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A Meta-Logical Framework for the Equivalence of Syntactic and Semantic Theories

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Abstract

This paper introduces a meta-logical framework—based on the *theory of institutions* (a categorical version of abstract model theory)—to be used as a tool for the formalization of the two main views regarding the structure of scientific theories, namely the *syntactic* and the *semantic views*, as they have emerged from the relevant contemporary discussion. The formalization leads to a proof of the equivalence of the two views, which supports the claim that the two approaches are not really in tension. The proof is based on the *Galois connection* between classes of sentences and classes of models defined over some institution. First, the history of the syntactic–semantic debate is recalled and the theory of institutions formally introduced. Secondly, the notions of syntactic and semantic theories are formalized within the institution and their equivalence proved. Finally, the novelty of the proposed framework is highlighted with respect to existing formalizations.

Keywords: scientific theories; theory of institutions; syntactic view; semantic view

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1. Introduction

It is widely accepted that a principal characteristic of the tradition of Western philosophy and science is the search for definitions. Definitions serve to clarify our concepts, leading to improved understanding of the world around us and ourselves by consciously refining, revising, and correcting our interim conceptions of theories and things. This process is inextricably linked to the time and place of research, as well as to the persons and institutions engaged in it. The very question of giving a definition of the notion of a *scientific theory* came to prominence in the 1960s, when the two main approaches, hence called the *syntactic* and the *semantic view*, were first formulated. Today, we witness a resurgence of interest in this crucial issue: the two sides in the current discussion representing the syntactic and the semantic approach—initially viewed as irreconcilable—focus their argumentation on equivalence issues. This paper aims at taking part in the current debate on the relation between the syntactic and semantic view of theories by introducing a category-based formal framework, the *theory of institutions* [1], and showing how the equivalence between the two approaches can be proved in a totally language-independent way.

Specifically, this section takes a closer look at the debate on the syntactic and semantic formulations of scientific theories. Section 2 formally introduces the theory of institutions. Section 3 formalizes, within the introduced meta-logical framework, the notions of syntactic and semantic theories and proves their equivalence, the main contribution of this paper.

Section 4 highlights how the novelty of the proposed framework, with respect to existing formalizations, relies on its language independence and in the opportunity of specifying many interesting relations among theories. Finally, Section 5 concludes the paper.

1.1. The Standard/Received View of Theories

The syntactic, or "standard", view of scientific theories is a product of logical empiricism[1]. As Hempel put it [2], a scientific theory can be formally considered as a class of sentences expressed in terms of a specific vocabulary. More precisely, a theory T is formalized as an axiomatized deductive system, or a calculus, formulated within a linguistic framework of perspicuous logical structure, which determines the relevant rules of inference. The basic constituents of this framework are as follows:

1. A vocabulary V of primitive terms, usually divided into logical, theoretical and observation terms.
2. A set of sentences, which include correspondence rules.
3. A set of axioms (primitive sentences or postulates).
4. A set of rules of inference (to derive theorems or derivative sentences from the axioms).

It is true that the deductive development of such a system does not require the assignment of meanings to its expressions—primitive or not—provided that the axioms have been—fully, or at least adequately—specified. But in order for it to function as a theory in empirical science, it must be given an interpretation by reference to empirical phenomena. Acknowledging this fact, the syntactic view takes a scientific theory to be an interpreted calculus, that is, (a) an uninterpreted calculus whose axioms correspond to the theory's basic principles, and (b) a set of sentences R, usually called "correspondence rules", which give empirical import to the calculus by interpreting some of its formulas in empirical terms. Schematically we could write—following Hempel [3]—$T = (C, R)$, where C is the set of all formulas of the calculus, and R the set of correspondence rules.

1.2. Suppes' Semantic Approach to Theories

In his unduly neglected *What is a scientific theory?* [4], Patrick Suppes criticizes the "received" view[2] of Carnap and Hempel, as being far too simplistic. More specifically, Suppes thinks it is more of a sketch of the notion of a scientific theory, which can lead to serious misconceptions; but it can be made more rigorous thanks to the tools of modern logic. In fact, Suppes warns against the possible consequences of such a compromise: omissions of important properties of theories, as well as significant distinctions that may be introduced between different theories. His critique of the standard sketch, as he calls it, concerns both a theory's formal part, and its empirical interpretation.

Suppes first turns to the question of the semantics for the theory's logical calculus; he finds that correspondence rules, or bridge principles, are inadequate as a complete specification of the semantics in question. Instead, he advocates further the extensive use of mathematical models, which he regards as (absolutely) necessary for the study of theories which do not have a manageable formalization in first order logic (*standard formalization*). For as far as theories with a complicated structure are concerned, like the theories of quantum mechanics, classical thermodynamics, or general relativity, whose axiomatization in first-order logic is impractical[3], it is neither easy nor natural to treat them as linguistic entities of high complexity. In fact, it is simpler to make assertions about their models than to talk about their sentences, mainly because the notion of a sentence is not well-defined[4] when the theory it belongs to is not given in standard formalization [4]. The simplicity of the model-theoretic characterization of such theories can lead to results of great generality; using models it can be proven, for example, that the two classical versions of quantum

mechanics—matrix mechanics and wave mechanics—are identical, as they are both realizations of a complex separable Hilbert space which is unique up to isomorphism[5] [7].

One main point made by Suppes concerns the empirical interpretation of theories, which (in Hempel's later formulation of the standard view) is given by the bridge principles. In particular, he argues that the actual practice of testing theories by relating them to empirical data is much more complicated than the construction of correspondence rules indicates. An experiment is a concrete experience which cannot be connected to a theory in any complete sense. By putting this experience through a—rather coarse—conceptual grinder the experimental data emerge in canonical form, giving rise to a model of the experiment. Correspondence rules—or bridge principles—are provided for this latter model, and not for a model of the formal theory. Furthermore, models of the experiment are of a different logical type than the models of the theory; while the first are highly discrete and finitistic, models of a theory usually include continuous functions and infinite sequences[6]. Neither can correspondence rules be thought of as establishing the right links between models of the theory and models of the experiment, as that would be an oversimplification; correspondence rules are unable, for example, to capture the elaborate methods used to estimate theoretical parameters in the model of the theory based on models of the experiment. Modern statistical methodology, in dealing with the fundamental problem of relating raw experimental data with a scientific theory, has developed an elaborate theory of experimentation which interposes a whole hierarchy of theories between them. Thus, contrary to what the concept of correspondence rules suggests, there is no simple way of giving an empirical interpretation of a theory.

These critical considerations led Suppes to postulate a hierarchy of models between a theory and its experimental testing. In this framework a theory's formal semantics given by mathematical models are connected through a representation theorem with empirical models of the most skeletal form. These highly abstract models of the experiment are in their turn connected with the raw experimental data through the mediation of a whole hierarchy of models (see Figure 1 below). In particular, models of the experiment are preceded by models of data, designed to include all and only those aspects of the experiment which have parametric analogues in the theory, so that the relevant information can be used in testing the theory's adequacy[7]. But models of data are still far removed from the concrete experimental experience. At least two more levels are postulated by Suppes in his discussion of a specific example taken from learning theory [9]: one concerning experimental design and another concerning the experiment's *ceteris paribus* conditions. Note that empirical meaning flows upwards in this hierarchy; the theory corresponding to the models of each level is given empirical meaning through the establishment of formal connections with the theory of the lower level. In this setting the logical (or statistical) relations holding between theories of different levels can be explored in a purely formal manner. For Suppes the existence of such a hierarchy of theories and models arising from the methodology of experimentation for testing a theory is an essential ingredient of any scientific discipline.

1.3. Contemporary Treatment of the Syntactic/Semantic Theories Debate

As logical empiricism waned and criticism of the syntactic view led to its eventual falling out of favor, the semantic approach as it was understood and presented by followers of Suppes (e.g., [10,11])—that is, as identifying theories with classes of model-theoretic structures—came into prominence. And although for a long period there was little direct discussion, today we witness a renewed interest in the subject among philosophers of science. The contemporary discussion begins with a distinction made by Halvorson [8] between two main versions of the semantic view, the 'semantic+L' (semantic plus language) or liberal semantic view, and the 'semantic-L' (semantic minus language) or strict seman-

tic view, followed by arguing that semantic+L collapses into the syntactic view, while semantic-L leads to absurdities.

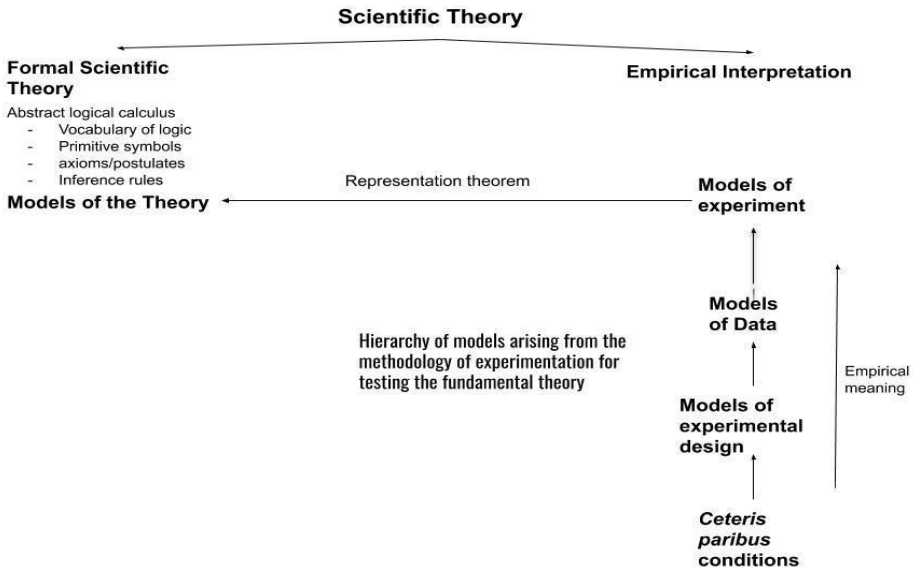


Figure 1. A diagram depicting Suppes' description of the structure of a scientific theory.

According to Halvorson semantic-L, which takes models to be completely non-linguistic entities (or pure set-theoretic structures) fails to distinguish or equate theories which, from a syntactic perspective are both intuitively and formally equivalent or inequivalent, respectively. More specifically, he gives examples of cases where two syntactically non-equivalent theories have equivalent semantic-L formalizations and, in a similar way, two syntactically equivalent theories are not model-theoretically equivalent. These examples are based on the relation of H-isomorphism, which Halvorson finds to be the only possible notion of model-theoretic isomorphism available to a semantic-L approach. Two (indexed) set-theoretic structures $\langle A; R_1, \ldots, R_n\rangle$, $\langle B; S_1 \ldots, S_n\rangle$ where R_s and S_s are relations of any order, are said to be H-isomorphic if and only if there exist two bijections $j : A \to B$ and k: $[\![\{Ri\}]\!]_{(i\in\{1,\ldots,n\})} \to [\![\{Si\}]\!]_{(i\in\{1,\ldots,n\})}$ such that if $\langle a_1, \ldots, a_n\rangle \in R_i$, then $\langle j(a_1), \ldots, j(a_n)\rangle \in k(R_i)$.

Glymour [12] criticizes the introduction of the concept of H-isomorphism by Halvorson as being too wide, treating relations as mere placeholders (since H-isomorphism allows for any nth-order relation to be mapped to any nth-order relation). Instead, he argues that Halvorson should have relied on the usual model-theoretic concept of isomorphism—referred to by Glymour as M-isomorphism—requiring models to be *elementarily equivalent*, that is, requiring predicates of one model (relations R_i interpreting predicates P_i) to be interpreted by the appropriate predicates of the other model (relations S_i interpreting the same predicates P_i). This classical understanding of isomorphism leads to the definition of a notion of model-theoretical equivalence which matches syntactical equivalence between theories, as shown by de Bouvere [13]. The definition is based on the notions of *definitional extension*[8] and *definitional expansion*[9] (or *definitional enrichment* in de Bouvere's terminology). In particular, two theories T, T' in disjoint languages are said to be *coalescent* if and only if their model classes can each be expanded to a class of relational structures for $L_{T'\cup T}$ so that every M-isomorphism of a T model extends uniquely to an M-isomorphism of an expanded structure, and every M-isomorphism of a T' model extends uniquely to an M-isomorphism of an expanded structure, and the expanded model classes are identical. de Bouvere has shown that two theories formulated in disjoined non-logical vocabularies[10] are equivalent (or *synonymous* in his own terminology) if and only if they

have a common definitional extension[11], or if their model classes are coalescent (or have a common definitional expansion)—that is, these alternative criteria are equivalent. Thus, equipped with de Bouvere's model-theoretic account of equivalence, the semantic view correctly reflects the—intuitively or formally established—syntactic equivalence relations holding between theories.

Replying to Glymour, Halvorson points out that his counterexamples were aimed at the semantic-L approach to theories, to which Glymour's notion of M-isomorphism is not available. These counterexamples were meant to show that purely semantic formalizations fail to individuate theories appropriately due to the fact that there is no good notion of isomorphism between classes of models of different theories if we preclude any reference to language. Glymour overcomes the counterexamples by adopting the standard notion of isomorphism between labelled structures which is only available to the semantic+L approach: M-isomorphism makes explicit mention of a first-order language L and of two nth-order relations belonging to two different relational structures both of which interpret the same predicate in L; and since labelled structures uniquely determine particular signatures, the use of M-isomorphism ensures that the relative semantic approach is essentially dependent on particular signatures [14]. Thus, Glymour's treatment can be viewed as supporting Halvorson's argument that the semantic-L approach is untenable. But once language is added to the semantic view, its purported advantages over the syntactic approach seem to be lost, argues Halvorson [15] quoting Van Fraassen: "the impact of Suppes' innovation is lost if models are defined, as in many standard logic texts, to be partially linguistic entities, each yoked to a particular syntax …" [11]. In other words, if we add language to the semantic view the latter seems to collapse into the syntactic view.

In light of Halvorson's critique of the semantic view, Van Fraassen admits that one cannot avoid reference to language when describing some given model, as models are always interpreted in a language [16]. His own semantic approach was rather addressed to draw attention away from language onto the representing structures, insofar as scientists present a theory by introducing a class of models which represent the empirical phenomena under investigation. But he stresses that models defining a scientific theory are not purely mathematical structures, but representations, that is, structures plus a theoretical hypothesis [16]. It follows that, while considering the equivalence of two semantic theories, their empirical adequacy [17] should also be considered. Two isomorphic structures representing different physical systems *do not* define equivalent theories; in order to do so it should also be possible for their representational relations to be established over the same empirical system. For Halvorson this point is in no tension with his argument, which solely concerns the formal/mathematical component of the semantic view and has nothing to do with a theory's empirical interpretation. But this does not seem to be true, because the semantic view described by Van Fraassen does not allow for a purely formal definition of equivalent theories. In fact, it can be seen as echoing Suppes' demand for a more rigorous account of scientific theories, as it advocates a more complete treatment of theoretical equivalence which will not rest exclusively on the formal part of theories.

Having established the untenability of a semantic-L view of theories, Halvorson focuses on the semantic+L approach. In particular, he observes that the latter seems to undermine many philosophical claims made by semanticists, like that the class of models identifying a theory is the invariant content lying behind its different possible linguistic formulations (or axiomatizations) [10,18]. This many-to-one relation between syntactic presentations of a theory and its class of models cannot be maintained if models are taken to be language-bound entities, because it then cannot account for equivalent theories with distinct classes of models (e.g., different axiomatizations of group theory, see [8])[12]. The semantic+L approach rather favors a many-to-many relation between syntactic structures

and semantic structures for a single theory, such that equivalent formulations of it in different languages reflect some kind of interconvertibility between their models (see Figure 2). But although this connection can be formally captured by de Bouvere's model-theoretic account of equivalence, the latter covers only first-order and single-sorted theories [15]. Thus, Halvorson concludes that correspondences between equivalences of linguistic formulations of theories and equivalences of their model classes should be deeply investigated, having the logical relations of Figure 2 be extended to any-order and multi-sorted logical systems.

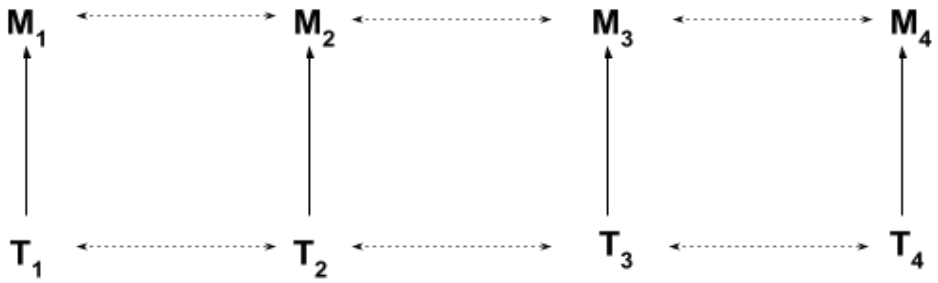


Figure 2. A diagram showing Halvorson's postulated connection between equivalent syntactic theories and their classes of models. The dashed arrows represent equivalence relations.

Overall, we could say that three main points emerge from the contemporary discussion outlined above. The first is the naiveté of the semantic-L view. Halvorson's arguments show that the strict semantic view is untenable, and both Glymour and Van Fraassen seem to agree. Moreover, Hudetz has recently shown that every family of set-theoretic structures has an—up to signature isomorphism—unique model-theoretic counterpart able to serve the same purposes [19]. In view of these results, we will focus exclusively on the liberal semantic view (since, in any case, it is arguable that any secure result transfers over to the strict semantic view). The second point is that the untenability of a semantic-L view of theories parallels the inadequacy of a syntactic view which excludes models, or a syntactic-M approach. This inadequacy has been clearly pointed out by Suppes, whose criticism of the received view shows that a syntactic account cannot go far without models. Thus, it turns out that a conception of theories as purely linguistic, or purely a-linguistic, entities cannot adequately capture their complex structure. Such oversimplifying accounts must be refined, and it is by now a common claim that their refined versions, the liberal syntactic and the liberal semantic view of theories, are no real alternatives [14,15,19–21], but merely distinguished by their focus on what they regard as primary. Finally, the third point is that syntactic and semantic formulations of a single theory are apparently in a many-to-many relation, and not in a many-to-one relation as many semanticists would have it. This many-to-many relation is such that the classes of models of equivalent syntactic formulations of a theory are also equivalent, in an appropriate sense of the term. But since de Bouvere's concept of model-theoretical equivalence captures this connection only for first-order and single-sorted theories, what is still missing is an extension of this concept of equivalence to cover any-order and multi-sorted cases.

In this article we give a formal proof of the equivalence of the liberal syntactic and semantic views of theories. The proof is based on a notion of model-theoretical equivalence which formally captures the equivalence of classes of models corresponding to equivalent syntactic formulations of a theory and is not limited to the first-order and single-sorted case. This notion of equivalence is provided by the logical framework of institutions, which allows for an adequately general formalization of the two approaches. In the next section we present the theory of institutions—a form of abstract model theory which uses the very basic concepts from category theory—and justify its choice for the task at hand. In par-

ticular, the provided notion of equivalence involves a notion of isomorphism based on signature morphisms in the category of both syntactic and semantic theories. This choice is motivated by the fact that such a notion of isomorphism goes beyond both H- and M-isomorphisms, in that on the one hand it is related to a language, i.e., a signature, but on the other hand it is independent from that given signature, being defined on signature morphisms. The use of abstract model theory as a promising tool for the formal study of philosophy of science in general has been extensively argued for in [22]. Finally, we should not fail to mention that our overall approach subscribes to Suppes' view of formalization as a primary method for the philosophical analysis of scientific concepts [7]. The resulting proof supports his transcendental argument in favor of formalization, which proclaims it necessary for the objective resolution of conflicts.

2. The Formal Framework of Institutions: Presentation and Justification

The idea of using institutions for the formalization of the liberal syntactic and the liberal semantic view of theories was motivated by their successful use in computer science for the formalization of program specifications (initially as syntactic theories, and recently also as semantic theories). The management of complex software systems specifications was in fact the driving force behind the original development of institutions theory in the 1980s [23]. Because the specification of different parts of complex programs required the use of (specification) languages based on a variety of logical systems, their combination (necessary for verification and proving correctness) was a very challenging task. The institutional framework solved this problem by providing highly abstract formal semantics for different specification languages, including a mechanism for the mutual translation and combination of specifications expressed in them. This solution is based on a well-founded formal explication of the notion of "logical system"[13].

The formal notion of institution explicates the informal notion of "logical system" in a way which makes it essentially independent of any particular underlying language. An institution consists of (1) a collection of signatures, representing the vocabularies out of which sentences are constructed, and (2) for each signature Σ the class of all Σ-sentences (its syntax), the category of all Σ-models (its semantics), and a satisfaction relation connecting the two, which is invariant under change of notation (see Definition 2 below). The satisfaction condition guarantees that when the vocabulary changes through some appropriate signature morphism, the sentences and models change accordingly so that satisfaction relations are preserved. In other words, the satisfaction condition captures (what is commonly accepted as) a basic fact of logic, namely, that the truth of a proposition is independent from its verbal expression[14]. We will be able to delve into the conception of a logical system underlying the theory of institutions in the last section, after we have given the definition of a triad over an institution.

Different specification languages were represented in the institutional framework as (liberal) syntactic theories over different institutions. Institution morphisms, a mechanism for the mutual truth-preserving inter-translatability of sentences (and models) of theories defined over different institutions, allowed for an easy combination of specifications expressed in different languages. The institutional framework facilitated not only the understanding and writing of complex specifications, but also the task of proving theorems about them (not having to produce an expensive theorem prover for each different specification language used) ([1,23]). Program specifications have also been recently represented as (liberal) semantic theories over institutions, in an attempt to capture the software verification practice of expressing program specifications in terms of state transition systems [27]. Previous work on the formalization of scientific theories using institutions was focused on complex social science theories [28].

The realization of the indispensability of both language and models for an adequate account of scientific theories leads, as we have seen above, to the replacement of the strictly syntactic and the strictly semantic view by their liberal counterparts. Both of these refined versions treat the formal part of a scientific theory as consisting of both a class of sentences and a class of models, defined over some particular logical system in the spirit of Suppes (Figure 1). In this new setting, the question of the relation of the two views becomes essentially a question of the relation between a theory's formalization as—and not its identification with—a class of sentences (the closed class of all the logical consequences of its postulates), and a theory's alternative formalization as a class of mathematical models. The liberal syntactic and the liberal semantic views can be aptly captured by the formalization of a theory as a class of Σ-sentences and a class of Σ-models, respectively, belonging to some institution. Their common content and differing focus will become clear in the relevant formal definitions, which will be based on the corresponding preliminary definitions of a syntactic and a semantic theory presentation.

In order to give a formal general proof of the contemporary claim of equivalence of the two approaches, the corresponding formalizations must provide relevant independence from any particular logical system, or language. The theory of institutions is thus an ideal choice of framework, because this type of generality lies at its very foundation. It is not confined to first order logic like model theory, but neither to extensions of first-order logic like other versions of abstract model theory. To the contrary, the theory of institutions[15] is concerned with the investigation of logical systems beyond first-order logic, and thus has a special bearing on the formalization of scientific theories, both in connection to Suppes' criticism of the standard view [7], and Halvorson's discussion of the semantic+L approach [15]. A wide variety of logics, e.g., equational logic, first-order logic, higher-order logic, intuitionistic logic, modal logic, temporal logic, as well as many-sorted versions thereof, can be represented as institutions. Furthermore, different formulations of the same theory are represented by classes of sentences which have the same content but are expressed in different signatures from the collection of signatures belonging to the underlying institution. The fact that the latter can, arguably, represent any logical system, combined with the generalized concept of model-theoretical equivalence available in the institutional framework, guarantees that any results we obtain regarding the relation between the classes of models corresponding to equivalent formulations of a single theory will not be confined to the first-order and single-sorted case. Finally, the relations of theories defined over different logical systems can be investigated using institution morphisms.

In the next section we give the liberal semantic and syntactic formalizations of theories within the formal framework of institutions, and a rigorous proof of their claimed equivalence. For purposes of completeness, we first present—in the remainder of this section—the definition of a category accompanied by some simple examples, and the definition of an institution, followed by an example of the simple case of propositional logic (a formulation and a detailed proof for the more complicated case of first-order logic can be found in [1]).

To achieve independence of the formalization of the notion of logical system from any particular choice of language, institutions use category theory. In particular, they parametrize a logical system's satisfaction relation by a structure of signatures; this allows for non-logical symbols to vary, while logical symbols and rules of inference are fixed. The structure of signatures is represented by the category **Sign**: the objects of **Sign** are signatures, or vocabularies, and its arrows are signature morphisms, or translations among vocabularies. The choice of category theory is crucial, as it is especially good at representing mathematical structures on a higher level of abstraction, where emphasis is not upon the internal structure of particular objects, but rather on relations between objects, as well

as relations of relations. To put it more precisely, category theory is essentially "a pure theory of functions" [29]. The definition of a category is given below[16].

Definition 1 (Category)**.** *a category C is defined by*

1. *a collection $|C|$ of objects;*
2. *a collection of arrows often called morphisms;*
3. *operations assigning to each arrow f an object $domf$, its domain, and an object $codf$, its codomain (we write $f : A \to B$ to show that $domf = A$ and $codf = B$). The collection of all arrows with domain A and codomain B is written as $C(A, B)$;*
4. *a composition operator $_ \circ _$, assigning to each pair of arrows f and g with $codf = domg$ a composite arrow $g \circ f : domf \to codg$, satisfying the associative law:for any arrows $f : A \to B$, $g : B \to C$, and $h : C \to D$ (with A, B, C, D not necessarily distinct)*

$$h \circ (g \circ f) = (h \circ g) \circ f$$

5. *for each object A, an identity arrow $id_A : A \to A$ satisfying the following identity law:*

$$\text{for any arrow } f : A \to B, id_B \circ f = f \text{ and } f \circ id_A = f$$

As the above definition shows, a category represents a mathematical structure as a class of relations—holding between elements of a given class of objects—which satisfy certain minimal conditions: they are composable, their composition—when defined—is associative, and for each object there exists an identity relation whose composition with any other relation—when defined—is equal to it.

Definition 2 (Functor)**.** *Given two categories C_1 and C_2, a functor $F : C_1 \to C_2$ maps objects to objects, $|C_1| \to |C_2|$, and morphisms to morphisms, $F_{A,B}$: $C_1(A, B) \to C_2(F(A), F(B))$, for all objects $A, B \in C_1$, while preserving the identity and composition of morphisms.*

Example 1 (Set Category)**.** *The structure of sets can be represented as the category **Set**. The objects of **Set** are sets, and its arrows are total functions between sets. Composition of arrows is the familiar set-theoretic composition of functions and identity arrows are identity functions. Note that there is no circularity here; we are merely presenting the well-known mathematical domain of sets as a category (we are not defining sets in terms of categories).*

Example 2 (Category of Partial Orders)**.** *A partial ordering $\leq_P$ on a set P is a reflexive $(\forall p \epsilon P : p \leq p)$, transitive $(\forall p, p', p'' \epsilon P : p \leq p' \leq p'' \to p \leq p'')$ and anti-symmetric $(\forall p, p' \epsilon P : p \leq p' and\ p' \leq p \to p = p')$ relation on the elements of P. An order-preserving (or monotone) function from $(P, \leq_P)$ to $(Q, \leq_Q)$ is a function $f : P \to Q$ such that if $p \leq_P p'$ then $f(p) \leq_Q f(p')$. The structure of partial orders is represented by the category **PoSet,** whose objects are all partially-ordered sets and whose arrows are all order-preserving total functions.*

Example 3 (Category of Preorders)**.** *A preorder $\leq_P$ on a set P is a reflexive and transitive relation on the elements of P. All equivalence relations and (non-strict) partial orders are preorders, but the latter are more general (they are neither necessarily anti-symmetric, nor necessarily symmetric). The category **Ord** has pre-ordered sets as objects and monotone functions as arrows.*

We now come to the definition of an institution, which formalizes the basic components of any logical system: syntax, semantics and a satisfaction relation. Focus on the satisfaction relation is achieved through the fully categorical abstraction of the concepts of sentence and model based on the category of signatures **Sign**. Thus, an institution is defined as consisting of *a category of signatures*, two functors mapping each signature Σ to a class of Σ-sentences and a category of Σ-models, respectively, and a satisfaction relation for

each signature Σ which preserves signature morphisms. Note that signatures can also be many-sorted: a many-sorted signature $\Sigma = \langle S, \Omega \rangle$ is defined by a set S of sort names and an $S* \times S$-sorted set Ω of operation names, where $S*$ is the set of all finite sequences of elements of S. An operation is of the form $f : s_1, s_2, \ldots, s_n \to s$, where $s_1, s_2, \ldots, s_n, \in S*, s \in S$ and $f \in \Omega_{\langle s_1 \ldots s_n, s \rangle}$. In what follows we formally define the one-sorted case for simplicity, as depicted in Figure 3.

Definition 3 (Institution). *An institution* $I = (Sign,\ Sen,\ Mod,\ \models_\Sigma)$ *is defined by*

1. *a category Sign having signatures as objects, and signature morphisms as arrows;*
2. *a functor* $Sen : Sign \to Set$ *mapping each signature* Σ *in Sign to a class of* Σ*-sentences;*
3. *a functor* $Mod : Sign \to Cat^{op}$ *assigning to each* Σ *in Sign a category whose objects are* Σ*-models and whose morphisms are* Σ*-model morphisms*[17]*;*
4. *a relation* $\models_\Sigma \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ *for each* $\Sigma \in |Sign|$*, called* Σ*-satisfaction, such that for each signature morphism* $\phi : \Sigma \to \Sigma'$*,* ***the satisfaction condition***

$$m' \models_{\Sigma'} Sen(\phi)(e) \text{ iff } Mod(\phi)(m') \models_\Sigma e$$

holds for each $m' \in |Mod(\Sigma')|$ *and each* $e \in Sen(\Sigma)$.

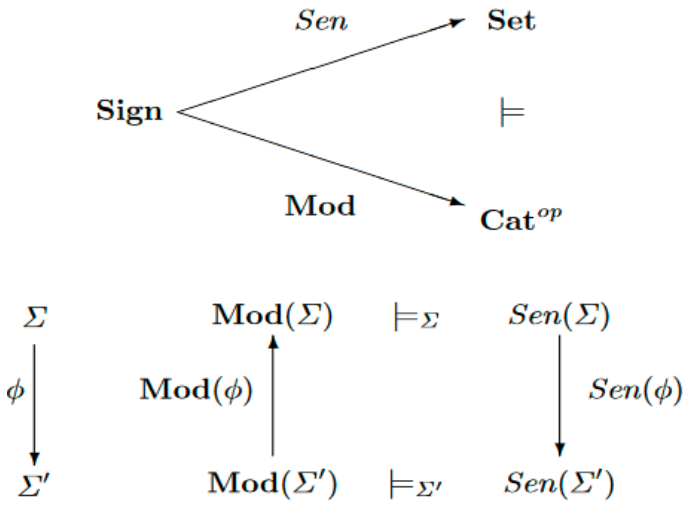


Figure 3. The structure of an institution[18].

Among the logical systems which have been shown to be institutions are many-sorted equational logic (with abstract algebras as models), first-order predicate logic (with first-order structures as models), many versions of higher-order and modal logics, infinitary logics, intuitionistic logic [26,31], and semantic networks [32]. The simple case of propositional logic is presented below.

Example 4 (Propositional logic). [19]*The institution* ***Prop*** *of propositional logic is defined by*

1. *a category* $Sign_{Prop}$ *of signatures corresponding to the category* ***Set*** *of sets, such that each element* $P \in |Set|$ *is a signature and elements of P are propositional variables;*
2. *a functor* $Sen_{Prop} : Set \to Set$*, such that*
   - *for each* $P \in |Set|$*,* $Sen_{Prop}(P)$ *is the least set that contains P together with 'true' and '*$false$*', and is closed under the propositional connectives*[20]*;*
   - *for each function* $\sigma : P \to P'$*,* $Sen_{Prop}(\sigma)$ *translates propositional sentences containing propositional variables in P to propositional sentences containing propositional variables in* P'*, without modifying the connectives in those sentences;*
3. *a functor* $Mod : Sign \to Cat^{op}$*, such that*

- *for each $P \in |Set|$, P-models are functions of the form: $M : P \to \{tt, ff\}$. Given two P-models M and M° there is a morphism from M to M° in case, for all $p \in P$, $M*(p) = tt$ iff $M(p) = tt$;*
- *for each function $\sigma : P \to P'$, $Mod_{Prop}(\sigma)$ maps any model $M' : p' \to \{tt, ff\}$ to the model $\sigma \circ M' : P \to \{tt, ff\}$;*

4. *for each $P \in |Set|$, the satisfaction relation $\models_{Prop,P} \subseteq \left|Mod_{Prop}(P)\right| \times Sen_{Prop}(P)$ corresponds to the usual satisfaction relation in propositional logic*[21].

3. Syntactic and Semantic Theories over Institutions

In what follows we give the definitions of a syntactic and a semantic theory over an institution, initially introduced to represent program specifications. In our setting they acquire a much more general role, that of formalizing the concepts of a liberal syntactic and a liberal semantic view of scientific theories.

3.1. The Liberal Syntactic View of Theories

Given an institution I, a syntactic theory over I is introduced by a signature Σ and a **closed** class *of Σ-sentences*.

Definition 4 (Syntactic)**.** *Theory over an institution*

1. *A **syntactic** Σ-theory **presentation** is a pair $\langle \Sigma, E \rangle$ where Σ is a signature and E is a class of Σ-sentences;*
2. *A Σ-model A satisfies a syntactic theory presentation $\langle \Sigma, E \rangle$ if A satisfies each sentence in E, in which case we write $A \models E$;*
3. *If E is a class of Σ-sentences, let $E*$ be the class of all Σ-models that satisfy each sentence in E;*
4. *If M is a class of Σ-models, let $M*$ be the class of all Σ-sentences that are satisfied by each model in M; $M*$ also denotes $\langle \Sigma, M* \rangle$;*
5. *By the closure of a class E of Σ-sentences we mean the class $E**$, written $E^{\blacksquare}$;*
6. *A class E of Σ-sentences is closed if $E = E^{\blacksquare}$. Then a syntactic Σ-theory is a theory presentation $\langle \Sigma, E \rangle$ such that E is closed;*
7. *The **syntactic theory** presented by the syntactic presentation $\langle \Sigma, E \rangle$ is $\langle \Sigma, E^{\blacksquare} \rangle$.*

The definition of a theory as a closed class of sentences in some signature is based on the view, also adopted by Hempel, that a theory already contains all the consequences of its sentences. Syntactic theories thus formalized can contain an infinite number of sentences, but they are defined by a finite presentation. Notice that the definition of closure given above is model-theoretic. But as Goguen and Burstall point out [1], logical systems for which there is a complete set of inference rules (that is, they can infer any valid conclusion), can be given a corresponding proof-theoretic definition of closure. In such cases, as, for example, in the case of equational logic, the proof-theoretic closure is easy to define by using inference rules embodying the equivalence properties of equality, including the substitution of equivalent terms for equivalent terms [34].

A structure $\langle \Sigma, E^{\blacksquare} \rangle$ over an institution I can be employed to introduce a scientific theory syntactically, as it satisfies the requirements of the liberal syntactic view. In this setting a syntactic theory is always defined over a logical system represented by an institution I, and is expressed in some signature from the category of signatures associated with I. The logical relations of theories defined over the same institution I, but formulated in differ-

ent languages, can be studied by introducing the category **Th** of (syntactic) theories over I, whose objects are theories over I and whose morphisms are theory morphisms.

Definition 5 (Category of syntactic theories Th)**.** *If T and T' are syntactic theories, say $\langle \Sigma, E^{\blacksquare} \rangle$ and $\langle \Sigma', E'^{\blacksquare} \rangle$, then a syntactic theory morphism $\varphi : T \to T'$ is the pair $\langle \phi, \supseteq \rangle$, where $\phi : \Sigma \to \Sigma'$ is a signature morphism and $\supseteq$ the inverse inclusion $E'^{\blacksquare} \supseteq Sen(\phi)(E^{\blacksquare})$. The category of syntactic theories **Th** has syntactic theories as objects, and syntactic theory morphisms as arrows. Composition and identities are defined as they are defined for signature morphisms.*

Morphisms in **Th** are *heteromorphisms*, in the sense that they connect theories expressed in different vocabularies. They are particularly important because they allow us to formally represent the possible logical relations of such theories like extension, refinement, abstraction, integration, composition, and *equivalence* [26]. In particular, two syntactic theories in **Th** are equivalent if and only if they are isomorphic.

Definition 6 (Isomorphic theories)**.** *A morphism $F : T \to T'$ in **Th** is an isomorphism if there exists another morphism $H : T' \to T$ such that $H \circ F = id_T$ and $F \circ H = id_{T'}$, where id_T and $id_{T'}$ are the identity morphisms of T and T', respectively.*

Definition 7 (Equivalence of syntactic theories)**.** *Given an institution I, two syntactic theories $T = \langle \Sigma, E^{\blacksquare} \rangle$ and $T' = \langle \Sigma', E'^{\blacksquare} \rangle$ over I are said to be equivalent if and only if there exists some $F : T \to T'$ such that F is an isomorphism.*

Since a morphism between two syntactic theories in **Th** is a signature morphism such that for any sentence $e \in E^{\blacksquare} \phi(e) \in E'^{\blacksquare}$, two theories over the same logical system but expressed in different vocabularies are equivalent only if they are mutually translatable. This definition captures the equivalence of axiomatizations of the same theory in different languages.

In the next subsection we give the dual definition of a semantic theory over an institution, and the corresponding model-theoretic notion of equivalence.

3.2. The Liberal Semantic View of Theories

In this subsection, we present the definition of a semantic theory over an institution I, in terms of a signature $\Sigma \in |Sign|$ and a class of Σ-models. This definition, used as a formalization of semantic theories, cannot be found guilty of treating them as a-linguistic entities, in the manner that the semantic-L approach arguably does [8,35]. Lutz in particular shows that semantic theories make use of syntax to the same extent that syntactic theories do, lending even greater support to the widely accepted central contention that the liberal semantic view is no real alternative to a refined syntactic view. Recognizing the self-evidence of this claim, the definition of a semantic theory over an institution I is always linked to a specific signature from the category of signatures associated with I, which also contains a collection of signature morphisms, or truth-preserving translations between different vocabularies. Thus, although semantic theories formalized in this way are not a-linguistic entities, they are in a sense language-independent. Furthermore, considering all semantic theories over an institution as a single category allows us to formally study the relations of semantic theories linked to different languages.

Definition 8 (Semantic theory over an institution)**.**

1. *A **semantic** Σ-theory **presentation** is a pair $\langle \Sigma, M \rangle$ where Σ is a signature and M is a class of Σ-models.*

2. *A Σ-sentence e is satisfied by a class M of Σ-models if e is satisfied by each model in M, in which case we write $M \models e$.*
3. *If M is a class of Σ-models, let $M*$ be the class of all Σ-sentences that are satisfied by each model in M.*
4. *If E is a class of Σ-sentences, let $E*$ be the class of all Σ-models that satisfy each sentence in E.*
5. *By the closure of a class M of Σ-models we mean the class $M**$, written $M^{\bullet}$*[22].
6. *A class M of Σ-models is closed if and only if $M = M^{\bullet}$. Then a semantic Σ-theory is a semantic theory presentation $\langle \Sigma, M \rangle$ such that M is closed.*
7. *The **semantic theory** presented by the semantic presentation $\langle \Sigma, M \rangle$ is $\langle \Sigma, M^{\circ} \rangle$.*

Let us now define the category of semantic theories **Vth**, whose objects are semantic theories and whose morphisms are semantic theory morphisms.

Definition 9 (Category of semantic theories Vth)**.** *If V and V' are semantic theories, say $\langle \Sigma, M^{\circ} \rangle$ and $\langle \Sigma', M'^{\circ} \rangle$, then a semantic theory morphism from V to V' is a signature morphism $\psi : \Sigma \to \Sigma'$ such that $Mod(\psi)^{-1}$*[23] *is in M'° for each m in M°; we will write $\psi : V \to V'$. The **category of semantic theories** has semantic theories as objects and semantic theory morphisms as morphisms, with their composition and identities defined as they are defined for signature morphisms; let us denote it **Vth.***

Using semantic theory morphisms, we can provide a categorical definition of model-theoretic equivalence in the following way:

Definition 10 (Equivalence of semantic theories)**.** *Given an institution I, two semantic theories $V = \langle \Sigma, M^{\circ} \rangle$ and $V' = \langle \Sigma', M'^{\circ} \rangle$ over I are said to be equivalent if and only if there exists $\psi : V \to V'$ such that ψ is an isomorphism.*

3.3. Proving the Equivalence of the Two Approaches

Once **Th** and **Vth** are introduced, the logical duality holding between equivalences of syntactic theories and equivalences of semantic theories is an immediate consequence of the *Galois connection*, a type of duality between classes of sentences and classes of models. Francis William Lawvere wrote about the "familiar Galois connection between classes of axioms and classes of models, for a fixed signature", which "reflects the view that syntax and semantics are adjoint" [36]. One of the earliest results about institutions is that the two * operations (see Definition 7 (3), 7(4)) define a Galois connection. In the institutional framework the Galois connection is generalized as follows:

Proposition 1 (Galois connection)**.** *The two functions *: sets of Σ − sentences $\to$ sets of Σ − models, *: sets of Σ − models $\to$ sets of Σ − sentences, given in Definitions 3 and 7, form what is known as a **Galois connection**, in that they satisfy the following properties, for any classes E, E' of Σ-sentences and classes M, M' of Σ-models:*

1. *$E \subseteq E'$ implies $E'* \subseteq E*$.*
2. *$M \subseteq M'$ implies $M'* \subseteq M*$.*
3. *$E \subseteq E**$.*
4. *$M \subseteq M**$.*

These imply the following properties:

1. *$E* = E***$.*
2. *$M* = M***$.*

3. *There is a dual (i.e., inclusion-reversing) isomorphism between the closed class of sentences and the closed class of models.*
4. $(\bigcup_n E_n)* = \bigcap E_n^*$.
5. $Mod(\psi)^{-1}(E*) = (Sen(\phi)E)^*$, *for* $\phi : \Sigma \to \Sigma'$ *a signature morphism.*

The following corollary states that any syntactic theory T determines an equivalent semantic theory $T*$, and every semantic theory V determines an equivalent syntactic theory $V*$.

Corollary 1 (from Galois connection)**.** *For every syntactic* Σ*-theory* $T = \langle \Sigma, E^{\blacksquare} \rangle$ *and every semantic* Σ*-theory* $V = \langle \Sigma, M^{\circ} \rangle$*:*

1. *T determines the semantic theory* $T* = \langle \Sigma, E* \rangle$*, consisting of all the* Σ*-models that satisfy all the sentences in T. (*$E* = E^{***}$*: Galois connection 5)*
2. *V determines the syntactic theory* $V* = \langle \Sigma, M* \rangle$*, consisting of those* Σ*-sentences satisfied by all models in V. (*$M* = M^{***}$*: Galois connection 6).*

We can now prove that there is a relation of logical duality between syntactic and semantic theoretical equivalences.

Lemma 1 (Galois Duality Lemma)**.** *Given an institution I, two syntactic theories* $T = \langle \Sigma, E^{\blacksquare} \rangle$ *and* $T' = \langle \Sigma', E'^{\blacksquare} \rangle$ *over I are equivalent if and only if the semantic theories* $T* = \langle \Sigma, E* \rangle$*, and* $T'* = \langle \Sigma', E'* \rangle$ *that they define are equivalent.*

Proof of Lemma 1. :" $\Rightarrow$ " Suppose that $T \cong T'$. From the corollary of Galois connection we know that T and T' determine the equivalent semantic theories $T*$, $T'*$, and that $T \cong T*$ and $T' \cong T'*$ (Galois Connection (7)). Because equivalence relations are symmetric, we have that $T* \cong T$. Thus $T* \cong T \cong T' \cong T'*$ from which we get $T* \cong T'*$ (from transitivity)." $\Leftarrow$ " Suppose that the semantic theories $T*$, $T'*$ determined by T, T' are equivalent, that is $T* \cong T'*$. We know (from Corollary 1 and Galois connection (7)) that $T \cong T*$ and $T' \cong T'*$. Because equivalence relations are symmetric, we have that $T'^* \cong T'$. Thus,$T \cong T* \cong T'* \cong T'$, from which we get $T \cong T'$ (from transitivity). □

The Galois duality lemma extends the many-to-many relation holding between syntactic and semantic formulations of a single theory to the higher-order and multi-sorted case, as it applies to theories defined over any institution. Thus, it formally proves the relevant claim made by Halvorson [8]. But in order to be able to answer the more general question of the relation between the (liberal) syntactic and semantic formulations of a theory we must take one more step. In particular, given the category of syntactic theories over an institution I and the category of semantic theories over I, we can construct a functor $F : Th \to Vth$ mapping each syntactic theory T to its corresponding semantic theory $F(T) = T*$ (which always exists due to Corollary 1), and each morphism between syntactic theories $\phi : T \to T'$ to a morphism between their corresponding semantic theories $F(\phi) : F(T) \to F(T') = T* \to T'*$.

In order for F to be a functor, the following two conditions must hold:

1. $F(id_T) = id_{F(T)}$, for every theory T in ***Th***.
2. $F(g \circ f) = F(g) \circ F(f)$, for all morphisms $f : T \to T'$ and $g : T' \to T''$ in ***Th***.

Proof. First, we prove the second condition:

Let us take a theory T in ***Th***. Then we have

$$F(g \circ f)(T) = F(g(f(T))) = F\big(g\big(T'\big)\big) = F(T'') = T''*$$

and

$$(F(g) \circ F(f))(T) = F(g)(F(f(T))) = F(g)(F(T'')) = F(T'') = T''*$$

(1) To prove the first condition, we have to show that for every $f: T \to T'$, $F(id_{T''}) \circ F(f) = F(f) = F(f) \circ F(id_T)$, we have $F(id_{T''}) \circ F(f) = F(id_{T'} \circ f) = F(f) = F(f \circ id_T) = F(f) \circ F(id_T)$, because of condition (2). □

Thus, F is a functor, and since functors preserve isomorphisms, F maps equivalences between syntactic theories to equivalences between semantic theories.

We are now ready to answer the question of the relation of categories **Th** and **Vth**. We have already shown that the categories of syntactic and semantic theories over an institution I are connected through a structure-preserving map: functor F. If there exists another functor G such that $FG = id_{Vth}$ and $GF = id_{Th}$ (identity functors on **Vth** and **Th**, respectively), then the two categories are isomorphic.[24] Using Corollary 1, we can construct a functor $G: Vth \to Th$, in the same way that F was constructed; G maps each semantic theory V to its corresponding syntactic theory $V*$, and each morphism between semantic theories to a morphism between their corresponding syntactic theories. Then, for any $V \in Vth$:

$$(F \circ G)(V) = F(G(V)) = F(V*) = V**$$

and for any $T \in Th$:

$$(G \circ F)(T) = G(F(T)) = G(T*) = T**$$

From the Galois connection (proposition 1) we have that $V \subseteq V**$ and $T \subseteq T**$. But according to the definitions of V and T we have that $V = V*$ and $T = T*$, and thus **Th** and **Vth** are isomorphic.

Having shown that the two categories **Th** and **Vth** are in a one-to-one correspondence, we can conclude that the duality between closed classes of sentences and closed classes of models captured by the Galois connection ultimately guarantees that the (liberal) syntactic and semantic formalizations of a theory over an institution are equivalent, as shown in Figure 4. This is a very general result, as it applies to theories defined over any logical system which has a corresponding institution (and this is arguably true for all logical systems (see [24]).

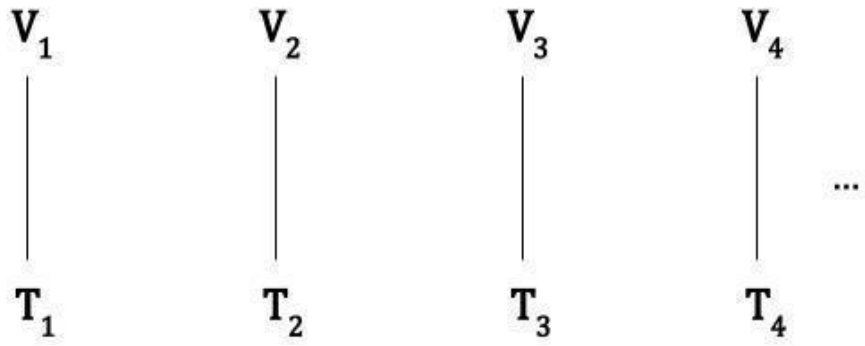


Figure 4. A diagram showing the one-to-one relation holding between the category of (liberal) syntactic theories and the category of (liberal) semantic theories.

4. Formalizing Suppes' Account of Scientific Theories

After having formally shown that the liberal syntactic and semantic views are equivalent, it is natural to attempt to replace them with a single formal account which will capture the common content they assign to theories, while allowing one to focus on either their syntactic, or their semantic formulation (depending on the task at hand). This common content—as described by Suppes in his account of the formal part of a theory—can

be formalized in the institutional framework in a very clear and straightforward way. The main benefit of such a formalization is that it has the potential to produce results of great generality, as the Galois Duality lemma proven in the previous section indicates.

In order to formalize Suppes' description of the formal part of a scientific theory we need a framework which will link each theory to (1) its underlying logical system, (2) the language used to express its sentences, (3) a closed class of sentences T containing all the logical consequences of the theory's axioms, and (4) a class of models M containing all models satisfying all sentences in T. The concept of a triad over an institution, whose definition is given below, satisfies all these requirements.

Definition 11 (Triad over an institution)**.** *A triad over an institution is a triple* $(\boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{T})$ *where* $\boldsymbol{M}$ *is a class of* $\boldsymbol{\Sigma}$ *m-models, and* $\boldsymbol{T}$ *is a syntactic* $\boldsymbol{\Sigma}$*-theory such that* $\boldsymbol{M} \models_{\boldsymbol{\Sigma}} \boldsymbol{T}$*. A triad morphism from* $(\boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{T})$ *to* $(\boldsymbol{M'}, \boldsymbol{\Sigma'}, \boldsymbol{T'})$ *is a pair of mappings* $\boldsymbol{\Phi} : \boldsymbol{T} \to \boldsymbol{T'}$ *and* $\boldsymbol{\Psi} : \boldsymbol{M'} \to \boldsymbol{M}$ *such that*

$$\boldsymbol{M'} \models_{\boldsymbol{\Sigma'}} \boldsymbol{\Phi}(\boldsymbol{T}) \textit{ iff } \boldsymbol{\Psi}(\boldsymbol{M'}) \models_{\boldsymbol{\Sigma}} \boldsymbol{T}$$

Triads support integration in situations that involve both syntactic and semantic information, just as the common structure of the two liberal views of theories does. Thus, we can define the formal part of a scientific theory as a triad $(\boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{T})$ over an institution $\boldsymbol{I}$, where $\boldsymbol{I}$ represents the underlying logical system, $\boldsymbol{\Sigma}$ the language in which the theory is expressed, $\boldsymbol{T}$ the closed class of sentences defining the theory syntactically, and $\boldsymbol{M}$ the class of mathematical models of $\boldsymbol{T}$ which defines the theory semantically. Theories over the same institution I but expressed in different languages can be studied using triad morphisms. Note that while every signature morphism in an institution's Sign category induces a triad morphism, triad morphisms are more general (see the skin color example in [26]). Finally, relations of theories underlined by different logical systems can be investigated using institution morphisms.

We can now take a closer look at the conception of "logical system" formalized by institutions. To do so we need to turn our attention to the Peircean triadic model of meaning on which triads, and ultimately institutions, are based. According to this model every sign is analyzed in three interrelated parts: a signifying element[25], an object, and an interpretant. A sign's signifying element is connected to the object it signifies through the "interpretant", which represents our interpretation, or understanding, of the sign–object relation.

> "I define a Sign as anything which is so determined by something else, called its Object, and so determines an effect upon a person, which effect I call its Interpretant, that the latter is thereby mediately determined by the former." [37], p. 478

The object determines a sign by setting the conditions it must satisfy in order to function as its signifier, and the sign determines an interpretant based on the way it signifies the object. The fundamental premise of the Peircean model is that *a sign signifies only when interpreted*; the sign's meaning lies in the interpretation generated in sign users by the sign. This approach to meaning applied to the concept of truth is at the heart of the formalization of a logical system as an institution; sentences are viewed as signs connected to their objects—models—through an interpretation, namely, the satisfaction relation, which fixes the meaning of truth in the system. But according to the logical maxim that truth is invariable under change of notation, changing the language which underlies the satisfaction condition—or the context of interpretation—should not affect the truth relations established by it. These considerations led Goguen and Burstall to treat the satisfaction relation of a logical system as parametrized by a structure of signatures, or contexts.

Thus, the formal representation of scientific theories as triads over institutions is based on a profound analysis of the notion of "logical system", according to which (1) the essence of a logical system lies in the satisfaction relation holding between sentences and models, which fixes the meaning of truth in the system, and (2) the satisfaction relation is language-independent, as we intuitively take truth to be. Such an approach satisfies the requirements of an adequate account of scientific theories arising from the relevant contemporary discussion, while capturing our common intuition of the language-independent character of truth. Using modern logic and its tools, we are thus led to a clearer and deeper understanding of the structure of scientific theories in the spirit of Suppes.

The language-independence of the satisfaction relation and, consequently, of the formalization of a theory constitutes one major difference with existing formal accounts of Suppes' approach on scientific theories, *scientific structuralism*, in the first place. According to scientific structuralism [38], a theory is introduced as a *theory-net*, that is, a network of *theory-elements* holding specialization relations. Each theory-element is defined semantically as the class of its models which can be divided into *potential models* and *actual models*. All actual models are required to satisfy the same axiom set, namely the axioms of the theory; each specialized theory-element of a theory-net is defined by the subclass of actual models that satisfy a superclass of the axioms satisfied by the theory-element higher in the structure. It should be emphasized here how models of distinct theory-elements in a theory-net, by satisfying consecutive superclasses of axioms, are defined over the same signature. By contrast, triad morphisms allow us to define theories over the same institution formalized using different signatures.

And the language-independence of the institution framework also marks a difference with other formal proofs of the equivalence of the syntactic and semantic approaches to scientific theories. For instance, in the work of Lutz [39], language independence is understood in terms of what he calls a *pure structure*, i.e., a structure containing arbitrary, well-ordered, index sets and mappings from those index sets to interpretations (based on a vocabulary). Language independence is guaranteed by the fact that there are many interpretations and index sets are arbitrary. Two syntactic theories can be said to be definitionally equivalent when they can be extended by using renaming of terms or extended definitions into two equivalent theories; in the same way, the two corresponding semantic theories become equivalent when their structures can be expanded to include new interpretations such that the two obtained structures be equivalent[26]. By contrast, the approach presented here understands both syntactic and semantic theories as linked to a specified vocabulary (signature Σ) and language-independence is guaranteed by signature morphisms in the category Th and Vth of syntactic and semantic theories. This allows the institutional approach to define equivalence of theories itself as directly language-independent.

Nevertheless, representing a theory as a triad $(\boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{T})$ over an institution I tells us nothing about its empirical interpretation, which constitutes an indispensable part of any scientific theory. We have already seen that giving empirical interpretation to a theory is no simple matter [4]; Hempel's bridge principles are a rather crude way of doing so, as they ignore the actual scientific practice of devising experiments to test theories. In Suppes' own account (depicted in Figure 1) a theory's formal part is connected to its empirical interpretation through a representation theorem connecting mathematical models of the theory to models of the experiment, which are in their turn connected to the raw experimental data through a hierarchy of experimental models (and theories) of increasing detail. This structure can be easily represented in an institutional framework: we simply have to add to the basic triad over an institution—representing the formal part of the theory—a second triad representing the theory of the experiment whose models are connected to the formal theory models through a representation theorem, plus a finite family of triads, each

representing an experimental theory in the relevant hierarchy. A scientific theory can then be defined as a category whose objects are the appropriate triads, and whose structure reflects the structure portrayed in Suppes' account.

Definition 12 (Scientific Theory)**.** *A scientific theory is a category ST, which contains as objects a triad* $\langle \boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{T} \rangle$ *over* $\boldsymbol{I}$ *representing the formal part of the theory, and a finite set of triads* $\langle \boldsymbol{M_i}, \boldsymbol{\Sigma_i}, \boldsymbol{T_i} \rangle_{\boldsymbol{I_i}}$ *over institutions* $\boldsymbol{I_i}$ *representing the hierarchy of theories comprising its empirical interpretation. The arrows of ST are triad morphisms. In particular, the representation theorem connecting the formal theory to its empirical interpretation is captured by a triad isomorphism* φ *between* $\langle \boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{T} \rangle$ *and the triad representing the most abstract theory of the experiment, while the various formal connections between theories in the hierarchy are represented by appropriate triad morphisms.*

Such a formalization would allow us to study a theory's internal structure using the mathematical framework best suited for this task, namely, category theory. In its setting we can also investigate the formal connections holding between members of the theory's hierarchy of empirical interpretations. The fact that the models of these theories are usually of a different logical type than their mathematical models is captured by the appropriate differences in their underlying institutions. Furthermore, many different relations between, and operations on, theories can be formally studied using category morphisms in ST, e.g., *inheritance*, *renaming*, *combination*, and *integration*, to mention some [26]. This constitutes a further, fundamental, difference with scientific structuralism. In the latter framework, relations between theory-elements in a theory-net are mainly specialization relations; if these can be defined in institution theory by means of *stepwise refinements* [33], inheritance, renaming, combination, and integration, they can be used to formalize other relations of interest for scientific theorizing. For instance, in the case of integration, given two theories, an integrated, "larger", theory can be construed by computing a pushout or a colimit (see [40] for technical details).

5. Conclusions

The main contribution of this article is the introduction of a powerful meta-logical formal framework in the philosophy of science to formally prove the equivalence of the liberal syntactic and semantic views of scientific theories. The proof independently verifies the conclusion of the contemporary philosophical discussion on the structure of scientific theories, according to which there is no real tension between the two approaches. Thus, it is shown how, by formalizing the two views in the proposed framework, one quickly arrives at a formal proof of their equivalence verifying the corresponding philosophical claim, which is the product of a much more arduous—but indispensable—process. Ultimately, the proposed formal framework is to be used as a catalyst in the process of philosophical discussion.

Furthermore, this article proposes a formalization of the notion of a scientific theory as a category, having the content and structure described in the relevant account given by Suppes [4]. The proposed formalization captures the common structure of the liberal views and is capable of producing formal results of great generality, owing to the use of institutions. In particular, and with respect to the structuralist approach, institution theory allows us to define a language-independent account of scientific theories. Also, morphisms in the appropriate category permit us to define many formal relations of interest among theories. The aim of this paper has been to show the relevance and potential of institution theory for the formal debate on the equivalence of syntactic and semantic theories, leaving to further investigations the object of developing further applications in the formal philosophy of science.

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Notes

1 We share the view that this appellation is preferable to the more widely used "logical positivism", but this is not the place to develop the point.

2 The term was coined by Putnam in a 1962 paper entitled *What theories are not* [5], which antedates Suppes' paper.

3 Quantum mechanics, classical thermodynamics, and general relativity involve set theory and many results concerning the real numbers; it follows that their axiomatization in first-order logic is impractical because they assume more than first-order logic.

4 A well-defined sentence is a sentence assigned a unique interpretation, or value, regardless of the formalism used to express it. In second-order logic sentences are not well-defined because the concept of truth is non-absolute, that is, its meaning is dependent upon the used formalism; while in first-order logic it can be proven that "$M \vDash_s \varphi$" is an absolute property of M, s and φ relative to ZFC, in second-order logic it is not [6].

5 It should be emphasized here that Suppes' [7] claim that the mathematical representations of matrix and wave mechanics are isomorphic is a quite debated one. For instance, Halvorson [8], discussed later in this paper, holds that matrix algebra is not isomorphic, and hence not equivalent, to a space of wave functions.

6 As Suppes points out, assessing the relation between these models is a problem of modern statistical methodology.

7 A theory's mathematical models can be linked to models of data without explicit consideration of the more abstract models of the experiment.

8 A definitional extension of a language L by a new predicate R is a formula $R(\boldsymbol{x}) \leftrightarrow \Phi(\boldsymbol{x})$ in the language of L extended with R ($L + R$), where $\boldsymbol{x}$ is a vector of variables such that no other variables occur free in the formula, and $\Phi(\boldsymbol{x})$ is a well-formed formula of L.

9 A definitional expansion by relation R of a model M of a theory T with language L_T expands M with R to form a structure for the language L_{T+R} satisfying a definitional extension of L_T by R.

10 If the non-logical vocabularies are not disjoint, we can always re-express one of the two theories so that the respective vocabularies contain no common non-logical symbols.

11 Two theories have a common definitional extension if each has a definitional extension such that the two extended theories (now in the same language) are logically equivalent.

12 Dropping this claim has further consequences for structural realists who identify belief in a theory with believing that the world is isomorphic to one of its models; if equivalent syntactic theories have different classes of models, then there is no way of choosing which theory to believe in [15].

13 For a formalization of the more general notion of a logic as an equivalence class of institutions based on an appropriate notion of equivalence see [24].

14 By formalizing the idea that truth is invariant under change of notation, the satisfaction condition extends Tarski's semantic definition of truth [25]; where Tarski defined truth for a given satisfaction relation, institutions extend this definition to a whole set of satisfaction relations, through the parametrization of truth by vocabulary [26].

15 In fact, the name "institution" was inspired by the dominant status of first-order-logic, resembling –in terms of its stability– institutions like the Bank of England and the US Treasury. This name choice was meant to convey "an implicit criticism of the tendency to downplay the importance of developing and using specialized logics (i.e., in computer science) for specialized tasks" [26].

[16] The definition of a category in terms of set theory and the relevant examples given here are taken from [30].

[17] Cat^{op} is the *opposite category* of category *Cat*; this means that Cat^{op} has the same objects of *Cat*, namey Σ-models, and reversed morphisms, that is, the morphisms of *Cat* but having opposite directions. This is why, in Figure 3 below, functor $Mod(\phi)$ goes in the opposite direction of $Sen(\phi)$.

[18] The figure is taken from [26].

[19] This formulation is taken from [33].

[20] This implies that, for any given signature Σ, if it holds that $\varphi, \psi \in Sen_{Prop}(\Sigma)$, then also holds that $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi) \in Sen_{Prop}(\Sigma)$.

[21] Given $\varphi, \psi \in Sen_{Prop}P)$, and for any *P*-model $M : P \rightarrow \{tt, ff\}$, the following holds: $M \vDash_{Prop,P} p$ iff $M(p) = tt$; $M \vDash_{Prop,P} \neg\varphi$ iff $M \nvDash_{Prop,P} \varphi$; $M \vDash_{Prop,P} (\varphi \wedge \psi)$ iff $M \vDash_{Prop,P} \varphi$ and $M \vDash_{Prop,P} \psi$; $M \vDash_{Prop,P} (\varphi \vee \psi)$ iff $M \vDash_{Prop,P} \varphi$ or $M \vDash_{Prop,P} \psi$; $M \vDash_{Prop,P} (\varphi \rightarrow \psi)$ iff $M \vDash_{Prop,P} \psi$ and $M \nvDash_{Prop,P} \varphi$.

[22] A set of models M is called closed if and only if it consists of all models satisfying the set of sentences satisfied by each element of M.

[23] $Mod(\psi)^{-1} : Mod(\Sigma) \rightarrow Mod(\Sigma')$ is the inverse image map of $Mod(\psi) : Mod(\Sigma') \rightarrow$ Mod(Σ).

[24] Another criterion for isomorphic categories is the following: a functor yields an isomorphism if and only if it is bijective on objects and on morphism sets. Corollary 1 tells us that every object in **Th** determines an object in **Vth** and vice versa. But this in no way guarantees that each theory in **Th** is paired with exactly one theory in **Vth** and vice versa.

[25] The signifying element refers to those parts of a sign which are mainly responsible for its function as a signifier.

[26] The reader should refer to [39] for formal definitions and proofs.

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