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Z_2 Topological Order and Topological Protection of Majorana Fermion Qubits

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Abstract: The Kitaev chain model exhibits topological order that manifests as topological degeneracy, Majorana edge modes and Z_2 topological invariant of the bulk spectrum. This model can be obtained from a transverse field Ising model (TFIM) using the Jordan–Wigner transformation. TFIM has neither topological degeneracy nor any edge modes. Topological degeneracy associated with topological order is central to topological quantum computation. In this paper, we explore topological protection of the ground state manifold in the case of Majorana fermion models which exhibit Z_2 topological order. We show that there are at least two different ways to understand this topological protection of Majorana fermion qubits: one way is based on fermionic mode operators and the other is based on anti-commuting symmetry operators. We also show how these two different ways are related to each other. We provide a very general approach to understanding the topological protection of Majorana fermion qubits in the case of lattice Hamiltonians. We then show how in topological phases in Majorana fermion models gives rise to new braid group representations. So, we give a unifying and broad perspective of topological phases in Majorana fermion models based on anti-commuting symmetry operators and braid group representations of Majorana fermions as anyons.

Keywords: topological protection; emergent Majorana modes; Majorana fermion models; Majorana fermion braiding



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1. Introduction

In this paper, we take an approach to understanding the topological protection of Majorana fermions qubits based on Majorana zero mode operators that are odd normalized zero modes. Similar to even mode operators in many-body localization (MBL), we show that odd zero-mode operators are also integrals of motion and the emergent symmetry operators of the Hamiltonian. Our approach brings out new aspects of the topological protection in Majorana fermion models especially the connection to emergent symmetry operators. What is novel about our approach to topological protection is that we relate Majorana zero mode operators with symmetries of the Hamiltonian which gives us more scope to understand the topological protection.

Normalized zero modes have attracted a lot of attention recently not only in topological phases of matter but also in many-body localization (MBL) where they offer a very intuitive language for understanding MBL phenomenology [1–7]. Normalized zero-modes can be even or odd depending on whether they commute or anti-commute with the parity operator for the quantum system. Even zero-modes are also called pseudo-spins in MBL literature. In this paper, we have explored Majorana zero mode operators which are an important indicator and manifestation of the topological phases in one-dimensional Hamiltonians like the Kitaev Chain model. Conventionally, phases of matter were introduced within the symmetry breaking framework in which ordered phases emerge out of the symmetry breaking. However, the quantum Hall effect provided a counterexample to this

framework. Quantization of Hall conductance in quantum Hall effect is associated with topological invariants and can not be understood based on Landau's symmetry breaking framework. Recently, a plethora of new topological phases has been discovered. One of the main interests in these topological phases comes from the quantum computation side because these phases offer the possibility of doing fault-tolerant computation. The ground state of the corresponding Hamiltonians has topological degeneracy. This degeneracy is topological in the sense that no local perturbation can lift it and hence there is topological protection of the ground state manifold. Topological degeneracy gives rise to a topological qubit for topological quantum computation. By degeneracy as used in this paper, we mean spectral degeneracy of a given Hamiltonian which refers to having more than one eigenstates corresponding to an eigenvalue of the Hamiltonian. In this paper, we will explore the protection of topological qubits in the context of Majorana fermions. Majorana fermions have recently emerged as promising candidates for topological quantum computation. Consequently, Majorana Fermions have been on the research frontiers in condensed matter physics, and especially in topological quantum computation. One main mathematical difference between Majorana fermions and standard fermions is that the latter satisfy Grassmann algebra while the operator algebra of Majorana Fermions is a Clifford algebra. Clifford algebra of Majorana fermions attributes them with anyonic statistics—which is central for their role in topological quantum computation. The Clifford Braiding Theorem [8] gives a rigorous way to understanding this relation of Clifford algebra and anyonic statistics.

In [9], Kitaev introduced a quadratic Hamiltonian for fermions in one dimension, which has a topological phase in which there are Majorana modes at the edges of the chain. Kitaev employed a Majorana fermion representation to diagonalize the Hamiltonian and showed that there are Majorana edge modes. The Kitaev model is not an entirely new model. It can be obtained from the transverse field Ising model (TFIM) using a Jordan-Wigner transformation (JWT). JWT is a non-local transformation that maps spin operators to fermion operators and has been very crucial for solving lattice spin models. So what Kitaev did is to take fermions as degrees of freedom instead of spins. This novel point of view opened up the way to understand topological order in a well-known system. TFIM exhibits only Landau order, and the ordered phase arises due to the symmetry breaking of the model. The immediate question that one can ask is what corresponds to which Landau order on the fermionic side once the Jordan–Wigner transformation is done?

The relation between these two models has been studied in a recent work [10]. There it is concluded that the spectral properties of the two models are the same, which we will find below is not correct. Using dualities and a bond-algebra approach and holographic symmetries, there is already an understanding of how topological order in the Kitaev chain model is related to Landau order in the corresponding spin model [11]. Under the duality transformation, local variables of spin models get transformed to non-local observables which are important for topological order in the fermionic model. The work [11] also shows that Hamiltonians that exhibit topological order have holographic symmetries. In another work, [1] Fendley has found an algebraic approach for topological order in a Majorana chain, and more generally for a parafermion chain. In Fendley's paper, a fermionic mode operator is defined and it is shown that in the topologically-ordered phase, there are these fermionic mode operators that, for the Kitaev chain model, are the Majorana mode operators. As with Fendley, we also find that in the topologically ordered case, there are emergent symmetry operators that are actually Majorana mode operators.

The rest of this paper is organized as follows: First, we briefly review the Kitaev chain model, its symmetries, and topological order. Then in the next section, we put together the various algebraic aspects of Majorana fermions with a special focus on Clifford algebra. This sets the algebraic background that we use later on to prove the new results. We have deliberately given background in the first two sections so that this paper can be both an exposition and a research paper. In Section 3.1, we discuss one of the important results of this paper. We explicitly write down the super-symmetry generator. A very interesting

aspect of this generator is that it is related to the emergent Majorana modes. In Section 4, we extend the analysis of Lee and Wilczek [12] and show that their emergent Majorana fermion is actually a fermionic zero mode operator. We then go ahead to establish the relation between emergent Majorana mode operator and the Z_2 topological order for the general case of odd-numbered Majorana fermion models. In Section 5.1, we extend our results for an odd number of Majorana fermions both without interaction as well with quartic interactions. In Section 6, we show how the fermionic mode operator-based approach can be put in the larger context of symmetry algebra. This unification is important because it opens the ways to understand the topological protection from symmetries of the lattice models. In Section 7, we explore the relation between topological order and the braiding representation of Majorana fermions. We show that this braid group representation has extra symmetry where the Majorana mode operators are the symmetry generators. Finally, we summarize our results in Section 8.

2. Kitaev p-Wave Chain

To study the relation between Landau order and topological order we introduce two Hamiltonians that are related to each other by a Jordan–Wigner transformation. The two models are a transverse field Ising model (TFIM) and the Kitaev p-wave chain model. Following Kitaev, we will diagonalize the Kitaev chain model using a Majorana fermion representation and show that, in its topological phase, the Kitaev chain model has a Majorana edge model and also topological degeneracy. Majorana fermions are very important in our study, and we will look closely at their algebra and how, as a quantum system, they are different from the standard (Dirac) fermions.

The Hamiltonian for the transverse field Ising model is [13]:

$$H = -J \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x - h_z \sum_{i=1}^N \sigma_i^z, \quad (1)$$

where J is the ferromagnetic exchange constant and h_z is the Zeeman field in the Z direction and σ_i is the Pauli spin matrix at the i -th site. This model has Z_2 symmetry, due to which the global symmetry operator $\prod_i \sigma_i^z$ commutes with the Hamiltonian.

$$\left[\prod_i \sigma_i^z, H \right] = 0. \quad (2)$$

The global symmetry operator flips all the spins. There is a doubly degenerate ground state. This model exhibits two phases that can be understood on the basis of Landau's symmetry-breaking theory. There is a ferromagnetic phase that arises when the symmetry of the model is broken. There is a disordered phase in which symmetry is intact. We will now apply the Jordan–Wigner transformation to this model to map it to a fermionic model that will turn out to be the Kitaev chain model. The Jordan–Wigner transform maps the spin operators into fermionic ones:

$$c_i = \sigma_i^+ \left(\prod_{j=1}^{i-1} \sigma_j^z \right) \quad c_i^\dagger = \sigma_i^- \left(\prod_{j=1}^{i-1} \sigma_j^z \right) \quad (3)$$

$$H = -t \sum_{i=0}^{N-1} (c_i^\dagger c_{i+1} + h.c.) + \Delta \sum_{i=0}^{N-1} c_i^\dagger c_{i+1}^\dagger + h.c. - \mu \sum_{i=0}^N c_i^\dagger c_i,$$

where t, Δ, μ are hopping strength, superconducting order parameter and chemical potential respectively. Kitaev employed a Majorana fermion representation to diagonalize this Hamiltonian.

$$c_i = \frac{\gamma_{1,i} - i\gamma_{2,i}}{\sqrt{2}} \quad c_i^\dagger = \frac{\gamma_{1,i} + i\gamma_{2,i}}{\sqrt{2}}. \quad (4)$$

In the Majorana fermion representation, the Hamiltonian gets transformed to:

$$H = it \sum_{i=0}^{N-1} (\gamma_{1,i}\gamma_{2,i+1} - \gamma_{2,i}\gamma_{1,i+1}) + i\Delta \sum_{i=0}^{N-1} (\gamma_{1,i}\gamma_{2,i+1} + \gamma_{2,i}\gamma_{1,i+1}) - \mu \sum_{i=0}^N \left(\frac{1}{2} - i\gamma_{1,i}\gamma_{2,i}\right). \quad (5)$$

The Hamiltonian has a trivial phase and topological phase. Trivial phase is obtained for the choice of parameters: $t = \Delta = 0$. In this case two Majorana fermions at each site couple together to form a complex fermion, and there is no topological phase as there are no Majorana edge modes. Choosing $\mu = 0$ and $t = \Delta$ the Hamiltonian becomes.

$$H = 2it \sum_{i=0}^{N-1} \gamma_{1,i}\gamma_{2,i+1}. \quad (6)$$

One can see that Majorana fermions from sites i and $i + 1$ are pairing together. Using the Majorana fermions from adjacent sites, we can define a complex fermion:

$$a_i = \frac{\gamma_{2,i+1} - i\gamma_{1,i}}{\sqrt{2}}. \quad (7)$$

The Hamiltonian becomes:

$$H_1 = \left(t \sum_{i=0}^{N-1} a_i^\dagger a_i - \frac{1}{2} \right). \quad (8)$$

Looking carefully through the Hamiltonian, we discover that two Majorana fermions $\gamma_{2,0}$ and $\gamma_{1,N}$ have been left out. These two Majorana fermions reside at the ends of the chain. These are the Majorana modes of the Hamiltonian which characterize the topological phase. Taking them together, we can form a non-local fermion.

$$\tilde{a} = \frac{\gamma_{1,N} - i\gamma_{2,0}}{\sqrt{2}}. \quad (9)$$

The two Majorana modes are the symmetries of the Hamiltonian. They commute with the Hamiltonian because they are not part of the Hamiltonian. $[H_1, \gamma_{2,0}] = 0 = [H_1, \gamma_{1,N}]$. The ground state of H_1 is doubly degenerate because the two states with Majorana modes present and absent have the same energy. These two ground states correspond to even and odd parity of the Majorana mode operators:

$$ib'b'' | \psi'_0 \rangle = | \psi'_0 \rangle \quad ib'b'' | \psi''_0 \rangle = | \psi''_0 \rangle, \quad (10)$$

where $b' = \gamma_{2,0}$ and $b'' = \gamma_{1,N}$. ψ'_0 and ψ''_0 are two degenerate ground states of the Hamiltonian H_1 . This is an example of topological degeneracy and it is preserved as long as the system has Majorana zero modes. In the long chain limit, there is no coupling between two edge modes and hence they are pinned at zero energy. However, for any finite system, the two Majorana edge modes get coupled together and hence pick up finite energy.

Majorana Fermions versus Complex Fermions

Majorana fermions can be taken algebraically as building blocks of (standard) fermions. The algebra of Majorana fermions makes them very different from the usual fermions.

$$\gamma = \gamma^\dagger \quad \gamma^2 = 1 \quad \gamma_1 \gamma_2 = -\gamma_2 \gamma_1 \quad P = i\gamma_1 \gamma_2, \quad (11)$$

where γ is the Majorana fermion operator and P is parity operator for two Majorana fermions. More compactly, we can write:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}. \quad (12)$$

Majoranas are very different from the complex fermions because they are self-Hermitian and hence the creation and annihilation operators are the same, which means that a Majorana fermion is its own anti-particle. A fermionic vacuum can not be defined for Majorana fermions because there is no well-defined number operator, or in other words, the number of Majorana fermions is not a well-defined quantity, and hence not a quantum number that can be used to label Majorana fermions. Majorana fermions don't have $U(1)$ symmetry, and hence a number operator can not be defined for them. However, they have Z_2 symmetry; parity is conserved for Majorana fermions.

Complex fermions can be mapped to Majorana fermion operators. Let γ_1 and γ_2 be two Majorana fermion operators, then corresponding complex fermion operators are:

$$c = \frac{1}{\sqrt{2}}(\gamma_1 + i\gamma_2) \quad c^\dagger = \frac{1}{\sqrt{2}}(\gamma_1 - i\gamma_2). \quad (13)$$

When taken abstractly, Majorana fermions appear to be very unrealistic particles, but physically they can appear as Bogolubov quasiparticles which exist as zero modes in topological superconductors. They are zero energy solutions of the BdG equation and are different from Majorana spinors which are solutions of the Dirac equation.

Fermions obey the Grassmann algebra:

$$\{c_i, c_i^\dagger\} = \delta_{ij} \quad c_i^2 = (c_i^\dagger)^2 = 0 \quad N = c^\dagger c \quad N^2 = N, \quad (14)$$

where c^\dagger , c and N are the creation, annihilation and number operators for a fermion.

$$|1\rangle = c^\dagger |0\rangle \quad |0\rangle = c |1\rangle \quad (15)$$

$$c |0\rangle = c^\dagger |1\rangle = 0. \quad (16)$$

Fermions have a vacuum state. Creation and annihilation operators are used to construct the states of fermions. Fermions have $U(1)$ symmetry, and hence the number of fermions is conserved, and the occupation number is a well-defined quantum number. The number of fermions in a state is given by the eigenvalue of the number operator. Here the number operator is idempotent, and hence there are only two eigenvalues: 0, 1. Different fermion operators anti-commute with each other and hence obey Fermi-Dirac statistics.

3. Algebra of Majorana Doubling

In this section, following [12] we will explore the full algebra of Majorana fermions. In [12], Lee and Wilczek gave an illuminating analysis of the doubled spectrum of the Kitaev chain model. They showed that the algebra that has been considered for the Kitaev chain model is conceptually incomplete. Using the case of three Majorana fermions that are at the edges of superconducting wires, we show that the Hamiltonian of these Majorana fermions has extra algebraic structure that is physically significant. The difference lies in another Majorana operator that has been called an Emergent Majorana for the reason that it obeys all the properties of a Majorana fermion.

In the Kitaev paper, the algebra of Majorana fermions is written as

$$\{a_i, a_j\} = 2\delta_{ij}. \quad (17)$$

This equation defines the Clifford algebra of Majorana fermions. The full algebra is generated by all the ordered products of these operators. For the case of three Majorana fermions the full Clifford algebra is described below:

$$\begin{aligned} \{1, \gamma_1 = a_1, \gamma_2 = a_2, \gamma_3 = a_3, \gamma_{12} = a_1 a_2, \gamma_{23} = a_2 a_3, \\ \gamma_{31} = a_3 a_1, \gamma_{123} = a_1 a_2 a_3\}. \end{aligned} \quad (18)$$

The Clifford algebra of three Majorana fermions is eight-dimensional, with these eight independent generators. There are three bivectors γ_{12} , γ_{23} , γ_{31} and one trivector (also called a pseudoscalar) γ_{123} . Bivectors are related to rotations. Trivector will turn out to be very important for our discussion on topological order because it is a chirality operator that distinguishes between even and odd parity. We refer to [14] for more discussion on Clifford algebra of spin. In that paper, the relation between spin and fermionic systems has been made clear. The reader will find seeds of the duality between Pauli matrices (spin) and Clifford algebra (Majorana fermions).

Lee and Wilczek showed that when we take the full Clifford algebra of the Majorana fermions into consideration, we arrive at very interesting mathematical and physical implications. Let b_1 , b_2 and b_3 be three Majorana fermions that can occur at the ends of three p-wave superconducting nano-wires. This situation is not artificial, rather it is crucial for the topological Kondo effect [15] where three Majorana fermions give rise to non-local spin-1/2 object which then couples to the conduction fermions in the leads. Similarly, three such wire junctions have also been explored to study the Kitaev spin model [16,17].

$$\{b_j, b_k\} = 2\delta_{jk}. \quad (19)$$

We can write down a Hamiltonian for these interacting Majoranas coming from three different wires.

$$H_m = -i(\alpha b_1 b_2 + \beta b_2 b_3 + \gamma b_3 b_1). \quad (20)$$

Now it is known that Majorana bilinears generate a spin algebra so one would naively think that it is a spin Hamiltonian. However, the spin Hamiltonian neither has edge modes nor any topological degeneracy. To understand this, one needs to realize that the Clifford algebra generated by Majorana fermions is larger than what is present in Equation (19). There are other generators of the algebra. The Hamiltonian given in Equation (20) has parity symmetry due to which fermion number N_e is conserved modulo 2.

$$[H_m, P] = 0 \quad P = (-1)^{N_e} \quad P^2 = 1. \quad (21)$$

Physically, the full implications of the parity operator need to be taken into consideration to conceptually complete the algebra. There is a special operator Γ in the algebra which we call as Emergent Majorana because it has all the properties of a Majorana fermion.

$$\Gamma \equiv -ib_1 b_2 b_3 \quad (22)$$

$$\Gamma^2 = 1 \quad [\Gamma, b_j] = 0 \quad [\Gamma, H_m] = 0 \quad \{\Gamma, P\} = 0. \quad (23)$$

The emergent Majorana operator commutes with the Hamiltonian, and hence there is an additional symmetry present, as it anti-commutes with the parity operator and hence it shifts among the parity states. Both the P and Γ operators commute with Hamiltonian but anti-commute with each other due to which there is a doubling of the spectrum. The presence of this extra symmetry leads to the doubled spectrum. This doubling is different from Kramer's doubling [18] because no time-reversal symmetry is needed. In the basis in

which P is diagonal with ± 1 eigenvalues, the Γ operator takes the states into degenerate eigenstates with eigenvalues ∓ 1 . It is very suggestive to write the parity operator for H_m in terms of a Γ operator in line with [19,20]. The case of an odd number of Majorana fermions can always be thought of as a combination of complex fermions (c fermions) and one Majorana fermion so that the parity operator can be written as $P = \prod_i (1 - 2c_i^\dagger c_i) \gamma$. This parity operator constitutes the local parity operator of reference [19].

Two Majorana fermions combine together to give a Dirac fermion. Three Majorana fermions are the simplest case that shows non-trivial Majorana physics. So Equation (20) gives the simplest Hamiltonian that we can write down for Majorana fermions having non-trivial spectral and hence physical properties. We have seen for this Hamiltonian that there are additional symmetries that lead not only to the topological degeneracy for the ground state, but the Hamiltonian has a doubling for the whole spectrum including excited states. This is a very powerful implication of the existence of emergent Majorana fermions and hence of the emergent symmetry operators. We can not talk about the topological order for three Majorana fermions, but later on, we will see that it is the presence of these emergent fermionic symmetries which leads to the topological order in Majorana fermion models.

Emergent Majorana fermions were considered in [19] and global parity was expressed in terms of these operators. In [20], authors have also studied the Hamiltonian given in Equation (20) and using the anti-symmetry property of the Hamiltonian arrived at what we call emergent Majorana fermions.

The doubled spectrum of the Kitaev chain Hamiltonian comes from this algebraic structure which leads to extra symmetries. This algebraic structure is non-perturbative, and hence is robust to perturbations as long as they preserve the discrete symmetry.

3.1. Emergent Supersymmetry in Majorana Fermions Models

In this section, we will present one of the novel results of this paper. We will show that the existence of the emergent Majorana operator, Γ , leads to the supersymmetry in the Majorana fermion model. We will explicitly write down the supersymmetry generator in terms of the Γ operator and parity operator.

Supersymmetry has been studied in lattice models for fermions [21] (and references therein). Recently, there has been some interest in searching for the supersymmetry in lattice models for Majorana fermions [22–24]. However, in those situations SUSY arises only at the critical point of the model. In our case, we find emergent supersymmetry which is tied to topological order because we need Γ operator which exists only in the topological phase. Emergent supersymmetry was also found in [25] for Majorana fermion models with translational symmetry. The difference with our case is that we need to have a Majorana mode operator Γ rather than translational symmetry to have emergent SUSY.

We once again consider a system of an odd number of Majorana fermions so that we can define the Γ operator. For the quadratic Hamiltonian for Majorana fermions, parity is conserved and hence we can define parity operator, P which commutes with Hamiltonian. We define the supersymmetry generator, Q which commutes with Hamiltonian and hence shows that Majorana fermion Hamiltonian has supersymmetry.

$$Q = \sqrt{H} \left(\frac{1+P}{2} \right) \Gamma \quad (24)$$

$$Q^2 = 0 \quad [Q, H] = 0 \quad \{Q, Q^\dagger\} = H. \quad (25)$$

It needs to be noted that we did not need to have translation symmetry to have supersymmetry in our Hamiltonian. So in that sense, we show that supersymmetry is a feature of Majorana fermion Hamiltonians and is also tied to their topological order because we need the existence of the Γ operator.

The supersymmetry that we have found in the section has been further explored in [26,27]. In [26], the authors have explored a special case of the supersymmetry generator which we have found. They consider the Hamiltonian for three Majorana fermions as

discussed in [12]. For this Hamiltonian, they have shown that our SUSY generator can be embedded within $Cl(1,3)$ which is Clifford algebra for a $3+1$ dimensional metric space with Lorentz signature $(+++-)$. In [27], the authors have extended the investigations of our supersymmetry generator for a finite temperature situation using the thermofield formalism.

4. Majorana Zero Modes and Γ Operator

In this section, we extend the analysis of Lee and Wilczek and present results about emergent Majorana fermion that go beyond their work [12]. In this section, we will introduce zero-mode operators and show that how emergent Majorana fermion as described in Section 3, is actually a zero-mode operator for a special case of a system of three Majorana fermions.

Zero modes are an important manifestation of the topological phases in one-dimensional lattice models [1]. What we have already found is that the complete algebra of Majorana fermions has extra operators that have been called emergent Majorana fermions and represented by Γ_{em} operators. For the case of three Majorana fermions we find that the Γ_{em} operator is the emergent Majorana fermion and is also the symmetry of the Hamiltonian. In this section, we will see that the Γ operator is also the zero-mode operator of the Hamiltonian. First, we give a definition of the zero-mode operator, and then we show how that is related to Emergent Majorana fermions or the Γ_{em} operator. Following [1], a fermionic zero mode is an operator Γ such that Γ

- Commutes with Hamiltonian: $[H, \Gamma] = 0$
- anti-commutes with parity: $\{P, \Gamma\} = 0$
- has finite “normalization” even in the $L \rightarrow \infty$ limit: $\Gamma^\dagger \Gamma = 1$.

The last condition may not be always satisfied for zero-mode operators. So, it can be relaxed in those cases. A zero-mode operator that satisfies all three conditions is called a strong zero-mode operator.

Now, let us consider a system of three Majorana fermions. We denote these Majorana fermions as γ_1, γ_2 and γ_3 . For this system of three Majorana fermions, the emergent Majorana Γ_{em} is defined as $\Gamma_{em} = -i\gamma_1\gamma_2\gamma_3$. Parity of Majorana fermions is defined by $P = (-1)^{N_e}$ where N_e is the number of the fermions. To write down the global parity operator for this system, we will add one more Majorana fermion f which does not have any pairwise interaction with other Majorana fermions and hence does not enter the Hamiltonian. Using f Majorana fermion, the global parity of our system is given by $P = i\Gamma_{em}f$. Now, we state our mathematical result about the emergent Majorana fermions as a theorem.

Theorem 1. *The emergent Majorana fermion Γ_{em} for a system of three Majorana fermions is the strong zero-mode of the system.*

Proof. As defined above, zero-mode is represented by an operator that commutes with the Hamiltonian and anti-commutes with the parity operator. Now, we show that for the case of three Majorana fermions, the emergent Majorana fermion Γ_{em} is the zero-mode operator. \square

The Hamiltonian for the three Majorana fermions can be written as

$$H_3 = -i(\gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1). \quad (26)$$

First, we show that Γ_{em} commutes with this Hamiltonian. To do that we show that Γ_{em} commutes with one of the terms of the Hamiltonian. The same result holds for the other two terms of the Hamiltonian.

$$[\Gamma_{em}, -i\gamma_1\gamma_2] \quad (27)$$

$$= [-i\gamma_1\gamma_2\gamma_3, -i\gamma_1\gamma_2] \quad (28)$$

$$= (-i)(-i)(\gamma_1\gamma_2\gamma_3\gamma_1\gamma_2 - \gamma_1\gamma_2\gamma_1\gamma_2\gamma_3) \quad (29)$$

$$= (-i)(-i)(\gamma_1\gamma_2\gamma_1\gamma_2\gamma_3 - \gamma_1\gamma_2\gamma_1\gamma_2\gamma_3) = 0. \quad (30)$$

In the last step, we have passed γ_3 through $\gamma_1\gamma_2$ because a Majorana fermion operator commutes with a bilinear of Majorana fermion operators.

Now we show that emergent Majorana operator Γ_{em} anti-commutes with Parity operator. To do that, we will also need to show that Γ_{em} anti-commutes with f Majorana fermion.

$$\{\Gamma_{em}, f\} \quad (31)$$

$$= -i\gamma_1\gamma_2\gamma_3f + f(-i\gamma_1\gamma_2\gamma_3) \quad (32)$$

$$= +i(f\gamma_1\gamma_2\gamma_3) - f(i\gamma_1\gamma_2\gamma_3) = 0, \quad (33)$$

where in the last line we have used the fact the f Majorana fermion will pick up a minus sign when swapped through odd Majorana fermions. Now using this relation $\{\Gamma_{em}, P\} = 0$, we can calculate the anti-commutator of Parity and Γ_{em} operators.

$$\{\Gamma_{em}, P\} \quad (34)$$

$$= \{\Gamma_{em}, i\Gamma_{em}f\} \quad (35)$$

$$= i\Gamma_{em}\Gamma_{em}f - i\Gamma_{em}f\Gamma_{em} \quad (36)$$

$$= if + if = 0. \quad (37)$$

In the second last line we have used the property of Γ_{em} operator that it squares to unity and also it anti-commutes with the f Majorana fermion. Using the anti-commutators which we calculated above through Equations (31) to (37), it not difficult to see $[\Gamma_{em}, P] = 2if$ where f is the Majorana fermion.

Hence we have showed that Γ_{em} satisfies the defining properties of the zero mode operator. Now, we show the Γ_{em} squares to unity and hence is normalized.

$$\Gamma_{em}^2 = (-i)(-i)\gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \quad (38)$$

$$(-1)\gamma_1\gamma_2\gamma_1\gamma_2\gamma_3\gamma_3 \quad (39)$$

$$(-1)(-1)\gamma_1\gamma_1\gamma_2\gamma_2 = 1. \quad (40)$$

That concludes our proof of all the conditions for Γ_{em} being a strong zero-mode operator. Even though we have explicitly proved this result only for the case of three Majorana fermions, it can be easily generalized to an arbitrary odd number of Majorana fermions and that proof has been given in Section 6. The reason being that the Clifford algebra of Majorana fermions allows that a given Majorana fermion can either commute or anti-commute with a string of Majorana fermions which exclude that particular Majorana fermion. This property of the Majorana fermions follows from the Z_2 grading of Clifford algebra in which we can define the various generators of the Clifford algebra into two classes with even and odd grading depending on whether they commute or anti-commute with a given Majorana fermion [28].

5. Zero-Mode Operators for a General Case

In this section, we generalize the results of the previous section and obtain new results about topological order in Majorana fermion chain models using Clifford algebraic methods.

We consider the general case of an odd number $N > 3$ of Majorana fermions. We will find that, as for the case of three Majorana fermions, there are emergent Majorana fermions and their Majorana mode operators are the symmetries of the Hamiltonian. For the case of three Majorana fermions, there was only one emergent Majorana mode operator. However, for the general case, there are many more Majorana mode operators as shown in [29]. We will focus just on the generalized case of the Γ operator considered in the previous section.

We consider a system that has $2N + 1$ Majorana fermions. These Majorana fermions will span a vector space of dimensionality 2^{2N+1} corresponding to the number of linearly independent generators of the Clifford algebra [28,30]. These generators can be written as

$$1, \gamma_1, \gamma_2, \dots, \gamma_{2N+1}, \quad (41)$$

$$\gamma_1 \gamma_2, \gamma_1 \gamma_3, \dots \quad (42)$$

$$\gamma_1 \gamma_2 \gamma_3, \dots \quad (43)$$

$$\vdots \quad (44)$$

$$\gamma_1 \gamma_2 \dots \gamma_{2N+1}. \quad (44)$$

In addition to the Majorana fermion operators, all the higher-order products are also the generators of the Clifford algebra corresponding to $2N + 1$ Majorana fermions. It is these higher-order generators of the Clifford algebra that also play a very important role in the dynamics of the Majorana fermion systems. Based on these generators of the Clifford algebra of Majorana fermions, we can write down the Hermitian operators which will commute with the Hamiltonian and hence are the symmetries of the Hamiltonian. These symmetries are the extra symmetries in addition to the Z_2 symmetry present in any quadratic Majorana fermion Hamiltonian. These symmetries are the emergent symmetries generated by higher-order Majorana fermion generators.

The most general local quadratic Hamiltonian for the Majorana fermions can be written as

$$H = i \sum_{ij} h_{ij} \gamma_i \gamma_j. \quad (45)$$

Due to the anti-commuting nature of the Majorana fermions, $h_{ij} = -h_{ji}$, this Hamiltonian has manifest Z_2 symmetry and consequently the Hamiltonian can be diagonalized in the parity eigenbasis. Since the Hamiltonian is bi-linear in Majorana fermion operators, all the elements of the algebra (of Equation (44)) with an even number of Majorana fermion (even grading) will commute with Hamiltonian and hence are the emergent symmetries of it. To have topological order, there must exist Majorana mode operators as defined in Section 3. As we defined the Majorana mode operator for the case of three Majorana fermions, similarly we find that there is a corresponding emergent Majorana fermion operator as written below:

$$\Gamma = i^{(2N+1)2N} \gamma_1 \gamma_2 \dots \gamma_{2N+1} \quad \Gamma^2 = 1 \quad [\Gamma, H] = 0. \quad (46)$$

Being the product of all Majorana fermions, it commutes with the Hamiltonian. It also squares to unity and anti-commutes with the parity operator. Hence it satisfies all the properties of a Majorana mode operator, and hence we show that the quadratic Majorana fermion model exhibits topological order. There are other Majorana mode operators in addition to the ones that we have written down above as shown in [20,29]. Since the emergent Majorana mode as written down in Equation (46) is a general one valid for any system of odd number of Majorana fermion, we will state our results as a theorem:

Theorem 2. *The emergent Majorana fermion for a systems of odd number of Majorana fermions is the zero mode operator of the system.*

Proof. Now, we show that the emergent Majorana mode satisfies the two defining properties of a zero mode operator. For the proof, we do not include the phase factor of the emergent Majorana mode because that is going to change. So, let us write down the generalized emergent Majorana operator without phase factors for system with n Majorana fermions where n being odd:

$$\Gamma_{em} = \gamma_1 \gamma_2 \dots \gamma_n \quad (47)$$

It is the operator form of this emergent Majorana which is going to be important for the proof of the theorem. First, we show that this operator commutes with the Hamiltonian as given in Equation (45) which basically translates to proving that Γ_{em} operator commutes with Majorana fermion bilinears.

$$\Gamma_{em} \gamma_i \gamma_j \quad (48)$$

$$= (\gamma_1 \gamma_2 \dots \gamma_i \dots \gamma_j \dots \gamma_n) \gamma_i \gamma_j \quad (49)$$

$$= (-1)^{n-i} (\gamma_1 \gamma_2 \dots \gamma_{i-1} \hat{\gamma}_i \gamma_{i+1} \dots \gamma_j \dots \gamma_n) \gamma_j \quad (50)$$

$$= (-1)^{n-i} (-1)^{n-j} (\gamma_1 \gamma_2 \dots \hat{\gamma}_i \dots \hat{\gamma}_j \dots \gamma_n) \quad (51)$$

$$= (-1)^{i+j} (\gamma_1 \gamma_2 \dots \hat{\gamma}_i \dots \hat{\gamma}_j \dots \gamma_n) \quad (52)$$

The operators with the hats are not the actual operators rather each one of them is unity which we get as the outside Majorana fermions on the rightmost are swapped until they reach the corresponding Majorana fermion with same index. Now, we need to prove other side of the equation,

$$\gamma_i \gamma_j \Gamma_{em} \quad (53)$$

$$= \gamma_i \gamma_j (\gamma_1 \gamma_2 \dots \gamma_i \dots \gamma_j \dots \gamma_n) \quad (54)$$

$$= (-1)^{j-1} \gamma_i (\gamma_1 \gamma_2 \dots \gamma_i \dots \hat{\gamma}_j \dots \gamma_n) \quad (55)$$

$$= (-1)^{j-1} (-1)^{i-1} \gamma_i (\gamma_1 \gamma_2 \dots \hat{\gamma}_i \dots \hat{\gamma}_j \dots \gamma_n) \quad (56)$$

$$= (-1)^{i+j} (\gamma_1 \gamma_2 \dots \hat{\gamma}_i \dots \hat{\gamma}_j \dots \gamma_n) \quad (57)$$

As can be seen from Equations (52) and (57), the emergent Majorana operator commutes with the Majorana fermion bilinears and hence with the Hamiltonian, that itself being a sum of Majorana fermion bilinears.

Now, we show that emergent Majorana operator anti-commutes with the parity operator. For that we need to recall from Section 4 that for odd number of Majorana fermions, parity operator itself can be defined in terms of the emergent Majorana fermion. $P = i\Gamma_{em}f$ where P is the parity operator and f is the extra Majorana fermion which is not part of the Hamiltonian. Once again we will drop the phase factors needed for the Hermiticity of the parity and emergent Majorana operators because they being scalars commute with all operators. So, for our purposes we just keep the operator structure of the parity operator which is $P = \gamma_1 \gamma_2 \dots \gamma_n f$. f being Majorana fermion squares to unity and anti-commutes with other Majorana fermions. $f^2 = 1$ $f\gamma = -\gamma f$

$$\Gamma_{em} P \quad (58)$$

$$= \gamma_1 \gamma_2 \dots \gamma_n \gamma_1 \gamma_2 \dots \gamma_n f \quad (59)$$

$$P \Gamma_{em} \quad (60)$$

$$= \gamma_1 \gamma_2 \dots \gamma_n f \gamma_1 \gamma_2 \dots \gamma_n \quad (61)$$

$$= (-1)^n \gamma_1 \gamma_2 \dots \gamma_n \gamma_1 \gamma_2 \dots \gamma_n f \quad (62)$$

In the last equation, f Majorana fermion has been passed through n Majorana fermions, to take it to the rightmost end which leads to the phase factor in front.

Combining two sides of the equation, we get

$$\Gamma_{em}P = (-1)^n P\Gamma_{em} \quad (63)$$

Now, n being an odd number in our case, we see that Parity operator anticommutes with the Γ_{em} .

$$\Gamma_{em}P = -P\Gamma_{em} \quad (64)$$

$$\Gamma_{em}P + P\Gamma_{em} \quad (65)$$

$$\{\Gamma_{em}, P\} = 0 \quad (66)$$

That completes the proof of our theorem. \square

We have proved that emergent Majorana operator Γ_{em} for the general case of n odd Majorana fermions commutes with the Hamiltonian and anti-commutes with the parity operator and hence Γ_{em} is the zero mode operator.

The quadratic Majorana fermion model as given in Equation (45) has manifest Z_2 symmetry but there are additional symmetries as shown above. We call these symmetries emergent symmetries while the Z_2 symmetry is the microscopic symmetry of the Hamiltonian, and it leads to the parity(fermion number) conservation. In the parity basis, the Hamiltonian takes quite a nice form as given in [16].

$$H = \epsilon(|e\rangle\langle e| + |o\rangle\langle o|), \quad (67)$$

where $|e\rangle, |o\rangle$ are even and odd parity states. In this notation, Majorana mode operators take the conceptually intuitive form:

$$\Gamma = (|e\rangle\langle o| + |o\rangle\langle e|). \quad (68)$$

This form of Γ brings out its property of flipping the parity state because it anti-commutes with the parity operator. So the matrix structure of the Majorana fermion model is not just block-diagonal with even and odd parity blocks, rather it has two more blocks corresponding to the identity operator and Γ operator, both of which occupy a single element block in the Hamiltonian matrix.

Based on our analysis of the quadratic Majorana model, we can now understand why there is topological order in the Kitaev chain model. Note first that Z_2 symmetry is present in both the Kitaev chain model and its Jordan–Wigner dual-spin model. We have also seen that for the special case of parameters of the Kitaev chain model, it reduces to a quadratic Hamiltonian of the kind that we have studied above, and hence the same analysis holds true for it also. We can now see that the topological phase arises in the Kitaev chain model because there are Majorana mode operators that commute with the Hamiltonian and anti-commute with the parity operator. For the spin model, there are no such symmetries that can lead to topological order. So we find that topological order in the Kitaev chain model arises due to the enrichment of the emergent symmetries generated by higher-order products of the Majorana fermion operators. What we have found goes beyond the duality between Landau order in TFIM and the Kitaev chain model. Based on duality, there is a transformation of local observables to non-local observables but we can not find the generators of the emergent symmetries that lead to topological order in the Kitaev chain model.

In other words, in the case of the Majorana fermion Hamiltonian, there are many conserved quantities whose Liouvillian vanishes and they are conserved under Liouvillian dynamics. This picture can be compared to the one presented in [31] where the protection of the Majorana edge modes has been shown to be related to prethermalization.

5.1. Interactions

Now we consider the Hamiltonian with quartic fermion terms and show that emergent Majorana operators exist in the presence of interactions as well.

$$H = i \sum_{ij} h_{ij} \gamma_i \gamma_j + \sum_{ijkl} V_{ijkl} \gamma_i \gamma_j \gamma_k \gamma_l. \quad (69)$$

The first term of the Hamiltonian is the same as Equation (45). The second term is the interaction term where V_{ijkl} is real and anti-symmetric under odd permutations. A first thing to be noted is that the interaction term does not break parity symmetry and hence we can still decompose the Hamiltonian into even and odd parity blocks. However, due to the interaction effects, Majorana mode operators are no longer the same as the ones for the mean-field Hamiltonian, as calculated in Section 2 above. In presence of the interactions, the zero-mode operator will get many-body character because it will be a zero-mode for an interacting Hamiltonian which means that the zero-mode operator has to commute with the interaction term as well. To arrive at the operator form for the many-body zero-mode, we will use Wegner's flow equation method which is a renormalization method for quantum many-body Hamiltonians [32,33]. This method has been recently used to calculate the zero-mode operator (also called l-bits) for Hamiltonians which exhibit many-body localization.

Within Wegner's method, the flow equation for the Hamiltonian is given by

$$\frac{dH}{dl} = [\eta, H], \quad (70)$$

where l is the flow parameter that runs from zero to infinity, zero index referring to the initial bare or unrenormalized Hamiltonian. η is the generator of the renormalization flow and within Wegner's scheme it is given by

$$\eta = [H_0, H_{int}]. \quad (71)$$

H_0 being the diagonal part of the Hamiltonian and H_{int} is the off-diagonal or the interaction part of the Hamiltonian. One of the appealing features of Wegner's flow equation method is that one can get the flow equations for the observables also using the same generator. The method also allows us to make the ansatz for the operator form of a given observable.

Using Wegner's flow equation method, we can write down the ansatz for the many-body Majorana zero mode operator:

$$\gamma_{MB} = \sum_i u_i \gamma_i + \sum_{ijk} u_{ijk} \gamma_i \gamma_j \gamma_k + \dots \quad (72)$$

where the three dots refer to the higher-order products of odd Majorana fermions. u_i and u_{ijk} and their higher-order versions are the coupling constants for this operator flow and their numerical values get determined by the flow equations. These coupling constants grow or decay depending on how physically relevant they are for the flow of γ_{MB} . The ansatz which we have arrived at using the flow equation method is in line with the numerical studies of such an operator [16]. The particular agreement with [16] is about the many-body character of this renormalized zero-mode operator.

6. Symmetry Algebra of Topological Protection

In the previous section, we have seen that topological order in one-dimensional lattice models of Majorana fermions, can be understood in terms of fermionic zero modes which is characteristic of such topological phases. These fermionic zero modes and the corresponding operators are the emergent symmetries of the Hamiltonian as well. In this section, we will put our earlier results in a larger mathematical framework to understanding

topological order especially the topological protection of the ground state, on the basis of symmetries.

Topological protection of the ground state manifold is an important feature of topologically ordered systems. The long-range topological order (true topological order) has been mostly approached from a topological approach which identifies a topological invariant using algebraic topology or using topological quantum field theories that offer the effective description of topologically-ordered systems. Recently, microscopic Hamiltonians have been introduced which exhibit topological order. Kitaev's Toric code and Levin–Wen model are two very well-known examples of these Hamiltonians. It is known that topological order can not be captured in terms of local order parameters because, in such systems, long-range quantum entanglement plays an essential role. Unlike long-range topological order, Short-range topological order is always protected by symmetries. We are interested in taking a symmetry algebra approach to short-range topological order especially the one exhibited by Majorana fermion Hamiltonians which are the focus of this paper. We show this symmetry algebra necessitates the ground state degeneracy which is topologically protected in the sense that no local perturbation can lift this degeneracy.

We will generalize our results from Section 3 and reformulate them in the symmetry algebra framework. Let us consider a system where we have N Majorana fermions, N being an odd number. Like in Section 4, we can write down a quadratic Hamiltonian that describes the nearest neighbor interactions between Majorana fermions. We can define a local parity operator $P_{loc} = i\gamma_j\gamma_k$ for any j -th and k -th Majorana fermion pair. However, to define global parity for this system, we will need a stray Majorana fermion as discussed in [20]. Global parity operator can be now defined as $P_{tot} = i\Gamma f$ where Γ is the emergent Majorana mode operator as defined in Equation (44) and f is the extra Majorana fermion. As parity operator, $P_{tot}^2 = 1$ and hence has two eigenvalues $\lambda = \pm 1$ which correspond to two parity states, even parity and odd parity. Now, we can easily identify two operators that both commute with the Hamiltonian and hence are symmetries of the Hamiltonian.

$$P = i\Gamma f \quad Q = \Gamma \quad [P, Q] = [Q, H] = 0. \quad (73)$$

If P and Q would also commute with each other, then we can simultaneously diagonalize all the three operators. However, P and Q do not commute; rather, they anti-commute with each other.

$$\{i\Gamma f, \Gamma\} = 0 \quad \{P, Q\} = 0. \quad (74)$$

So, Hamiltonian can be simultaneously diagonalized only with either of P and Q operators. Since P and Q operators do not commute, so the ground state of the Hamiltonian can not be non-degenerate. In the eigen-basis of the parity operator P_{tot} , the Hamiltonian will have two ground states corresponding to the even and odd parity. The Hamiltonian will become block diagonal in the parity basis. Additionally, if we also require P and Q operators to square to unity, then the order of the degeneracy will be two only and we will have a double degenerate ground state.

$$\{P, Q\} = 0 \quad P^2 = Q^2 = 1. \quad (75)$$

It is interesting to note that P and Q operators generate Clifford algebra among themselves. So, the symmetry operators of the Hamiltonian are actually Clifford algebra generators.

To elucidate, how the symmetry algebra of Γ and P_{tot} lead to degeneracy for the Hamiltonian H_{Maj} , let's take a closer look at the physical implications of the algebra. Let $|e\rangle$

and $|o\rangle$ be even and odd parity eigenstates of the parity operator P_{tot} . We can diagonalize the Hamiltonian in this parity basis.

$$H|e\rangle = \epsilon_e|e\rangle \quad H|o\rangle = \epsilon_o|o\rangle. \quad (76)$$

Since P and Q anti-commute with each other, we would like to apply both of them to the parity eigenstates:

$$P|e\rangle = +1|e\rangle \quad P|o\rangle = -1|o\rangle \quad (77)$$

$$Q(P|e\rangle) = Q(+1|e\rangle) \quad (78)$$

$$-P(Q|e\rangle) = Q|e\rangle \quad (79)$$

$$P(Q|e\rangle) = -(Q|e\rangle). \quad (80)$$

Since there is already a unique parity state corresponding to -1 eigenvalue, therefore $Q|e\rangle$ is actually odd parity state, $Q|e\rangle = |o\rangle$. So, the effect of the Q operator is to take an even parity state to an odd one and vice versa. In that sense, the Q operator is a cyclic permutation operator. It is the Q operator that leads to the transitions between even and odd parity sectors; otherwise, the parity symmetry of the Hamiltonian does not allow any such transitions between even and odd parity sectors.

We have seen that symmetry algebra necessitates that eigenstates of the Hamiltonian come in pairs. Corresponding to each eigenvalue of the Hamiltonian, there will be two eigenstates with even and odd parity respectively. Now, we will show that this pair of states actually has the same energy eigenvalue and hence are degenerate. To do that, all we need to look at is the fact that the Q operator which permutes even and odd parity states, is also the symmetry operator of the Hamiltonian.

$$QH|e\rangle = \epsilon_e(Q|e\rangle) \quad (81)$$

$$H(Q|e\rangle) = \epsilon_e(Q|e\rangle) \quad (82)$$

$$H(|o\rangle) = \epsilon_e|o\rangle, \quad (83)$$

where in the last line we have we have used the fact that Q operator changes even parity to odd parity state. From Equation (80), it is obvious that $\epsilon_e = \epsilon_o$, that even and odd parity states have same energy and hence are degenerate.

So, we have shown that for the quadratic Hamiltonian for a system of an odd number of Majorana fermions, the symmetry algebra generated by global symmetry operator Γ and parity operator P_{tot} leads to the doubly degenerate ground state. However, what is even more interesting is that this double degeneracy is present not only in the ground state but the whole excitation spectrum of the Hamiltonian such that corresponding to each eigenstate of the Hamiltonian with even parity there is another eigenstate with odd parity.

This degeneracy which we have obtained from the symmetry algebra is different than the one which we have for a quantum system whose Hamiltonian commutes with various symmetry operators. Those degeneracies are susceptible to a local perturbation that leads to the lifting of the degeneracy. This degeneracy is protected by non-local symmetry operators and these symmetries can not be broken by local perturbations. The algebraic structure of Equations (73) and (74) ensures the topological protection of the ground state and the encoded information. As long as these symmetries as represented by P and Q operators are present, quantum information stored in the degenerate ground state is also preserved.

The symmetry algebra which we have introduced in this section is very general and is not only applicable for Majorana fermion Hamiltonians which exhibit Z_2 topological order but it can also be applied for lattice spin models which have exhibit long-range topological order as considered in [34]. Our route to the symmetry algebra is different than [34] because our motivation and starting point is Majorana zero modes which are important elements in our symmetry algebra. It should be noted that the Majorana zero

modes which enter symmetry algebra are not only the ones that are edge modes but more importantly the ones that are the global symmetry operators of the Hamiltonian.

7. Topological Order and Yang-Baxter Equation

Majorana fermions have been the focus of interest in research in topological quantum computation because as shown in [35–37] that Majorana fermions have non-abelian braid statistics and generate a representation of braid group. Kitaev chain realization of Majorana fermions have given ways to engineer Majorana fermions and there has already been some progress on that front [38]. It has also been realized [39] that the Majorana representation of the braid group is different than the ones known in the literature. This representation has been called a type-II representation. Now the question which has been asked is that is the topological order which arises from quantum entanglement is also related to topological entanglement which arises from the solutions of the Yang–Baxter equation. Majorana fermions give new solutions to Yang–Baxter equations and hence the new type of topological entanglement. When there is topological order, we get a representation of the braid group and also solutions to YBE. We will first briefly review the Majorana fermion representation of the braid group [37,40]. Then we will discuss the new solution of Yang–Baxter equation (YBE) which has been called the type-II solution. One important property of the type-II solution is its R matrix commutes with the Γ operator which is the Majorana edge mode operator.

Braiding operators arise from a row of Majorana Fermions $\{\gamma_1, \dots, \gamma_n\}$ as follows: Let

$$\sigma_i = (1/\sqrt{2})(1 + \gamma_{i+1}\gamma_i). \quad (84)$$

Note that if we define

$$\lambda_k = \gamma_{i+1}\gamma_i \quad (85)$$

for $i = 1, \dots, n$ with $\gamma_{n+1} = \gamma_1$, then

$$\lambda_i^2 = -1 \quad (86)$$

and

$$\lambda_i\lambda_j + \lambda_j\lambda_i = 0, \quad (87)$$

where $i \neq j$. From this it is easy to see that

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (88)$$

for all i and that

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad (89)$$

when $|i - j| > 2$. Thus we have constructed a representation of the Artin braid group from a row of Majorana fermions. This construction is due to Ivanov [37] and he notes that

$$\sigma_i = e^{(\pi/4)\gamma_{i+1}\gamma_i} \quad (90)$$

In [39] authors make the further observation that if we define

$$\check{R}_i(\theta) = e^{\theta\gamma_{i+1}\gamma_i}. \quad (91)$$

Then $\check{R}_i(\theta)$ satisfies the full Yang–Baxter equation with rapidity parameter θ . That is, we have the equation

$$\check{R}_i(\theta_1)\check{R}_{i+1}(\theta_2)\check{R}_i(\theta_3) = \check{R}_{i+1}(\theta_3)\check{R}_i(\theta_2)\check{R}_{i+1}(\theta_1). \quad (92)$$

This makes it very clear that $\check{R}_i(\theta)$ has physical significance, and suggests examining the physical process for a temporal evolution of the unitary operator $\check{R}_i(\theta)$.

In fact, following [39], we can construct a Kitaev chain based on the solution $\check{R}_i(\theta)$ of the Yang–Baxter Equation. Let a unitary evolution be governed by $\check{R}_i(\theta)$. When θ in the unitary operator $\check{R}_i(\theta)$ is time-dependent, we define a state $|\psi(t)\rangle$ by $|\psi(t)\rangle = \check{R}_i|\psi(0)\rangle$. With the Schrödinger equation $i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$ one obtains:

$$i\hbar \frac{\partial}{\partial t} [\check{R}_i|\psi(0)\rangle] = \hat{H}(t)\check{R}_i|\psi(0)\rangle. \quad (93)$$

Then the Hamiltonian $\hat{H}_i(t)$ related to the unitary operator $\check{R}_i(\theta)$ is obtained by the formula:

$$\hat{H}_i(t) = i\hbar \frac{\partial \check{R}_i}{\partial t} \check{R}_i^{-1}. \quad (94)$$

Substituting $\check{R}_i(\theta) = \exp(\theta\gamma_{i+1}\gamma_i)$ into Equation (94), we have:

$$\hat{H}_i(t) = i\hbar \dot{\theta} \gamma_{i+1} \gamma_i. \quad (95)$$

This Hamiltonian describes the interaction between i -th and $(i+1)$ -th sites via the parameter $\dot{\theta}$. When $\theta = n \times \frac{\pi}{4}$, the unitary evolution corresponds to the braiding progress of two nearest Majorana fermion sites in the system as we have described it above. Here n is an integer and signifies the time of the braiding operation. We remark that it is interesting to examine this periodicity of the appearance of the topological phase in the time evolution of this Hamiltonian. For applications, one may consider processes that let the Hamiltonian take the system right to one of these topological points and then this Hamiltonian cuts off. One may also think of a mode of observation that is tuned in frequency with the appearances of the topological phase.

In [39] authors also point out that if we only consider the nearest-neighbour interactions between Majorana Fermions, and extend Equation (95) to an inhomogeneous chain with $2N$ sites, the derived model is expressed as:

$$\hat{H} = i\hbar \sum_{k=1}^N (\dot{\theta}_1 \gamma_{2k} \gamma_{2k-1} + \dot{\theta}_2 \gamma_{2k+1} \gamma_{2k}), \quad (96)$$

with $\dot{\theta}_1$ and $\dot{\theta}_2$ describing odd-even and even-odd pairs, respectively.

They then analyze the above chain model in two cases:

1. $\dot{\theta}_1 > 0, \dot{\theta}_2 = 0$.

In this case, the Hamiltonian is:

$$\hat{H}_1 = i\hbar \sum_k^N \dot{\theta}_1 \gamma_{2k} \gamma_{2k-1}. \quad (97)$$

The Majorana operators γ_{2k-1} and γ_{2k} come from the same ordinary fermion site k , $i\gamma_{2k}\gamma_{2k-1} = 2a_k^\dagger a_k - 1$ (a_k^\dagger and a_k are spinless ordinary fermion operators). \hat{H}_1 simply means the total occupancy of ordinary fermions in the chain and has $U(1)$ symmetry, $a_j \rightarrow e^{i\phi} a_j$. Specifically, when $\theta_1(t) = \frac{\pi}{4}$, the unitary evolution $e^{\theta_1 \gamma_{2k} \gamma_{2k-1}}$ corresponds to the braiding operation of two Majorana sites from the same k -th ordinary fermion site. The ground state represents the ordinary fermion occupation number 0. In comparison to 1D Kitaev model, this Hamiltonian corresponds to the trivial case of Kitaev's. This Hamiltonian is described by the intersecting lines above the dashed line, where the intersecting lines correspond to interactions. The unitary evolution of the system $e^{-i\int \hat{H}_1 dt}$ stands for the exchange process of odd-even Majorana sites.

2. $\dot{\theta}_1 = 0, \dot{\theta}_2 > 0$.

In this case, the Hamiltonian is:

$$\hat{H}_2 = i\hbar \sum_k^N \dot{\theta}_2 \gamma_{2k+1} \gamma_{2k}. \quad (98)$$

This Hamiltonian corresponds to the topological phase of 1D Kitaev model and has \mathbb{Z}_2 symmetry, $a_j \rightarrow -a_j$. Here the operators γ_1 and γ_{2N} are absent in \hat{H}_2 . The Hamiltonian has two degenerate ground state, $|0\rangle$ and $|1\rangle = d^\dagger |0\rangle$, $d^\dagger = e^{-i\varphi/2}(\gamma_1 - i\gamma_{2N})/2$. This mode is the so-called Majorana mode in 1D Kitaev chain model. When $\theta_2(t) = \frac{\pi}{4}$, the unitary evolution $e^{\theta_2 \gamma_{2k+1} \gamma_{2k}}$ corresponds to the braiding operation of two Majorana sites γ_{2k} and γ_{2k+1} from k -th and $(k+1)$ -th ordinary fermion sites, respectively.

Thus the Hamiltonian derived from $\check{R}_i(\theta(t))$ corresponding to the braiding of nearest Majorana fermion sites is exactly the same as the 1D wire proposed by Kitaev, and $\dot{\theta}_1 = \dot{\theta}_2$ corresponds to the phase transition point in the “superconducting” chain. By choosing different time-dependent parameters θ_1 and θ_2 , one finds that the Hamiltonian \hat{H} corresponds to different phases. These observations of Mo-Lin Ge give physical substance and significance to the Majorana Fermion braiding operators discovered by Ivanov [37], putting them into a robust context of Hamiltonian evolution via the simple Yang–Baxterization $\check{R}_i(\theta) = e^{\theta \gamma_{i+1} \gamma_i}$. Yu and Mo-lin Ge [39] make another observation, that we wish to point out. In [40], Kauffman and Lomonaco observe that the Bell Basis Change Matrix in the quantum information context is a solution to the Yang–Baxter equation. Remarkably this solution can be seen as a 4×4 matrix representation for the operator $\check{R}_i(\theta)$.

This lets one can ask whether there is a relation between topological order and quantum entanglement and braiding [40] which is the case for the Kitaev chain where non-local Majorana modes are entangled and also braiding.

The Bell-Basis Matrix B_{II} is given as follows:

$$B_{II} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(I + M) \quad (M^2 = -1) \quad (99)$$

and

$$M_i M_{i\pm 1} = -M_{i\pm 1} M_i, \quad M^2 = -I, \quad (100)$$

$$M_i M_j = M_j M_i, \quad |i - j| \geq 2. \quad (101)$$

Remark 1. The operators M_i take the place here of the products of Majorana Fermions $\gamma_{i+1} \gamma_i$ in the Ivanov picture of braid group representation in the form

$$\sigma_i = (1/\sqrt{2})(1 + \gamma_{i+1} \gamma_i).$$

This observation of authors in [39] gives a concrete interpretation of these braiding operators and relates them to a Hamiltonian for the physical system. This goes beyond the work of Ivanov [37], who examines the representation of Majoranas obtained by conjugating by these operators. The Ivanov representation is of order two, while this representation is of order eight. The reader may wish to compare this remark with the contents of [41] where we associate Majorana fermions with elementary periodic processes. These processes can be regarded as prior to the periodic process associated with the Hamiltonian of Yu and Mo-Lin Ge [39].

Remark 2. We write down a Majorana fermion representation of Temperley–Lieb algebra (TLA) which is related to the Braid group representation discussed above. We define A and B as $A = \gamma_i \gamma_{i+1}$, $B = \gamma_{i-1} \gamma_i$ where $A^2 = B^2 = -1$. Note the following relations:

$$U = (1 + iA) \quad V = (1 + iB), \quad (102)$$

$$U^2 = 2U \quad V^2 = 2V, \quad (103)$$

$$UVU = V \quad VUV = U. \quad (104)$$

Thus, a Majorana fermion representation of TLA is given by:

$$U_k = \frac{1}{\sqrt{2}}(1 + i\gamma_{k+1}\gamma_k), \quad (105)$$

$$U_k^2 = \sqrt{2}U_k, \quad (106)$$

$$U_k U_{k\pm 1} U_k = U_k, \quad (107)$$

$$U_k U_j = U_j U_k \quad \text{for } |k - j| \geq 2. \quad (108)$$

Hence we have a representation of the Temperley–Lieb algebra with loop value $\sqrt{2}$. Using this representation of the Temperley–Lieb algebra [8,42], we can construct (via the Jones representation of the braid group to the Temperley–Lieb algebra) another representation of the braid group that is based on Majorana Fermions. It remains to be seen what is the physical significance of this new representation.

Topological Order and Topological Entanglement

One of the main aims of this paper is to understand the relation between the topological order which comes from quantum entanglement and topological entanglement which comes from braid group representation and Yang–Baxter equation. This is an extension of the work [40,41] in which it was shown how quantum entanglement is related to the braid group and Yang–Baxter equation. So it became natural to understand topological order based on the approach of [8,40,41]. In this section, we will show that the Yang–Baxter equation-based approach opens new ways to understand the topological order in case of Majorana fermion models. However, this approach is not restricted to Majorana fermions only rather it is a very general approach that is not limited to quadratic Hamiltonians and hence offers a new method for the classification of the topological phases, which goes beyond K theory and Berry phase-based methods.

To understand the relation between quantum entanglement in the Kitaev chain model and the corresponding topological entanglement which manifests as braid group representation, we point out that it is only in the topological phase of the Kitaev chain model that braid group representation arises while as in topologically trivial phase there are no Majorana edge modes and hence no braid group representation. To see this relation mathematically, we rewrite the Kitaev chain Hamiltonian corresponding to the topological phase.

$$H = 2it \sum_{i=0}^{N-1} \gamma_{1,i+1} \gamma_{2,i}, \quad (109)$$

and now find out that for Majorana representation as shown by Ivanov we need the operator of the form $\frac{1+\gamma_{i+1}\gamma_i}{\sqrt{2}}$ which arises only in the topological phase. So this brings out the relation between topological order and topological entanglement (braiding). The solution of the Yang–Baxter equation which arises in the topologically ordered phases of the Majorana fermion models is different than the ones which arise from the other Hamiltonians which do not exhibit topological order. Majorana fermion solutions are called type-II while the other ones are called type-I solutions [39]. Hence the Majorana fermion braiding solutions of the Yang–Baxter equation characterizes and classifies the topologically ordered phases. Using this relation we give a new characterization of topological order. A system

is said to be topologically ordered if it gives a type-II solution to Yang–Baxter equation. This characterization is very general and just depends on the braiding properties of anyons (Majorana fermions in this case) and hence should apply to other systems as well.

8. Summary

In this paper, we have taken a novel approach to explore the topological protection of qubits in Majorana fermion systems. We have explored the topological protection based on the fermionic mode operators which lead to the emergent symmetries of the Majorana fermion Hamiltonians. We have shown the existence of these fermionic mode operators is related to the Clifford algebra of Majorana fermions. The existence of the fermionic mode operators as integrals of motion for the Hamiltonian has been shown to an essential feature for topological protection. These fermionic mode operators correspond to the non-local symmetries which can not be broken by local perturbations and hence guarantee the topological protection of the qubits. For the generalized case of Hamiltonian with an odd number of Majorana fermions, we have explicitly written down the fermionic mode operator(s) which are necessary for the topological protection of the qubits.

In this paper, we have also unified the fermionic mode operators in the larger mathematical approach based on symmetries. This unified approach helps us to better understand the topological protection based on the symmetries of the Majorana fermion Hamiltonians. Our approach can be easily extended to other lattice models of Majorana fermion in higher dimensions.

We have also explored the effect of interactions on the Majorana mode operators and hence on the topological order. We find that in the presence of the interactions, the Majorana mode operators become dressed and become the linear combination of all Majorana fermion operators with odd parity. In the presence of the interactions, the Majorana fermion Hamiltonian has a four-block structure, two of them being even and odd parity blocks. The other single element blocks are those of the identity and the Γ operator, which both commute with Hamiltonian. We also notice the similarity between the proliferation of the conserved quantities in the topological order in Majorana fermion models and many-body localized phase (MBL) which is characterized by the existence of the local integrals of motion. This similarity is suggestive of some similar phenomenon leading to these seemingly unrelated phenomena. In that regard, we find the recent work of Fendley and collaborators quite insightful. They have shown that the existence of Majorana mode operators and hence the topological order in these models is a non-equilibrium dynamical phenomenon and is related to pre-thermalization.

We also explored the implications of emergent symmetries of Majorana fermion models on the braid group representation. We find that braid group generators commute with the Majorana mode operator and hence give a new solution to the Yang–Baxter equation, which has been called a type-II solution. We also write down a Majorana Fermion presentation of Temperley–Lieb algebra. The fact that a Majorana fermion representation of the braid group is different from the type-I solution, which is for spin models, shows that we can distinguish between topological order and Landau order based on the solutions of the Yang–Baxter equation. This gives a nice mathematical procedure to check for the topological order. For the Majorana fermion models including the Kitaev chain model, we notice that solutions to the Yang–Baxter equation exist only in the topological phase. So the Yang–Baxter equation can be used to explore topological order in the quantum Hamiltonians. The relation between the Yang–Baxter equation and topological order show that the topological order that arises due to the quantum entanglement is related to the topological entanglement of the braid group.

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