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Optimal Boundary Control of Non-Isothermal Viscous Fluid Flow

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Abstract: We study an optimal control problem for the mathematical model that describes steady non-isothermal creeping flows of an incompressible fluid through a locally Lipschitz bounded domain. The control parameters are the pressure and the temperature on the in-flow and out-flow parts of the boundary of the flow domain. We propose the weak formulation of the problem and prove the existence of weak solutions that minimize a given cost functional. It is also shown that the marginal function of this control system is lower semi-continuous.

Keywords: non-isothermal flows; creeping flows; viscous fluid; optimal control; boundary control; pressure boundary conditions; weak solution; existence theorem; marginal function

1. Introduction and Problem Formulation

In this work, we consider the following optimal control problem for the model describing steady non-isothermal creeping flows of an incompressible fluid through a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with the locally Lipschitz boundary $\partial\Omega$:

$$\left\{ \begin{array}{l} -\operatorname{div} [\mu(\theta) D(\mathbf{u})] + \nabla p = \mathbf{f}(\mathbf{x}, \theta) \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \\ (\mathbf{u} \cdot \nabla) \theta - \operatorname{div} [\kappa(\theta) \nabla \theta] = \omega(\mathbf{x}, \theta) \text{ in } \Omega, \\ \mathbf{u} = \mathbf{0}, \quad \kappa(\theta) \frac{\partial \theta}{\partial \mathbf{n}} = -\alpha \theta \text{ on } \partial\Omega \setminus S, \\ \mathbf{u}_\tau = \mathbf{0}, \quad p = \pi, \quad \theta = \zeta \text{ on } S, \\ (\pi, \zeta) \in \mathbb{U}_1 \times \mathbb{U}_2, \\ J(\mathbf{u}, \theta, \pi, \zeta) \rightarrow \min, \end{array} \right. \quad (1)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is the velocity of the fluid at a point $\mathbf{x} \in \Omega$, $\theta = \theta(\mathbf{x})$ is the deviation from the average temperature value, $\mathbf{f}(\mathbf{x}, \theta)$ denotes the external forces, $D(\mathbf{u})$ is the deformation rate tensor, $D(\mathbf{u}) \stackrel{\text{def}}{=} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$, the function $p = p(\mathbf{x})$ represents the pressure field, $\mu(\theta) > 0$ is the viscosity, $\kappa(\theta) > 0$ is the thermal conductivity, $\alpha > 0$ is a coefficient characterizing the heat transfer on solid walls of the flow domain, $\omega(\mathbf{x}, \theta)$ stands for the heat source intensity, $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit outward normal to the surface $\partial\Omega$, S is a flat (straight for $d = 2$) portion of $\partial\Omega$ or the union of several such portions. Functions $\zeta: S \rightarrow \mathbb{R}$ and $\pi: S \rightarrow \mathbb{R}$ play the role of controls, $\mathbb{U}_1 \times \mathbb{U}_2$ is the set of admissible controls, while $J = J(\mathbf{u}, \theta, \pi, \zeta)$ is a given cost functional. By the symbol τ we denote the tangential component of a vector, i.e., $\mathbf{u}_\tau \stackrel{\text{def}}{=} \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$.

The main scope of the present paper is to prove the solvability of problem (1) in the weak formulation. The proof is based on the Galerkin procedure, methods of the topological degree theory, compactness arguments, and the well-known theorem of Krasnoselskii on the continuity of a superposition operator acting in Lebesgue spaces. In addition, for control system (1), we introduce the concept of the marginal function, which shows how the optimal value of the cost functional changes with a change in the set of admissible controls. In this paper, it is proved that the marginal function of (1) is lower semi-continuous.

It should be mentioned that many mathematical works are devoted to optimization and control problems for non-isothermal flows of a viscous fluid. The results from the existing literature mainly deal with the case where the flow occurs inside a bounded domain and external body forces and/or heat sources are used as control parameters (see, for instance, [1–8]). Moreover, the Dirichlet-type boundary control by the velocity field is quite well studied (for details, see References [9–13]). However, for the analysis of real applications, it is very important to consider models of pressure driven flows, which typically occur in control problems for the heat and mass transfer in pipeline networks. For this purpose, following the approach from [14], we employ mixed boundary conditions, including the inhomogeneous Dirichlet condition for the pressure on a portion of the boundary of the flow domain; but at the same time, we formulate the optimal control problem without using the **curl** operator and boundary conditions associated with this operator. Another distinguishing feature of the present paper is that it takes into account the dependence of both the viscosity and the thermal conductivity coefficient on the temperature. This expands the range of possible applications of our results.

2. Preliminaries

In this section, we present some notations and function spaces utilized in the paper.

Let E_1, E_2 be Banach spaces. By $\mathcal{L}(E_1, E_2)$ we denote the space of all bounded linear mappings from E_1 to E_2 . The space $\mathcal{L}(E_1, E_2)$ is equipped with the norm

$$\|A\|_{\mathcal{L}(E_1, E_2)} \stackrel{\text{def}}{=} \sup_{\|v\|_{E_1} \neq 0} \frac{\|A(v)\|_{E_2}}{\|v\|_{E_1}}.$$

The strong (weak) convergence in a Banach space is denoted by $\rightarrow$ ($\rightharpoonup$).

Let $\mathcal{U}$ and $\mathcal{W}$ be subsets of a Banach space E . By definition, put

$$d_E(\mathcal{U}, \mathcal{W}) \stackrel{\text{def}}{=} \sup_{u \in \mathcal{U}} \inf_{w \in \mathcal{W}} \|u - w\|_E.$$

This quantity is termed as the directed Hausdorff distance (or one-sided Hausdorff distance) from the set $\mathcal{U}$ to the set $\mathcal{W}$.

Throughout this paper, boldface symbols denote vector- and matrix-valued quantities. For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{d \times d}$, by $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{X} : \mathbf{Y}$, we denote the scalar products, respectively:

$$\mathbf{x} \cdot \mathbf{y} \stackrel{\text{def}}{=} \sum_{i=1}^d x_i y_i, \quad \mathbf{X} : \mathbf{Y} \stackrel{\text{def}}{=} \sum_{i,j=1}^d X_{ij} Y_{ij}.$$

As usual, $|\cdot|$ denotes the Euclidean norm:

$$|\mathbf{x}| \stackrel{\text{def}}{=} (\mathbf{x} \cdot \mathbf{x})^{1/2}, \quad |\mathbf{X}| \stackrel{\text{def}}{=} (\mathbf{X} : \mathbf{X})^{1/2}.$$

The symbol ∇ stands for the gradient with respect to the variables $x_1, \dots, x_d$. The divergence operator is defined as follows:

$$\operatorname{div} \boldsymbol{w} \stackrel{\text{def}}{=} \sum_{i=1}^d \frac{\partial w_i}{\partial x_i}, \quad \operatorname{div} \boldsymbol{Q} \stackrel{\text{def}}{=} \left(\sum_{i=1}^d \frac{\partial Q_{i1}}{\partial x_i}, \dots, \sum_{i=1}^d \frac{\partial Q_{id}}{\partial x_i} \right)^T,$$

for $\boldsymbol{w}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\boldsymbol{Q}: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$.

We employ the standard notation for Lebesgue spaces, such as $L^q(\Omega)$, $L^q(\partial\Omega)$, where $q \geq 1$, and the Sobolev space $H^1(\Omega) \stackrel{\text{def}}{=} W^{1,2}(\Omega)$. When it comes to classes of $\mathbb{R}^d$ -valued functions, we use bold face letters, for instance, $\boldsymbol{L}^q(\Omega) \stackrel{\text{def}}{=} L^q(\Omega)^d$, $\boldsymbol{H}^1(\Omega) \stackrel{\text{def}}{=} H^1(\Omega)^d$, etc.

Recall that the restriction of a function $w \in H^1(\Omega)$ to the surface $\partial\Omega$ is defined by the rule $w|_{\partial\Omega} \stackrel{\text{def}}{=} \gamma_{\partial\Omega} w$, where $\gamma_{\partial\Omega}: H^1(\Omega) \rightarrow L^q(\partial\Omega)$ is the trace operator (see, e.g., [15] Section 2.4.2), $q = 4$ if $d = 3$, and q is an arbitrary number from $[1, +\infty)$ if $d = 2$.

Let us introduce two sets:

$$\begin{aligned} \mathbb{V}_S(\Omega) &\stackrel{\text{def}}{=} \{v \in C^\infty(\overline{\Omega}) : \operatorname{div} v = 0, \ v|_{\partial\Omega \setminus S} = \mathbf{0}, \ v_\tau|_S = \mathbf{0}\}, \\ \mathbb{Y}_S(\Omega) &\stackrel{\text{def}}{=} \{\eta \in C^\infty(\overline{\Omega}) : \eta|_S = 0\}, \end{aligned}$$

and two basic spaces for handling problem (1):

$$\begin{aligned} \boldsymbol{V}_S(\Omega) &\stackrel{\text{def}}{=} \text{the closure of the set } \mathbb{V}_S(\Omega) \text{ in the space } \boldsymbol{H}^1(\Omega), \\ \boldsymbol{Y}_S(\Omega) &\stackrel{\text{def}}{=} \text{the closure of the set } \mathbb{Y}_S(\Omega) \text{ in the space } \boldsymbol{H}^1(\Omega). \end{aligned}$$

In our study, it is convenient to use the following scalar product and norm in $\boldsymbol{V}_S(\Omega)$:

$$(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{V}_S(\Omega)} \stackrel{\text{def}}{=} \int_{\Omega} \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{w}) \, d\boldsymbol{x}, \quad \|\boldsymbol{v}\|_{\boldsymbol{V}_S(\Omega)} \stackrel{\text{def}}{=} (\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{V}_S(\Omega)}^{1/2}.$$

From ([16] Chap. I, Theorems 2.2 and 2.3) it follows that the scalar product $(\cdot, \cdot)_{\boldsymbol{V}_S(\Omega)}$ is well defined and the norm $\|\cdot\|_{\boldsymbol{V}_S(\Omega)}$ is equivalent to the standard $\boldsymbol{H}^1$ -norm if the set $\partial\Omega \setminus S$ has the positive $(d-1)$ -dimensional measure.

In the space $\boldsymbol{Y}_S(\Omega)$, we introduce the scalar product and the norm as follows:

$$(\eta, \xi)_{\boldsymbol{Y}_S(\Omega)} \stackrel{\text{def}}{=} \int_{\Omega} \nabla \eta \cdot \nabla \xi \, d\boldsymbol{x}, \quad \|\eta\|_{\boldsymbol{Y}_S(\Omega)} \stackrel{\text{def}}{=} (\eta, \eta)_{\boldsymbol{Y}_S(\Omega)}^{1/2}.$$

By the Friedrichs inequality (see, e.g., [15] Section 1.1.8, Theorem 1.9), it is easy to show that the scalar product $(\cdot, \cdot)_{\boldsymbol{Y}_S(\Omega)}$ is well defined and the norm $\|\cdot\|_{\boldsymbol{Y}_S(\Omega)}$ is equivalent to the standard H^1 -norm.

3. Main Assumptions and Some Examples of Cost Functionals

Let us assume that the following conditions are fulfilled:

- (C1) the inequalities $\operatorname{meas}_{d-1}(S) > 0$ and $\operatorname{meas}_{d-1}(\partial\Omega \setminus S) > 0$ hold, where $\operatorname{meas}_{d-1}(\cdot)$ denotes the Lebesgue $(d-1)$ -dimensional measure of a set;
- (C2) the functions $\mu: \mathbb{R} \rightarrow \mathbb{R}$ and $\kappa: \mathbb{R} \rightarrow \mathbb{R}$ are continuous and there exist constants μ_i, κ_i , $i = 0, 1$, such that $0 < \mu_0 \leq \mu(y) \leq \mu_1$, $0 < \kappa_0 \leq \kappa(y) \leq \kappa_1$ for every $y \in \mathbb{R}$;
- (C3) the functions $f_i(\cdot, y): \Omega \rightarrow \mathbb{R}$, $i = 1, \dots, d$, and $\omega(\cdot, y): \Omega \rightarrow \mathbb{R}$ are measurable for every $y \in \mathbb{R}$;
- (C4) the functions $f_i(\boldsymbol{x}, \cdot): \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, \dots, d$, and $\omega(\boldsymbol{x}, \cdot): \mathbb{R} \rightarrow \mathbb{R}$ are continuous for almost every $\boldsymbol{x} \in \Omega$;
- (C5) there exist functions g_0 and ω_0 from the space $L^2(\Omega)$ and a positive constant M such that

$$|f(\boldsymbol{x}, y)| \leq g_0(\boldsymbol{x}) + M|y|^2, \quad |\omega(\boldsymbol{x}, y)| \leq \omega_0(\boldsymbol{x}),$$

for every $y \in \mathbb{R}$ and for almost every $\boldsymbol{x} \in \Omega$;

- (C6) the set $\mathbb{U}_1$ is sequentially weakly closed in the space $L^2(S)$;
 (C7) the set $\mathbb{U}_2$ is closed in the space $L^2(S)$ and $0 \in \mathbb{U}_2$;
 (C8) the functional $J: V_S(\Omega) \times H^1(\Omega) \times L^2(S) \times L^2(S) \rightarrow \mathbb{R}$ is lower weakly semi-continuous; that is, for any sequence $\{(\mathbf{u}^n, \theta^n, \pi^n, \zeta^n)\}_{n=1}^\infty$ such that $\mathbf{u}^n \rightharpoonup \mathbf{u}^0$ weakly in $V_S(\Omega)$, $\theta^n \rightharpoonup \theta^0$ weakly in $H^1(\Omega)$, $\pi^n \rightharpoonup \pi^0$ weakly in $L^2(S)$, and $\zeta^n \rightharpoonup \zeta^0$ weakly in $L^2(S)$ as $n \rightarrow \infty$, we have

$$J(\mathbf{u}^0, \theta^0, \pi^0, \zeta^0) \leq \liminf_{n \rightarrow \infty} J(\mathbf{u}^n, \theta^n, \pi^n, \zeta^n);$$

- (C9) the coercivity condition [17] holds for J ; that is, for every $R > 0$ the set

$$\mathcal{G}_R \stackrel{\text{def}}{=} \{(\mathbf{u}, \theta, \pi, \zeta) \in V_S(\Omega) \times H^1(\Omega) \times L^2(S) \times L^2(S) : J(\mathbf{u}, \theta, \pi, \zeta) \leq R\}$$

is bounded in $V_S(\Omega) \times H^1(\Omega) \times L^2(S) \times L^2(S)$.

Here, we also give two examples of cost functionals satisfying conditions (C8) and (C9):

$$\begin{aligned} J = J_1(\mathbf{u}, \theta, \pi, \zeta) &\stackrel{\text{def}}{=} \lambda_1 \int_{\Omega} |\mathbf{u} - \mathbf{u}_0|^2 d\mathbf{x} + \lambda_2 \sum_{i=1}^d \int_{\Omega} \left| \frac{\partial(\mathbf{u} - \mathbf{u}_0)}{\partial x_i} \right|^2 d\mathbf{x} \\ &+ \lambda_3 \int_{\Omega} |\theta - \theta_0|^2 d\mathbf{x} + \lambda_4 \sum_{i=1}^d \int_{\Omega} \left| \frac{\partial(\theta - \theta_0)}{\partial x_i} \right|^2 d\mathbf{x} \\ &+ \lambda_5 \int_S |\pi|^2 d\sigma + \lambda_6 \int_S |\zeta|^2 d\sigma, \end{aligned}$$

$$\begin{aligned} J = J_2(\mathbf{u}, \theta, \pi, \zeta) &\stackrel{\text{def}}{=} \lambda_1 \int_{\Omega} |\mathbf{D}(\mathbf{u})|^2 d\mathbf{x} + \lambda_2 \int_{\Omega} |\nabla \theta|^2 d\mathbf{x} + \lambda_3 \int_{\partial\Omega \setminus S} |\theta|^2 d\sigma \\ &+ \lambda_4 \int_S |\pi - \pi_0|^2 d\sigma + \lambda_5 \int_S |\zeta - \zeta_0|^2 d\sigma, \end{aligned}$$

where $\mathbf{u}_0, \theta_0, \pi_0, \zeta_0$ are given functions, and $\lambda_1, \lambda_2, \dots, \lambda_6$ are positive parameters (weight coefficients).

4. Weak Formulation of Problem (1) and Marginal Function

Definition 1. We shall say that $(\mathbf{u}, \theta, \pi, \zeta) \in V_S(\Omega) \times H^1(\Omega) \times L^2(S) \times L^2(S)$ is an *admissible quadruplet* to problem (1) if $(\pi, \zeta) \in \mathbb{U}_1 \times \mathbb{U}_2$, $\theta|_S = \zeta$, and

$$\int_{\Omega} \mu(\theta) \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) d\mathbf{x} + \int_S \pi(\mathbf{v} \cdot \mathbf{n}) d\sigma = \int_{\Omega} \mathbf{f}(\mathbf{x}, \theta) \cdot \mathbf{v} d\mathbf{x}, \quad (2)$$

$$\sum_{i=1}^d \int_{\Omega} u_i \frac{\partial \theta}{\partial x_i} \eta d\mathbf{x} + \int_{\Omega} \kappa(\theta) \nabla \theta \cdot \nabla \eta d\mathbf{x} + \alpha \int_{\partial\Omega \setminus S} \theta \eta d\sigma = \int_{\Omega} \omega(\mathbf{x}, \theta) \eta d\mathbf{x} \quad (3)$$

for any vector-valued function $\mathbf{v} \in V_S(\Omega)$ and function $\eta \in Y_S(\Omega)$.

The following lemma shows that relations (2) and (3) are natural in the weak formulation of problem (1).

Lemma 1. Suppose $(\mathbf{u}, \theta, p)$ is a classic solution of boundary-value problem (1)₁–(1)₅, then equalities (2) and (3) hold.

Proof. Let us fix an arbitrary vector-valued function $\mathbf{v} \in V_S(\Omega)$. On taking the scalar product of both the left-hand and right-hand sides of equality (1)₁ with $\mathbf{v}$ and integrating over the domain Ω , we obtain

$$-\underbrace{\int_{\Omega} \operatorname{div} [\mu(\theta) \mathbf{D}(\mathbf{u})] \cdot \mathbf{v} d\mathbf{x}}_{I_1} + \underbrace{\int_{\Omega} \nabla p \cdot \mathbf{v} d\mathbf{x}}_{I_2} = \int_{\Omega} \mathbf{f}(\mathbf{x}, \theta) \cdot \mathbf{v} d\mathbf{x}. \quad (4)$$

By using integration by parts, one can show that

$$I_1 = \int_{\partial\Omega} [\mu(\theta) \mathbf{D}(\mathbf{u}) \mathbf{n}] \cdot \mathbf{v} \, d\sigma - \int_{\Omega} \mu(\theta) \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) \, d\mathbf{x},$$

from which, bearing in mind that $\mathbf{v} = (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}$ on S and $\mathbf{v} = \mathbf{0}$ on $\partial\Omega \setminus S$, we derive

$$I_1 = \int_S \mu(\theta) ([\mathbf{D}(\mathbf{u}) \mathbf{n}] \cdot \mathbf{n}) (\mathbf{v} \cdot \mathbf{n}) \, d\sigma - \int_{\Omega} \mu(\theta) \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) \, d\mathbf{x}. \quad (5)$$

Note that

$$[\mathbf{D}(\mathbf{u}) \mathbf{n}] \cdot \mathbf{n} = \frac{\partial(\mathbf{u} \cdot \mathbf{n})}{\partial \mathbf{n}} \quad \text{on } S.$$

Since

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}_\tau = 0 \quad \text{on } S,$$

we see that

$$\frac{\partial(\mathbf{u} \cdot \mathbf{n})}{\partial \mathbf{n}} = 0 \quad \text{on } S,$$

and hence,

$$[\mathbf{D}(\mathbf{u}) \mathbf{n}] \cdot \mathbf{n} = 0 \quad \text{on } S.$$

Substituting the last relation into (5), we obtain, obviously,

$$I_1 = - \int_{\Omega} \mu(\theta) \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) \, d\mathbf{x}. \quad (6)$$

To handle the term I_2 , we again use integration by parts and get

$$I_2 = \int_{\partial\Omega} p(\mathbf{v} \cdot \mathbf{n}) \, d\sigma - \int_{\Omega} p \operatorname{div} \mathbf{v} \, d\mathbf{x}. \quad (7)$$

Since

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad \mathbf{v} = \mathbf{0} \quad \text{on } \Omega \setminus S, \quad p = \pi \quad \text{on } S,$$

from (7) it follows that

$$I_2 = \int_S \pi(\mathbf{v} \cdot \mathbf{n}) \, d\sigma. \quad (8)$$

Finally, by substituting (6) and (8) into equality (4), we arrive at (2).

Next, we take the scalar product in $L^2(\Omega)$ of each term from (1)₃ with a function $\eta \in Y_S(\Omega)$ and get

$$\sum_{i=1}^d \int_{\Omega} u_i \frac{\partial \theta}{\partial x_i} \eta \, d\mathbf{x} - \underbrace{\int_{\Omega} \operatorname{div} [\kappa(\theta) \nabla \theta] \eta \, d\mathbf{x}}_{I_3} = \int_{\Omega} \omega(\mathbf{x}, \theta) \eta \, d\mathbf{x}.$$

Integrating by parts the term I_3 , we can rewrite the last equality as follows:

$$\sum_{i=1}^d \int_{\Omega} u_i \frac{\partial \theta}{\partial x_i} \eta \, d\mathbf{x} - \int_{\partial\Omega} \kappa(\theta) \frac{\partial \theta}{\partial \mathbf{n}} \eta \, d\sigma + \int_{\Omega} \kappa(\theta) \nabla \theta \cdot \nabla \eta \, d\mathbf{x} = \int_{\Omega} \omega(\mathbf{x}, \theta) \eta \, d\mathbf{x}.$$

Taking into account the boundary conditions

$$\kappa(\theta) \frac{\partial \theta}{\partial \mathbf{n}} = -\alpha \theta \quad \text{on } \partial\Omega \setminus S, \quad \eta = 0 \quad \text{on } S,$$

we arrive at equality (3). This finishes the proof of Lemma 1. $\square$

We are now ready to accurately define the concept of solutions to the considered optimal control problem. Let $\mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)$ be the set of admissible quadruplets to problem (1).

Definition 2. By a *solution* of problem (1) we mean a quadruplet $(u_*, \theta_*, \pi_*, \zeta_*) \in \mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)$ at which the functional J attains the minimum:

$$J(u_*, \theta_*, \pi_*, \zeta_*) = \inf_{(u, \theta, \pi, \zeta) \in \mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)} J(u, \theta, \pi, \zeta).$$

By $\mathfrak{M}_{\text{opt}}(\mathbb{U}_1, \mathbb{U}_2)$ denote the set of solutions to problem (1). Assuming that $\mathfrak{M}_{\text{opt}}(\mathbb{U}_1, \mathbb{U}_2) \neq \emptyset$, we define

$$\Phi(\mathbb{U}_1, \mathbb{U}_2) \stackrel{\text{def}}{=} J(u_*, \theta_*, \pi_*, \zeta_*), \quad (9)$$

where $(u_*, \theta_*, \pi_*, \zeta_*)$ is a quadruplet that belongs to the set $\mathfrak{M}_{\text{opt}}(\mathbb{U}_1, \mathbb{U}_2)$. It is obvious that the value of $\Phi(\mathbb{U}_1, \mathbb{U}_2)$ does not depend on the choice of an element from the set $\mathfrak{M}_{\text{opt}}(\mathbb{U}_1, \mathbb{U}_2)$ in the right-hand sides of (9).

Definition 3. The function Φ that is defined by equality (9) is called the *marginal function* of control system (1).

5. Main Results

The main results of the present work are summarized as follows:

Theorem 1. Suppose conditions (C1)–(C9) hold. Then:

- (a) optimal control problem (1) has at least one solution;
- (b) the marginal function Φ is lower semi-continuous in the following sense:
if $\mathfrak{M}_{\text{opt}}(\mathbb{U}_1^n, \mathbb{U}_2^n) \neq \emptyset$, for any $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} d_{L^2(S)}(\mathbb{U}_1^n, \mathbb{U}_1) = 0, \quad \lim_{n \rightarrow \infty} d_{L^2(S)}(\mathbb{U}_2^n, \mathbb{U}_2) = 0, \quad (10)$$

then

$$\Phi(\mathbb{U}_1, \mathbb{U}_2) \leq \liminf_{n \rightarrow \infty} \Phi(\mathbb{U}_1^n, \mathbb{U}_2^n).$$

The proof of this theorem is given in Section 7.

6. Auxiliary Propositions

For the reader's convenience, we state here some preparatory results on which the proof of Theorem 1 is based.

Proposition 1. Suppose $\mathfrak{B}_r \stackrel{\text{def}}{=} \{a \in \mathbb{R}^N : |a| < r\}$ and $F: \overline{\mathfrak{B}_r} \times [0, 1] \rightarrow \mathbb{R}^N$ is a continuous mapping such that

- (i) $F(a, \lambda) \neq 0$ for any pair $(a, \lambda) \in \partial \mathfrak{B}_r \times [0, 1]$;
- (ii) $F(\cdot, 0): \overline{\mathfrak{B}_r} \rightarrow \mathbb{R}^N$ is an odd mapping; that is, $F(-a, 0) = -F(a, 0)$ for any vector $a \in \overline{\mathfrak{B}_r}$.

Then, for any $\lambda \in [0, 1]$, there exists a vector $a_\lambda \in \mathfrak{B}_r$ such that $F(a_\lambda, \lambda) = 0$; in other words, the equation $F(a, \lambda) = 0$ is solvable with respect to a in the ball $\mathfrak{B}_r$.

Proof. Employing the homotopy invariance property of Brouwer's degree and condition (i), one can show that $\deg(F(\cdot, \lambda), \mathfrak{B}_r, 0)$ does not depend on $\lambda \in [0, 1]$ and, consequently,

$$\deg(F(\cdot, \lambda), \mathfrak{B}_r, 0) = \deg(F(\cdot, 0), \mathfrak{B}_r, 0),$$

for any $\lambda \in [0, 1]$. Besides, since $F(\cdot, 0): \overline{\mathfrak{B}}_r \rightarrow \mathbb{R}^N$ is an odd mapping, we see that $\deg(F(\cdot, 0), \mathfrak{B}_r, \mathbf{0})$ is an odd number. This follows from Borsuk's theorem (see, e.g., [18] Chap. 1, Theorem 1.2.11). Thereby, for any $\lambda \in [0, 1]$,

$$\deg(F(\cdot, \lambda), \mathfrak{B}_r, \mathbf{0}) \neq 0.$$

Then, by [18] (Chap. 1, Theorem 1.2.6), we deduce that the equation $F(\mathbf{a}, \lambda) = \mathbf{0}$ has at least one solution $\mathbf{a}_\lambda \in \mathfrak{B}_r$. $\square$

Proposition 2. Suppose $G \subset \mathbb{R}^N$ is a Lebesgue measurable set and a given function $\varphi: G \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following conditions:

- the function $\varphi(\cdot, y): G \rightarrow \mathbb{R}$ is measurable for every $y \in \mathbb{R}$;
- the function $\varphi(x, \cdot): \mathbb{R} \rightarrow \mathbb{R}$ is continuous for almost every $x \in G$;
- there exist constants $q_1 \geq 1, q_2 \geq 1, K > 0$ and a function $\phi_0 \in L^{q_2}(G)$ such that

$$|\varphi(x, y)| \leq \phi_0(x) + K|y|^{q_1/q_2},$$

for every $y \in \mathbb{R}$ and for almost every $x \in G$.

Then, the superposition operator T_φ defined by

$$T_\varphi: L^{q_1}(G) \rightarrow L^{q_2}(G), \quad T_\varphi[w](x) \stackrel{\text{def}}{=} \varphi(x, w(x))$$

is a bounded and continuous mapping.

This proposition was proved by Krasnoselskii (see [19] Chap. 1).

7. Proof of Theorem 1

Let us prove the existence result (a). Our first step is to show that $\mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2) \neq \emptyset$.

Fix a pair $(\pi^0, 0) \in \mathbb{U}_1 \times \mathbb{U}_2$. Let $\{v^j\}_{j=1}^\infty \subset \mathbb{V}_S(\Omega)$ be an orthonormal basis of the space $\mathbb{V}_S(\Omega)$, $\{\eta^j\}_{j=1}^\infty \subset \mathbb{Y}_S(\Omega)$ an orthonormal basis of the space $\mathbb{Y}_S(\Omega)$, and m an arbitrary fixed integer.

Consider the following one-parameter problem in space $\mathbb{R}^{2m}$:

Find a vector $(a_{m1}, \dots, a_{mm}, b_{m1}, \dots, b_{mm})^T \in \mathbb{R}^{2m}$ such that

$$\int_{\Omega} \mu(\lambda \theta^m) D(\mathbf{u}^m) : D(\mathbf{v}^j) \, d\mathbf{x} + \lambda \int_S \pi^0(\mathbf{v}^j \cdot \mathbf{n}) \, d\sigma = \lambda \int_{\Omega} \mathbf{f}(\mathbf{x}, \theta^m) \cdot \mathbf{v}^j \, d\mathbf{x}, \quad j = 1, \dots, m, \quad (11)$$

$$\begin{aligned} & \lambda \sum_{i=1}^d \int_{\Omega} u_i^m \frac{\partial \theta^m}{\partial x_i} \eta^j \, d\mathbf{x} + \int_{\Omega} \kappa(\lambda \theta^m) \nabla \theta^m \cdot \nabla \eta^j \, d\mathbf{x} + \lambda \alpha \int_{\partial\Omega \setminus S} \theta^m \eta^j \, d\sigma \\ & = \lambda \int_{\Omega} \omega(\mathbf{x}, \theta^m) \eta^j \, d\mathbf{x}, \quad j = 1, \dots, m, \end{aligned} \quad (12)$$

where $\mathbf{u}^m$ and θ^m are defined as follows:

$$\mathbf{u}^m \stackrel{\text{def}}{=} \sum_{j=1}^m a_{mj} \mathbf{v}^j, \quad \theta^m \stackrel{\text{def}}{=} \sum_{j=1}^m b_{mj} \eta^j, \quad (13)$$

and λ is a parameter, $\lambda \in [0, 1]$.

We shall establish *a priori* estimates of solutions to problem (11)–(13). Suppose a vector $(a_{m1}, \dots, a_{mm}, b_{m1}, \dots, b_{mm})^T$ satisfies (11)–(13). Let us multiply (11) by a_{mj} and add the obtained equalities for $j = 1, \dots, m$. This yields

$$\int_{\Omega} \mu(\lambda \theta^m) |D(\mathbf{u}^m)|^2 \, d\mathbf{x} + \lambda \int_S \pi^0(\mathbf{u}^m \cdot \mathbf{n}) \, d\sigma = \lambda \int_{\Omega} \mathbf{f}(\mathbf{x}, \theta^m) \cdot \mathbf{u}^m \, d\mathbf{x}.$$

Then, using conditions **(C2)** and **(C5)**, the Cauchy–Bunyakovsky–Schwarz inequality, and $0 \leq \lambda \leq 1$, we derive

$$\begin{aligned} \mu_0 \|\mathbf{u}^m\|_{V_S(\Omega)}^2 &= \mu_0 \int_{\Omega} |\mathbf{D}(\mathbf{u}^m)|^2 \, d\mathbf{x} \\ &\leq \int_{\Omega} \mu(\lambda \theta^m) |\mathbf{D}(\mathbf{u}^m)|^2 \, d\mathbf{x} \\ &= -\lambda \int_S \pi^0(\mathbf{u}^m \cdot \mathbf{n}) \, d\sigma + \lambda \int_{\Omega} \mathbf{f}(\mathbf{x}, \theta^m) \cdot \mathbf{u}^m \, d\mathbf{x} \\ &\leq \left(\int_S |\pi^0|^2 \, d\sigma \right)^{1/2} \left(\int_S |\mathbf{u}^m|^2 \, d\sigma \right)^{1/2} + \left(\int_{\Omega} |g_0|^2 \, d\mathbf{x} \right)^{1/2} \left(\int_{\Omega} |\mathbf{u}^m|^2 \, d\mathbf{x} \right)^{1/2} \\ &\quad + M \left(\int_{\Omega} |\theta^m|^4 \, d\mathbf{x} \right)^{1/2} \left(\int_{\Omega} |\mathbf{u}^m|^2 \, d\mathbf{x} \right)^{1/2} \\ &= \|\pi^0\|_{L^2(S)} \|\mathbf{u}^m\|_{L^2(S)} + \|g_0\|_{L^2(\Omega)} \|\mathbf{u}^m\|_{L^2(\Omega)} + M \|\theta^m\|_{L^4(\Omega)}^2 \|\mathbf{u}^m\|_{L^2(\Omega)}. \end{aligned} \quad (14)$$

Note that

$$\|\mathbf{u}^m\|_{L^2(S)} \leq \|\gamma_S\|_{\mathcal{L}(V_S(\Omega), L^2(S))} \|\mathbf{u}^m\|_{V_S(\Omega)}, \quad (15)$$

$$\|\mathbf{u}^m\|_{L^2(\Omega)} \leq \|\mathcal{I}\|_{\mathcal{L}(V_S(\Omega), L^2(\Omega))} \|\mathbf{u}^m\|_{V_S(\Omega)}, \quad (16)$$

$$\|\theta^m\|_{L^4(\Omega)} \leq \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^4(\Omega))} \|\theta^m\|_{Y_S(\Omega)}, \quad (17)$$

where γ_S is the trace operator, the symbol $\mathcal{I}$ stands for the identity mapping. Combining (14) with (15)–(17), we get

$$\begin{aligned} \mu_0 \|\mathbf{u}^m\|_{V_S(\Omega)}^2 &\leq \|\pi^0\|_{L^2(S)} \|\gamma_S\|_{\mathcal{L}(V_S(\Omega), L^2(S))} \|\mathbf{u}^m\|_{V_S(\Omega)} \\ &\quad + \|g_0\|_{L^2(\Omega)} \|\mathcal{I}\|_{\mathcal{L}(V_S(\Omega), L^2(\Omega))} \|\mathbf{u}^m\|_{V_S(\Omega)} \\ &\quad + M \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^4(\Omega))}^2 \|\mathcal{I}\|_{\mathcal{L}(V_S(\Omega), L^2(\Omega))} \|\theta^m\|_{Y_S(\Omega)}^2 \|\mathbf{u}^m\|_{V_S(\Omega)}. \end{aligned}$$

It immediately follows that

$$\begin{aligned} \|\mathbf{u}^m\|_{V_S(\Omega)} &\leq \mu_0^{-1} \|\pi^0\|_{L^2(S)} \|\gamma_S\|_{\mathcal{L}(V_S(\Omega), L^2(S))} + \mu_0^{-1} \|g_0\|_{L^2(\Omega)} \|\mathcal{I}\|_{\mathcal{L}(V_S(\Omega), L^2(\Omega))} \\ &\quad + M \mu_0^{-1} \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^4(\Omega))}^2 \|\mathcal{I}\|_{\mathcal{L}(V_S(\Omega), L^2(\Omega))} \|\theta^m\|_{Y_S(\Omega)}^2. \end{aligned} \quad (18)$$

Now multiply (12) by b_{mj} and add the results for $j = 1, \dots, m$; this gives

$$\lambda \underbrace{\sum_{i=1}^d \int_{\Omega} u_i^m \frac{\partial \theta^m}{\partial x_i} \, d\mathbf{x}}_{I_4} + \int_{\Omega} \kappa(\lambda \theta^m) |\nabla \theta^m|^2 \, d\mathbf{x} + \lambda \alpha \int_{\partial\Omega \setminus S} |\theta^m|^2 \, d\sigma = \lambda \int_{\Omega} \omega(\mathbf{x}, \theta^m) \theta^m \, d\mathbf{x}. \quad (19)$$

Using the integration by parts formula, it is easy to show that the term I_4 vanishes. Indeed,

$$\begin{aligned} I_4 &= \frac{1}{2} \sum_{i=1}^d \int_{\Omega} u_i^m \frac{\partial |\theta^m|^2}{\partial x_i} \, d\mathbf{x} \\ &= \frac{1}{2} \sum_{i=1}^d \int_{\partial\Omega} u_i^m n_i |\theta^m|^2 \, d\sigma - \frac{1}{2} \sum_{i=1}^d \int_{\Omega} \frac{\partial u_i^m}{\partial x_i} |\theta^m|^2 \, d\mathbf{x} \\ &= \frac{1}{2} \int_S (\mathbf{u}^m \cdot \mathbf{n}) \underbrace{|\theta^m|^2}_{=0} \, d\sigma + \frac{1}{2} \int_{\partial\Omega \setminus S} \underbrace{(\mathbf{u}^m \cdot \mathbf{n})}_{=0} |\theta^m|^2 \, d\sigma - \frac{1}{2} \int_{\Omega} \underbrace{\operatorname{div} \mathbf{u}^m}_{=0} |\theta^m|^2 \, d\mathbf{x} \\ &= 0. \end{aligned}$$

Therefore, equality (19) can be rewritten as follows:

$$\int_{\Omega} \kappa(\lambda \theta^m) |\nabla \theta^m|^2 dx = -\lambda \alpha \int_{\partial\Omega \setminus S} |\theta^m|^2 d\sigma + \lambda \int_{\Omega} \omega(x, \theta^m) \theta^m dx,$$

from which, by using conditions (C2) and (C5), the Cauchy–Bunyakovsky–Schwarz inequality, and $0 \leq \lambda \leq 1$, we derive

$$\begin{aligned} \kappa_0 \|\theta^m\|_{Y_S(\Omega)}^2 &= \kappa_0 \int_{\Omega} |\nabla \theta^m|^2 dx \\ &\leq \int_{\Omega} \kappa(\lambda \theta^m) |\nabla \theta^m|^2 dx \\ &= -\lambda \alpha \int_{\partial\Omega \setminus S} |\theta^m|^2 d\sigma + \lambda \int_{\Omega} \omega(x, \theta^m) \theta^m dx \\ &\leq \left(\int_{\Omega} |\omega_0|^2 dx \right)^{1/2} \left(\int_{\Omega} |\theta^m|^2 dx \right)^{1/2} \\ &= \|\omega_0\|_{L^2(\Omega)} \|\theta^m\|_{L^2(\Omega)} \\ &\leq \|\omega_0\|_{L^2(\Omega)} \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^2(\Omega))} \|\theta^m\|_{Y_S(\Omega)}, \end{aligned}$$

whence

$$\|\theta^m\|_{Y_S(\Omega)} \leq \kappa_0^{-1} \|\omega_0\|_{L^2(\Omega)} \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^2(\Omega))}. \quad (20)$$

With this majoration, we deduce from (18) that

$$\begin{aligned} \|\mathbf{u}^m\|_{V_S(\Omega)} &\leq \mu_0^{-1} \|\pi^0\|_{L^2(S)} \|\gamma_S\|_{\mathcal{L}(V_S(\Omega), L^2(S))} + \mu_0^{-1} \|g_0\|_{L^2(\Omega)} \|\mathcal{I}\|_{\mathcal{L}(V_S(\Omega), L^2(\Omega))} \\ &\quad + M \mu_0^{-1} \kappa_0^{-2} \|\omega_0\|_{L^2(\Omega)}^2 \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^4(\Omega))}^2 \|\mathcal{I}\|_{\mathcal{L}(V_S(\Omega), L^2(\Omega))} \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^2(\Omega))}^2. \end{aligned} \quad (21)$$

Furthermore, since $\{v^j\}_{j=1}^{\infty}$ is an orthonormal basis of the space $V_S(\Omega)$ and $\{\eta^j\}_{j=1}^{\infty}$ is an orthonormal basis of the space $Y_S(\Omega)$, from (13) it follows that

$$\|\mathbf{u}^m\|_{V_S(\Omega)}^2 = \sum_{j=1}^m a_{mj}^2, \quad \|\theta^m\|_{Y_S(\Omega)}^2 = \sum_{j=1}^m b_{mj}^2.$$

Taking into account (20) and (21), we obtain

$$\begin{aligned} \sum_{j=1}^m a_{mj}^2 &\leq \left\{ \mu_0^{-1} \|\pi^0\|_{L^2(S)} \|\gamma_S\|_{\mathcal{L}(V_S(\Omega), L^2(S))} + \mu_0^{-1} \|g_0\|_{L^2(\Omega)} \|\mathcal{I}\|_{\mathcal{L}(V_S(\Omega), L^2(\Omega))} \right. \\ &\quad \left. + M \mu_0^{-1} \kappa_0^{-2} \|\omega_0\|_{L^2(\Omega)}^2 \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^4(\Omega))}^2 \|\mathcal{I}\|_{\mathcal{L}(V_S(\Omega), L^2(\Omega))} \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^2(\Omega))}^2 \right\}^2, \\ \sum_{j=1}^m b_{mj}^2 &\leq \kappa_0^{-2} \|\omega_0\|_{L^2(\Omega)}^2 \|\mathcal{I}\|_{\mathcal{L}(Y_S(\Omega), L^2(\Omega))}^2. \end{aligned}$$

Note that these estimates do not depend on $m \in \mathbb{N}$ as well as $\lambda \in [0, 1]$. Therefore, we can apply Proposition 1 to justify the solvability of problem (11)–(13). Here, of course, we used Proposition 2 and conditions (C2)–(C5) to establish the continuity of the corresponding mappings.

Let $(\tilde{a}_{m1}, \dots, \tilde{a}_{mm}, \tilde{b}_{m1}, \dots, \tilde{b}_{mm})$ be a solution to (11)–(13) with $\lambda = 1$. Setting

$$\tilde{\mathbf{u}}^m \stackrel{\text{def}}{=} \sum_{j=1}^m \tilde{a}_{mj} v^j, \quad \tilde{\theta}^m \stackrel{\text{def}}{=} \sum_{j=1}^m \tilde{b}_{mj} \eta^j,$$

we have

$$\int_{\Omega} \mu(\tilde{\theta}^m) D(\tilde{\mathbf{u}}^m) : D(\mathbf{v}^j) \, d\mathbf{x} + \int_S \pi^0(\mathbf{v}^j \cdot \mathbf{n}) \, d\sigma = \int_{\Omega} \mathbf{f}(\mathbf{x}, \tilde{\theta}^m) \cdot \mathbf{v}^j \, d\mathbf{x}, \quad j = 1, \dots, m, \quad (22)$$

$$\sum_{i=1}^d \int_{\Omega} \tilde{u}_i^m \frac{\partial \tilde{\theta}^m}{\partial x_i} \eta^j \, d\mathbf{x} + \int_{\Omega} \kappa(\tilde{\theta}^m) \nabla \tilde{\theta}^m \cdot \nabla \eta^j \, d\mathbf{x} + \alpha \int_{\partial\Omega \setminus S} \tilde{\theta}^m \eta^j \, d\sigma = \int_{\Omega} \omega(\mathbf{x}, \tilde{\theta}^m) \eta^j \, d\mathbf{x}, \quad j = 1, \dots, m. \quad (23)$$

Consider the sequence $\{(\tilde{\mathbf{u}}^m, \tilde{\theta}^m)\}_{m=1}^{\infty}$. It is clear that estimate (21) remains valid if we replace $\mathbf{u}^m$ with $\tilde{\mathbf{u}}^m$. Consequently, the sequence $\{\tilde{\mathbf{u}}^m\}_{m=1}^{\infty}$ is bounded in the space $V_S(\Omega)$. Likewise, taking into account estimate (20), we see that the sequence $\{\tilde{\theta}^m\}_{m=1}^{\infty}$ is bounded in the space $Y_S(\Omega)$. Therefore, there exists a pair $(\tilde{\mathbf{u}}^0, \tilde{\theta}^0) \in V_S(\Omega) \times Y_S(\Omega)$ such that $\tilde{\mathbf{u}}^{m_k}$ converges to $\tilde{\mathbf{u}}^0$ weakly in $V_S(\Omega)$ and $\tilde{\theta}^{m_k}$ converges to $\tilde{\theta}^0$ weakly in $Y_S(\Omega)$, for some subsequence $m_k \rightarrow \infty$ as $k \rightarrow \infty$. Without loss of generality, it can be assumed that

$$\tilde{\mathbf{u}}^m \rightharpoonup \tilde{\mathbf{u}}^0 \text{ weakly in } V_S(\Omega) \text{ as } m \rightarrow \infty, \quad (24)$$

$$\tilde{\theta}^m \rightharpoonup \tilde{\theta}^0 \text{ weakly in } Y_S(\Omega) \text{ as } m \rightarrow \infty. \quad (25)$$

Since the trace operator $\gamma_{\partial\Omega}: H^1(\Omega) \rightarrow L^2(\partial\Omega)$ is compact (see [15] Section 2.6.2, Theorem 6.2), we have

$$\tilde{\theta}^m|_{\partial\Omega \setminus S} \rightarrow \tilde{\theta}^0|_{\partial\Omega \setminus S} \text{ strongly in } L^2(\partial\Omega \setminus S) \text{ as } m \rightarrow \infty. \quad (26)$$

Moreover, from (24) and (25) and the compactness theorem for the identity mapping $\mathcal{I}: H^1(\Omega) \rightarrow L^4(\Omega)$ (see [15] Section 2.6.1, Theorem 6.1), it follows that

$$\tilde{\mathbf{u}}^m \rightarrow \tilde{\mathbf{u}}^0 \text{ strongly in } L^4(\Omega) \text{ as } m \rightarrow \infty, \quad (27)$$

$$\tilde{\theta}^m \rightarrow \tilde{\theta}^0 \text{ strongly in } L^4(\Omega) \text{ as } m \rightarrow \infty. \quad (28)$$

Using (24)–(28) and Proposition 2, we can pass to the limit $m \rightarrow \infty$ in (22) and (23); this gives

$$\int_{\Omega} \mu(\tilde{\theta}^0) D(\tilde{\mathbf{u}}^0) : D(\mathbf{v}^j) \, d\mathbf{x} + \int_S \pi^0(\mathbf{v}^j \cdot \mathbf{n}) \, d\sigma = \int_{\Omega} \mathbf{f}(\mathbf{x}, \tilde{\theta}^0) \cdot \mathbf{v}^j \, d\mathbf{x}, \quad (29)$$

$$\sum_{i=1}^d \int_{\Omega} \tilde{u}_i^0 \frac{\partial \tilde{\theta}^0}{\partial x_i} \eta^j \, d\mathbf{x} + \int_{\Omega} \kappa(\tilde{\theta}^0) \nabla \tilde{\theta}^0 \cdot \nabla \eta^j \, d\mathbf{x} + \alpha \int_{\partial\Omega \setminus S} \tilde{\theta}^0 \eta^j \, d\sigma = \int_{\Omega} \omega(\mathbf{x}, \tilde{\theta}^0) \eta^j \, d\mathbf{x}, \quad (30)$$

for each $j \in \mathbb{N}$. Because $\{\mathbf{v}^j\}_{j=1}^{\infty}$ is a basis of $V_S(\Omega)$ and $\{\eta^j\}_{j=1}^{\infty}$ is a basis of $Y_S(\Omega)$, equalities (29) and (30) remain valid if we replace $\mathbf{v}^j$ and η^j with arbitrary vector function $\mathbf{v} \in V_S(\Omega)$ and function $\eta \in Y_S(\Omega)$, respectively:

$$\int_{\Omega} \mu(\tilde{\theta}^0) D(\tilde{\mathbf{u}}^0) : D(\mathbf{v}) \, d\mathbf{x} + \int_S \pi^0(\mathbf{v} \cdot \mathbf{n}) \, d\sigma = \int_{\Omega} \mathbf{f}(\mathbf{x}, \tilde{\theta}^0) \cdot \mathbf{v} \, d\mathbf{x},$$

$$\sum_{i=1}^d \int_{\Omega} \tilde{u}_i^0 \frac{\partial \tilde{\theta}^0}{\partial x_i} \eta \, d\mathbf{x} + \int_{\Omega} \kappa(\tilde{\theta}^0) \nabla \tilde{\theta}^0 \cdot \nabla \eta \, d\mathbf{x} + \alpha \int_{\partial\Omega \setminus S} \tilde{\theta}^0 \eta \, d\sigma = \int_{\Omega} \omega(\mathbf{x}, \tilde{\theta}^0) \eta \, d\mathbf{x}.$$

Thus, we established that $(\tilde{\mathbf{u}}^0, \tilde{\theta}^0, \pi^0, 0)$ is an admissible quadruplet to problem (1) and hence $\mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2) \neq \emptyset$.

Now, consider a sequence $\{(\hat{\mathbf{u}}^s, \hat{\theta}^s, \hat{\pi}^s, \hat{\zeta}^s)\}_{s=1}^{\infty} \subset \mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)$ such that

$$\lim_{s \rightarrow \infty} J(\hat{\mathbf{u}}^s, \hat{\theta}^s, \hat{\pi}^s, \hat{\zeta}^s) = \inf_{(\mathbf{u}, \theta, \pi, \zeta) \in \mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)} J(\mathbf{u}, \theta, \pi, \zeta). \quad (31)$$

Owing to coercivity condition **(C9)**, we deduce from (31) that the set $\{(\hat{u}^s, \hat{\theta}^s, \hat{\pi}^s, \hat{\zeta}^s)\}_{s=1}^\infty$ is bounded in the space $V_S(\Omega) \times H^1(\Omega) \times L^2(S) \times L^2(S)$. Hence, there exists a subsequence $\{s_k\}_{k=1}^\infty$ such that

$$\hat{u}^{s_k} \rightharpoonup u_* \text{ weakly in } V_S(\Omega) \text{ as } k \rightarrow \infty, \quad (32)$$

$$\hat{\theta}^{s_k} \rightharpoonup \theta_* \text{ weakly in } Y_S(\Omega) \text{ as } k \rightarrow \infty, \quad (33)$$

$$\hat{\pi}^{s_k} \rightharpoonup \pi_* \text{ weakly in } L^2(S) \text{ as } k \rightarrow \infty, \quad (34)$$

$$\hat{\zeta}^{s_k} \rightharpoonup \zeta_* \text{ weakly in } L^2(S) \text{ as } k \rightarrow \infty, \quad (35)$$

for some quadruplet $(u_*, \theta_*, \pi_*, \zeta_*) \in V_S(\Omega) \times H^1(\Omega) \times L^2(S) \times L^2(S)$.

The inclusion $\{(\hat{u}^{s_k}, \hat{\theta}^{s_k}, \hat{\pi}^{s_k}, \hat{\zeta}^{s_k})\}_{k=1}^\infty \subset \mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)$ implies that

$$\begin{aligned} & \int_{\Omega} \mu(\hat{\theta}^{s_k}) D(\hat{u}^{s_k}) : D(v) \, dx + \int_S \hat{\pi}^{s_k} (v \cdot \mathbf{n}) \, d\sigma = \int_{\Omega} f(x, \hat{\theta}^{s_k}) \cdot v \, dx, \quad \forall v \in V_S(\Omega), \\ & \sum_{i=1}^d \int_{\Omega} \hat{u}_i^{s_k} \frac{\partial \hat{\theta}^{s_k}}{\partial x_i} \eta \, dx + \int_{\Omega} \kappa(\hat{\theta}^{s_k}) \nabla \hat{\theta}^{s_k} \cdot \nabla \eta \, dx + \alpha \int_{\partial\Omega \setminus S} \hat{\theta}^{s_k} \eta \, d\sigma = \int_{\Omega} \omega(x, \hat{\theta}^{s_k}) \eta \, dx, \quad \forall \eta \in Y_S(\Omega), \end{aligned}$$

from which, by (32)–(35), one can derive the following equalities:

$$\begin{aligned} & \int_{\Omega} \mu(\theta_*) D(u_*) : D(v) \, dx + \int_S \pi_* (v \cdot \mathbf{n}) \, d\sigma = \int_{\Omega} f(x, \theta_*) \cdot v \, dx, \quad \forall v \in V_S(\Omega), \\ & \sum_{i=1}^d \int_{\Omega} u_{*i} \frac{\partial \theta_*}{\partial x_i} \eta \, dx + \int_{\Omega} \kappa(\theta_*) \nabla \theta_* \cdot \nabla \eta \, dx + \alpha \int_{\partial\Omega \setminus S} \theta_* \eta \, d\sigma = \int_{\Omega} \omega(x, \theta_*) \eta \, dx, \quad \forall \eta \in Y_S(\Omega). \end{aligned}$$

Note that

$$\hat{\zeta}^{s_k} = \hat{\theta}^{s_k}|_S \rightarrow \theta_*|_S \text{ strongly in } L^2(S) \text{ as } k \rightarrow \infty.$$

Comparing this convergence with (35), we conclude that $\theta_*|_S = \zeta_*$. Moreover, since $\{\hat{\zeta}^{s_k}\}_{k=1}^\infty \subset \mathbb{U}_2$ and the set $\mathbb{U}_2$ is closed in the space $L^2(S)$, we see that the function ζ_* belongs to $\mathbb{U}_2$. Next, taking into account condition **(C6)**, the inclusion $\{\hat{\pi}^{s_k}\}_{k=1}^\infty \subset \mathbb{U}_1$, and convergence (34), we get $\pi_* \in \mathbb{U}_1$. Thus, $(u_*, \theta_*, \pi_*, \zeta_*) \in \mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)$.

In addition, by condition **(C8)** and (31)–(35), we can obtain

$$\begin{aligned} J(u_*, \theta_*, \pi_*, \zeta_*) & \leq \liminf_{k \rightarrow \infty} J(\hat{u}^{s_k}, \hat{\theta}^{s_k}, \hat{\pi}^{s_k}, \hat{\zeta}^{s_k}) \\ & = \inf_{(u, \theta, \pi, \zeta) \in \mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)} J(u, \theta, \pi, \zeta). \end{aligned}$$

This means that the quadruplet $(u_*, \theta_*, \pi_*, \zeta_*)$ belongs to the set $\mathfrak{M}_{\text{opt}}(\mathbb{U}_1, \mathbb{U}_2)$, and hence $\mathfrak{M}_{\text{opt}}(\mathbb{U}_1, \mathbb{U}_2) \neq \emptyset$. Therefore, we have shown that optimal control problem (1) is solvable in the sense of Definition 2.

We now turn to proving assertion (b). Assume the converse. Then, there exists a subsequence $\{n_k\}_{k=1}^\infty$ such that

$$\lim_{k \rightarrow \infty} \Phi(\mathbb{U}_1^{n_k}, \mathbb{U}_2^{n_k}) < \Phi(\mathbb{U}_1, \mathbb{U}_2). \quad (36)$$

Consider a sequence $\{(u_*^{n_k}, \theta_*^{n_k}, \pi_*^{n_k}, \zeta_*^{n_k})\}_{k=1}^\infty$ such that

$$(u_*^{n_k}, \theta_*^{n_k}, \pi_*^{n_k}, \zeta_*^{n_k}) \in \mathfrak{M}_{\text{opt}}(\mathbb{U}_1^{n_k}, \mathbb{U}_2^{n_k}), \quad \forall k \in \mathbb{N}.$$

By Definition 1, we get

$$\int_{\Omega} \mu(\theta_*^{n_k}) D(\mathbf{u}_*^{n_k}) : D(\mathbf{v}) \, d\mathbf{x} + \int_S \pi_*^{n_k} (\mathbf{v} \cdot \mathbf{n}) \, d\sigma = \int_{\Omega} \mathbf{f}(\mathbf{x}, \theta_*^{n_k}) \cdot \mathbf{v} \, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{V}_S(\Omega), \quad (37)$$

$$\sum_{i=1}^d \int_{\Omega} u_{*i}^{n_k} \frac{\partial \theta_*^{n_k}}{\partial x_i} \eta \, d\mathbf{x} + \int_{\Omega} \kappa(\theta_*^{n_k}) \nabla \theta_*^{n_k} \cdot \nabla \eta \, d\mathbf{x} + \alpha \int_{\partial\Omega \setminus S} \theta_*^{n_k} \eta \, d\sigma = \int_{\Omega} \omega(\mathbf{x}, \theta_*^{n_k}) \eta \, d\mathbf{x},$$

$$\forall \eta \in Y_S(\Omega). \quad (38)$$

Moreover, by Definition 3, we have

$$\Phi(\mathbb{U}_1^{n_k}, \mathbb{U}_2^{n_k}) = J(\mathbf{u}_*^{n_k}, \theta_*^{n_k}, \pi_*^{n_k}, \zeta_*^{n_k}), \quad \forall k \in \mathbb{N}. \quad (39)$$

It follows from (36) and (39) that there exists a number k_0 such that

$$J(\mathbf{u}_*^{n_k}, \theta_*^{n_k}, \pi_*^{n_k}, \zeta_*^{n_k}) < \Phi(\mathbb{U}_1, \mathbb{U}_2)$$

for each $k \geq k_0$. Hence,

$$\{(\mathbf{u}_*^{n_k}, \theta_*^{n_k}, \pi_*^{n_k}, \zeta_*^{n_k})\}_{k=k_0}^{\infty} \subset \mathcal{G}_{|\Phi(\mathbb{U}_1, \mathbb{U}_2)|}.$$

In view of condition (C9), the set $\mathcal{G}_{|\Phi(\mathbb{U}_1, \mathbb{U}_2)|}$ is bounded in $\mathbf{V}_S(\Omega) \times H^1(\Omega) \times L^2(S) \times L^2(S)$. Therefore, without loss of generality, it can be assumed that

$$\mathbf{u}_*^{n_k} \rightharpoonup \mathbf{u}_*^0 \text{ weakly in } \mathbf{V}_S(\Omega) \text{ as } k \rightarrow \infty, \quad (40)$$

$$\theta_*^{n_k} \rightharpoonup \theta_*^0 \text{ weakly in } Y_S(\Omega) \text{ as } k \rightarrow \infty, \quad (41)$$

$$\pi_*^{n_k} \rightharpoonup \pi_*^0 \text{ weakly in } L^2(S) \text{ as } k \rightarrow \infty, \quad (42)$$

$$\zeta_*^{n_k} \rightharpoonup \zeta_*^0 \text{ weakly in } L^2(S) \text{ as } k \rightarrow \infty, \quad (43)$$

for some quadruplet $(\mathbf{u}_*^0, \theta_*^0, \pi_*^0, \zeta_*^0) \in \mathbf{V}_S(\Omega) \times H^1(\Omega) \times L^2(S) \times L^2(S)$.

Besides, since both the identity mapping $\mathcal{I}: H^1(\Omega) \rightarrow L^4(\Omega)$ and the trace operator $\gamma_{\partial\Omega}: H^1(\Omega) \rightarrow L^2(\partial\Omega)$ are compact, the following convergences hold:

$$\mathbf{u}_*^{n_k} \rightarrow \mathbf{u}_*^0 \text{ strongly in } L^4(\Omega) \text{ as } k \rightarrow \infty, \quad (44)$$

$$\theta_*^{n_k} \rightarrow \theta_*^0 \text{ strongly in } L^4(\Omega) \text{ as } k \rightarrow \infty, \quad (45)$$

$$\zeta_*^{n_k} = \theta_*^{n_k}|_S \rightarrow \theta_*^0|_S \text{ strongly in } L^2(S) \text{ as } k \rightarrow \infty. \quad (46)$$

Comparing (43) with (46), one can infer that $\theta_*^0|_S = \zeta_*^0$ and

$$\zeta_*^{n_k} \rightarrow \zeta_*^0 \text{ strongly in } L^2(S) \text{ as } k \rightarrow \infty. \quad (47)$$

In view of (10), there exist sequences $\{\bar{\pi}^{n_k}\}_{k=1}^{\infty} \subset \mathbb{U}_1$ and $\{\bar{\zeta}^{n_k}\}_{k=1}^{\infty} \subset \mathbb{U}_2$ such that

$$\lim_{k \rightarrow \infty} \|\pi_*^{n_k} - \bar{\pi}^{n_k}\|_{L^2(S)} = 0, \quad \lim_{k \rightarrow \infty} \|\zeta_*^{n_k} - \bar{\zeta}^{n_k}\|_{L^2(S)} = 0. \quad (48)$$

From (42), (47), and (48), it follows that

$$\bar{\pi}^{n_k} \rightharpoonup \pi_*^0 \text{ weakly in } L^2(S) \text{ as } k \rightarrow \infty,$$

$$\bar{\zeta}^{n_k} \rightarrow \zeta_*^0 \text{ strongly in } L^2(S) \text{ as } k \rightarrow \infty,$$

and, by conditions (C6) and (C7), we get $(\pi_*^0, \zeta_*^0) \in \mathbb{U}_1 \times \mathbb{U}_2$.

Next, taking into account the convergence results (40)–(42), (44), and (45), we pass to the limit $k \rightarrow \infty$ in equalities (37) and (38) and obtain

$$\int_{\Omega} \mu(\theta_*^0) D(u_*^0) : D(v) \, dx + \int_S \pi_*^0 (v \cdot n) \, d\sigma = \int_{\Omega} f(x, \theta_*^0) \cdot v \, dx, \quad \forall v \in V_S(\Omega),$$

$$\sum_{i=1}^d \int_{\Omega} u_{*i}^0 \frac{\partial \theta_*^0}{\partial x_i} \eta \, dx + \int_{\Omega} \kappa(\theta_*^0) \nabla \theta_*^0 \cdot \nabla \eta \, dx + \alpha \int_{\partial\Omega \setminus S} \theta_*^0 \eta \, d\sigma = \int_{\Omega} \omega(x, \theta_*^0) \eta \, dx, \quad \forall \eta \in Y_S(\Omega).$$

Therefore, we have established that the quadruplet $(u_*^0, \theta_*^0, \pi_*^0, \zeta_*^0)$ belongs to the set $\mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)$. This is a key point of the proof of assertion (b). Indeed, using condition (C8) and (39)–(43), we can deduce the following relations:

$$\begin{aligned} \Phi(\mathbb{U}_1, \mathbb{U}_2) &= \inf_{(u, \theta, \pi, \zeta) \in \mathfrak{M}(\mathbb{U}_1, \mathbb{U}_2)} J(u, \theta, \pi, \zeta) \\ &\leq J(u_*^0, \theta_*^0, \pi_*^0, \zeta_*^0) \\ &\leq \liminf_{k \rightarrow \infty} J(u_*^{n_k}, \theta_*^{n_k}, \pi_*^{n_k}, \zeta_*^{n_k}) \\ &= \lim_{k \rightarrow \infty} \Phi(\mathbb{U}_1^{n_k}, \mathbb{U}_2^{n_k}), \end{aligned}$$

thus contradicting inequality (36). This contradiction concludes the proof.

8. Concluding Remarks

In this study, we considered an optimal control problem for the system of nonlinear equations describing steady non-isothermal creeping flows of an incompressible fluid through a locally Lipschitz bounded domain. Using the discussion presented above, we established the existence of weak solutions that minimize a given cost functional. For the considered control system, we also introduced the concept of the marginal function and proved that this function is lower semi-continuous. This means that it is impossible to achieve a significant improvement in the optimal value of the cost functional by small changes in the set of admissible controls.

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