

## Article

# Interaction Features of Internal Wave Breathers in a Stratified Ocean

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**Abstract:** Oscillating wave packets (breathers) are a significant part of the dynamics of internal gravity waves in a stratified ocean. The formation of these waves can be provoked, in particular, by the decay of long internal tidal waves. Breather interactions can significantly change the dynamics of the wave fields. In the present study, a series of numerical experiments on the interaction of breathers in the frameworks of the etalon equation of internal waves—the modified Korteweg–de Vries equation (mKdV)—were conducted. Wave field extrema, spectra, and statistical moments up to the fourth order were calculated.

**Keywords:** breathers; internal waves; moments of the wave field; spectrum; modified Korteweg–de Vries equation

## 1. Introduction

Localized wave formations (solitons or breathers) play a key role in the dynamics of different nonlinear integrable systems (the nonlinear Schrödinger equation, the Korteweg–de Vries (KdV) equation, etc.), which describe physical phenomena in the environment, including surface and internal water waves and wave movements in optics and plasma. The classical problem of soliton theory about their interaction with each other began to develop back in the last century [1], but is still relevant [2–5], also for non-integrable equations [6,7]. Multiple soliton interactions lead to so-called soliton turbulence or soliton gas [8–14], and such interactions are blamed for the formation of abnormally large (freak or rogue) waves [15–18]. The impact of pairwise soliton interactions and multiple soliton interactions on soliton gas dynamics and statistics has been considered in [19–24].

Like solitons, breathers also have particle-like behavior in some sense, but they have been studied much less. The existence of breathers has been proved in many experiments, and some recent studies have been devoted to the dynamics of breathers on the surface of deep water [25–27] and inside the ocean [28–31]. The most common models to describe the dynamics of internal waves in a stratified ocean are the extended KdV (the Gardner equation) when both cubic and quadratic nonlinearities are essential, and the modified Korteweg–de Vries equation, when quadratic nonlinearity can be neglected. The existence of solitons and breathers within these models is well predicted by weakly nonlinear theory [32,33]. These waves are often observed in the ocean [34,35]. Such waves complicate human economic activity in the shelf area, affect the propagation of acoustic signals, the motion of submersibles and mixing stratified water, and the transfer of admixtures and pollution; they lead to soil erosion under oil and gas platforms and change the plankton distribution. Interactions of energy-carrying

waves play a special role in these processes. Some features of the interaction between a soliton and a breather have been considered in [36] using the Gardner equation and in [37] using the mKdV equation. Breather interactions are a complex physical process, the results of which depend on several wave parameters: amplitudes, positions of waves (initial phases) and the number of waves in the oscillating packets. Such a variation of parameters leads to a wide variety of resulting impulses at the time of interaction. Pairwise breather interactions using the Sine-Gordon equations have been considered in [38–40]. The interaction of two breathers within the Gardner equation has been considered in [41]. In the present work, the study of interaction features between two breathers has been carried out using the mKdV equation.

## 2. The mKdV Equation and Its Breather Solution

The mKdV equation can be derived using the asymptotic technique for waves in a liquid medium with a specific stratification (both continuous and multilayer), and in a moving frame of reference, it takes the form:

$$\frac{\partial u}{\partial t} + \alpha_1 u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0, \quad (1)$$

where  $\alpha_1$  is the coefficient of the cubic nonlinear term, and  $\beta$  is the coefficient of dispersion. In the Boussinesq approximation, the coefficients are determined in terms of given vertical distributions of the field [42]:

$$\beta = \frac{1}{2} \frac{\int_0^H (c - U)^2 \Phi^2 dz}{\int_0^H (c - U) (d\Phi/dz)^2 dz}, \quad \alpha_1 = -\frac{3}{2} \frac{\int_0^H (c - U)^2 \left[ 2 \left( \frac{d\Phi}{dz} \right)^4 - 3 \left( \frac{dT}{dz} \right) \left( \frac{d\Phi}{dz} \right)^2 \right] dz}{\int_0^H (c - U) \left( \frac{d\Phi}{dz} \right)^2 dz} \quad (2)$$

Here,  $c$  is the phase velocity of long waves,  $U$  is a shear flow, and the function  $T$  is a nonlinear correction to the modal function  $\Phi$ , determined by the equation:

$$\frac{d}{dz} \left[ (c - U)^2 \frac{dT}{dz} \right] + N^2 T = \frac{3}{2} \frac{d}{dz} \left[ (c - U)^2 \left( \frac{d\Phi}{dz} \right)^2 \right] \quad (3)$$

with zero boundary conditions  $T(0) = T(H) = 0$  and normalization conditions  $T(z_{max}) = 0$ , where  $z_{max}$  is the coordinate of the maximum point of the modal function  $\Phi(z_{max}) = 1$ ,  $N(z)$  is the Brunt–Vaisala frequency.

In the particular case of a three-layer model, the coefficients in Equation (1) have been found in [42].

After scaling, the coefficients are:  $\alpha_1 = 6$  and  $\beta = 1$ . Equation (1) with such coefficients is canonical and completely integrable by means of the inverse scattering method.

The single-breather solution of the mKdV equation is:

$$u(x, t) = -4q \operatorname{sech}(R) \left[ \frac{\cos(f) - \frac{q}{p} \sin(f) \tanh(R)}{1 + \left( \frac{q}{p} \right)^2 \sin^2(f) \operatorname{sech}^2(R)} \right], \quad (4)$$

where

$$R = 2qx + 8q(3p^2 - q^2)t + R_0, \quad (5)$$

$$f = 2px + 8p(p^2 - 3q^2)t + f_0, \quad (6)$$

$p$ ,  $q$ ,  $R_0$ ,  $f_0$  are free parameters:  $p$  affects the number of waves in a packet, and  $q$  determines the breather amplitude; constants  $R_0$  and  $f_0$  determine the initial positions of the envelope and carrier, and their meaning is obvious in the case of a linear wave packet. If  $q \gg p$ , a breather has few cycles and resembles the superposition of the two mKdV-solitons. A breather containing many oscillations ( $p \gg q$ ) can be interpreted as an envelope soliton of the nonlinear Schrödinger equation [43].

Breathers are sometimes called pulsating solitons because they propagate as isolated disturbances without losses and have an additional internal “oscillatory” degree of freedom. In contrast to solitons, all breather solutions (4) have zero mass. According to the relation between  $p$  and  $q$ , a breather may propagate in any direction.

### 3. Pairwise Breather Interaction

The features of two-breather interactions are studied as an elementary act of soliton-breather turbulence for a further understanding of their impact on multi-wave dynamics. A series of 150 numerical experiments with different breather phases was conducted. The initial conditions corresponded to the sum of two breathers (4) with parameters  $f_{01}$ ,  $R_{01}$ ,  $p_1$ ,  $q_1$  and  $f_{02}$ ,  $R_{02}$ ,  $p_2$ ,  $q_2$ , respectively. In each experiment, breathers were initially separated by the level  $10^{-5}$ , and the calculations stopped when the distance between waves after their interaction became the same as at  $t = 0$ . Equation (1) is solved numerically by the pseudo-spectral method with periodic boundary conditions. This method replaces the initial partial differential equations with ordinary differential equations for the coefficients of the expansion of the desired functions for a certain basis. At each time, the coefficients allow one to retrieve the desired solution with the help of a fast Fourier transform. This method is described in more detail in [44]. Numerical simulations are controlled by retaining the first and second moments with precisions of  $10^{-14}$  and  $10^{-7}$ , respectively.

In each experiment there are always two breathers with fixed parameters  $f_{01} = 0$ ;  $p_{01} = 0.05$ ;  $q_{01} = 0.25$  and  $f_{02} = 0$ ;  $R_{02} = 0$ ;  $p_{02} = 2$ ;  $q_{02} = 0.25$ . The first breather has an N-shape (a pair of coupled solitons), and the second one contains many oscillations. The only parameter which is changed from one experiment to another is the position of the first breather,  $R_{01}$ . In the experiments, it changes from 30 to 61.5, and this range covers all possible shapes of the first breather (during one period). Several examples of initial conditions are shown in Figure 1. Red curves correspond to the boundary positions of the first breather. The initial form and phase of the second breather remain the same.

The chosen parameters  $p_{01}$ ,  $q_{01}$ ,  $p_{02}$ ,  $q_{02}$  correspond to the counter propagation of the breathers. The interaction from a single experiment is demonstrated in Figure 2. Breathers retain their identity after the interaction; there is no energy loss after the interaction. A spatiotemporal diagram of the interaction shows the difference in the velocities of the breathers, which remain constant before and after the interaction (Figure 3). The first breather is much slower than the second one. There are phase shifts, which are observed in Figure 3.

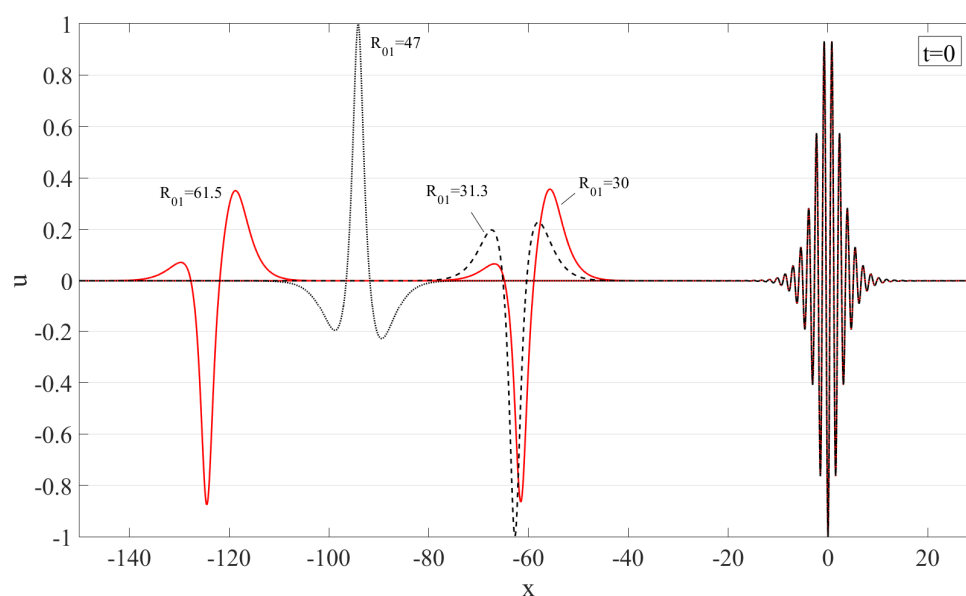
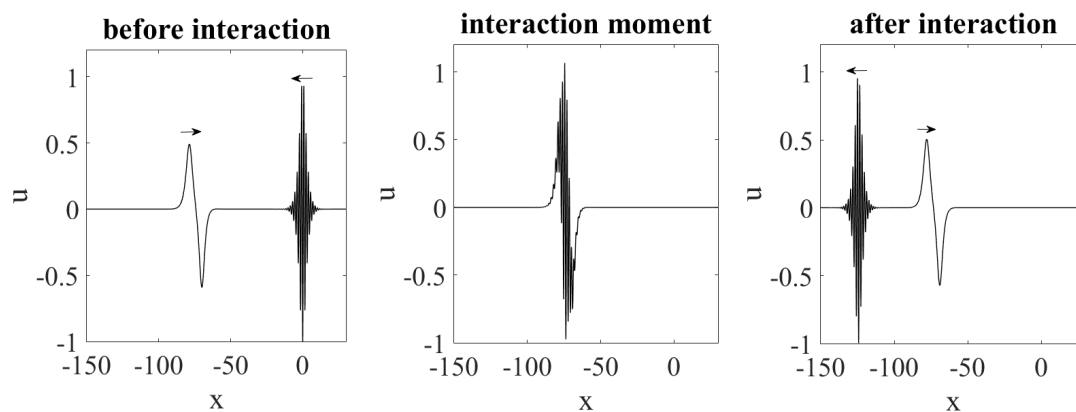
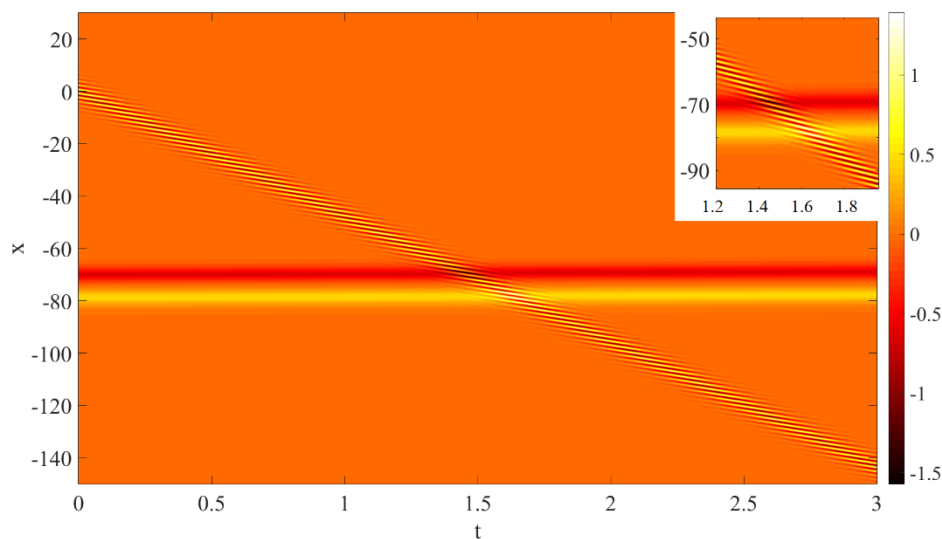


Figure 1. Examples of the initial conditions.



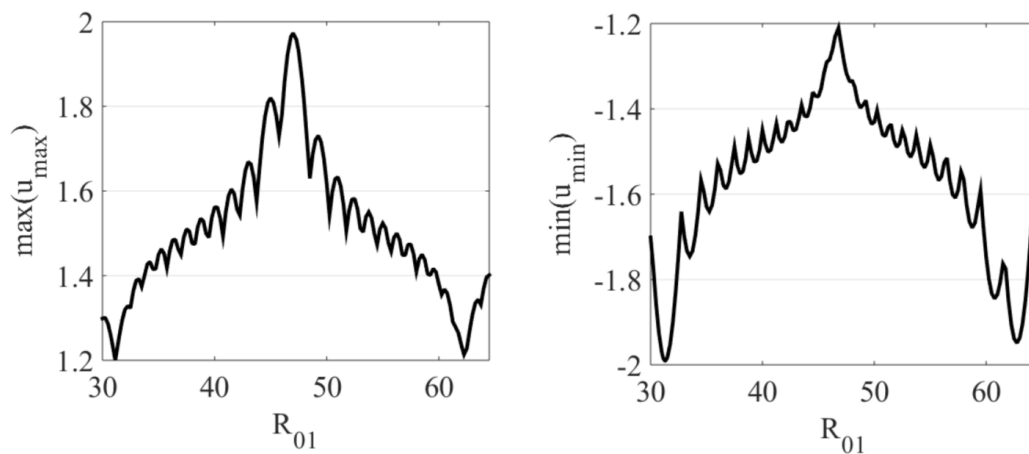
**Figure 2.** Pairwise breather interaction with the following parameters:  $f_{01} = 0$ ;  $R_{01} = 37$ ;  $p_{01} = 0.05$ ;  $q_{01} = 0.25$ ; and  $f_{02} = 0$ ;  $R_{02} = 0$ ;  $p_{02} = 2$ ;  $q_{02} = 0.25$ .



**Figure 3.** Spatiotemporal diagram of the breather-breather interaction.

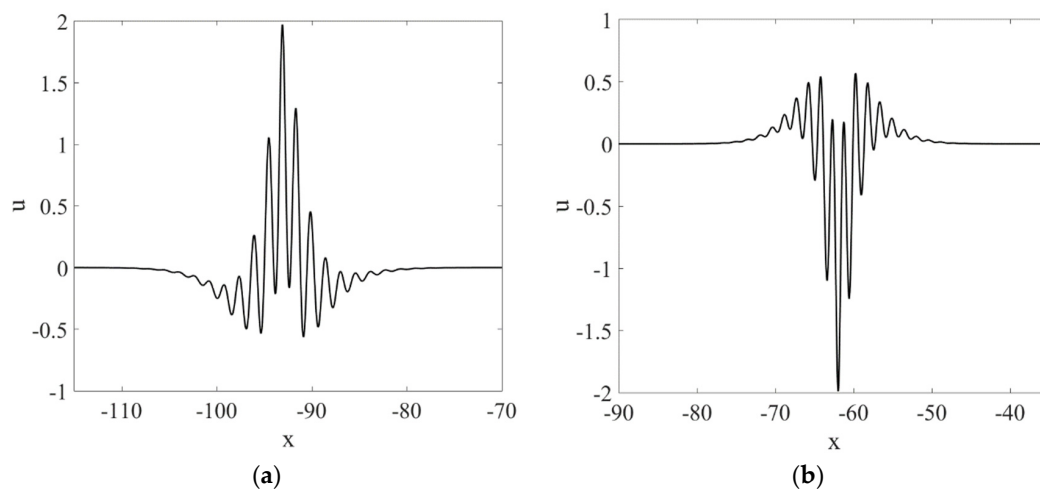
#### 4. Wave Field Extrema

Breathers “pulsate” during their propagation, and the internal oscillations can greatly change their amplitude. These changes intensify as the number of oscillations in the wave packet decreases. Both breathers participating in the interaction reach the maximum positive and negative amplitude equal to 1 during the propagation. For optimal phases of interacting breathers, the amplification of the wave field is equal to the sum of the maximum amplitudes of the breathers [23], i.e., 2 in this case. However, such situations are extremely rare in real physical systems, and non-optimal interactions without maximum wave amplification often occur. In Figure 4, the maximum maximorums and minimum minimorums are plotted for each numerical experiment (for each  $R_{01}$ ), up to 64.5, to demonstrate the periodicity of the functions. Because both breathers oscillate during the propagation, changing their shape, the number of unique forms of the resulting impulses at the time of the interaction is extremely large. The series of experiments covering all possible initial forms of the first breather was considered. Although this does not give all the possible interactions, these experiments show a typical picture of the interaction between breathers with given parameters, which may undergo small quantitative but not qualitative changes. Figure 4 demonstrates the change in the maximum field values from 1.2 to close to 2, and the minimum values from  $-2$  to  $-1.2$ ; the maxima and minima on both graphs relate to each other.



**Figure 4.** Maximum and minimum values of  $u_{\max}$  and  $u_{\min}$  in each experiment.

The pulse shapes at the instant of interaction with the largest positive and negative amplification of the wave field are shown in Figure 5. Both figures correspond to near-optimal focusing, and their shapes are almost symmetrical about the vertical axis. In the case of optimal focusing, the resulting pulse is asymmetric about the horizontal axis: it is significantly extended towards the amplification of the wave field (in contrast to non-optimal focusing, see the middle plot in Figure 2).



**Figure 5.** Shapes of resulting pulses at the moment of optimal focusing: (a)  $f_{01} = 0$ ;  $R_{01} = 47$ ;  $p_{01} = 0.05$ ;  $q_{01} = 0.25$ ; and  $f_{02} = 0$ ;  $R_{02} = 0$ ;  $p_{02} = 2$ ;  $q_{02} = 0.25$ , (b)  $f_{01} = 0$ ;  $R_{01} = 31.3$ ;  $p_{01} = 0.05$ ;  $q_{01} = 0.25$ ; and  $f_{02} = 0$ ;  $R_{02} = 0$ ;  $p_{02} = 2$ ;  $q_{02} = 0.25$ .

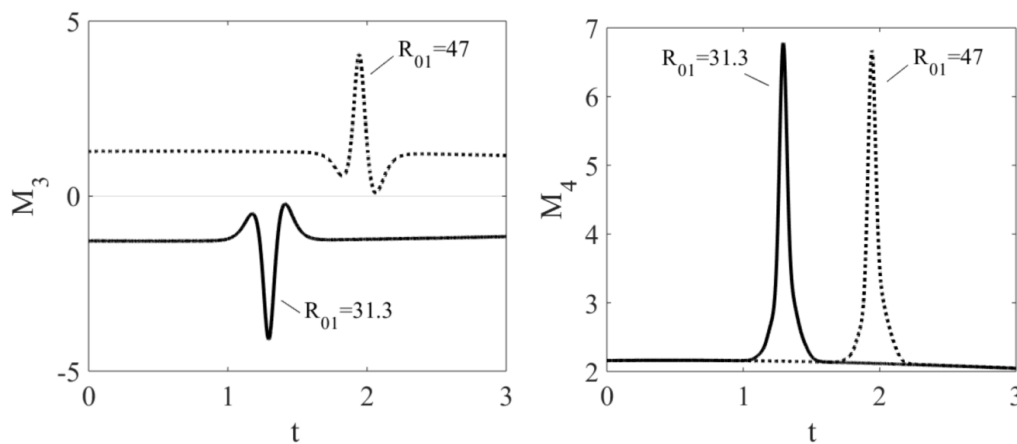
## 5. Moments and Spectra of the Wave Fields

A statistical approach, when the probability distribution functions of the wave parameters, statistical moments, etc., are calculated to describe the stochastic wave dynamics, is often used in the theory of turbulence. This section examines the behavior of the moments of the wave fields consisting of the two breathers discussed in the previous sections. The moments of the wave fields corresponding to the mean field, variance, skewness and kurtosis, see, for example, [45], are calculated by:

$$M_n(t) = \int_{-\infty}^{+\infty} u^n(x, t) dx, \quad n = 1, 2, 3, 4. \quad (7)$$

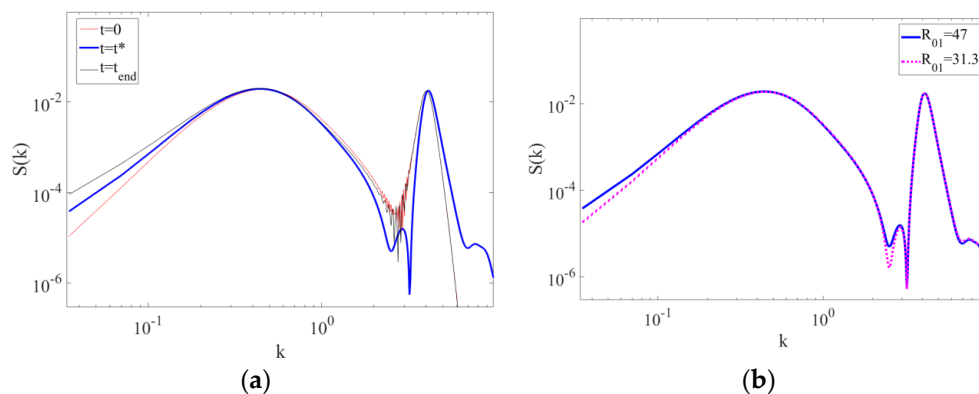
The first two of them correspond to invariants (conservation laws of the mKdV equation); therefore, they are preserved in the process of breather interaction. The most interesting is the third moment, which is responsible for the asymmetry of the wave field and the fourth, which is responsible

for the amplification of the wave field. Figure 6 shows the graphs of changes in  $M_3$  and  $M_4$  for two experiments corresponding to the largest negative and positive field amplification during the interaction of breathers:  $R_{01} = 31.3$  and  $R_{01} = 47$ . First, when breathers are separated, the left (the slowest) breather gives the largest impact on  $M_3$  and  $M_4$  (because the moment changes of the right breather are close to constant; see [43]). During the considered time equals to 3, the form of the left breather changes very little, so it gives small changes in the “background” in Figure 6. At the same time, we are interested in the interaction process, which happens very quickly because of the high speed of the second breather. For longer times, this function will be periodic. Second, we see a pronounced stage of the wave interaction on these graphs. The third moment increases significantly at  $R_{01} = 47$ , signaling a positive amplification of the wave field, and significantly decreases at  $R_{01} = 31.3$ , signaling a negative amplification of the wave field. The fourth moment behaves in approximately the same way during the positive and negative wave amplification.



**Figure 6.** Evolution of  $M_3$  and  $M_4$  in time for two experiments with maximum (dotted curve) and minimum (solid curve) amplification.

The Fourier spectra of the wave field, corresponded to the positive “optimal” interaction ( $R_{01} = 47$ ), before interaction ( $t = 0$ ), at the moment of the strongest breather interaction ( $t = t^*$ ) and after the interaction ( $t = t_{end}$ ) shown in Figure 7a. At  $t = t^*$ , the spectrum is smoother, and it decreases slower; there are two pronounced depressions. Before and after the interaction, the spectrum is rougher. Different relative breather phases influence the shape of two depressions at the moment of breather interaction (Figure 7b).



**Figure 7.** (a) Fourier spectra of the wave field corresponded to the positive “optimal” interaction ( $R_{01} = 47$ ) at  $t = 0$ ,  $t = t^*$  and  $t = t_{end}$ , (b) Fourier spectra of the wave fields corresponded to positive “optimal” interaction ( $R_{01} = 47$ ) and negative “optimal” interaction ( $R_{01} = 31.3$ ) at the interaction moment.

## 6. Conclusions

A series of 150 numerical experiments on the interaction of two breathers using the mKdV equation was carried out. In each experiment, one breather was N-shaped, and the other had several oscillations. In each experiment, the phase of the first breather was changed. The considered series of experiments covers all possible initial forms of the first breather. Although this does not give all the possible variants of interaction, these experiments show a typical picture of the interaction of breathers with given parameters, which may undergo small quantitative changes, but not qualitative ones. As a result of the calculations, cases with “optimal” and “non-optimal” interaction of breathers were analyzed. It is shown that, as a result of the interaction, the amplitude of the resulting pulse can reach double the amplification of the initial amplitudes of the breathers. These results are important for further study of multi-soliton-breather dynamics.

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