fluids

MDPI

Article

Modelling Bidispersive Local Thermal Non-Equilibrium Flow

Franca Franchi [1], Roberta Nibbi [1,*] and Brian Straughan [2]

[1] Department of Mathematics, University of Bologna, 5 Piazza di Porta S. Donato, 40126 Bologna, Italy; franca.franchi@unibo.it

[2] Department of Mathematics, University of Durham, Durham DH1 3LE, UK; brian.straughan@durham.ac.uk

* Correspondence: roberta.nibbi@unibo.it; Tel.: +39-051-209-4419

Received: 1 August 2017; Accepted: 13 September 2017; Published: 18 September 2017

Abstract: In this work, we present a system of equations which describes non-isothermal flow in a bidispersive porous medium under conditions of local thermal non-equilibrium. The porous medium consists of macro pores, and in the solid skeleton are cracks or fissures which give rise to micro pores. The temperatures in the solid skeleton and in the fluids in the macro and micro pores are all allowed to be independent. After presenting the general model, we derive a result of universal stability, which guarantees exponential decay of the solution for all initial data. We further present a concrete example by specializing the model to the problem of thermal convection in a layer heated from below.

Keywords: bidispersive porous flows; local thermal non-equilibrium; universal stability; thermal convection

1. Introduction

Fluid flow in a porous medium is a subject with a long history which is currently very active; for example, see Nield and Bejan [1], Straughan [2]. In particular, recent attention has often focussed on local thermal non-equilibrium, which is where there is a single porosity but the fluid and solid skeleton have different temperatures (e.g., Banu and Rees [3], Barletta and Rees [4,5], Eltayeb [6], Nield [7,8], Nield and Bejan [1], Nield and Kuznetsov [9,10], Nouri-Borujerdi et al. [11], Postelnicu and Rees [12], Rees [13,14], Rees and Bassom [15], Rees et al. [16], Straughan [17–19]).

In addition, another area which has separately attracted much attention is flow in a bidispersive (or double porosity) material. Here, the usual macro pores are present, but in addition, the solid skeleton contains cracks or fissures which give rise to micro pores. This is studied in detail in, for example, Falsaperla et al. [20], Nield [21], Nield and Bejan [1], Nield and Kuznetsov [9,10], Straughan [19,22].

The goal of this paper is to combine both approaches and develop a theory for a porous material with a double porosity structure, but one which allows for different temperatures in the solid skeleton, the fluid in the macro pores, and the fluid in the micro pores.

It is very important to realize that there are situations where the local thermal non-equilibrium theory predicts very different temperature evolution profiles from what is observed with a single temperature; see, for example, the very interesting analyses of Rees and Bassom [15] and Rees et al. [16]. Furthermore, in a very useful work, Rees [13,14] has shown how one may derive estimates for the interaction coefficients for the temperatures in a local thermal non-equilibrium theory, thus rendering this theory to be of practical value. In addition, David et al. [23], Homand-Etienne and Houpert [24], and Siratovich et al. [25] show where the inclusion of thermal effects can lead to stress-induced micro cracking in rocks such as granite, and so we believe that the inclusion of local thermal non-equilibrium effects in a bidispersive porous medium will be useful in real life.

We thus consider a porous body which has a porosity ϕ, i.e., the ratio of the volume of the macro pores to the total volume of the saturated porous material. The solid skeleton contains much smaller micro pores which may be cracks, or may even be due to a man-made structure (e.g., the picture in Nield and Kuznetsov [10], p. 3069). The micro pores give rise to a porosity ϵ, which is defined as being the ratio of the volume occupied by the micro pores to the volume of the porous body which remains after the macro pores are removed. This means that the fraction of volume occupied by the micro pores is $\epsilon(1-\phi)$ and the fraction occupied by the solid skeleton is $(1-\epsilon)(1-\phi)$.

It is worth pointing out that perhaps the major reason for the recent interest in double porosity materials is due to the many applications in real engineering and geophysics situations. For example, landslides and land movement due to thermal gradients (e.g., Hammond and Barr [26], Montrasio et al. [27]); provision of clean and safe drinking water (e.g., Ghasemizadeh et al. [28], Zuber and Motyka [29]); the controversial area of hydraulic fracturing for natural gas (e.g., Huang et al. [30], Kim and Moridis [31]); and many other applications may be found in Straughan [19].

2. Basic Model

As we have written in the introduction and reiterate now, the porous body is composed of a solid skeleton, macro pores, and micro pores. So, besides a macroporosity ϕ, there is a microporosity ϵ such that $\epsilon(1-\phi)$ and $(1-\epsilon)(1-\phi)$ denote the fractions of the body occupied by the micro pores and the solid skeleton, respectively. We follow Nield and Kuznetsov [10] and denote by U_i^f and U_i^p the pore-averaged velocities in the macro and micro pores. In addition, Nield and Kuznetsov [10] denote by T^f and T^p the temperature in the macro and micro pores. The solid skeleton does not move, but we denote the temperature there by T^s.

The momentum and continuity equations in the macro and micro pores are given by Nield and Kuznetsov [10] for a Brinkman porous body. We follow these writers, but employ a Darcy theory and so omit the Brinkman terms (cf. Straughan [19]). Thus, the momentum and continuity equations for the macro pores are:

$$-\frac{\mu}{K_f}U_i^f - \zeta(U_i^f - U_i^p) - p^f_{,i} - \frac{g_i\rho_0\alpha\phi}{D}T^f - \frac{g_i\rho_0\alpha(1-\phi)\epsilon}{D}T^p = 0\,, \tag{1}$$

$$U^f_{i,i} = 0\,, \tag{2}$$

where μ is the dynamic viscosity, K_f, p^f are the permeability and the pressure in the macro pores, g_i is the gravity vector, α is the coefficient of thermal expansion in the fluid, and ρ_0 is a constant. The term $D = \phi + (1-\phi)\epsilon$, ζ is an interaction coefficient, and standard indicial notation is used throughout with subscript $,i$ denoting $\partial/\partial x_i$.

In writing (1), we have followed Nield and Kuznetsov [10] and used a Boussinesq approximation to write the buoyancy term as linear in the weighted temperatures so that the buoyancy term has the form

$$g_i\rho_0\{1-\alpha(T-T_0)\}\,, \tag{3}$$

where $T = \left(\phi T^f + (1-\phi)\epsilon T^p\right)/D$, T_0 is a reference temperature, and in (1) the constant terms in (3) have been absorbed in p^f.

In a similar manner, the momentum and continuity equations in the micro pores have the form

$$-\frac{\mu}{K_p}U_i^p - \zeta(U_i^p - U_i^f) - p^p_{,i} - \frac{g_i\rho_0\alpha\phi}{D}T^f - \frac{g_i\rho_0\alpha(1-\phi)\epsilon}{D}T^p = 0\,, \tag{4}$$

$$U^p_{i,i} = 0\,. \tag{5}$$

To write the balance of energy equations for the temperature fields, we let V_i^f and V_i^p be the actual velocities in the macro and micro pores. These are connected to the pore-averaged velocities by $U_i^f = \phi V_i^f$ and $U_i^p = \epsilon(1-\phi)V_i^p$. Thus, the equations for the temperature fields T^s, T^f, and T^p are

$$\epsilon_1(\rho c)_s T^s_{,t} = \epsilon_1 \kappa_s \Delta T^s + s_1(T^f - T^s) + s_2(T^p - T^s)\,, \tag{6}$$

$$\phi(\rho c)_f T^f_{,t} + \phi(\rho c)_f V_i^f T^f_{,i} = \phi\kappa_f \Delta T^f + h(T^p - T^f) + s_1(T^s - T^f)\,, \tag{7}$$

$$\epsilon_2(\rho c)_p T^p_{,t} + \epsilon_2(\rho c)_p V_i^p T^p_{,i} = \epsilon_2\kappa_p \Delta T^p + h(T^f - T^p) + s_2(T^s - T^p)\,, \tag{8}$$

where

$$\epsilon_1 = (1-\epsilon)(1-\phi), \qquad \epsilon_2 = (1-\phi)\epsilon,$$

and where $(\rho c)_s$, $(\rho c)_f$, $(\rho c)_p$ are the products of the density and the specific heat at constant pressure in the solid, in the fluid in the macro pores, and in the fluid in the micro pores, respectively. The terms κ_s, κ_f, and κ_p are thermal conductivities in the solid, and in the fluid in the macro and micro pores, respectively. We denote by s, f, and p the solid, the macro pores, and the micro pores. The terms h, s_1, and s_2 are interaction coefficients, and we have here assumed that the interactions are linear in the temperature differences.

The governing equations are (1), (2), and (4)–(8). These equations hold in a bounded regular domain $\Omega \subset \mathbb{R}^3$, for $t > 0$.

Remark 1. *As we believe this is the first article to develop a local thermal non-equilibrium theory for a bidispersive porous material, we have followed the lead of Banu and Rees [3] and Nield and Kuznetsov [10] in ignoring fluid acceleration terms in Equations (1) and (4). For a Darcy porous material where we expect relatively slow flow, we believe that this is realistic. We could easily include acceleration (inertia) terms in (1) and (4), and very little would change in the rest of the article.*

Remark 2. *In this work, we treat the thermal coupling terms h, s_1 and s_2 as constants. This is in line with the treatment for a local thermal non-equilibrium single porosity model by Banu and Rees [3], and also the interaction coefficients are treated as constant by Nield and Kuznetsov [10] in their development of a bidispersive porous model. For the present scenario, it is conceivable that the thermal interaction coefficients may depend on the solution, but this would lead to an extremely complicated model. In connection with this, we point out that Franchi et al. [32] study continuous dependence on the interaction parameters in a single-temperature bidispersive porous medium. We believe such a continuous dependence (or structural stability) result could be established for the model presented here, but the calculations will be technically involved and this is deferred to a later article.*

3. Universal Stability

In this section we define the governing equations on $\Omega \times \{t > 0\}$ with conditions on the boundary Γ of Ω. Suppose on $\Gamma \times \{t > 0\}$

$$U_i^f n_i = 0\,,\ U_i^p n_i = 0\,,\ T^s = T_1(\mathbf{x},t)\,,\ T^f = T_2(\mathbf{x},t)\,,\ T^p = T_3(\mathbf{x},t)\,, \quad \mathbf{x} \in \Gamma\,, t > 0\,, \tag{9}$$

for given functions T_1, T_2, T_3, where $\mathbf{n}$ is the unit outward normal to Γ.

The initial conditions are

$$T^s(\mathbf{x},0) = T_0^s(\mathbf{x})\,, \quad T^f(\mathbf{x},0) = T_0^f(\mathbf{x})\,, \quad T^p(\mathbf{x},0) = T_0^p(\mathbf{x})\,, \tag{10}$$

for prescribed functions T_0^s, T_0^f, T_0^p. Let the boundary-initial value problem comprising (1), (2), and (4)–(10) be denoted by $\mathcal{P}$.

We let $\bar{U}_i^f$, $\bar{U}_i^p$, $\bar{p}^f$, $\bar{p}^p$, $\bar{T}^s$, $\bar{T}^f$, and $\bar{T}^p$ be a basic solution to $\mathcal{P}$. We wish to analyze the stability of this basic solution. Hence, we define perturbations u_i^f, u_i^p, π^f, π^p, θ^s, θ^f, and θ^p by

$$\begin{aligned}
&U_i^f = \bar{U}_i^f + u_i^f\,, \qquad U_i^p = \bar{U}_i^p + u_i^p\,,\\
&p^f = \bar{p}^f + \pi^f\,, \qquad p^p = \bar{p}^p + \pi^p\,,\\
&T^s = \bar{T}^s + \theta^s\,, \qquad T^f = \bar{T}^f + \theta^f\,, \qquad T^p = \bar{T}^p + \theta^p\,.
\end{aligned}$$

We now non-dimensionalize the governing equations for the perturbations u_i^f, u_i^p, π^f, π^p, θ^s, θ^f, and θ^p. We choose length and time scales L and $\mathcal{T} = \phi(\rho c)_f/h$, and we define non-dimensional numbers Λ_h, Λ_p, and Λ_s by

$$\Lambda_h = \frac{(\rho c)_p}{(\rho c)_f}\,, \qquad \Lambda_p = \frac{(\rho c)_p \epsilon_2}{(\rho c)_f \phi}\,, \qquad \Lambda_s = \frac{(\rho c)_s \epsilon_1}{(\rho c)_f \phi}\,.$$

We further define non-dimensional parameters R_1, R_2, μ_1, μ_2, ϕ_1, ϕ_2, S_1, S_2, Γ_s, Γ_p, Γ_f by

$$R_1 = \frac{(\rho c)_f U}{Lh} \qquad R_2 = \frac{\alpha \rho_0 T^\sharp}{\zeta U} \qquad \mu_1 = \frac{\mu}{\zeta K_f}\,, \qquad \mu_2 = \frac{\mu}{\zeta K_p}\,,$$

$$\phi_1 = \frac{\phi}{D}\,, \qquad \phi_2 = \frac{(1-\phi)\epsilon}{D}\,, \qquad S_1 = \frac{s_1}{h}\,, \qquad S_2 = \frac{s_2}{h}\,,$$

$$\Gamma_s = \frac{\epsilon_1 \kappa_s}{L^2 h}\,, \qquad \Gamma_f = \frac{\phi \kappa_f}{L^2 h}\,, \qquad \Gamma_p = \frac{\epsilon_2 \kappa_p}{L^2 h}\,,$$

where U and $T^\sharp$ are velocity and temperature scales. The quantities L, U, and $T^\sharp$ are specified exactly when one considers a particular problem of thermal flow, as for example in Section 4.

Hence, the non-dimensional perturbation equations may be shown to have the form

$$\begin{aligned}
\mu_1 u_i^f + (u_i^f - u_i^p) &= -\pi^f_{,i} - R_2\phi_1 g_i \theta^f - R_2 \phi_2 g_i \theta^p\,, && \text{(11a)}\\
u_{i,i}^f &= 0\,, && \text{(11b)}\\
\mu_2 u_i^p + (u_i^p - u_i^f) &= -\pi^p_{,i} - R_2\phi_1 g_i \theta^f - R_2 \phi_2 g_i \theta^p\,, && \text{(11c)}\\
u_{i,i}^p &= 0\,, && \text{(11d)}\\
\Lambda_s \theta^s_{,t} &= \Gamma_s \Delta\theta^s + S_1(\theta^f - \theta^s) + S_2(\theta^p - \theta^s)\,, && \text{(11e)}\\
\theta^f_{,t} + R_1(\bar{U}_i^f \theta^f_{,i} + \bar{T}^f_{,i} u_i^f + u_i^f \theta^f_{,i}) &= \Gamma_f \Delta\theta^f + (\theta^p - \theta^f) + S_1(\theta^s - \theta^f)\,, && \text{(11f)}\\
\Lambda_p \theta^p_{,t} + \Lambda_p R_1(\bar{U}_i^p \theta^p_{,i} + \bar{T}^p_{,i} u_i^p + u_i^p \theta^p_{,i}) &= \Gamma_p \Delta\theta^p + (\theta^f - \theta^p) + S_2(\theta^s - \theta^p)\,. && \text{(11g)}
\end{aligned}$$

From the definition of the perturbations and the conditions (9), the boundary conditions are

$$u_i^f n_i = 0\,, \quad u_i^p n_i = 0\,, \quad \theta^s = 0\,, \quad \theta^f = 0\,, \quad \theta^p = 0\,, \qquad \text{on} \quad \Gamma \times \{t > 0\}\,. \qquad (12)$$

The initial conditions are of form

$$\theta^s(\mathbf{x},0) = \theta_0^s(\mathbf{x})\,, \quad \theta^f(\mathbf{x},0) = \theta_0^f(\mathbf{x})\,, \quad \theta^p(\mathbf{x},0) = \theta_0^p(\mathbf{x})\,, \qquad \mathbf{x} \in \Omega\,,$$

for prescribed functions θ_0^s, θ_0^f, θ_0^p.

Our goal is to now develop a result of universal stability in the sense of Serrin [33]. That is, we find conditions on the base flow to ensure u_i^f, u_i^p, θ^s, θ^f, and θ^p decay exponentially in a suitable measure for all initial data (i.e., a global stability estimate).

Let $\|\cdot\|$ and $(\cdot,\cdot)$ be the norm and inner product on $L^2(\Omega)$. To achieve the above aim, we commence by multiplying Equation (11a) by u_i^f and we integrate over Ω. Further, multiply (11c) by u_i^p and integrate over Ω. Use integration by parts and the boundary conditions (12), and then add the resulting equations to obtain

$$\mu_1\|\mathbf{u}^f\|^2 + \mu_2\|\mathbf{u}^p\|^2 + \|\mathbf{u}^f - \mathbf{u}^p\|^2 = -R_2\phi_1(g_i\theta^f, u_i^f + u_i^p) - R_2\phi_2(g_i\theta^p, u_i^f + u_i^p)\,. \tag{13}$$

We use the arithmetic-geometric mean inequality on the right hand side of this identity, and one may thus show

$$\frac{\mu_1}{2}\|\mathbf{u}^f\|^2 + \frac{\mu_2}{2}\|\mathbf{u}^p\|^2 \le \kappa_1\|\theta^f\|^2 + \kappa_2\|\theta^p\|^2\,, \tag{14}$$

where κ_1, κ_2 are given by

$$\kappa_1 = \phi_1\ell\,, \quad \kappa_2 = \phi_2\ell\,,$$

with

$$\ell = |g|^2R_2^2\left(\frac{\phi_1+\phi_2}{2}\right)\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\,.$$

The next step is to multiply (11e) by θ^s, (11f) by θ^f, and (11g) by θ^p, integrate each over Ω, use integration by parts and (12), and then add the results to find

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\Lambda_s}{2}\|\theta^s\|^2 + \frac{1}{2}\|\theta^f\|^2 + \frac{\Lambda_p}{2}\|\theta^p\|^2\right) + \Gamma_s\|\nabla\theta^s\|^2 + \Gamma_f\|\nabla\theta^f\|^2 + \Gamma_p\|\nabla\theta^p\|^2 + \|\theta^f - \theta^p\|^2\\ &+ S_1\|\theta^f - \theta^s\|^2 + S_2\|\theta^p - \theta^s\|^2 = R_1(\bar{T}^f u_i^f, \theta^f_{,i}) + \Lambda_p R_1(\bar{T}^p u_i^p, \theta^p_{,i})\,.\end{aligned} \tag{15}$$

Denote by $\mathcal{I}^f$ and $\mathcal{I}^p$

$$\mathcal{I}^f = \max_{\Omega\times\{t>0\}}|\bar{T}^f|\,, \quad \mathcal{I}^p = \max_{\Omega\times\{t>0\}}|\bar{T}^p|\,, \tag{16}$$

and then we may employ the arithmetic-geometric mean inequality on the right of (15) to show that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\Lambda_s}{2}\|\theta^s\|^2 + \frac{1}{2}\|\theta^f\|^2 + \frac{\Lambda_p}{2}\|\theta^p\|^2\right) + \Gamma_s\|\nabla\theta^s\|^2 + \frac{\Gamma_f}{2}\|\nabla\theta^f\|^2 + \frac{\Gamma_p}{2}\|\nabla\theta^p\|^2\\ &\le \frac{R_1^2\mathcal{I}_f^2}{2\Gamma_f}\|\mathbf{u}^f\|^2 + \frac{\Lambda_p^2R_1^2\mathcal{I}_p^2}{2\Gamma_p}\|\mathbf{u}^p\|^2\,.\end{aligned} \tag{17}$$

We now employ (14) to estimate the right hand side of (17), and thus we arrive at

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\Lambda_s}{2}\|\theta^s\|^2 + \frac{1}{2}\|\theta^f\|^2 + \frac{\Lambda_p}{2}\|\theta^p\|^2\right) + \Gamma_s\lambda_1\|\theta_s\|^2\\ &\quad + \left(\frac{\Gamma_f\lambda_1}{2} - \kappa_1R_1^2Q\right)\|\theta^f\|^2 + \left(\frac{\Gamma_p\lambda_1}{2} - \kappa_2R_1^2Q\right)\|\theta^p\|^2 \le 0\,,\end{aligned} \tag{18}$$

where

$$Q = \frac{\mathcal{I}_f^2}{\mu_1\Gamma_f} + \frac{\Lambda_p^2\mathcal{I}_p^2}{\mu_2\Gamma_p}\,,$$

where we have also employed Poincaré's inequality and λ_1 is the first eigenvalue in the membrane problem for Ω.

From inequality (18), we see that a condition for universal stability is that the coefficients of the last two terms are positive. If we return to the definition of κ_1 and κ_2, then we find that a sufficient condition for universal stability is that

$$\frac{\Gamma_f \lambda_1}{2} > \phi_1 H R_1^2 R_2^2 \left[\frac{\mathcal{I}_f^2}{\mu_1 \Gamma_f} + \frac{\Lambda_p^2 \mathcal{I}_p^2}{\mu_2 \Gamma_p} \right]$$

and

$$\frac{\Gamma_p \lambda_1}{2} > \phi_2 H R_1^2 R_2^2 \left[\frac{\mathcal{I}_f^2}{\mu_1 \Gamma_f} + \frac{\Lambda_p^2 \mathcal{I}_p^2}{\mu_2 \Gamma_p} \right]$$

where

$$H = \frac{|\mathbf{g}|^2(\phi_1 + \phi_2)}{2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) .$$

It is likely that $\mathcal{I}_f$ and $\mathcal{I}_p$ are measurable quantities in a given situation and the numbers R_1 andR_2 may be calculated. Thus, this universal stability estimate should be of use. However, we do not expect it to be optimum, and for a particular flow situation it is better to return to the precise equations at hand. In the next section we present the appropriate equations for thermal convection in a plane layer.

4. Thermal Convection

We now specialize Equation (11) to the case of thermal convection. Suppose that the saturated porous medium is contained in the layer

$$\{(x, y) \in \mathbb{R}^2\} \times \{0 < z < d\} \tag{19}$$

with the temperatures

$$T^s = T^f = T^p = T_L , \qquad \text{at} \quad z = 0 , \tag{20}$$

$$T^s = T^f = T^p = T_U , \qquad \text{at} \quad z = d , \tag{21}$$

where T_L, T_U are constants with $T_L > T_U$. The velocity boundary conditions are $U_i^f n_i = 0$, $U_i^p n_i = 0$ at $z = 0, d$. The steady solution in whose stability we are interested has form

$$\bar{T}^s = \bar{T}^f = \bar{T}^p = T_L - \beta z = 0 ; \qquad \bar{U}_i^f \equiv 0 , \quad \bar{U}_i^p \equiv 0 ,$$

where

$$\beta = \frac{T_L - T_U}{d}$$

is the temperature gradient.

If $u_i^f, u_i^p, \pi^f, \pi^p, \theta^s, \theta^f, \theta^p$ denote perturbations to the steady solution, then one verifies that these quantities satisfy the equations

$$
\begin{aligned}
&\frac{\mu}{K_f}u_i^f + \zeta(u_i^f - u_i^p) = -\pi^f_{,i} + \frac{g\rho_0\alpha\phi}{D}\theta^f k_i + \frac{g\rho_0\alpha(1-\phi)\epsilon}{D}\theta^p k_i\,,\\
&\frac{\mu}{K_p}u_i^p + \zeta(u_i^p - u_i^f) = -\pi^p_{,i} + \frac{g\rho_0\alpha\phi}{D}\theta^f k_i + \frac{g\rho_0\alpha(1-\phi)\epsilon}{D}\theta^p k_i\,,\\
&\epsilon_1(\rho c)_s\theta^s_{,t} = \epsilon_1\kappa_s\Delta\theta^s + s_1(\theta^f - \theta^s) + s_2(\theta^p - \theta^s)\,,\\
&\phi(\rho c)_f\theta^f_{,t} + (\rho c)_f u_i^f\theta^f_{,i} = (\rho c)_f\beta w^f + \phi\kappa_f\Delta\theta^f + h(\theta^p - \theta^f) + s_1(\theta^s - \theta^f)\,,\\
&\epsilon_2(\rho c)_p\theta^p_{,t} + (\rho c)_p u_i^p\theta^p_{,i} = (\rho c)_p\beta w^p + \epsilon_2\kappa_p\Delta\theta^p + h(\theta^f - \theta^p) + s_2(\theta^s - \theta^p)\,,
\end{aligned}
\tag{22}
$$

where g is the gravity constant, $k = (0,0,1)$, $w^f = u_3^f$, and $w^p = u_3^p$.

We non-dimensionalize (22) with the time, length, pressure, and velocity scales $(\rho c)_f\, d^2/\kappa_f$, d, $d\zeta U$, $\kappa_f/(\rho c)_f\, d$; the temperature scale is

$$
T^\sharp = U\sqrt{\frac{\zeta(\rho c)_f\beta d^2}{\phi\kappa_f g\rho_0\alpha}}
\tag{23}
$$

and the Rayleigh number R_a is

$$
R_a = R^2 = \frac{(\rho c)_f\beta d^2 g\alpha\rho_0}{\zeta\kappa_f\phi}\,.
$$

Define $\mu_1, \mu_2, \Lambda_s, \Lambda_p, \Lambda_{Tp}, \Lambda_{Ts}, S_1, S_2$, and H by

$$
\begin{aligned}
&\mu_1 = \frac{\mu}{\zeta K_f}\,,\quad \mu_2 = \frac{\mu}{\zeta K_p}\,,\quad \Lambda_s = \frac{(\rho c)_s}{(\rho c)_f}\,,\quad \Lambda_p = \frac{(\rho c)_p}{(\rho c)_f}\,,\\
&\Lambda_{Tp} = \Lambda_p\frac{\epsilon_2}{\phi}\,,\quad \Lambda_{Ts} = \Lambda_s\frac{\epsilon_1}{\phi}\,,\quad S_1 = \frac{s_1 d^2}{\phi\kappa_f}\,,\quad S_2 = \frac{s_2 d^2}{\phi\kappa_f}\,,\quad H = \frac{hd^2}{\phi\kappa_f}\,.
\end{aligned}
$$

Then, (22) may be rewritten in the non-dimensional form

$$
\mu_1 u_i^f + (u_i^f - u_i^p) = -\pi^f_{,i} + R\tfrac{\phi}{D}\theta^f k_i + R\tfrac{(1-\phi)\epsilon}{D}\theta^p k_i\,,
\tag{24a}
$$

$$
\mu_2 u_i^p + (u_i^p - u_i^f) = -\pi^p_{,i} + R\tfrac{\phi}{D}\theta^f k_i + R\tfrac{(1-\phi)\epsilon}{D}\theta^p k_i\,,
\tag{24b}
$$

$$
\Lambda_{Ts}\theta^s_{,t} = K_{SV}\Delta\theta^s + S_1(\theta^f - \theta^s) + S_2(\theta^p - \theta^s)\,,
\tag{24c}
$$

$$
\theta^f_{,t} + \tfrac{1}{\phi}u_i^f\theta^f_{,i} = Rw^f + \Delta\theta^f + H(\theta^p - \theta^f) + S_1(\theta^s - \theta^f)\,,
\tag{24d}
$$

$$
\Lambda_{Tp}\theta^p_{,t} + \tfrac{\Lambda_p}{\phi}u_i^p\theta^p_{,i} = R\Lambda_p w^p + K_{PV}\Delta\theta^p + H(\theta^f - \theta^p) + S_2(\theta^s - \theta^p)\,,
\tag{24e}
$$

where $K_{SV} = \epsilon_1\kappa^s/\phi\kappa^f$ and $K_{PV} = \epsilon_2\kappa_p/\phi\kappa_f$, and $u^f_{i,i} = 0\,,\ u^p_{i,i} = 0$.

Equations (24) are defined on $\mathbb{R}^2 \times \{z \in (0\,,1)\} \times \{t > 0\}$, with

$$
u_i^f n_i = 0\,,\quad u_i^p n_i = 0\,,\quad \theta^s = 0\,,\quad \theta^f = 0\,,\quad \theta^p = 0\,.
$$

on $z = 0\,, 1$.

In addition, we suppose u_i^f, u_i^p, π^f, π^p, θ^s, θ^f, θ^p satisfy a plane tiling periodicity in the (x, y) directions (cf. Straughan [19], p. 189) for a typical hexagonal cell structure.

We do not attempt a complete analysis of the nonlinear stability and linear instability of system (24) here. While Straughan [22] has numerically verified that the linear instability results of Nield and Kuznetsov [10]—for the case where θ^s is neglected and the Brinkman equations are considered—are close to the nonlinear stability ones, the same is not true when the Darcy equations are taken into consideration (see Straughan [19]). Since we might expect oscillatory instabilities with $\theta^s \equiv 0$, we believe a careful instability analysis of (24) will require substantial numerical computation. This, together with a careful nonlinear energy stability analysis, will be a lengthy article, and we defer this to the future. However, we may commence a nonlinear energy stability here to give a view as to how things will behave.

Let the period cell of the solution be V, and let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and inner product on $L^2(V)$. To develop a fully nonlinear energy stability analysis, we multiply (24a) by u_i^f, (24b) by u_i^p, and integrate each over V. We likewise multiply (24c) by θ^s, (24d) by θ^f, (24e) by θ^p, and integrate each over V. After addition of the resulting identities and use of the boundary conditions, one may show that

$$\frac{dE}{dt} = RI - D\,, \tag{25}$$

where

$$E(t) = \frac{1}{2}\Lambda_{Ts}\|\theta^s\|^2 + \frac{1}{2}\|\theta^f\|^2 + \frac{1}{2}\Lambda_{Tp}\|\theta^p\|^2\,,$$

$$\begin{aligned} I(t) =& \Big(1+\frac{\phi}{D}\Big)(w^f,\theta^f) + \frac{(1-\phi)\epsilon}{D}(\theta^p,w^f) \\ &+ \frac{\phi}{D}(\theta^f,w^p) + \Big[\Lambda_p + \frac{\epsilon(1-\phi)}{D}\Big](w^p,\theta^p)\,, \end{aligned}$$

and

$$\begin{aligned} D(t) =& K_{SV}\|\nabla\theta^s\|^2 + \|\nabla\theta^f\|^2 + K_{PV}\|\nabla\theta^p\|^2 + S_1\|\theta^f-\theta^s\|^2 + S_2\|\theta^p-\theta^s\|^2 + H\|\theta^p-\theta^f\|^2 \\ &+ \mu_1\|\mathbf{u}^f\|^2 + \mu_2\|\mathbf{u}^p\|^2 + \|\mathbf{u}^f-\mathbf{u}^p\|^2\,. \end{aligned}$$

The global nonlinear stability threshold follows by computing $\max_{\mathcal{H}} I/D$ over a suitable space $\mathcal{H}$ (cf. Straughan [19]), and is then compared to the equivalent linear instability threshold.

5. Conclusions

We have produced a system of equations for a saturated double porosity body under conditions of local thermal non-equilibrium. This is effectively a generalization of the bidispersive theory of Nield and Kuznetsov [9,10], and the local thermal non-equilibrium theory for a single porosity material of Nield [7,8] and Banu and Rees [3].

We have shown that one may derive a universal stability estimate when a general base flow is considered. This is in the spirit of the original universal stability estimate for the Navier–Stokes equations by Serrin [33]. When one requires accurate quantitative stability estimates, then one must address a specific flow problem as we show in Section 4 when we introduce thermal convection.

Acknowledgments: This paper was performed under the auspices of Gruppo Nazionale per la Fisica Matematica-Istituto Nazionale di Alta Matematica (G.N.F.M.-I.N.d.A.M.) and partially supported by Italian Ministero dell'Istruzione dell'Università e della Ricerca (M.I.U.R.) We should like to thank two anonymous referees for incisive remarks which have led to improvements in the paper.

Author Contributions: This is a joint research by all the three authors. Brian Straughan conceived the idea of investigating the properties of the mathematical modelling. All authors contributed fruitfully to the study. All authors gave final approval for publication.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Nield, D.A.; Bejan, A. *Convection in Porous Media*, 5th ed.; Springer: New York, NY, USA, 2017.

2. Straughan, B. *Stability and Wave Motion in Porous Media*; Serious Applied Mathematical Sciences; Springer: New York, NY, USA, 2008; Volume 165.
3. Banu, N.; Rees, D.A.S. Onset of Darcy-Benard convection using a thermal non-equilibrium model. *Int. J. Heat Mass Transf.* **2002**, *145*, 2221–2228.
4. Barletta, A.; Rees, D.A.S. Local thermal non-equilibrium effects in the Darcy-Benard instability with isoflux boundary conditions. *Int. J. Heat Mass Transf.* **2012**, *55*, 384–394.
5. Barletta, A.; Rees, D.A.S. Local thermal non-equilibrium analysis of the thermoconvective instability in an inclined porous layer. *Int. J. Heat Mass Transf.* **2015**, *83*, 327–336.
6. Eltayeb, I.A. Stability of a porous Benard-Brinkman layer in local thermal non-equilibrium with Cattaneo effects in solid. *Int. J. Therm. Sci.* **2015**, *98*, 208–218.
7. Nield, D.A. Effects of local thermal non-equilibrium in steady convection processes in saturated porous media: Forced convection in a channel. *J. Porous Media* **1998**, *1*, 181–186.
8. Nield, D.A. A note on modelling of local thermal non-equilibrium in a structured porous medium. *Int. J. Heat Mass Transf.* **2002**, *45*, 4367–4368.
9. Nield, D.A.; Kuznetsov, A.V. A two-velocity two temperature model for a bi-dispersed porous medium: Forced convection in a channel. *Transp. Porous Media* **2005** , *59*, 325–339.
10. Nield, D.A.; Kuznetsov, A.V. The onset of convection in a bidisperse porous medium. *Int. J. Heat Mass Transf.* **2006** , *49*, 3068–3074.
11. Nouri-Borujerdi, A.; Noghrehabadi, A.R.; Rees, D.A.S. The linear stability of a developing thermal front in a porous medium: The effect of local thermal non-equilibrium. *Int. J. Heat Mass Transf.* **2007** , *50*, 3090–3099.
12. Postelnicu, A; Rees, D.A.S. The onset of Darcy-Bénard convection in a porous layer using a thermal non-equilibrium model-Part I: Stress-free boundaries. *Int. J. Energy Res.* **2003**, *27*, 961–973.
13. Rees, D.A.S. Microscopic modelling of the two-temperature model for conduction in heterogeneous media: Three dimensional media. In Proceedings of the 4th International Conference on Application of Porous Media, Istanbul, Turkey, 10–12 August 2009.
14. Rees, D.A.S. Microscopic modelling of the two-temperature model for conduction in heterogeneous media. *J. Porous Media* **2010**, *13*, 125–143.
15. Rees, D.A.S.; Bassom, A.P. Radial injection of a hot fluid into a cold porous medium: The effects of local thermal non-equilibrium. *Comput. Therm. Sci.* **2010**, *2*, 221–230.
16. Rees, D.A.S.; Bassom, A.P.; Siddheshwar, P.G. Local thermal non-equilibrium effects arising from the injection of a hot fluid into a porous medium. *J. Fluid Mech.* **2008**, *594*, 379–398.
17. Straughan, B. Global nonlinear stability in porous convection with a thermal non-equilibrium model. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **2006**, *462*, 409–418.
18. Straughan, B. Porous convection with local thermal non-equilibrium temperatures and with Cattaneo effects in the solid. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **2013**, *469*, doi:10.1098/rspa.2013.0187.
19. Straughan, B. *Convection with Local Thermal Non-Equilibrium and Microfluidic Effects*; Serious Advances in Mechanics and Mathematics; Springer: Berlin, Germany, 2015; Volume 32.
20. Falsaperla, P.; Mulone, G.; Straughan, B. Bidispersive-inclined convection. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **2016**, *472*, doi:10.1098/rspa.2016.0480.
21. Nield, D.A. A Note on the Modelling of Bidisperse Porous Media. *Transp. Porous Media* **2016**, *111*, 51–520.
22. Staughan, B. On the Nield-Kuznetsov theory for convection in bidispersive porous media. *Transp. Porous Media* **2009**, *77*, 159–168.
23. David, C.; Menéndez, B.; Darot, M. Influence of stress-induced and thermal cracking on physical properties and microstructure of La Peyratte granite. *Int. J. Rock Mech. Min. Sci.* **1999**, *36*, 433–448.
24. Homand-Etienne, F.; Houpert, R. Thermally Induced Microcracking in Granites: Characterisation and Analysis. *Int. J. Rock Mech. Min. Sci. Geomech. Abstr.* **1989**, *26*,125–134.
25. Siratovich, P.A.; Villeneuve, M.C.; Cole, J.W.; Kennedy, B.M.; Bégué, F. Saturated heating and quenching of three crustal rocks and implications for thermal stimulation of permeability in geothermal reservoirs. *Int. J. Rock Mech. Min. Sci.* **2015**, *80*, 265–280.
26. Hammond, N.P.; Barr, A.C. Global resurfacing of Uranus's moon Miranda by convection. *Geology* **2014**, *42*, 931–934
27. Montrasio, L.; Valentino, R.; Losi, G.L. Rainfall infiltration in a shallow soil: A numerical simulation of the double-porosity effect. *Electron. J. Geotechnol. Eng.* **2011**, *16*, 1387–1403.

28. Ghasemizadeh, R.; Hellweger, F.; Butscher, C.; Padilla, I.; Vesper, D.; Field, M.; Alshawabkeh, A. Groundwater flow and transport modeling of karst aquifers, with particular reference to the North Coast Limestone aquifer system of Puerto Rico. *Hydrogeol. J.* **2012**, *20*, 1441–1461.
29. Zuber, A.; Motyka, J. Hydraulic parameters and solute velocities in triple-porosity karstic-fissured-porous carbonate aquifers: Case studies in southern Poland. *Environ. Geol.* **1998**, *34*, 243–250.
30. Huang, T.; Guo, X.; Chen, F. Modeling transient flow behavior of a multiscale triple porosity model for shale gas reservoirs. *J. Nat. Gas Sci. Eng.* **2015**, *23*, 33–46.
31. Kim, J.; Moridis, G.J. Numerical analysis of fracture propagation during hydraulic fracturing operations in shale gas systems. *Int. J. Rock Mech. Min. Sci.* **2015**, *76*, 127–137.
32. Franchi, F.; Nibbi, R.; Straughan, B. Structural stability for temperature dependent bidispersive flow. Unpublished work, 2017.
33. Serrin, J. On the stability of viscous fluid motions. *Arch. Rational Mech. Anal.* 1959, *3*, 1–13.