \left(Y_2 \right. \right. \\ &- \left. \left. \frac{X_2}{T^2} \right) + e^{\theta(T-t_1)} \left(\frac{X_2}{T} + Y_2 T + Y_1 C_H \right) \right] - X_2] \\ J &= C_H \left[X_1 \left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta} \right) - \frac{b_1 t_1^2}{2\theta} - \frac{c_1 t_1^3}{3\theta} + \frac{c_1 t_1^2}{\theta^2} \right] - (P - C_P)\mu t_1 - PU_c \left[(a_1 - hP)\frac{K_i^2}{2} \right. \\ &+ \left. b_1\frac{K_i^3}{6} + c_1\frac{K_i^4}{12} \right] + C_1 - X_2 t_1 - \frac{Y_1 t_1^2}{2} - \frac{Y_2 t_1^3}{3}. \end{aligned}$$

Similarly for second decision variable which is selling price P ,

$$\begin{aligned} \frac{\partial AVP_{1i}}{\partial P} &= \frac{\mu t_1}{T} + \frac{U_c K_i^2}{2T} \left(a_1 + \frac{b_1 K_i}{3} + \frac{c_1 K_i^2}{6} \right) - \frac{U_c h P K_i^2}{T} \\ &= 0. \end{aligned}$$

By obtaining $\frac{\partial AVP_{1i}}{\partial P} = 0$, P^* will be observed as

$$P^* = \frac{T}{U_c h K_i^2} \left[\frac{\mu t_1}{T} + \frac{U_c K_i^2}{2T} \left(a_1 + \frac{b_1 K_i}{3} + \frac{c_1 K_i^2}{6} \right) \right].$$

The sufficient consideration for the optimum solution of AVP_{1i} is two leading principal minors are alternative in sign, that is, if the leading principal minors $\frac{\partial^2 AVP_{1i}}{\partial T^2} < 0$ and $\frac{\partial^2 AVP_{1i}}{\partial T^2} \frac{\partial^2 AVP_{1i}}{\partial P^2} - \left(\frac{\partial^2 AVP_{1i}}{\partial P \partial T} \right)^2 > 0$ at the optimal point, then H is negative definite and the function AVP_{1i} is strictly concave.

For this case, the partial derivatives of average profit function AVP_{1i} are

$$\begin{aligned} \frac{\partial^2 AVP_{1i}}{\partial T^2} = & \frac{-2}{T^3} \left[C_1 + C_H X_1 \left(t_1 + \frac{e^{-\theta t_1}}{\theta} \right) - C_H \frac{b_1 t_1^2}{2\theta} - C_H \frac{c_1 t_1^3}{3\theta} + C_H \frac{c_1 t_1^2}{\theta^2} \right] + \frac{2C_H}{T^3} \left[X_2(T \right. \\ & - t_1) + \frac{Y_1(T^2 - t_1^2)}{2} + \frac{Y_2(T^3 - t_1^3)}{3} \left. \right] + \frac{(X_2 + Y_1 T + Y_2 T^2)}{T} \left[\frac{U_f C_P}{T} \left(\frac{e^{\theta(T-K_i)} - 1}{T\theta} \right. \right. \\ & - \left. \left. e^{\theta(T-K_i)} \right) + \frac{C_H}{T} e^{\theta(T-t_1)} - \frac{C_H}{T^2} \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) - C_H \theta e^{\theta(T-t_1)} - U_f C_P e^{\theta(T-K_i)} \theta \right] \\ & + \frac{2U_f C_P}{T^3} \left[X_2(T - K_i) - \frac{Y_1(T^2 - K_i^2)}{2} - \frac{Y_2(T^3 - K_i^3)}{3} \right] - \frac{(Y_1 + 2Y_2 T)}{T} \left[C_H e^{\theta(T-t_1)} \right. \\ & + \left. U_f C_P e^{\theta(T-K_i)} \right] - \frac{(X_2 + Y_1 T + Y_2 T^2)}{T^2} (C_H + U_f C_P) + \frac{Y_1}{T^2} \left[C_H \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) \right. \\ & + \left. U_f C_P \frac{e^{\theta(T-K_i)} - 1}{\theta} \right] \end{aligned}$$

$$\frac{\partial^2 AVP_{1i}}{\partial P^2} = -\frac{hK_i^2 U_c}{T},$$

and

$$\begin{aligned} \frac{\partial^2 AVP_{1i}}{\partial P \partial T} = & \frac{(hK_i^2 P U_c - \mu t_1)}{T^2} - \frac{U_c}{T^2} \left(\frac{a_1 K_i^2}{2} + \frac{b_1 K_i^3}{6} + \frac{c_1 K_i^4}{12} \right) + \frac{C_H h}{\theta T^2} \left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta} \right) \\ & + \frac{C_H h}{\theta} - \frac{U_f C_P h}{T^2 \theta} (1 + K_i). \end{aligned}$$

The Hessian matrix for AVP_{2i} is H as follows

$$H = \begin{bmatrix} \frac{\partial^2 AVP_{2i}}{\partial T^2} & \frac{\partial^2 AVP_{2i}}{\partial T \partial P} \\ \frac{\partial^2 AVP_{2i}}{\partial P \partial T} & \frac{\partial^2 AVP_{2i}}{\partial P^2} \end{bmatrix}.$$

The necessary condition of the optimal solution AVP_{2i} is the first order partial derivatives of AVP_{2i} with respect to inventory cycle's duration T and selling price P are equating to zero. that is, $\frac{\partial AVP_{2i}}{\partial T} = 0$ and $\frac{\partial AVP_{2i}}{\partial P} = 0$. Now,

$$\begin{aligned} \frac{\partial AVP_{2i}}{\partial T} = & -\frac{(P - C_P)\mu t_1}{T^2} + P U_c \left[\frac{(a_1 - hP)}{2} + \frac{b_1 T}{3} + \frac{c_1 T^2}{4} - \frac{\mu t_1 K_i}{T^2} \right] + \frac{C_H}{T^2} \left[X_1 \left(t_1 \right. \right. \\ & + \left. \left. \frac{e^{-\theta t_1} - 1}{\theta} \right) - t_1^2 \left(\frac{b_1}{2\theta} - \frac{c_1}{\theta^2} \right) - \frac{c_1 t_1^3}{3\theta} \right] + \frac{c_1}{T^2} + C_H \left[\frac{X_2 t_1}{T^2} + \frac{Y_1}{2} \left(1 + \frac{t_1^2}{T^2} \right) \right. \\ & + \left. \frac{Y_2}{3} \left(2T + \frac{t_1^3}{T^2} \right) - \left(Y_2 - \frac{X_2}{T^2} \right) \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) - \left(\frac{X_2}{T} + Y_1 + Y_2 T \right) e^{\theta(T-t_1)} \right] \\ = & 0. \end{aligned}$$

Now, T^* will be obtained from $\frac{\partial AVP_{2i}}{\partial T} = 0$ that is, from $GT^4 + VT^3 + f(T) + B = 0$. where,

$$\begin{aligned} G &= \frac{PU_c C_1}{4} \\ V &= \frac{PU_c b_1}{3} + \frac{2C_H Y_2}{3} \\ f(T) &= T^2 \left[\frac{C_H Y_2}{\theta} + \frac{PU_c(a_1 - hP)}{2} + \frac{C_H Y_1}{2} + \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) \left(\frac{C_H X_2}{T^2} - C_H Y_2 \right) - C_H \left(\frac{X_2}{T} \right. \right. \\ &\quad \left. \left. + Y_1 + Y_2 T \right) e^{\theta(T-t_1)} \right] \\ B &= \left[C_H \left(\left(X_1 \left(t_1 + \frac{e^{-\theta t_1} - 1}{\theta} \right) - t_1^2 \left(\frac{b_1}{2\theta} - \frac{c_1}{\theta^2} \right) - \frac{c_1 t_1^3}{3} \right) + X_2 t_1 + \frac{Y_1 t_1}{2} + \frac{Y_2 t_1^3}{3} \right. \right. \\ &\quad \left. \left. - \frac{X_2}{\theta} \right) - (P - C_P) \mu t_1 - PU_c \mu t_1 K_i + C_1 \right]. \end{aligned}$$

Now, for the second decision variable, that is, selling price P ,

$$\begin{aligned} \frac{\partial AVP_{2i}}{\partial P} &= \frac{\mu t_1}{T} + \frac{U_c T}{2} \left(a_1 + \frac{b_1 T}{3} + \frac{c_1 T^2}{6} \right) + \frac{U_c}{T} \mu t_1 (K_i - T) - U_c h P T \\ &= 0. \end{aligned}$$

By determining $\frac{\partial AVP_{2i}}{\partial P} = 0$, P^* will be found as

$$P^* = \frac{1}{U_c h T} \left[\frac{\mu t_1}{T} + \frac{U_c T}{2} \left(a_1 + \frac{b_1 T}{3} + \frac{c_1 T^2}{6} \right) + \frac{U_c}{T} \mu t_1 (K_i - T) \right].$$

Therefore, the sufficient condition for the optimum solution of AVP_{2i} are two leading principal minors are opposite in sign that means the leading principal minors $\frac{\partial^2 AVP_{2i}}{\partial T^2} < 0$ and $\frac{\partial^2 AVP_{2i}}{\partial T^2} \frac{\partial^2 AVP_{2i}}{\partial P^2} - \left(\frac{\partial^2 AVP_{2i}}{\partial P \partial T} \right)^2 > 0$ at the optimal point, then H is negative definite and the function AVP_{2i} is strictly concave.

For this case, partial derivatives of average profit function AVP_{2i} are

$$\begin{aligned} \frac{\partial^2 AVP_{2i}}{\partial T^2} &= \frac{2(P - C_P) \mu t_1}{T^3} + PU_c \left(\frac{b_1}{3} + \frac{c_1 T}{2} \right) - \frac{2c_1}{T^3} - \frac{2C_H X_1}{T^3} - \frac{C_H X_2}{T^2} + C_H Y_2 \left[1 \right. \\ &\quad \left. - e^{\theta(T-t_1)} + \frac{1}{T} \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right) \right] + C_H \left[\theta e^{\theta(T-t_1)} + \left(\frac{e^{\theta(T-t_1)}}{T} \right) \left(\frac{X_2}{T} + Y_1 \right. \right. \\ &\quad \left. \left. + Y_2 T \right) - \left(\frac{e^{\theta(T-t_1)} - 1}{\theta T^2} \right) \right] - \frac{C_H}{T} (Y_1 + 2Y_2 T) e^{\theta(T-t_1)} + \left[\frac{C_H}{T^2} (Y_1 + 2Y_2 T) \right. \\ &\quad \left. - \frac{2Y_2 C_H}{T} \right] \left(\frac{e^{\theta(T-t_1)} - 1}{\theta} \right), \end{aligned}$$

$$\frac{\partial^2 AVP_{2i}}{\partial P^2} = -U_c T h,$$

and

$$\frac{\partial^2 AVP_{2i}}{\partial P \partial T} = \frac{U_c a_1}{2} - \frac{\mu t_1}{T^2} - PU_c h + \frac{U_c b_1 T}{3} + \frac{c U_c T^2}{4}.$$

As the above mentioned average profit functions given by Equations (2) and (4) of the system are not linear equations and all the partial derivatives of second order of AVP_{1i} and AVP_{2i} while $i \in \{1, 2, 3\}$ for both P^* and T^* are extremely complicated. Hence the closed form solution cannot be determined. However, by means of empirical methodologies, one can observe that the above equations are concave for lower values of P^* and T^* . Due to highly non-linearity of the principal minors, the optimality of the analytical method cannot be shown. The optimality of the system's average profit AVP_{1i} and AVP_{2i} when $i \in \{1, 2, 3\}$ is demonstrated numerically in the numerical example section.

5. Numerical Example

By applying the best fitted numerical data from Sana and Chaudhuri [48] model, the average profit of the system, selling-price per unit, and duration of inventory cycle are calculated.

Example 1. Let $C_1 = \$10$ per order, $\theta = 0.2$, $C_H = \$0.5/\text{unit/month}$, $K_1 = 2$ months, $K_2 = 4$ months, $K_3 = 6$ months, $\delta_1 = 20\%$, $\delta_2 = 10\%$, $\delta_3 = 0\%$, $a_1 = 80$, $b_1 = 5$, $c_1 = 0.5$, $h = 1.71$, $C_M = \$120/\text{unit}$, $U_c = \frac{0.13}{12} / \$/\text{month}$, $U_f = \frac{0.16}{12} / \$/\text{month}$, and $\mu = 300$ units/month.

Then the optimal solutions are $\{AVP_{11} = \$319.101, T^* = 6.8$ months, and $P^* = \$90.11/\text{unit}\}$, $\{AVP_{12} = \$452.971, T^* = 7$ months, and $P^* = \$95.19/\text{unit}\}$, $\{AVP_{13} = \$670.85, T^* = 7.39$ months, and $P^* = \$100.93/\text{unit}\}$, $\{AVP_{21} = \$171.71, T^* = 5.2$ months, and $P^* = \$84.22/\text{unit}\}$, $\{AVP_{22} = \$361.11, T^* = 6$ months, and $P^* = \$91.5/\text{unit}\}$, $\{AVP_{23} = \$634.286, T^* = 6.67$ months, and $P^* = \$98.57/\text{unit}\}$.

Among the all optimal solutions, the better optimal solutions are $\{AVP_{13} = \$670.85, T^* = 7.39$ months, and $P^* = \$100.93/\text{unit}\}$. In Sana and Chaudhuri [48] model, the optimal solutions were $\{AVP_{11} = \$91.5398, T^* = 4.88$ months, and $P^* = \$49.57/\text{unit}\}$. If these two optimal solutions are compared to each other, then it is obvious that this proposed model maximized more profit from the inventory system in comparison to Sana and Chaudhuri [48] model (See Figure 3).

Figure 4 indicates the average profit AVP_{11} is globally maximum for $T = 6.8$ months, and $P = \$90.11/\text{unit}$. Figure 5 shows the average profit AVP_{12} is globally maximum for $T = 7$ months, and $P = \$95.19/\text{unit}$. Figure 6 provided the average profit AVP_{13} is globally maximum for $T = 7.39$ months, and $P = \$100.93/\text{unit}$. Figure 7 indicates the average profit AVP_{21} is globally maximum for $T = 5.2$ months, and $P = \$84.22/\text{unit}$. Figure 8 shows that the average profit AVP_{22} is globally maximum for $T = 6$ months, and $P = \$91.5/\text{unit}$. Figure 9 describes that the average profit AVP_{23} is globally maximum for $T = 6.67$ months, and $P = \$98.57/\text{unit}$.

For AVP_{11} , Hessian matrix H 's leading principal minors are $\frac{\partial^2 AVP_{11}}{\partial T^2} = -15.79 < 0$ and $\frac{\partial^2 AVP_{11}}{\partial T^2} \frac{\partial^2 AVP_{11}}{\partial P^2} - \left(\frac{\partial^2 AVP_{11}}{\partial P \partial T}\right)^2 = 18.47 > 0$ for optimal results of inventory cycle's duration T and selling-price P . Hence, Hessian matrix H for AVP_{11} is negative definite also AVP_{11} is strictly concave.

For AVP_{12} , Hessian matrix H 's leading principal minors are $\frac{\partial^2 AVP_{12}}{\partial T^2} = -16.48 < 0$ and $\frac{\partial^2 AVP_{12}}{\partial T^2} \frac{\partial^2 AVP_{12}}{\partial P^2} - \left(\frac{\partial^2 AVP_{12}}{\partial P \partial T}\right)^2 = 12.2 > 0$ for optimal results of inventory cycle's duration T and selling-price P . Hence, Hessian matrix H for AVP_{12} is negative definite also AVP_{12} is strictly concave.

For AVP_{13} , Hessian matrix H 's leading principal minors are $\frac{\partial^2 AVP_{13}}{\partial T^2} = -19.5 < 0$ and $\frac{\partial^2 AVP_{13}}{\partial T^2} \frac{\partial^2 AVP_{13}}{\partial P^2} - \left(\frac{\partial^2 AVP_{13}}{\partial P \partial T}\right)^2 = 4.5 > 0$ for optimal results of inventory cycle's duration T and selling-price P . Hence, Hessian matrix H for AVP_{13} is negative definite also AVP_{13} is strictly concave.

For AVP_{21} , Hessian matrix H 's leading principal minors are $\frac{\partial^2 AVP_{21}}{\partial T^2} = -7.32 < 0$ and $\frac{\partial^2 AVP_{21}}{\partial T^2} \frac{\partial^2 AVP_{21}}{\partial P^2} - \left(\frac{\partial^2 AVP_{21}}{\partial P \partial T}\right)^2 = 10.23 > 0$ for optimal results of inventory cycle's duration T and selling-price P . Hence, Hessian matrix H for AVP_{21} is negative definite also AVP_{21} is strictly concave.

For AVP_{22} , Hessian matrix H 's leading principal minors are $\frac{\partial^2 AVP_{22}}{\partial T^2} = -19.5 < 0$ and $\frac{\partial^2 AVP_{22}}{\partial T^2} \frac{\partial^2 AVP_{22}}{\partial P^2} - \left(\frac{\partial^2 AVP_{22}}{\partial P \partial T}\right)^2 = 4.5 > 0$ for optimal results of inventory cycle's duration T and

selling-price P . Hence, Hessian matrix H for AVP_{22} is negative definite also AVP_{22} is strictly concave.

For AVP_{23} , Hessian matrix H 's leading principal minors are $\frac{\partial^2 AVP_{23}}{\partial T^2} = -2.98 < 0$ and $\frac{\partial^2 AVP_{23}}{\partial T^2} \frac{\partial^2 AVP_{23}}{\partial P^2} - \left(\frac{\partial^2 AVP_{23}}{\partial P \partial T}\right)^2 = 9.43 > 0$ for optimal results of inventory cycle's duration T and selling-price P . Hence, Hessian matrix H for AVP_{23} is negative definite also AVP_{23} is strictly concave.

Figure 3. Comparison of the average profit function for this model and Sana and Chaudhuri [48] model.

Figure 4. Average profit AVP_{11} versus inventory cycle's duration (T) and selling price (P).

Figure 5. Average profit AVP_{12} versus inventory cycle's duration (T) and selling-price (P).

Figure 6. Average profit AVP_{13} versus inventory cycle's duration (T) and selling-price (P).

Figure 7. Average profit AVP_{21} against selling-price (P) and inventory cycle's duration (T).

Figure 8. Average profit AVP_{22} versus inventory cycle's duration (T) and selling-price (P).

Figure 9. Average profit AVP_{23} versus inventory cycle's duration (T) and selling-price (P).

6. Sensitivity Analysis

This section highlights the impact on the system's average profit, that is, AVP_{1i} and AVP_{2i} , $i \in \{1, 2, 3\}$ while there is some deviation in various parameters C_H , C_1 , U_c , and U_f (See Table 2).

- For C_H , which is the retailer's carrying cost, it can be seen that whenever the parameter C_H enhances, under various circumstances the system's average profit functions are AVP_{1i} and AVP_{2i} , where $i \in \{1, 2, 3\}$ decreases.
- If the unit ordering cost C_1 increased, then the system's average profit, that is, AVP_{1i} and AVP_{2i} , $i \in \{1, 2, 3\}$ diminish rapidly. With this observation, it can be found that the variation in both positive percentage changes along with negative percentage changes are quite equal for system's average profit, that is, AVP_{1i} and AVP_{2i} , $i \in \{1, 2, 3\}$.
- As the retailer's interest rate gaining for credit-balances which is U_c increases, the system's average profit which are AVP_{1i} and AVP_{2i} where $i \in \{1, 2, 3\}$ raises automatically. That means the retailers may increase their benefit level as much as they can to gain more interest. U_c is the key parameter to increase the system's profitability.
- The parameter U_f which is defined as the retailer's interest rate for financing inventory which means that the retailer has to pay that much interest to the supplier. For this parameter U_f , it is clearly observed from the sensitivity table that system's average profit, that is, AVP_{1i} and AVP_{2i} , when $i \in \{1, 2, 3\}$ always decreases whenever the parameter U_f changes from a negative percentage to a positive percentage.

Table 2. Sensitivity Analysis for distinct parameters.

Parameters	Changes (in %)	AVP_{11}	AVP_{12}	AVP_{13}	AVP_{21}	AVP_{22}	AVP_{23}
C_H	−50%	20.4	16.51	12.69	25.2	22.53	10.22
	−25%	10.28	7.87	8.54	12.53	11.28	4.9
	+25%	−2.54	−5.32	−1.79	−4.1	−3.32	−5.25
	+50%	−7.76	−9.65	−7.51	−11.74	−8.93	−14.61
C_1	−50%	0.23	0.16	0.1	0.56	0.23	0.34
	−25%	0.11	0.08	0.05	0.28	0.11	0.4
	+25%	−0.11	−0.08	−0.05	−0.28	−0.11	−0.4
	+50%	−0.23	−0.16	−0.1	−0.56	−0.23	−0.34
U_c	−50%	3.17	9.74	16.12	24.57	20.23	12.9
	−25%	4.53	12.87	24.76	30.77	37.12	15.84
	+25%	7.21	15.98	32.55	45.21	43.7	22.76
	+50%	9.56	20.66	47.87	58.1	59.31	30.32
U_f	−50%	−14.19	−4.04	−0.54	−0.41	−0.03	−5.45
	−25%	−20.3	−6.87	−2.45	−1.43	−0.45	−8.9
	+25%	−25.7	−10.54	−4.76	−3.48	−0.64	−14.27
	+50%	−31.45	−16.16	−5.89	−5.71	−1.2	−19.64

All the impact of variations for changes of distinct parameters on system's average profit AVP_{1i} and AVP_{2i} where $i \in \{1, 2, 3\}$ are demonstrated in Figures 10–15, respectively.

Figure 10. Impact of variations for changes of distinct parameters on system's average profit AVP_{11} . **Figure 11.** Impact of variations for changes of distinct parameters on system's average profit AVP_{12} .

Figure 12. Impact of variations for changes of distinct parameters on system's average profit AVP_{13} .

Figure 13. Impact of variations for changes of distinct parameters on system's average profit AVP_{21} .

Figure 14. Impact of variations for changes of distinct parameters on system's average profit AVP_{22} .

Figure 15. Impact of variations for changes of distinct parameters on system's average profit AVP_{23} .

Case Study

This model discussed the concept of delaying the payments of the retailer and several pricing discount policy on purchasing cost to the retailer allowed by supplier. Any marketing companies are a practical example of this model. They offer both a delay-in-payments policy and several discount strategies to customers for increasing their sales of products. For example, the supplier offers 30 days to the retailer for settling payment. If the retailer is unable to pay the amount within 5 days, then the supplier will provide 20% discount on purchasing cost to the retailer. In the case the retailer takes 15 days to pay that amount, then the supplier will provide 10% discount on purchasing cost. Additionally after 15 days, supplier will not provide any price discount on purchasing cost to the retailer.

Based on this above case study, a numerical example is discussed which is as follows:

Let $C_1 = \$25$ per order, $\theta = 0.12$, $C_H = \$0.6/\text{unit}/\text{month}$, $K_1 = 2$ months, $\delta_1 = 20\%$, $a_1 = 90$, $b_1 = 5.4$, $c_1 = 0.4$, $h = 1.6$, $C_M = \$90/\text{unit}$, $U_c = \frac{0.14}{12}/\$/\text{month}$, $U_f = \frac{0.17}{12}/\$/\text{month}$, and $\mu = 210$ units/month.

Then the optimal solutions are $\{AVP_{11} = \$211.68, T^* = 0.5$ months, and $P^* = \$23.98/\text{unit}\}$, $\{AVP_{12} = \$370.97, T^* = 0.8$ months, and $P^* = \$36.8/\text{unit}\}$, $\{AVP_{13} = \$658.46, T^* = 1.1$ months, and $P^* = \$41.43/\text{unit}\}$, $\{AVP_{21} = \$169.50, T^* = 5.2$ months, and $P^* = \$94.63/\text{unit}\}$, $\{AVP_{22} = \$267.72, T^* = 5.6$ months, and $P^* = \$100.84/\text{unit}\}$, and $\{AVP_{23} = \$514.38, T^* = 6.4$ months, and $P^* = \$109.25/\text{unit}\}$.

Average profit function AVP_{11} is globally maximum when $T = 0.5$ months, and $P = \$23.98/\text{unit}$ in Figure 16. Average profit function AVP_{12} is globally maximum when $T = 0.8$ months, and $P = \$36.8/\text{unit}$ in Figure 17. Average profit function AVP_{13} is globally maximum when $T = 1.1$ months, and $P = \$41.43/\text{unit}$ in Figure 18. Average profit function AVP_{21} is globally maximum when $T = 5.2$ months, and $P = \$94.63/\text{unit}$ in Figure 19. Average profit function AVP_{22} is globally maximum when $T = 5.6$ months, and $P = \$100.84/\text{unit}$ in Figure 20. Average profit function AVP_{23} is globally maximum when $T = 6.4$ months, and $P = \$109.25/\text{unit}$ in Figure 21.

Figure 16. Average profit AVP_{11} versus inventory cycle's duration (T) and selling-price (P).

Figure 17. Average profit AVP_{12} versus inventory cycle's duration (T) and selling-price (P).

Figure 18. Average profit AVP_{13} versus inventory cycle's duration (T) and selling-price (P).

Figure 19. Average profit AVP_{21} versus inventory cycle's duration (T) and selling-price (P).

Figure 20. Average profit AVP_{22} versus inventory cycle's duration (T) and selling-price (P).

Figure 21. Average profit AVP_{23} versus inventory cycle's duration (T) and selling-price (P).

7. Managerial Insights

This model presented a better optimal profit which would serve the manufacturing company to obtain the benefit at the level of optimality. There are various components in this model those are responsible to increase the profit level of any industry sector. Those components are price discount strategy and delay-in-payments. By providing price discount scheme, managers are easily can attract the attention of their consumers.

1. Furthermore, delay-in-payments also act as an affecting parameter to enrich the profit matter for managers. They often allow this delay-in-payments system to their customers. For this scheme, managers are much more capable of enhancing their profit level which is the basic business concern in any companies.

2. Sensitivity analysis of this research is also portrays the same matter of how the rate of interest gain for any inventory system is the primary factor to achieve higher profit margin. This study serves the managers to highlight their average profit by evaluating inventory cycle's duration and selling-price practically.

8. Conclusions

An inventory model was extended from the extension of Sana and Chaudhuri's [48] model. Using the concept of model of Sana and Chaudhuri [48], the supplier offers several discount systems to the retailer for stimulating their business. This study generated a production model with deterioration for time-dependent as well as selling-price-related demand. Deterioration is also considered in this study. In this model, the optimal results of selling-price and the inventory cycle's length for better optimality of an average profit function of the inventory system are calculated briefly. At the end of the numerical example, a comparison is done with the optimal findings of the Sana and Chaudhuri [48] model. This model enhances higher profit for the inventory system more than previous literature. Further, this work can be expanded to a multi-item inventory model (Tamjidzad and Mirmohammadi [53], Tayyab et al. [54]) with probabilistic demand (Khosravikia and Clayton [55], Guo et al. [56]), machine breakdown (Peymankar et al. [57], Chiu et al. [58]), imperfect production (Sarkar and Saren [59], Sarkar et al. [60]) and solving by any meta-heuristic procedure (Vahdani et al. [61]).

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Appendix A

$$\begin{aligned} Y_1 &= \left(\frac{b_1}{\theta} - \frac{2c_1}{\theta^2} \right) \\ Y_2 &= \frac{c_1}{\theta} \end{aligned}$$

Appendix B

$$\begin{aligned}
 S &= X_1\theta + [(\theta + T\theta^2)(X_2 + Y_1T + Y_2T^2)] \\
 E &= -\frac{(X_1 + X_2 + Y_1T + Y_2T^2)\theta^2}{2} \\
 F &= \left[\left(1 + \frac{\theta^2 T^2}{2} \right) (X_2 + Y_1T + Y_2T^2) \right] - X_2
 \end{aligned}$$

Decision Variables

- P selling-price (\$/unit)
 T inventory cycle's length (months)

Parameters

- $I_1(t)$ inventory's level throughout the time interval $0 \leq t \leq t_1$ (units)
 $I_2(t)$ inventory's level throughout $t_1 \leq t \leq T$ (units)
 μ production rate (unit/unit time)
 K delay-period (unit time)
 K_i i th allowable delay-duration (unit time)
 $D(P, t)$ demand is related to both price, time, $D(P, t) = a_1 + b_1t + c_1t^2 - hP$, a_1, b_1, c_1 , and $h > 0$ all are scaling parameters.
 $\theta(t)$ decaying rate, $0 < \theta(t) < 1$
 δ_i rate of discount on MRP at the i th allowable delay-period (%)
 C_P cost of purchasing (\$/unit)
 C_M maximum retail price (MRP) (\$/unit)
 C_H carrying cost (\$/unit)
 C_1 ordering cost (\$/order)
 U_c interest rate gaining for credit-balance (/\$/unit time)
 U_f interest rate for financing inventory (/\$/unit time)
 t_1 production time
 T^* optimal inventory cycle's length (months)
 P^* optimal selling-price (\$/unit)
 AVP_{1i} average profit function when $T \geq K_i$ (\$)
 AVP_{2i} system's average profit during $T \leq K_i$ (\$)

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