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The Kinetic-Energy–Momentum–Mass 5-Flux of a Baryon Fluid in Bargmann Spacetimes †

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Abstract

A Bargmann spacetime is a constrained five-dimensional setting that, while introducing no new physical degrees of freedom beyond those of ordinary four-dimensional spacetime, permits Galilei physics to be expressed with a tensor formalism that respects the distinction between mass and energy while affording the conceptual and technical advantages of a spacetime metric. This framework offers a route to a strong-field 'Galilei general relativity' approximating the usual Poincaré general relativity introduced by Einstein. In preparation for modeling core-collapse supernovae, where such an approximation would be useful, this work generalizes the kinetic-energy–momentum–mass 5-flux $\mathcal{T}$ and its associated spacetime tensor law from a simple fluid of constant particle mass to a baryon fluid whose multiple nuclear species can interconvert rest mass and internal energy. The spacetime tensor law on Bargmann–Galilei spacetime $B\mathcal{G}$ and its decompositions relative to comoving ('Lagrangian') and fiducial ('Eulerian') observers are derived in detail. The formalism is rendered more suitable for core-collapse supernova modeling by an extension from strict $B\mathcal{G}$ to a regime that might be denoted as $B\mathcal{G}+$: microscopically Poincaré yet macroscopically Galilei. This extension accommodates energy generation by nuclear composition changes and allows comoving energy density and pressure to contribute relative to mass density, while preserving the simplifications of Galilei bulk fluid flow and the streamlined geometry governed by the Bargmann–Galilei spacetime metric.

Keywords: relativistic fluids; Poincaré group; Galilei group; Bargmann group; core-collapse supernovae

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1. Introduction

An ongoing series of papers ([1], hereafter Paper I; [2], hereafter Paper IIA; and Paper IIB, forthcoming) proposes a path to a theory of strong-field 'Galilei general relativity' that would serve as a useful approximation to 'Poincaré general relativity'—the familiar theory of general relativity introduced by Einstein. In common parlance the label 'relativity' commonly means 'physics according to Einstein', with 'special relativity' and 'general relativity' featuring flat and curved spacetimes respectively. But what distinguishes these from 'physics according to Newton' is not relativity per se, that is, the mixing of time and space by a spacetime symmetry. No, what distinguishes physics according to Einstein from physics according to Newton is the particular spacetime symmetry group: the Poincaré

group, which mixes both time and space into each other, as opposed to the Galilei group, which mixes time into space but not space into time. As to the modifiers 'special' and 'general', these ought to refer to whether the spacetime is flat or curved—and therefore whether the spacetime symmetry group applies globally or only locally—regardless of which spacetime symmetry group is imposed. Thus, one might speak of 'Poincaré special relativity' and 'Poincaré general relativity' for physics governed by the Poincaré group, and 'Galilei special relativity' and 'Galilei general relativity' for physics governed by the Galilei group.

Paper I treats 'special relativity' as conceived here, that is, physics on flat spacetime. Poincaré physics is perfectly at home in a flat four-dimensional spacetime, Minkowski spacetime $\mathbb{M}$, whose metric $\boldsymbol{g}$ embodying Poincaré invariance furnishes a comprehensive spacetime tensor formalism. But Galilei physics is less at home in an affine four-dimensional spacetime $\mathbb{G}$: as the speed of light $c \to \infty$, the Einstein metric $\boldsymbol{g}$ and its inverse $\overleftrightarrow{\boldsymbol{g}}$ degenerate into a time 1-form $\boldsymbol{\tau}$ and inverse 3-metric $\overleftrightarrow{\boldsymbol{\gamma}}$ that respect Galilei invariance, but do not afford the technical benefits of a spacetime metric; the consequent spacetime tensor formalism is thus more restricted. Moreover, absolute time embodied in $\boldsymbol{\tau}$ as fundamental to the spacetime structure breaks the unification of the mass and energy characteristic of Poincaré physics, with the result that the transformation properties of kinetic energy in Galilei physics cannot be manifest in a four-dimensional formalism. However, the linear part of the Bargmann group (a central extension of the Galilei group important to the Galilei invariance of non-relativistic quantum mechanics [3–7]) furnishes a five-dimensional representation of the Galilei group that does exhibit the transformation of kinetic energy [7–12]. While it is not necessary to Poincaré physics, Paper I highlights a similar central extension of the Poincaré group (see also [7]); this can be referred to as the 'Bargmann-Poincaré' group to distinguish it from the original, which might now be called the 'Bargmann–Galilei' group. Employing these Bargmann groups as spacetime symmetry groups, and allowing particle kinematics governed by $\boldsymbol{g}$ and $\overleftrightarrow{\boldsymbol{g}}$ on the one hand or by $\boldsymbol{\tau}$ and $\overleftrightarrow{\boldsymbol{\gamma}}$ on the other to define an 'action coordinate' η in addition to time t and space position (x^i), results in extended five-dimensional spacetimes $B\mathbb{M}$ and $B\mathbb{G}$. Both of these feature a 'Bargmann metric' $\boldsymbol{G}$, with that on $B\mathbb{G}$ being the $c \to \infty$ limit of that on $B\mathbb{M}$. These 'Bargmann spacetimes' do not involve any new physical degrees of freedom but merely serve as a mathematical setting in which Poincaré relativity and Galilei relativity can be treated on a common footing more congenial to the full expression of Galilei physics in a spacetime tensor formalism, because a spacetime metric is now available in both cases [7–9,11,12].

Paper IIA and Paper IIB then address 'general relativity' as conceived here, that is, physics on curved spacetime. Picking up where Paper I leaves off, Paper IIA introduces the curved spacetime counterparts of the flat spacetimes $\mathbb{M}$, $B\mathbb{M}$, and $B\mathbb{G}$: Einstein spacetime $\mathcal{E}$, Bargmann–Einstein spacetime $B\mathcal{E}$, and Bargmann–Galilei spacetime $B\mathcal{G}$. (A curved counterpart of four-dimensional flat spacetime $\mathbb{G}$ is not considered as interest narrows to spacetimes with a metric.) The idea is to extend the usual 1 + 3 (traditionally 3 + 1) formulation of Poincaré general relativity on $\mathcal{E}$ to an equivalent 1 + 3 + 1 formulation on $B\mathcal{E}$, from which a strong-field Galilei general relativity on $B\mathcal{G}$ is to be deduced (see Figure 1 of Paper I). Paper IIA focuses first on the 'kinematics' of general relativity, in the sense of identifying the gravitational degrees of freedom (basically, the 3-metric $\boldsymbol{\gamma}$ induced on the position-space leaves of the 1 + 3 or 1 + 3 + 1 foliation) and their 'velocities' (basically, the extrinsic curvature $\boldsymbol{K}$, another symmetric bilinear form tangent to the leaves of the foliation, carrying information about how the leaves are embedded in spacetime). The basic strategy of starting with known physics on $\mathcal{E}$, encoding it on $B\mathcal{E}$, and then deducing the corresponding results on $B\mathcal{G}$ is illustrated in Paper IIA for two systems: the motion of an

elementary classical material particle, and the dynamics of a simple fluid. The results serve as examples of Galilei physics that allow for gravitational fields of arbitrary strength. This same approach (moving from $\mathcal{E}$ to $B\mathcal{E}$ to $B\mathcal{G}$) is to be applied in Paper IIB to the 'dynamics' of general relativity, in the sense of determining the Einstein equation relating the spacetime geometry to the energy–momentum–mass content on spacetime, which in turn yields the evolution of the extrinsic curvature $\boldsymbol{K}$.

This paper serves as a supplement to Paper IIA, preparatory to application of the forthcoming results of Paper IIB to astrophysical systems of interest, such as core-collapse supernovae. In Paper IIA, the usual total-energy–momentum four-flux $\boldsymbol{T}$ (traditionally 'energy–momentum tensor' or 'stress-energy tensor') appearing on the right-hand side of the Einstein equation on $\mathcal{E}$ becomes a kinetic-energy–momentum–mass 5-flux $\mathcal{T}$ on $B\mathcal{E}$ or $B\mathcal{G}$. This reflects the strict separation of mass and energy required by Galilei physics. One result in Paper IIA is that instead of the vanishing spacetime four-divergence $\boldsymbol{\nabla} \cdot \boldsymbol{T} = 0$ on $\mathcal{E}$, the Bargmann approach effects a partial reversion to a Newton-like perspective in which, even formulated as a spacetime tensor equation, the motion of fluid elements is not geodesic but subject to an explicit gravitational force:

$$\mathcal{D} \cdot \mathcal{T} = m \left(\mathcal{n} \cdot \boldsymbol{D}\phi \right) \underline{\chi} - m \left(\mathcal{n} \cdot \underline{\chi} \right) \boldsymbol{D}\phi. \tag{1}$$

Here the left-hand side is a 5-divergence on $B\mathcal{E}$ or $B\mathcal{G}$, with $\mathcal{D}$ the Levi-Civita connection associated with the Bargmann metric $\boldsymbol{G}$. On the right-hand side—whose first and second terms represent gravitational sources of energy and 3-momentum respectively—the 1-form $\underline{\chi}$ is normal to the position-space leaves. In the case of $B\mathcal{E}$, this explicit dependence on the foliation shows that the Bargmann construction obscures the fact that Poincaré relativity admits freedom in the choice of foliation. For this reason $B\mathcal{E}$ is not promoted as an end in itself, but serves primarily as a conceptual bridge to $B\mathcal{G}$. Indeed, on $B\mathcal{G}$, the 1-form $\underline{\chi}$ coincides with the fundamental time form $\boldsymbol{\tau}$; its presence, along with that of the Levi-Civita connection $\boldsymbol{D}$ associated with the 3-metric γ, reflects the expected absolute nature of the foliation of spacetime $B\mathcal{G}$ associated with the absolute time of Galilei relativity. Also present in this spacetime tensor equation are a nonlinear gravitational potential ϕ associated with the usual lapse function of the 1 + 3 and 1 + 3 + 1 formulations, as well as the particle mass m and particle number 5-flux $\mathcal{n}$. This equation was derived with the assumption of a 'simple fluid', that is, one composed microscopically of a single particle type of constant mass m. The present work lifts this restriction by generalizing the 5-flux $\mathcal{T}$ and the associated spacetime tensor law to a baryon fluid in which the reactions of multiple species convert rest mass into internal energy and vice versa. Such a generalization will allow this formalism to be used in the modeling of core-collapse supernovae, for which enhanced gravity is desirable but the complications of 'Minkowski' bulk fluid flow and the back-reaction of gravitational radiation can reasonably be set aside. A secondary purpose, which facilitates the primary generalization, is to consider more carefully the decomposition of $\mathcal{T}$ and of Equation (1) relative to comoving (Lagrangian) observers, complementing the fiducial (Eulerian) observer decomposition already presented in Paper IIA.

This paper is organized as follows. Section 2 reviews the geometry of the usual Einstein spacetime $\mathcal{E}$ and the Bargmann–Einstein and Bargmann–Galilei spacetimes $B\mathcal{E}$ and $B\mathcal{G}$, and the kinematics of material particles on these. The kinematics and dynamics of a simple fluid are presented in Section 3, including equations decomposed relative to comoving observers. These results are then extended to a baryon fluid in Section 4 before conclusions are drawn in Section 5. Sections A and B contain some details related to decompositions with respect to comoving observers on $B\mathcal{E}$.

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2. Spacetime Geometry and Particle Kinematics

Consider basic spacetime geometry and particle kinematics on Einstein spacetime $\mathcal{E}$ and on the Bargmann–Einstein and Bargmann–Galilei spacetimes $B\mathcal{E}$ and $B\mathcal{G}$. The goal is to reformulate Poincaré physics—without altering the physics itself—in such a way that the $c \to \infty$ limit is well-defined and yields a tensor formulation of Galilei physics in which the separation of mass from energy is respected and the transformation properties of kinetic energy are manifest in a tensor formalism that conveniently includes a full spacetime metric. See Paper I and Paper IIA for more detailed discussion of the overview presented here. For reference, selected tensor fields associated with the 1 + 3 and 1 + 3 + 1 decompositions of spacetimes $\mathcal{E}$ and $B\mathcal{E}/B\mathcal{G}$ are displayed in Table 1.

Table 1. Selected tensor fields relevant to the 1 + 3 and 1 + 3 + 1 decompositions of spacetimes $\mathcal{E}$ and $B\mathcal{E}/B\mathcal{G}$ respectively.

Description	On $\mathcal{E}$	On $B\mathcal{E}/B\mathcal{G}$	Introduced	Remarks
Spacetime metric	$\boldsymbol{g}$	$\boldsymbol{G}$	Equations (2) and (15)	Index lowering: $\underline{(\,)}$
Inverse metric	$\overleftrightarrow{\boldsymbol{g}}$	$\overleftrightarrow{\boldsymbol{G}}$	Equations (2) and (16)	Index raising: $\overleftarrow{(\,)}$, $\overrightarrow{(\,)}$, $\overleftrightarrow{(\,)}$
Fiducial observer 4- or 5-velocity	$\boldsymbol{n}$	$\boldsymbol{n}$	Equations (3) and (17)	Not the same symbol
Dual fiducial observer 1-form	$\underline{\boldsymbol{\chi}}$	$\underline{\boldsymbol{\chi}}$	Equations (6) and (18)	Same symbol
Comoving observer 4- or 5-velocity	$\boldsymbol{u}$	$\mathcal{U}$	Equations (11) and (25)	Not the same symbol
Dual fiducial observer 1-form	$\underline{\boldsymbol{\chi}}_{\boldsymbol{u}}$	$\underline{\boldsymbol{\chi}}_{\mathcal{U}}$	Equations (12) and (26)	(Essentially the) same symbol
Action vector	–	$\boldsymbol{\xi}$	Equations (19) and (27)	Associated with the 5th dimension
3-metric on foliation leaves	$\boldsymbol{\gamma}$	$\boldsymbol{\gamma}$	Equations (9) and (24)	Same symbol
Fiducial projector to position space	$\overleftarrow{\boldsymbol{\gamma}}$	$\overleftarrow{\boldsymbol{\gamma}}$	Equations (8) and (23)	Same symbol
Comoving projector to position space	$\overleftarrow{\boldsymbol{\gamma u}}$	$\overleftarrow{\boldsymbol{\gamma u}}$	Equations (13) and (28)	(Essentially the) same symbol
Governor of causality	$\boldsymbol{g}$	$\boldsymbol{g}$ ($B\mathcal{E}$) or $\boldsymbol{\tau}$ ($B\mathcal{G}$)	Equations (2) and (29)	Set in $B\mathcal{E}/B\mathcal{G}$ with vanishing action components

2.1. Spacetime Geometry and Particle Kinematics on Einstein Spacetime $\mathcal{E}$

Einstein spacetime $\mathcal{E}$ is a four-dimensional Lorentz manifold, that is, a pseudo-Riemann manifold with metric $\boldsymbol{g}$ of signature $(-,+,+,+)$ and associated Levi-Civita connection $\boldsymbol{\nabla}$. The 1 + 3 formulation applies to spacetimes that admit a foliation into the position-space leaves $\mathcal{S}_t$ of co-dimension 1 (hypersurfaces), labeled by a global time coordinate t, with induced 3-metric $\boldsymbol{\gamma}$ of signature $(+,+,+)$ on each leaf with the associated Levi-Civita connection $\boldsymbol{D}$. On each local chart position-space coordinates (x^i) are chosen with $i \in (1,2,3)$ adapted to the foliation. The coordinate basis frame field is denoted by $(\boldsymbol{\partial}t, \boldsymbol{\partial}x_j)$, for example $\boldsymbol{\partial}x_j = \partial/\partial x^j$. In terms of the coordinate basis vector fields $(\boldsymbol{\partial}t, \boldsymbol{\partial}x_j)$ and the dual coordinate basis 1-forms $(\mathbf{d}t, \mathbf{d}x^i)$, the spacetime metric $\boldsymbol{g}$ and its inverse $\overleftrightarrow{\boldsymbol{g}}$ decompose as

$$\begin{aligned} \boldsymbol{g} &= -c^2\,\alpha^2\,\mathbf{d}t\otimes\mathbf{d}t + \gamma_{ab}\Big(\mathbf{d}x^a\ +\beta^a\,\mathbf{d}t\Big)\otimes\Big(\mathbf{d}x^b+\beta^b\,\mathbf{d}t\Big),\\ \overleftrightarrow{\boldsymbol{g}} &= -\frac{1}{c^2\,\alpha^2}\Big(\boldsymbol{\partial}t-\beta^a\,\boldsymbol{\partial}x_a\Big)\otimes\Big(\boldsymbol{\partial}t-\beta^b\,\boldsymbol{\partial}x_b\Big)+\gamma^{ab}\,\boldsymbol{\partial}x_a\otimes\boldsymbol{\partial}x_b. \end{aligned} \tag{2}$$

Here c is the speed of light and α is the lapse function. The shift vector field $\boldsymbol{\beta}=\beta^a\,\boldsymbol{\partial}x_a$ is tangent to the leaves $\mathcal{S}_t$. Restriction to the leaves $\mathcal{S}_t$ yields the position-space metric $\boldsymbol{\gamma}=\gamma_{ab}\,\mathbf{d}x^a\otimes\mathbf{d}x^b$.

The vector field

$$\boldsymbol{n}=\frac{1}{\alpha}(\boldsymbol{\partial}t-\boldsymbol{\beta}) \tag{3}$$

is interpreted as the 4-velocity field of fiducial (Eulerian) observers. As the metric dual of the 1-form

$$\underline{\boldsymbol{n}}=-c^2\,\alpha\,\boldsymbol{\nabla}t=-c^2\,\alpha\,\mathbf{d}t,$$

that is,

$$\boldsymbol{n}=-c^2\,\alpha\,\overleftarrow{\boldsymbol{\nabla}t}=-c^2\,\alpha\,\overleftrightarrow{\boldsymbol{g}}\cdot\boldsymbol{\nabla}t,$$

it is manifestly orthogonal to the leaves $\mathcal{S}_t$, and the pairing of $\mathbf{d}t$ and $\boldsymbol{\partial}x_j$ vanishes:

$$0=\underline{\boldsymbol{n}}\cdot\boldsymbol{\partial}x_j=\boldsymbol{g}\big(\boldsymbol{n},\boldsymbol{\partial}x_j\big)=0.$$

It also satisfies the normalization

$$\boldsymbol{g}(\boldsymbol{n},\boldsymbol{n})=\underline{\boldsymbol{n}}\cdot\boldsymbol{n}=-c^2$$

required of any 4-velocity. Note the notation $\underline{\boldsymbol{n}}=\boldsymbol{g}(\boldsymbol{n},\cdot)=\boldsymbol{g}\cdot\boldsymbol{n}$ relating a vector field to its metric dual 1-form, and $\overleftarrow{\boldsymbol{\nabla}t}=\overleftrightarrow{\boldsymbol{g}}(\boldsymbol{\nabla}t,\cdot)=\overleftrightarrow{\boldsymbol{g}}\cdot\boldsymbol{\nabla}t$ relating a 1-form to its metric dual vector field. Note also that in this paper the infix dot operator (for instance in $\underline{\boldsymbol{n}}\cdot\boldsymbol{n}$) always denotes a dual pairing of a 1-form and vector field (contraction of a lower index and an upper index) and never a scalar product of vector fields; the latter will always be explicitly displayed with a metric. This definition of $\boldsymbol{n}$ gives physical meaning to the lapse function and shift vector: $\delta\tau=\alpha\,\delta t$ is the proper time measured by a fiducial observer during an infinitesimal time coordinate interval δt, and β^i is the coordinate 3-velocity field of a curve of constant x^i relative to the worldline of a fiducial observer. The (1-form version of the) 4-acceleration of fiducial observers is

$$\boldsymbol{\nabla}_{\boldsymbol{n}}\underline{\boldsymbol{n}}=c^2\,\boldsymbol{D}\ln\alpha=\boldsymbol{D}\phi, \tag{4}$$

in which

$$\alpha=\exp\left(\frac{\phi}{c^2}\right),\qquad\qquad \boldsymbol{D}\ln\alpha=\frac{1}{c^2}\,\boldsymbol{D}\phi \tag{5}$$

defines a nonlinear gravitational potential ϕ.

Decomposition of tensor fields relative to fiducial observers is accomplished with the spatial projection operator $\overleftarrow{\boldsymbol{\gamma}}$. Define the dual fiducial observer 1-form

$$\underline{\boldsymbol{\chi}}=-\frac{1}{c^2}\,\underline{\boldsymbol{n}}, \tag{6}$$

which satisfies

$$\underline{\boldsymbol{\chi}}\cdot\boldsymbol{n}=\boldsymbol{g}(\boldsymbol{\chi},\boldsymbol{n})=1,\qquad\qquad \underline{\boldsymbol{\chi}}\cdot\boldsymbol{\chi}=\boldsymbol{g}(\boldsymbol{\chi},\boldsymbol{\chi})=-\frac{1}{c^2}. \tag{7}$$

From these properties it is clear that

$$\overleftarrow{\gamma} = \delta - \boldsymbol{n} \otimes \underline{\chi}, \qquad \overrightarrow{\gamma} = \delta - \underline{\chi} \otimes \boldsymbol{n}, \tag{8}$$

where δ is the identity tensor on $\mathcal{E}$, projects vector fields and 1-forms to $\mathcal{S}_t$, annihilating components along $\boldsymbol{n}$ and $\underline{\chi}$ respectively:

$$\overleftarrow{\gamma} \cdot \boldsymbol{n} = 0, \qquad \overrightarrow{\gamma} \cdot \underline{\chi} = 0,$$
$$\underline{\chi} \cdot \overleftarrow{\gamma} = 0, \qquad \boldsymbol{n} \cdot \overrightarrow{\gamma} = 0.$$

Contraction of vector fields and 1-forms with $\underline{\chi}$ and $\boldsymbol{n}$ yields the components of vector fields and 1-forms along $\boldsymbol{n}$ and $\underline{\chi}$ respectively. The notation $\overleftarrow{\gamma}$ arises from the fact that it is the first-index-raised version of the (extension from $\mathcal{S}_t$ to $\mathcal{E}$ of the) 3-metric γ, or the second-index-lowered version of its inverse $\overleftrightarrow{\gamma}$:

$$\gamma = g - \underline{\boldsymbol{n}} \otimes \underline{\chi}, \qquad \overleftrightarrow{\gamma} = \overleftrightarrow{g} - \boldsymbol{n} \otimes \chi,$$

as confirmed from the above definitions. Note the use of a left-arrow ($\overleftarrow{}$) or right-arrow ($\overrightarrow{}$) to denote the raising of a first or last index respectively. Note also that writing

$$g = -c^2\, \underline{\chi} \otimes \underline{\chi} + \gamma, \qquad \overleftrightarrow{g} = -\frac{1}{c^2}\, \boldsymbol{n} \otimes \boldsymbol{n} + \overleftrightarrow{\gamma} \tag{9}$$

instead of Equation (2) is a significant simplification, that, for example, makes the relations $\overleftrightarrow{g} \cdot g = \delta$ and $g \cdot \overleftrightarrow{g} = \delta$ more transparent.

The 4-velocity $\boldsymbol{U}$ of a material particle, which satisfies the normalization

$$g(\boldsymbol{U}, \boldsymbol{U}) = \underline{\boldsymbol{U}} \cdot \boldsymbol{U} = -c^2, \tag{10}$$

decomposes as

$$\boldsymbol{U} = \Lambda_{\boldsymbol{V}}(\boldsymbol{n} + \boldsymbol{V}), \qquad \Lambda_{\boldsymbol{V}}^{-1} = \sqrt{1 - \frac{1}{c^2}\, \gamma(\boldsymbol{V}, \boldsymbol{V})}, \tag{11}$$

where $\boldsymbol{V}$ is the physical 3-velocity measured by fiducial observers and $\Lambda_{\boldsymbol{V}}$ is the corresponding Lorentz factor. By defining

$$\underline{\chi}_{\boldsymbol{U}} = -\frac{1}{c^2}\, \underline{\boldsymbol{U}} \tag{12}$$

which satisfies

$$\underline{\chi}_{\boldsymbol{U}} \cdot \boldsymbol{U} = g(\chi u, \boldsymbol{U}) = 1, \qquad \underline{\chi}_{\boldsymbol{U}} \cdot \chi u = g(\chi u, \chi u) = -\frac{1}{c^2}$$

in analogy with Equations (6) and (7), one sees that

$$\overleftarrow{\gamma u} = \delta - \boldsymbol{U} \otimes \underline{\chi}_{\boldsymbol{U}}, \qquad \overrightarrow{\gamma u} = \delta - \underline{\chi}_{\boldsymbol{U}} \otimes \boldsymbol{U} \tag{13}$$

in analogy with Equation (8) provides for the decomposition of tensor fields relative to comoving (Lagrangian) observers.

Finally, consider the 4-momentum $\boldsymbol{P}$ of a particle of mass m, which is most naturally regarded as a 1-form. It is defined by

$$\overleftarrow{\boldsymbol{P}} = m\, \boldsymbol{U}, \qquad \boldsymbol{P} = m\, \underline{\boldsymbol{U}}. \tag{14}$$

Decompose with respect to fiducial observers, writing

$$\overleftarrow{\boldsymbol{P}} = \frac{\mathcal{E}_p}{c^2}\,\boldsymbol{n} + \overleftarrow{\boldsymbol{p}}\,, \qquad\qquad \boldsymbol{P} = -\mathcal{E}_p\,\underline{\boldsymbol{\chi}} + \boldsymbol{p},$$

and utilize the decomposition of $\boldsymbol{U}$ in Equation (11) to find

$$\boldsymbol{p} = m\,\Lambda_{V_p}\,\underline{\boldsymbol{V}}_p = \frac{\mathcal{E}_p}{c^2}\,\underline{\boldsymbol{V}}_p,$$

$$\mathcal{E}_p = mc^2\,\Lambda_{V_p} = c^2\sqrt{m^2 + \overleftrightarrow{\boldsymbol{\gamma}}(\boldsymbol{p},\boldsymbol{p})}$$

for the particle 3-momentum $\boldsymbol{p}$ and total energy $\mathcal{E}_p$.

2.2. Spacetime Geometry and Particle Kinematics on Bargmann Spacetimes B$\mathcal{E}$ and B$\mathcal{G}$

A Bargmann spacetime $B\mathcal{E}$ or $B\mathcal{G}$ is a five-dimensional Lorentz manifold, that is, a pseudo-Riemann manifold with metric $\boldsymbol{G}$ of signature $(-,+,+,+,+)$ and associated Levi-Civita connection $\mathcal{D}$. A 1 + 3 + 1 formulation applies to spacetimes that admit a foliation into the position-space leaves $\mathcal{S}_{(t,\eta)}$ of co-dimension 2, labeled by a global time coordinate t and a global action coordinate η associated with kinetic energy, with induced 3-metric $\boldsymbol{\gamma}$ of signature $(+,+,+)$ on each leaf with associated Levi-Civita connection $\boldsymbol{D}$. As before on $\mathcal{E}$, on each local chart choose position-space coordinates (x^i) with $i \in (1,2,3)$ adapted to the foliation. Denote the coordinate basis frame field by $(\partial t, \partial x_j, \partial\eta)$. In terms of the coordinate basis vector fields $(\partial t, \partial x_j, \partial\eta)$ and the dual coordinate basis 1-forms $(\mathbf{d}t, \mathbf{d}x^i, \mathbf{d}\eta)$ the spacetime metric $\boldsymbol{G}$ decomposes as

$$\boldsymbol{G} = \begin{cases} \gamma_{ab}\Big(\mathbf{d}x^a + \beta^a\,\mathbf{d}t\Big) \otimes \Big(\mathbf{d}x^b + \beta^b\,\mathbf{d}t\Big) - \alpha(\mathbf{d}t \otimes \mathbf{d}\eta + \mathbf{d}\eta \otimes \mathbf{d}t) & \\ \qquad + \dfrac{1}{c^2}\,\mathbf{d}\eta \otimes \mathbf{d}\eta & (\text{on } B\mathcal{E}) \\ \gamma_{ab}\Big(\mathbf{d}x^a + \beta^a\,\mathbf{d}t\Big) \otimes \Big(\mathbf{d}x^b + \beta^b\,\mathbf{d}t\Big) - \alpha(\mathbf{d}t \otimes \mathbf{d}\eta + \mathbf{d}\eta \otimes \mathbf{d}t) & (\text{on } B\mathcal{G}), \end{cases} \tag{15}$$

and its inverse $\overleftrightarrow{\boldsymbol{G}}$ decomposes as

$$\overleftrightarrow{\boldsymbol{G}} = \begin{cases} -\dfrac{1}{c^2\,\alpha^2}\Big(\partial t - \beta^a\,\partial x_a\Big) \otimes \Big(\partial t - \beta^b\,\partial x_b\Big) & \\ \quad + \gamma^{ab}\,\partial x_a \otimes \partial x_b - \dfrac{1}{\alpha}\Big(\partial t - \beta^a\,\partial x_a\Big) \otimes \partial\eta - \dfrac{1}{\alpha}\,\partial\eta \otimes \Big(\partial t - \beta^b\,\partial x_b\Big) & (\text{on } B\mathcal{E}) \\ \gamma^{ab}\,\partial x_a \otimes \partial x_b - \dfrac{1}{\alpha}\Big(\partial t - \beta^a\,\partial x_a\Big) \otimes \partial\eta - \dfrac{1}{\alpha}\,\partial\eta \otimes \Big(\partial t - \beta^b\,\partial x_b\Big) & (\text{on } B\mathcal{G}). \end{cases} \tag{16}$$

The lapse function α, the shift vector field $\boldsymbol{\beta} = \beta^a\,\partial x_a$, and the position-space metric $\boldsymbol{\gamma} = \gamma_{ab}\,\mathbf{d}x^a \otimes \mathbf{d}x^b$ have precisely the same meanings on $B\mathcal{E}$ and $B\mathcal{G}$ as they do on $\mathcal{E}$. Important to the formalism is the requirement that no fields are allowed to depend on the new coordinate η: all partial derivatives with respect to this coordinate vanish. (The requirement that no field depend on η is a symmetry condition ensuring that the five-dimensional geometry introduces no new physical degrees of freedom. It is independent of the fact that the action coordinate advances along the worldline of an individual material particle according to its kinetic energy. The latter relation leads to the definition of the Bargmann metric, while the former is an additional restriction on the continuum fields and geometry themselves.)

The vector field

$$\boldsymbol{n} = -\overleftarrow{\mathcal{D}\eta} = \frac{1}{\alpha}(\partial t - \boldsymbol{\beta}) \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}) \tag{17}$$

is interpreted as the 5-velocity field of fiducial observers. (Note that on $B\mathcal{E}$ and $B\mathcal{G}$ the over-arrow notation now refers to metric duality via $\overleftrightarrow{\boldsymbol{G}}$ (index raising).) Like $\boldsymbol{n}$ on $\mathcal{E}$, the fiducial observer 5-velocity $\boldsymbol{n}$ is orthogonal to the position-space leaves $\mathcal{S}_{(t,\eta)}$; but it is denoted by the script character $\boldsymbol{n}$ instead of the plain italic $\boldsymbol{n}$ as on $\mathcal{E}$, not least because it is the metric dual of $\mathcal{D}\eta$ instead of $\mathcal{D}t$. Indeed, the dual fiducial observer 1-form, given by

$$\underline{\boldsymbol{\chi}} = \alpha\, \mathcal{D}t = \alpha\, \mathbf{d}t \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}) \tag{18}$$

using the same notation as on $\mathcal{E}$, is not collinear with $\underline{\boldsymbol{n}}$. (Note that on $B\mathcal{E}$ and $B\mathcal{G}$ the underbar notation now refers to metric duality via $\boldsymbol{G}$ (index lowering).) Instead it is convenient to define the action vector field

$$\boldsymbol{\xi} = -\boldsymbol{\partial}\eta = \begin{cases} \dfrac{1}{c^2}\,\boldsymbol{n} + \boldsymbol{\chi} & (\text{on } B\mathcal{E}) \\ \boldsymbol{\chi} & (\text{on } B\mathcal{G}), \end{cases} \tag{19}$$

and compare/contrast the variant of $\boldsymbol{\xi}$ on $B\mathcal{E}$ with (the metric dual of) Equation (6) on $\mathcal{E}$. Also in contrast with $\mathcal{E}$, it turns out that

$$\mathcal{D}_{\boldsymbol{n}}\underline{\boldsymbol{n}} = 0 \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}) \tag{20}$$

in contrast with Equation (4). That is, on $B\mathcal{E}$ and $B\mathcal{G}$ the fiducial observers are geodesic rather than accelerated as on $\mathcal{E}$. The gravitational force emerges elsewhere, for instance in the important relation

$$c^2\, \mathcal{D}\underline{\boldsymbol{\xi}} = -\frac{1}{2}\, \boldsymbol{D}\phi \otimes \underline{\boldsymbol{\chi}} + \frac{1}{2}\, \underline{\boldsymbol{\chi}} \otimes \boldsymbol{D}\phi. \tag{21}$$

Other identities involving the action vector field $\boldsymbol{\xi}$ that will be used later are

$$\begin{aligned} \mathcal{D} \cdot \boldsymbol{\xi} &= 0, & (\boldsymbol{\xi} \cdot \mathcal{D})b &= 0, \\ (\boldsymbol{\xi} \cdot \mathcal{D})\boldsymbol{\xi} &= 0, & (\boldsymbol{\xi} \cdot \mathcal{D})\boldsymbol{B} &= (\boldsymbol{B} \cdot \mathcal{D})\boldsymbol{\xi}, \end{aligned} \tag{22}$$

in which b is any differentiable scalar field and $\boldsymbol{B}$ is any differentiable vector field. Note the alternative notation $\boldsymbol{\xi} \cdot \mathcal{D} = \mathcal{D}_{\boldsymbol{\xi}}$ for a directional derivative along $\boldsymbol{\xi}$.

Decomposition of tensor fields relative to fiducial observers is accomplished with the vector fields $\boldsymbol{n}$ and $\boldsymbol{\xi}$ and the 1-forms $\underline{\boldsymbol{\chi}}$ and $\underline{\boldsymbol{n}}$, along with the spatial projection operator

$$\overleftarrow{\boldsymbol{\gamma}} = \boldsymbol{\delta} - \boldsymbol{n} \otimes \underline{\boldsymbol{\chi}} - \boldsymbol{\xi} \otimes \underline{\boldsymbol{n}} \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}), \tag{23}$$

where, in this context, $\boldsymbol{\delta} = \overleftarrow{\boldsymbol{G}}$ is the identity tensor on $B\mathcal{E}$ or $B\mathcal{G}$. The norms

$$\boldsymbol{G}(\boldsymbol{n}, \boldsymbol{n}) = \underline{\boldsymbol{n}} \cdot \boldsymbol{n} = 0 \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}),$$

$$\boldsymbol{G}(\boldsymbol{\chi}, \boldsymbol{\chi}) = \underline{\boldsymbol{\chi}} \cdot \boldsymbol{\chi} = \begin{cases} -\dfrac{1}{c^2} & (\text{on } B\mathcal{E}) \\ 0 & (\text{on } B\mathcal{G}), \end{cases}$$

$$\boldsymbol{G}(\boldsymbol{\xi}, \boldsymbol{\xi}) = \underline{\boldsymbol{\xi}} \cdot \boldsymbol{\xi} = \begin{cases} \dfrac{1}{c^2} & (\text{on } B\mathcal{E}) \\ 0 & (\text{on } B\mathcal{G}) \end{cases}$$

and mutual contractions

$$\begin{aligned} \boldsymbol{G}(\chi, \boldsymbol{n}) &= \underline{\chi} \cdot \boldsymbol{n} = 1, \\ \boldsymbol{G}(\boldsymbol{n}, \xi) &= \underline{\boldsymbol{n}} \cdot \xi = 1, \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}) \\ \boldsymbol{G}(\chi, \xi) &= \underline{\chi} \cdot \xi = 0 \end{aligned}$$

ensure that

$$\begin{aligned} 0 = \overleftarrow{\gamma} \cdot \boldsymbol{n} = \overleftarrow{\gamma} \cdot \chi = \overleftarrow{\gamma} \cdot \xi, \\ 0 = \underline{\boldsymbol{n}} \cdot \overleftarrow{\gamma} = \underline{\chi} \cdot \overleftarrow{\gamma} = \underline{\xi} \cdot \overleftarrow{\gamma} \end{aligned} \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G})$$

as required. Note also the relative simplicity of

$$\boldsymbol{G} = \underline{\boldsymbol{n}} \otimes \underline{\chi} + \gamma + \underline{\xi} \otimes \underline{\boldsymbol{n}} \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}) \tag{24}$$

(clearly symmetric when taking into account Equation (19)) compared with Equation (15).

The 5-velocity $\mathcal{U}$ of a material particle decomposes as

$$\mathcal{U} = \begin{cases} \Lambda_V(\boldsymbol{n} + \boldsymbol{V}) - c^2(\Lambda_V - 1)\,\xi & (\text{on } B\mathcal{E}) \\ \boldsymbol{n} + \boldsymbol{V} - \dfrac{1}{2}\gamma(\boldsymbol{V}, \boldsymbol{V})\,\xi & (\text{on } B\mathcal{G}), \end{cases} \tag{25}$$

in which $\boldsymbol{V}$ and Λ_V have the same meaning as on $\mathcal{E}$. Dual to $\mathcal{U}$ is the 1-form

$$\underline{\chi}_{\mathcal{U}} = \begin{cases} \Lambda_V\left(\underline{\chi} - \dfrac{1}{c^2}\,\underline{\boldsymbol{V}}\right) & (\text{on } B\mathcal{E}) \\ \underline{\chi} & (\text{on } B\mathcal{G}). \end{cases} \tag{26}$$

Moreover the relation

$$\xi = \begin{cases} \dfrac{1}{c^2}\,\mathcal{U} + \chi_{\mathcal{U}} & (\text{on } B\mathcal{E}) \\ \chi_{\mathcal{U}} & (\text{on } B\mathcal{G}) \end{cases} \tag{27}$$

analogous to Equation (19) holds, with no change to the action vector field ξ, as well the analogous norms and mutual contractions, with the result that

$$\overleftarrow{\gamma_{\mathcal{U}}} = \delta - \mathcal{U} \otimes \underline{\chi}_{\mathcal{U}} - \xi \otimes \underline{\mathcal{U}} \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}) \tag{28}$$

is the projection operator to position space as locally reckoned by a comoving observer.

It is important to note that while it is useful to have a spacetime metric $\boldsymbol{G}$ on $B\mathcal{E}$ or $B\mathcal{G}$, it does not play a direct role in governing causality, unlike $\boldsymbol{g}$ on $\mathcal{E}$. Indeed $\boldsymbol{g}$ is also a fundamental structure on $B\mathcal{E}$ governing causality, being imported from $\mathcal{E}$ with vanishing action components (Paper I), with the consequence for example that as for any 4-velocity on $\mathcal{E}$, so also

$$-c^2 = \boldsymbol{g}(\boldsymbol{n}, \boldsymbol{n}) = \boldsymbol{g}(\mathcal{U}, \mathcal{U}) \qquad (\text{on } B\mathcal{E})$$

for any 5-velocity on $B\mathcal{E}$. On $B\mathcal{G}$ the fundamental time 1-form

$$\tau = \underline{\chi} = \underline{\chi}_{\mathcal{U}} \qquad (\text{on } B\mathcal{G}) \tag{29}$$

plays the analogous role, with the condition

$$1 = \tau(\boldsymbol{n}) = \tau(\mathcal{U}) \qquad (\text{on } B\mathcal{G})$$

required of any 5-velocity on $B\mathcal{G}$. Note that while $\overleftarrow{\gamma}$ and $\overleftarrow{\gamma_{\mathcal{U}}}$ both project to the same (universal) position-space leaves characterized by the normal 1-form $\boldsymbol{\tau}$ in the case of $B\mathcal{G}$, they yield different projections to those same leaves. The 5-metric $\boldsymbol{G}$ does play a role in completing the normalization of a 5-velocity, the additional component requiring an additional constraint:

$$0 = \boldsymbol{G}(\boldsymbol{n}, \boldsymbol{n}) = \boldsymbol{G}(\mathcal{U}, \mathcal{U}) \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}) \tag{30}$$

for any 5-velocity on $B\mathcal{E}$ or $B\mathcal{G}$. Indeed this may be regarded as one of the defining conditions that summons the Bargmann metric $\boldsymbol{G}$ and the notion of Bargmann spacetime into existence.

Finally, consider the 5-momentum $\boldsymbol{\Pi}$ of a particle of mass m, a 1-form, as is $\boldsymbol{P}$ on $\mathcal{E}$. It is defined by

$$\overleftarrow{\boldsymbol{\Pi}} = m\,\mathcal{U}, \qquad \boldsymbol{\Pi} = m\,\underline{\mathcal{U}}. \tag{31}$$

Decompose with respect to fiducial observers, writing

$$\boldsymbol{\Pi} = -\epsilon_{\boldsymbol{p}}\,\underline{\boldsymbol{\chi}} + \boldsymbol{p} + m\,\underline{\boldsymbol{n}} \qquad (\text{on } B\mathcal{E} \text{ or } B\mathcal{G}),$$

and utilize the decomposition of $\mathcal{U}$ in Equation (25) to find

$$\boldsymbol{p} = \begin{cases} m\,\Lambda_{\boldsymbol{V}_p}\,\underline{\boldsymbol{V}}_p & (\text{on } B\mathcal{E}) \\ m\,\underline{\boldsymbol{V}}_p & (\text{on } B\mathcal{G}) \end{cases}$$

and

$$\epsilon_{\boldsymbol{p}} = \begin{cases} mc^2\left(\Lambda_{\boldsymbol{V}_p} - 1\right) & (\text{on } B\mathcal{E}) \\ \dfrac{m}{2}\,\gamma(\boldsymbol{V}_p, \boldsymbol{V}_p) & (\text{on } B\mathcal{G}) \end{cases}$$

for the particle 3-momentum $\boldsymbol{p}$ and kinetic energy $\epsilon_{\boldsymbol{p}}$.

3. A Simple Fluid

Before generalizing to a baryon fluid, consider first a simple fluid governed by Poincaré relativity (on four-dimensional spacetime $\mathcal{E}$ or five-dimensional spacetime $B\mathcal{E}$) or Galilei relativity (on five-dimensional spacetime $B\mathcal{G}$). Here a 'simple fluid' consists microscopically of a single type of particle of constant mass m. The fluid kinematics—the mere *description* of the fluid, that is, how dense it is and how fast it is moving—is given by defining the particle number flux on spacetime, denoted as $\boldsymbol{N}$ on $\mathcal{E}$ and $\mathcal{N}$ on $B\mathcal{E}$ or $B\mathcal{G}$. The fluid dynamics—the governing *prescription* of how its motion evolves—is given by a law obeyed by the total-energy–momentum flux $\boldsymbol{T}$ on $\mathcal{E}$ or the kinetic-energy–momentum–mass flux $\mathcal{T}$ on $B\mathcal{E}$ or $B\mathcal{G}$.

The kinematics described in Section 2 can be applied at two levels in order to physically interpret the decompositions of $\boldsymbol{N}$ and $\mathcal{N}$, and of $\boldsymbol{T}$ and $\mathcal{T}$, relative to fiducial and comoving observers. First, from a macroscopic perspective, the tangent vector field along the worldline of each infinitesimal control volume of a fiducial observer is a 4-velocity $\boldsymbol{n}$ on $\mathcal{E}$ or 5-velocity $\boldsymbol{n}$ on $B\mathcal{E}$ or $B\mathcal{G}$, and the tangent vector field along the worldline of each infinitesimal fluid element (a fixed collection of microscopic particles) is a 4-velocity $\boldsymbol{U}$ on $\mathcal{E}$ or 5-velocity $\mathcal{U}$ on $B\mathcal{E}$ or $B\mathcal{G}$. Second, from a microscopic perspective, each fiducial control volume or comoving fluid element encompasses a large number of microscopic particles, each with its own individual velocity relative to $\boldsymbol{n}$ or $\boldsymbol{n}$, or $\boldsymbol{U}$ or $\mathcal{U}$, characterizing the motion of each control volume or fluid element as a whole.

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Having decomposed the flux tensors characterizing the fluid, the spacetime tensor laws governing the elements of those fluxes can also be decomposed. The equations decomposed relative to fiducial observers were presented in Paper IIA; the greater focus here is on the decompositions relative to comoving observers.

3.1. Particle Number Flux on Einstein Spacetime $\mathcal{E}$

On $\mathcal{E}$, the 4-flux of particle number is a vector field

$$\boldsymbol{N} = \int f \overleftarrow{\boldsymbol{P}} \, \mathrm{d}P_m, \tag{32}$$

where f is the Lorentz-invariant particle distribution function (particle density in phase space restricted to the mass shell), $\mathrm{d}P_m$ is the Lorentz-invariant 3-volume element in momentum space for particles of mass m, and $\overleftarrow{\boldsymbol{P}}$ is the inertia-momentum of each microscopic particle, given by Equation (14).

Decomposition of $\overleftarrow{\boldsymbol{P}}$, and thereby $\boldsymbol{N}$, yields fluid densities and bulk velocities measured by fiducial and comoving observers. Relative to a fiducial observer,

$$\overleftarrow{\boldsymbol{P}} = \frac{\mathcal{E}_{\boldsymbol{p}}}{c^2}(\boldsymbol{n} + \boldsymbol{V}_{\boldsymbol{p}}), \qquad \mathrm{d}P_m = \frac{c^2 \, \mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3 \, \mathcal{E}_{\boldsymbol{p}}},$$

where $\boldsymbol{p}$ is the individual particle 3-momentum measured by a fiducial observer, that is, one with 4-velocity $\boldsymbol{n}$. Insertion into Equation (32) yields

$$\boldsymbol{N} = N\,\boldsymbol{n} + N\,\boldsymbol{V}, \tag{33}$$

where

$$N = \int f \frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3}, \qquad \boldsymbol{V} = \frac{1}{N} \int f \, \boldsymbol{V}_{\boldsymbol{p}} \frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3}$$

are the particle number density and 3-velocity of the fluid measured by fiducial observers. Alternatively, relative to a comoving observer,

$$\overleftarrow{\boldsymbol{P}} = \frac{\mathcal{E}_{\boldsymbol{s}}}{c^2}(\boldsymbol{U} + \boldsymbol{V}_{\boldsymbol{s}}), \qquad \mathrm{d}P_m = \frac{c^2 \, \mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3 \, \mathcal{E}_{\boldsymbol{s}}},$$

where $\boldsymbol{s}$ is the individual particle 3-momentum measured by a comoving observer, that is, one with the 4-velocity $\boldsymbol{U}$ assigned to the fluid element as a whole. It is important to note that $\boldsymbol{U}$ of the fluid element is not the same as the individual 4-velocities of the microscopic particles of which the fluid element is composed. Insertion into Equation (32) yields

$$\boldsymbol{N} = n\,\boldsymbol{U} \tag{34}$$

where

$$n = \int f \frac{\mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3}, \qquad 0 = \frac{1}{n} \int f \, \boldsymbol{V}_{\boldsymbol{s}} \frac{\mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3}$$

are the particle number density and vanishing 3-velocity of the fluid measured by comoving observers, the latter condition imposed as the definition of what is meant by 'comoving observer'. Contraction of both Equations (33) and (34) with $\underline{\boldsymbol{\chi}}$ yields

$$N = \Lambda_{\boldsymbol{V}}\, n$$

relating the number densities N and n measured by fiducial and comoving observers.

With decompositions of $\boldsymbol{N}$ relative to fiducial and comoving observers in hand, the spacetime tensor relation

$$\boldsymbol{\nabla} \cdot \boldsymbol{N} = 0 \tag{35}$$

that governs it, expressing the conservation of particles on $\mathcal{E}$, can be expressed in terms of quantities measured by these observers. The particle conservation law

$$\frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial t}\Big(\sqrt{\gamma}\, N\Big) + \frac{1}{\alpha}\, \boldsymbol{D} \cdot \Big(N(\alpha\, \boldsymbol{V} - \boldsymbol{\beta})\Big) = 0 \tag{36}$$

relevant to fiducial observers is worked out in Paper IIA. To obtain the perspective of comoving observers, insert Equation (34) into Equation (35) and find

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = -n\, \boldsymbol{\nabla} \cdot \boldsymbol{u}. \tag{37}$$

The comoving observers' worldlines being parametrized by their proper time τ,

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \boldsymbol{u} \cdot \boldsymbol{\nabla} \tag{38}$$

is the directional derivative along their worldlines. The message of Equation (37) is that the particle density measured by a comoving observer increases or decreases along its worldline in proportion with the convergence or divergence of its neighboring worldlines. The picture to have in mind is of an infinitesimal fluid element consisting of a fixed set of particles whose volume changes along its worldline.

3.2. Particle Number Flux on Bargmann Spacetimes $B\mathcal{E}$ and $B\mathcal{G}$

On $B\mathcal{E}$ or $B\mathcal{G}$ the 5-flux of particle number is a vector field

$$\boldsymbol{n} = \int f\, \overleftarrow{\boldsymbol{\Pi}}\, \mathrm{d}P_m \tag{39}$$

similar to Equation (32), but with $\overleftarrow{\boldsymbol{\Pi}}$ of Equation (31) replacing $\overleftarrow{\boldsymbol{P}}$. Decompositions parallel those on $\mathcal{E}$ but acquire additional action components. Relative to fiducial observers,

$$\boldsymbol{n} = N\, \boldsymbol{n} + N\, \boldsymbol{V} - Z\, \boldsymbol{\xi}, \tag{40}$$

with

$$Z = \begin{cases} \displaystyle\int f\, \frac{c^2\, \epsilon_{\boldsymbol{p}}}{\mathcal{E}_{\boldsymbol{p}}}\, \frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3} & (\text{on } B\mathcal{E}) \\ \displaystyle\int f\, \frac{\epsilon_{\boldsymbol{p}}}{m}\, \frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3} & (\text{on } B\mathcal{G}), \end{cases}$$

a scalar field not encountered in fluid dynamics as traditionally formulated. Relative to comoving observers,

$$\boldsymbol{n} = n\, \boldsymbol{u} - z\, \boldsymbol{\xi}, \tag{41}$$

with

$$z = \begin{cases} \displaystyle\int f\, \frac{c^2\, \epsilon_{\boldsymbol{s}}}{\mathcal{E}_{\boldsymbol{s}}}\, \frac{\mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3} & (\text{on } B\mathcal{E}) \\ \displaystyle\int f\, \frac{\epsilon_{\boldsymbol{s}}}{m}\, \frac{\mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3} & (\text{on } B\mathcal{G}). \end{cases}$$

Contraction of both Equations (40) and (41) with $\underline{\boldsymbol{\chi}}$ and $-\underline{\boldsymbol{n}}$ yields

$$N = \begin{cases} \Lambda_{\boldsymbol{V}}\, n & (\text{on } B\mathcal{E}) \\ n & (\text{on } B\mathcal{G}), \end{cases} \qquad Z = \begin{cases} z + n\, c^2(\Lambda_{\boldsymbol{V}} - 1) & (\text{on } B\mathcal{E}) \\ z + \dfrac{1}{2}\, n\, \boldsymbol{\gamma}(\boldsymbol{V}, \boldsymbol{V}) & (\text{on } B\mathcal{G}) \end{cases}$$

relating quantities pertaining to fiducial and comoving observers. (The scalar fields Z and z are, up to a factor of c^2, the average of the ratio of kinetic energy to total energy measured by fiducial and comoving observers respectively. As will be seen, their appearance is something of a transient quirk of the Bargmann approach, since Z and z will not explicitly appear in the fluid equations resulting from decomposition of the spacetime tensor laws governing the fluid, Equations (42) and (62) below. Instead they will turn out always to contribute to the more recognizable kinetic/internal energy E or e measured by fiducial or comoving observers respectively according to Equations (59) and (61) below.)

In Paper IIA, it is shown that the spacetime tensor relation

$$\mathcal{D} \cdot \mathcal{N} = 0, \tag{42}$$

a vanishing 5-divergence expressing particle conservation, governs $\mathcal{N}$ on $B\mathcal{E}$ and $B\mathcal{G}$. The particle conservation law of Equation (36) found on $\mathcal{E}$, relevant to fiducial observers, holds as well (adopting the appropriate definitions of N for $B\mathcal{E}$ and $B\mathcal{G}$). As to comoving observers, note that Equation (41) differs from Equation (34) in that $\mathcal{N}$ is not parallel to $\mathcal{U}$. However, the action component in Equation (41) does not end up affecting the form of the equation governing the evolution of particle number density as reckoned by comoving observers. Because no dependence on action coordinate η is allowed, the extra term vanishes in the spacetime 5-divergence:

$$\mathcal{D} \cdot \mathcal{N} = \mathcal{D} \cdot (n\, \mathcal{U}) = 0,$$

with the result that

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = -n\, \mathcal{D} \cdot \mathcal{U} \tag{43}$$

in which

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \mathcal{U} \cdot \mathcal{D} \tag{44}$$

in parallel with Equations (37) and (38) on $\mathcal{E}$, thus demonstrating the similarity of the comoving description of particle number evolution across $\mathcal{E}$, $B\mathcal{E}$, and $B\mathcal{G}$.

3.3. Total-Energy–Momentum Flux on Einstein Spacetime $\mathcal{E}$

On $\mathcal{E}$, the 4-flux of total-energy–momentum carried by the fluid is a manifestly self-adjoint tensor field

$$\boldsymbol{T} = \int f\, \overleftarrow{\boldsymbol{P}} \otimes \boldsymbol{P}\, \mathrm{d}P_m \tag{45}$$

of type $\binom{1}{1}$. While the particle number 4-flux $\boldsymbol{N}$ is the first momentum moment of the particle distribution, $\boldsymbol{T}$ is a second moment.

Decomposition relative to fiducial observers yields

$$\boldsymbol{T} = -\mathfrak{E}\, \boldsymbol{n} \otimes \underline{\boldsymbol{\chi}} + \boldsymbol{n} \otimes \boldsymbol{S} - \boldsymbol{\mathfrak{Q}} \otimes \underline{\boldsymbol{\chi}} - \boldsymbol{\Sigma}, \tag{46}$$

in which the total energy density scalar field $\mathfrak{E}$, the 3-momentum density 1-form $\boldsymbol{S}$, the total energy flux 3-vector field $\boldsymbol{\mathfrak{Q}}$, and the 3-stress $(1,1)$ tensor field $\boldsymbol{\Sigma}$ measured by fiducial observers are

$$\mathfrak{E} = \int f\, \mathcal{E}_p\, \frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3}, \qquad \boldsymbol{S} = \int f\, \boldsymbol{p}\, \frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3},$$
$$\boldsymbol{\mathfrak{Q}} = \int f\, \mathcal{E}_p\, \boldsymbol{V}_p\, \frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3}, \qquad \boldsymbol{\Sigma} = -\int f\, (\boldsymbol{V}_p \otimes \boldsymbol{p})\, \frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3}.$$

A consequence of $\boldsymbol{T}$ being self-adjoint is that

$$\boldsymbol{\mathfrak{Q}} = c^2\, \overleftarrow{\boldsymbol{S}}, \qquad \boldsymbol{S} = \frac{1}{c^2}\, \underline{\boldsymbol{\mathfrak{Q}}}.$$

The decomposition

$$\boldsymbol{T} = -\mathfrak{e}\, \boldsymbol{U} \otimes \underline{\boldsymbol{\chi}}_{\boldsymbol{U}} + \frac{1}{c^2}\, \boldsymbol{U} \otimes \underline{\boldsymbol{q}} - \boldsymbol{q} \otimes \underline{\boldsymbol{\chi}}_{\boldsymbol{U}} - \boldsymbol{\sigma}$$

relative to comoving observers is important because constitutive relations (for instance an equation of state) apply to quantities measured by comoving observers—the total energy density $\mathfrak{e}$, heat 3-flux $\boldsymbol{q}$, and 3-stress $\boldsymbol{\sigma}$. Assume for simplicity a perfect fluid, characterized by

$$\boldsymbol{q} = 0, \qquad \boldsymbol{\sigma} = -p\, \overleftarrow{\boldsymbol{\gamma}\boldsymbol{u}},$$

where p is the pressure, so that

$$\boldsymbol{T} = -\mathfrak{e}\, \boldsymbol{U} \otimes \underline{\boldsymbol{\chi}}_{\boldsymbol{U}} + p\, \overleftarrow{\boldsymbol{\gamma}\boldsymbol{u}} \qquad (\text{on } \mathcal{E}) \tag{47}$$

with

$$\mathfrak{e} = \int f\, \mathcal{E}_s\, \frac{\mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3}, \qquad 0 = \int f\, \boldsymbol{s}\, \frac{\mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3},$$
$$0 = \int f\, \mathcal{E}_s\, \boldsymbol{V}_s\, \frac{\mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3}, \qquad p\, \overleftarrow{\boldsymbol{\gamma}\boldsymbol{u}} = \int f\, (\boldsymbol{V}_s \otimes \boldsymbol{s})\, \frac{\mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3}.$$

Contraction of both Equations (46) and (47) with appropriate combinations of $\underline{\boldsymbol{\chi}}$ and $\boldsymbol{n}$, and $\overleftarrow{\boldsymbol{\gamma}}$, yields

$$\mathfrak{E} = \Lambda_V{}^2\, (\mathfrak{e} + p) - p, \qquad \boldsymbol{S} = \frac{1}{c^2}\, \Lambda_V{}^2\, (\mathfrak{e} + p)\, \underline{\boldsymbol{V}},$$
$$\boldsymbol{\mathfrak{Q}} = \Lambda_V{}^2\, (\mathfrak{e} + p)\, \boldsymbol{V}, \qquad \boldsymbol{\Sigma} = -\frac{1}{c^2}\, \Lambda_V{}^2\, (\mathfrak{e} + p)\, \boldsymbol{V} \otimes \underline{\boldsymbol{V}} - p\, \overleftarrow{\boldsymbol{\gamma}}$$

in the case of a perfect fluid, relating quantities measured by fiducial observers to those measured by comoving observers.

With decompositions of $\boldsymbol{T}$ relative to fiducial and comoving observers in hand, the spacetime tensor relation

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = 0 \tag{48}$$

that governs it, expressing the conservation of total-energy–momentum on $\mathcal{E}$, can be expressed in terms of quantities measured by these observers.

The laws of total energy and 3-momentum balance

$$\frac{1}{\alpha\sqrt{\gamma}}\, \frac{\partial}{\partial t}\Big(\sqrt{\gamma}\, \mathfrak{E}\Big) + \frac{1}{\alpha}\, \boldsymbol{D} \cdot \Big(\alpha\, \boldsymbol{\mathfrak{Q}} - \mathfrak{E}\, \boldsymbol{\beta}\Big) = -\overleftarrow{\boldsymbol{S}} \cdot \boldsymbol{D}\phi - \overrightarrow{\boldsymbol{\Sigma}} : \boldsymbol{K} \tag{49}$$

$$\frac{1}{\alpha\sqrt{\gamma}}\, \frac{\partial}{\partial t}\Big(\sqrt{\gamma}\, S_j\Big) + \frac{1}{\alpha}\, D_a\Big(-\alpha\, \Sigma^a{}_j - \beta^a\, S_j\Big) = -\frac{\mathfrak{E}}{c^2}\, D_j\phi + \frac{1}{\alpha}\, S_a\, D_j\beta^a \tag{50}$$

relevant to fiducial observers are worked out in Paper IIA; note that $\boldsymbol{K}$ is the extrinsic curvature characterizing the embedding of the position-space leaves. To make contact with physics on Bargmann spacetimes $B\mathcal{E}$ and $B\mathcal{G}$ it is convenient to derive a balance equation for kinetic energy rather than total energy. In particular,

$$E = \int f\,\epsilon_{\boldsymbol{p}}\,\frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3}, \qquad \boldsymbol{Q} = \int f\,\epsilon_{\boldsymbol{p}}\,\boldsymbol{V}_{\boldsymbol{p}}\,\frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3}$$

are the kinetic energy density and 3-flux measured by fiducial observers, satisfying

$$\begin{aligned} E &= \mathfrak{E} - mc^2\,N \\ &= {\Lambda_{\boldsymbol{V}}}^2\,(e+p) - p + mc^2\,n\,\Lambda_{\boldsymbol{V}}(\Lambda_{\boldsymbol{V}}-1) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{\mathfrak{Q}} - mc^2\,N\,\boldsymbol{V} \\ &= \Big({\Lambda_{\boldsymbol{V}}}^2\,(e+p) + mc^2\,n\,\Lambda_{\boldsymbol{V}}(\Lambda_{\boldsymbol{V}}-1)\Big)\,\boldsymbol{V}, \end{aligned}$$

where

$$e = \int f\,\epsilon_{\boldsymbol{s}}\,\frac{\mathrm{d}\boldsymbol{s}}{(2\pi\hbar)^3}$$

is the kinetic energy measured by comoving observers, here called the internal energy density. The terms in which c^2 appears in these expressions for E and $\boldsymbol{Q}$ cause no trouble as $c \to \infty$ because they limit nicely to the Galilei bulk or macroscopic kinetic energy density $m\,n\,\boldsymbol{\gamma}(\boldsymbol{V},\boldsymbol{V})/2$. With constant particle mass m, subtraction of mc^2 times Equation (36) for particle conservation from Equation (49) for total energy balance yields the balance equation

$$\frac{1}{\alpha\sqrt{\gamma}}\,\frac{\partial}{\partial t}\Big(\sqrt{\gamma}\,E\Big) + \frac{1}{\alpha}\,\boldsymbol{D}\cdot\Big(\alpha\,\boldsymbol{Q} - E\,\boldsymbol{\beta}\Big) = -\overleftarrow{\boldsymbol{S}}\cdot\boldsymbol{D}\phi - \overrightarrow{\boldsymbol{\Sigma}} : \boldsymbol{K} \tag{51}$$

for the kinetic energy density.

Now for decomposition according to comoving observers. Insert Equation (47) into Equation (48) and, with the help of Equations (37) and (38), find the 1-form equation

$$0 = \left(-\frac{\mathrm{d}\mathfrak{e}}{\mathrm{d}\tau} + \frac{\mathfrak{e}+p}{n}\,\frac{\mathrm{d}n}{\mathrm{d}\tau}\right)\underline{\chi}_{\boldsymbol{U}} - (\mathfrak{e}+p)\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\underline{\chi}_{\boldsymbol{U}} + \boldsymbol{\mathcal{D}}p, \tag{52}$$

where

$$\boldsymbol{\mathcal{D}} = \overrightarrow{\boldsymbol{\gamma}\boldsymbol{U}}\cdot\boldsymbol{\nabla} \tag{53}$$

projects the gradient to position space as reckoned by comoving observers. The last two terms in Equation (52) are aligned with position space as reckoned by comoving observers. This is obvious in the case of $\boldsymbol{\mathcal{D}}p$, thanks to the comoving position-space projection operator $\overrightarrow{\boldsymbol{\gamma}\boldsymbol{U}}$ appearing in Equation (53). As to the second to last term in Equation (52), note from the definition of the dual comoving observer covector field $\underline{\chi}_{\boldsymbol{U}}$ that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,\underline{\chi}_{\boldsymbol{U}} = -\frac{1}{c^2}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\underline{\boldsymbol{U}} = -\frac{1}{c^2}\,\underline{\boldsymbol{A}},$$

where $\underline{\boldsymbol{A}}$ is the covector version of the 4-acceleration of comoving observers. Recall that this is spatial according to comoving observers, since 4-acceleration is orthogonal to the 4-velocity defining the time direction of comoving observers:

$$\underline{\boldsymbol{A}}\cdot\boldsymbol{U} = g(\boldsymbol{A},\boldsymbol{U}) = g\left(\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\tau},\boldsymbol{U}\right) = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,g(\boldsymbol{U},\boldsymbol{U}) = 0$$

because of the symmetry of the metric and the constant normalization in Equation (10). Thus the comoving position-space projection of Equation (52) yields

$$\frac{\mathfrak{e}+p}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\underline{\boldsymbol{U}} = -\boldsymbol{\mathcal{D}}p. \tag{54}$$

That is, comoving observer worldlines bend in a direction pointing from higher pressure to lower pressure, resisted by an inertia or 'total mass' composed of total energy and pressure. The picture to have in mind is of Newton's second law applied to an infinitesimal comoving fluid element consisting of a fixed set of particles.

Contraction of Equation (52) with $\boldsymbol{U}$ yields the component associated with the time direction as reckoned by comoving observers:

$$\frac{\mathrm{d}\mathfrak{e}}{\mathrm{d}\tau} = \frac{\mathfrak{e}+p}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau}. \tag{55}$$

This is a manifestation of the first law of thermodynamics, as follows. Regard each comoving fluid element—a particular, fixed collection of particles—as a thermodynamic system unto itself. Because the number of particles in the element is fixed, in terms of total energy per particle $\mathfrak{e}/n$, entropy per particle s/n, and volume per particle $1/n$, the first law of thermodynamics reads

$$\mathrm{d}\left(\frac{\mathfrak{e}}{n}\right) = T\,\mathrm{d}\left(\frac{s}{n}\right) - p\,\mathrm{d}\left(\frac{1}{n}\right),$$

where s is the entropy density and T is the temperature of the element. This is equivalent to Equation (55) for constant entropy per particle s/n, characteristic of a perfect fluid (modulo shocks). For comparison with results on the Bargmann spacetimes $B\mathcal{E}$ and $B\mathcal{G}$, it is useful to define the internal energy density e by separating out the particle rest mass density $mc^2\,n$:

$$\mathfrak{e} = mc^2\,n + e. \tag{56}$$

The microscopic particle mass m being constant, the first law can also be expressed in terms of the internal energy density e:

$$\mathrm{d}\left(\frac{e}{n}\right) = T\,\mathrm{d}\left(\frac{s}{n}\right) - p\,\mathrm{d}\left(\frac{1}{n}\right).$$

Then for a perfect fluid,

$$\frac{\mathrm{d}e}{\mathrm{d}\tau} = \frac{e+p}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau} \tag{57}$$

may replace Equation (55).

3.4. Kinetic-Energy–Momentum–Mass Flux on Bargmann Spacetimes $B\mathcal{E}$ and $B\mathcal{G}$

On $B\mathcal{E}$ or $B\mathcal{G}$ the 5-flux of kinetic-energy–momentum–mass is a $\binom{1}{1}$ self-adjoint tensor field

$$\mathcal{T} = \int f\,\overleftarrow{\boldsymbol{\Pi}}\otimes\boldsymbol{\Pi}\,\mathrm{d}P_m$$

similar to Equation (45), but with $\overleftarrow{\boldsymbol{\Pi}}$ of Equation (31) replacing $\overleftarrow{\boldsymbol{P}}$. In the decompositions, kinetic energy is split from mass, a separation inherited from the difference between $\boldsymbol{\Pi}$ on $B\mathcal{E}$ and $B\mathcal{G}$ and $\boldsymbol{P}$ on $\mathcal{E}$. Relative to fiducial observers,

$$\begin{aligned}\mathcal{T} = &-E\,\boldsymbol{n}\otimes\underline{\boldsymbol{\chi}} + \boldsymbol{n}\otimes\boldsymbol{S} + m\,N\,\boldsymbol{n}\otimes\underline{\boldsymbol{n}}\\ &-\boldsymbol{Q}\otimes\underline{\boldsymbol{\chi}} - \boldsymbol{\Sigma} + m\,N\,\boldsymbol{V}\otimes\underline{\boldsymbol{n}}\\ &+Y\,\boldsymbol{\xi}\otimes\underline{\boldsymbol{\chi}} - \boldsymbol{\xi}\otimes\underline{\boldsymbol{Q}} - m\,Z\,\boldsymbol{\xi}\otimes\underline{\boldsymbol{n}},\end{aligned} \tag{58}$$

with

$$Y = \begin{cases} \int f \, \frac{c^2 \, {\epsilon_p}^2}{\mathcal{E}_p} \, \frac{\mathrm{d}p}{(2\pi\hbar)^3} & (\text{on } B\mathcal{E}) \\ \int f \, \frac{{\epsilon_p}^2}{m} \, \frac{\mathrm{d}p}{(2\pi\hbar)^3} & (\text{on } B\mathcal{G}) \end{cases}$$

another scalar field not encountered in fluid dynamics as traditionally formulated, like Z introduced previously. However these are related to the kinetic energy density E by

$$E = \begin{cases} \frac{1}{c^2} \, Y + m \, Z & (\text{on } B\mathcal{E}) \\ m \, Z & (\text{on } B\mathcal{G}), \end{cases} \tag{59}$$

which follows from their definitions in terms of integrals over momentum space. Relative to comoving observers,

$$\begin{aligned} \mathcal{T} = & - e \, \mathcal{U} \otimes \underline{\chi}_{\mathcal{U}} + \frac{1}{c^2} \, \mathcal{U} \otimes \underline{q} \; + \; m \, n \, \mathcal{U} \otimes \underline{\mathcal{U}} \\ & - q \otimes \underline{\chi}_{\mathcal{U}} \; - \quad \sigma \\ & + y \, \xi \otimes \underline{\chi}_{\mathcal{U}} \; - \quad \xi \otimes \underline{q} \; - \; m \, z \, \xi \otimes \underline{\mathcal{U}}, \end{aligned}$$

or

$$\begin{aligned} \mathcal{T} = & - e \, \mathcal{U} \otimes \underline{\chi}_{\mathcal{U}} \qquad\qquad + \; m \, n \, \mathcal{U} \otimes \underline{\mathcal{U}} \\ & \qquad + \quad p \, \overleftarrow{\gamma} u \\ & + y \, \xi \otimes \underline{\chi}_{\mathcal{U}} \qquad\qquad - \; m \, z \, \xi \otimes \underline{\mathcal{U}} \end{aligned} \tag{60}$$

in the case of a perfect fluid, with

$$y = \begin{cases} \int f \, \frac{c^2 \, {\epsilon_s}^2}{\mathcal{E}_s} \, \frac{\mathrm{d}s}{(2\pi\hbar)^3} & (\text{on } B\mathcal{E}) \\ \int f \, \frac{{\epsilon_s}^2}{m} \, \frac{\mathrm{d}s}{(2\pi\hbar)^3} & (\text{on } B\mathcal{G}) \end{cases}$$

and

$$e = \begin{cases} \frac{1}{c^2} \, y + m \, z & (\text{on } B\mathcal{E}) \\ m \, z & (\text{on } B\mathcal{G}). \end{cases} \tag{61}$$

(The scalar fields Y and y are, up to a factor of c^2, the average of the ratio of the square of kinetic energy to the total energy measured by fiducial and comoving observers respectively. As mentioned previously in connection with Z and z, their appearance is but a passing curiosity, as they always combine according to Equations (59) and (61) to yield E and e in the fluid equations that result from the decomposition of Equations (42) and (62) governing the fluid.) Contraction of both Equations (58) and (60) with appropriate combinations of $\underline{\chi}$ and n, and $\overleftarrow{\gamma}$, and $\underline{n}$ and ξ, yields quantities measured by fiducial observers in terms of quantities measured by comoving observers. Such relations for N and Z on $B\mathcal{E}$ and $B\mathcal{G}$ have already been given in the discussion of the particle number 5-flux. The relations for

E and Q and for $\boldsymbol{S}$ and $\boldsymbol{\Sigma}$ have already been given on $\mathcal{E}$ and are the same on $B\mathcal{E}$, but are displayed again here for comparison with $B\mathcal{G}$. For the density and 3-flux of kinetic energy,

$$E = \begin{cases} {\Lambda_V}^2\,(e+p) - p + m\,n\,c^2\,\Lambda_V(\Lambda_V - 1) & (\text{on } B\mathcal{E}) \\ e + \dfrac{1}{2}\,m\,n\,\boldsymbol{\gamma}(\boldsymbol{V},\boldsymbol{V}) & (\text{on } B\mathcal{G}), \end{cases}$$

$$\boldsymbol{Q} = \begin{cases} \Big({\Lambda_V}^2\,(e+p) + m\,n\,c^2\,\Lambda_V(\Lambda_V - 1)\Big)\,\boldsymbol{V} & (\text{on } B\mathcal{E}) \\ \Big(e + p + \dfrac{1}{2}\,m\,n\,\boldsymbol{\gamma}(\boldsymbol{V},\boldsymbol{V})\Big)\,\boldsymbol{V} & (\text{on } B\mathcal{G}). \end{cases}$$

For the density and 3-flux of 3-momentum,

$$\boldsymbol{S} = \begin{cases} {\Lambda_V}^2\left(m\,n + \dfrac{1}{c^2}(e+p)\right)\underline{\boldsymbol{V}} & (\text{on } B\mathcal{E}) \\ m\,n\,\underline{\boldsymbol{V}} & (\text{on } B\mathcal{G}), \end{cases}$$

$$\boldsymbol{\Sigma} = \begin{cases} -{\Lambda_V}^2\left(m\,n + \dfrac{1}{c^2}(e+p)\right)\boldsymbol{V}\otimes\underline{\boldsymbol{V}} - p\,\overleftarrow{\boldsymbol{\gamma}} & (\text{on } B\mathcal{E}) \\ -m\,n\,\boldsymbol{V}\otimes\underline{\boldsymbol{V}} - p\,\overleftarrow{\boldsymbol{\gamma}} & (\text{on } B\mathcal{G}). \end{cases}$$

For the new scalar field arising from action components,

$$Y = \begin{cases} y + (e+p){\Lambda_V}^2\,\boldsymbol{\gamma}(\boldsymbol{V},\boldsymbol{V}) + m\,n\,c^4(\Lambda_V - 1)(\Lambda_V - 1) & (\text{on } B\mathcal{E}) \\ y + (e+p)\boldsymbol{\gamma}(\boldsymbol{V},\boldsymbol{V}) + \dfrac{1}{4}\,m\,n\,\boldsymbol{\gamma}(\boldsymbol{V},\boldsymbol{V})\,\boldsymbol{\gamma}(\boldsymbol{V},\boldsymbol{V}) & (\text{on } B\mathcal{G}), \end{cases}$$

in which Equation (61) has been employed. The consistency of these results with Equation (59) can also be directly confirmed.

In Paper IIA, it is shown that the spacetime tensor relation

$$\mathcal{D}\cdot\mathcal{T} = m\left(\boldsymbol{n}\cdot\boldsymbol{D}\phi\right)\underline{\boldsymbol{\chi}} - m\left(\boldsymbol{n}\cdot\underline{\boldsymbol{\chi}}\right)\boldsymbol{D}\phi \tag{62}$$

governs $\mathcal{T}$ on $B\mathcal{E}$ and $B\mathcal{G}$. Unlike the vanishing divergence in Equation (48) on $\mathcal{E}$, the Bargmann perspective results in an explicit gravitational force on the right-hand side. It is apparent from the arrangement of terms in the decomposition according to fiducial observers in Equation (58) that contractions from the right with $-\boldsymbol{n}$, with $\overleftarrow{\boldsymbol{\gamma}}$, and with $\boldsymbol{\xi}$ respectively yield a kinetic energy vector, a 3-momentum vector, and a mass flux

$$\mathcal{T}\cdot\boldsymbol{\xi} = m\,\mathcal{N}, \tag{63}$$

in which the particle number 5-flux $\mathcal{N}$ appears. As shown in Paper IIA, the above divergence then yields the balance equations encountered previously on $\mathcal{E}$: Equation (51) for kinetic energy, Equation (50) for 3-momentum balance, and Equation (36) for particle number, multiplied by constant particle mass m.

Before embarking on a decomposition relative to comoving observers, and focusing for now on $B\mathcal{E}$, it is useful to recast the right-hand side of Equation (62) in a slightly different form. Just before this equation is obtained in Paper IIA, it is written as

$$\mathcal{D}\cdot\mathcal{T} = 2\,mc^2\,(\mathcal{N}\cdot\mathcal{D})\underline{\boldsymbol{\xi}}, \tag{64}$$

with Equation (21) providing the link to Equation (62). (Be careful: the 'slot' opened by $\mathcal{D}$ in taking the gradient of $\underline{\xi}$ is associated with the second factor in the tensor products of 1-forms on the right-hand side of Equation (21). Failure to get this index ordering right results in a sign error.) Substituting Equation (41) yields

$$\mathcal{D} \cdot \mathcal{T} = 2\, m\, n\, c^2 (\mathcal{U} \cdot \mathcal{D}) \underline{\xi} \tag{65}$$

after the appropriate identity from Equation (22) has been applied.

The divergence $\mathcal{D} \cdot \mathcal{T}$ on the left-hand side of Equation (65) is decomposed relative to comoving observers on $B\mathcal{E}$ in Appendix A. Three of the terms in Equation (A1) have the same structure that appears in Equation (52) on $\mathcal{E}$, while one of its terms combines naturally with the right-hand side of Equation (65), resulting in

$$\left(-\frac{\mathrm{d}e}{\mathrm{d}\tau} + \frac{e+p}{n} \frac{\mathrm{d}n}{\mathrm{d}\tau} \right) \underline{\chi}_{\mathcal{U}} + \left(m\, n \, + \, \frac{e+p}{c^2} \right) \frac{\mathrm{d}}{\mathrm{d}\tau} \underline{\mathcal{U}} + \boldsymbol{\mathcal{D}} p$$
$$= 2\, c^2 \left(m\, n \, + \, \frac{e+p}{c^2} \right) (\mathcal{U} \cdot \mathcal{D})\, \underline{\xi}.$$

Make manifest once again the character of the right-hand side as a gravitational force by returning to Equation (21) for $\mathcal{D}\underline{\xi}$ and Equation (25) for $\mathcal{U}$, finding

$$\left(-\frac{\mathrm{d}e}{\mathrm{d}\tau} + \frac{e+p}{n} \frac{\mathrm{d}n}{\mathrm{d}\tau} \right) \underline{\chi}_{\mathcal{U}} + \left(m\, n \, + \, \frac{e+p}{c^2} \right) \frac{\mathrm{d}}{\mathrm{d}\tau} \underline{\mathcal{U}} + \boldsymbol{\mathcal{D}} p$$
$$= \Lambda_{\boldsymbol{V}} \left(m\, n \, + \, \frac{e+p}{c^2} \right) \Big((\boldsymbol{V} \cdot \boldsymbol{D}\phi)\, \underline{\chi} \, - \, \boldsymbol{D}\phi \Big), \tag{66}$$

in which the appropriate identity from Equation (22) has again been invoked.

The right-hand side of Equation (66) remains partially expressed in terms of fiducial frame quantities, which are to be recast from the perspective comoving observers. Begin with $\boldsymbol{D}\phi$, and remember throughout that $\underline{\xi} \cdot \mathcal{D}\phi = 0$ is in accord with Equation (22), because there is no explicit dependence on the action coordinate η. Because the 3-gradient is the position-space projection of the 5-gradient, we have

$$\boldsymbol{D}\phi = \overrightarrow{\gamma} \cdot \mathcal{D}\phi$$
$$= \Big(\delta - \underline{\chi} \otimes \boldsymbol{n} \Big) \cdot \mathcal{D}\phi.$$

Substitute for $\boldsymbol{n}$ from Equation (25) and find

$$\boldsymbol{D}\phi = \left(\delta - \underline{\chi} \otimes \frac{1}{\Lambda_{\boldsymbol{V}}} \mathcal{U} \right) \cdot \mathcal{D}\phi \, + \, (\boldsymbol{V} \cdot \boldsymbol{D}\phi)\, \underline{\chi}.$$

The second term on the right cancels the first term in the last factor on the right-hand side of Equation (66), so this factor becomes

$$(\boldsymbol{V} \cdot \boldsymbol{D}\phi)\, \underline{\chi} - \boldsymbol{D}\phi = \left(\delta - \underline{\chi} \otimes \frac{1}{\Lambda_{\boldsymbol{V}}} \mathcal{U} \right) \cdot \mathcal{D}\phi$$
$$= \left(\delta - \Lambda_{V_{\mathcal{U}}} \left(\underline{\chi}_{\mathcal{U}} - \frac{1}{c^2} \underline{V}_{\mathcal{U}} \right) \otimes \frac{1}{\Lambda_{\boldsymbol{V}}} \mathcal{U} \right) \cdot \mathcal{D}\phi$$

after substitution of Equation (A6) for $\underline{\chi}$. As detailed in Appendix B, $\boldsymbol{V}_{\mathcal{U}}$ is the 3-velocity of fiducial observers measured by comoving observers (note its orthogonality to $\mathcal{U}$), and $\Lambda_{V_{\mathcal{U}}}$ is the corresponding Lorentz factor. According to Equations (A4) and (A5) the mutual

Lorentz factors measured by fiducial and comoving observers are equal even though the relationship between the 3-velocities they measure for each other cannot be as simple as $\boldsymbol{V}_{\mathcal{U}} = -\boldsymbol{V}$ since, for observers subject to Poincaré physics, relative motion induces a relative rotation of their position spaces. Cancellation of the Lorentz factors allows the spatial gradient $\overrightarrow{\gamma_{\mathcal{U}}} \cdot \mathcal{D}\phi = \boldsymbol{\mathcal{D}}\phi$ according to comoving observers to emerge. Thus

$$\left(-\frac{\mathrm{d}e}{\mathrm{d}\tau} + \frac{e+p}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau}\right)\underline{\boldsymbol{\chi}}_{\mathcal{U}} + \left(m\,n + \frac{e+p}{c^2}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\underline{\mathcal{U}} + \boldsymbol{\mathcal{D}}p$$
$$= -\Lambda_{\boldsymbol{V}_{\mathcal{U}}}\left(m\,n + \frac{e+p}{c^2}\right)\left(\boldsymbol{\mathcal{D}}\phi + \frac{1}{c^2}\,\underline{\boldsymbol{V}}_{\mathcal{U}}\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right) \quad (\text{on } B\mathcal{E})$$

and

$$\left(-\frac{\mathrm{d}e}{\mathrm{d}\tau} + \frac{e+p}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau}\right)\underline{\boldsymbol{\chi}}_{\mathcal{U}} + m\,n\,\frac{\mathrm{d}}{\mathrm{d}\tau}\underline{\mathcal{U}} + \boldsymbol{\mathcal{D}}p = -m\,n\,\boldsymbol{\mathcal{D}}\phi \qquad (\text{on } B\mathcal{G})$$

are the final results for the full decomposition of Equation (62) according to comoving observers on $B\mathcal{E}$ and $B\mathcal{G}$.

The projections of these results to position space according to comoving observers (via contraction with $\overrightarrow{\gamma_{\mathcal{U}}}$) and parallel to the worldline of comoving observers (via contraction with $\mathcal{U}$) are now immediate. Along position space,

$$\left.\begin{matrix}\left(m\,n + \frac{e+p}{c^2}\right)\\ m\,n\end{matrix}\right\}\frac{\mathrm{d}}{\mathrm{d}\tau}\underline{\mathcal{U}} = -\boldsymbol{\mathcal{D}}p - \begin{cases}\Lambda_{\boldsymbol{V}_{\mathcal{U}}}\left(m\,n + \frac{e+p}{c^2}\right)\left(\boldsymbol{\mathcal{D}}\phi + \frac{1}{c^2}\,\underline{\boldsymbol{V}}_{\mathcal{U}}\,\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right) & (\text{on } B\mathcal{E})\\ m\,n\,\boldsymbol{\mathcal{D}}\phi & (\text{on } B\mathcal{G}).\end{cases} \tag{67}$$

The difference with Equation (54) on $\mathcal{E}$ is that in the geometry of the Bargmann spacetimes $B\mathcal{E}$ and $B\mathcal{G}$ an explicit gravitational force on comoving observers is manifest, as a result of the difference between the Levi-Civita connection $\mathcal{D}$ associated with the 5-metric $\boldsymbol{G}$ and the Levi-Civita connection $\boldsymbol{\nabla}$ associated with the 4-metric $\boldsymbol{g}$ appearing in the comoving time derivatives of Equations (38) and (44) respectively. (This is the flipside of the distinction between accelerated fiducial observers on $\mathcal{E}$ and geodesic fiducial observers on $B\mathcal{E}$ and $B\mathcal{G}$ evident in the contrast between Equations (4) and (20).) However, the connection coefficients do not come into play in the gradients of scalar fields, and the energy equation

$$\frac{\mathrm{d}e}{\mathrm{d}\tau} = \frac{e+p}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau} \tag{68}$$

emerging from the projection along comoving observers' worldlines is the same on $B\mathcal{E}$ and $B\mathcal{G}$ as Equation (57) on $\mathcal{E}$, with no gravitational energy source: the nature of a comoving fluid element as a local thermodynamic system unto itself obeying the first law is undisturbed by the transition from Einstein spacetime to Bargmann spacetime.

4. A Baryon Fluid

Generalize now from a simple fluid to a baryon fluid in the context of, for example, core-collapse supernova modeling. For recent overviews of core-collapse supernova modeling see for instance [13–16]; for more detail on baryonic matter under the conditions of interest here, see for instance [17–19]. Whereas the motion of the simple fluid of Section 3 is characterized by the number flux of particles of a single type with fixed mass, the motion of a baryon fluid is characterized by the flux of baryon number, which is shared among various nuclei (including free nucleons) and perhaps more exotic species—hyperons, Δ resonances, and possibly even deconfined quarks. Labeled by the Gothic index $\mathfrak{i}$, each baryonic species is characterized by its own rest mass $m_{\mathfrak{i}}$, baryon number $A_{\mathfrak{i}}$ (mass number in the case of nuclei), and electric charge number $Z_{\mathfrak{i}}$ (atomic number in the case of nuclei). While baryon

number is the principal quantum number characterizing the fluid, attention must also be paid to electron and muon lepton numbers as well as strangeness, because electric charge neutrality is enabled through the presence of charged leptons (primarily electrons, with a tiny admixture of muons at sufficiently high density) as well as exotic baryonic species that carry negative electric charge (also at sufficiently high density), including species with a nonzero strangeness number $S_{\mathfrak{i}}$. The relevant charged lepton numbers will be indexed by $\ell \in \{e, \mu\}$; these, together with strangeness, denoted by $\mathcal{s}$, will collectively be indexed by $\mathcal{X} \in \{\ell, \mathcal{s}\}$.

Under the hot, dense conditions of interest, strong and electromagnetic interactions maintain tight coupling as perceived on macroscopic scales, so that the various baryonic species together with photons and charged leptons may be treated as a single fluid component in thermal equilibrium. Chemical equilibrium among baryonic species, however, is not necessarily assumed: baryon number is conserved and electric charge neutrality is maintained, but the abundances of individual species may evolve according to a reaction network. Such reactions convert rest mass into internal energy (and vice versa), so in the context of Bargmann spacetimes $B\mathcal{E}$ and $B\mathcal{G}$ the kinetic-energy–momentum–mass 5-flux $\mathcal{T}$ and the associated spacetime tensor law must be generalized beyond the constant-mass simple-fluid expressions of Section 3. Hot and dense baryonic matter also gives rise to neutrinos; but these are excluded from the baryon fluid, because they only couple by weak interactions and range from near-equilibrium diffusion to free streaming relative to the fluid. Neutrinos exchange energy–momentum and lepton number with the fluid and can induce changes in strangeness; consequently they give rise to source terms in the baryon fluid equations for energy and momentum, and for the net number densities of the quantum numbers $\mathcal{X}$ (lepton numbers and strangeness) that are carried by the leptons and strange baryons participating in electric charge neutrality.

4.1. Baryon, Lepton, and Strangeness Number Fluxes

On $\mathcal{E}$ the 4-flux of baryon number is a vector field

$$\boldsymbol{N} = \sum_{\mathfrak{i}} A_{\mathfrak{i}}\, \boldsymbol{N}_{\mathfrak{i}}, \tag{69}$$

the sum of the 4-fluxes $\boldsymbol{N}_{\mathfrak{i}}$ of the various baryonic species weighted by their respective baryon numbers $A_{\mathfrak{i}}$. Characterization of the baryonic component of the total system as a single 'baryon fluid', rather than a 'multi-fluid' consisting of several mutually interflowing baryonic components, implies that each 4-flux $\boldsymbol{N}_{\mathfrak{i}}$ can be expressed in terms of a common 4-velocity $\boldsymbol{U}$:

$$\boldsymbol{N}_{\mathfrak{i}} = n_{\mathfrak{i}}\, \boldsymbol{U},$$

in which $n_{\mathfrak{i}}$ is the number density of species $\mathfrak{i}$ measured by comoving observers with 4-velocity $\boldsymbol{U}$. That is, diffusion of the various baryonic species relative to the average flow of baryon number is neglected, as is standard when strong and electromagnetic interactions maintain tight coupling on macroscopic scales of interest. Then Equation (69) reads

$$\boldsymbol{N} = n\, \boldsymbol{U}, \qquad\qquad n = \sum_{\mathfrak{i}} A_{\mathfrak{i}}\, n_{\mathfrak{i}}$$

in analogy with Equation (34), where n is the total baryon number density measured by comoving observers.

The picture to have in mind is of macroscopically tiny fluid elements of 4-velocity $\boldsymbol{U}$, each an electrically neutral thermodynamic system unto itself with a fixed total baryon number. Each fluid element consists microscopically of a large number of baryonic particles of various interacting species, along with photons and charged leptons, all in thermal equi-

librium. Allowing the baryonic species to be out of chemical equilibrium, their abundances Y_i are governed by a reaction network expressed by comoving observers as

$$\frac{\mathrm{d}Y_\mathrm{i}}{\mathrm{d}\tau} = r_\mathrm{i}, \qquad Y_\mathrm{i} = \frac{n_\mathrm{i}}{n}. \tag{70}$$

The baryon reaction rates r_i are due to strong, electromagnetic, and weak interactions, and depend on the baryon density n, temperature T, and baryon abundances $\{Y_\mathrm{i}\}$, with (almost all) weak reactions depending also on the net quantum number abundances $\{Y_\mathcal{X}\}$ and on the neutrino distributions. Such a network satisfies the baryon number constraints

$$\sum_\mathrm{i} A_\mathrm{i}\, Y_\mathrm{i} = 1, \qquad \sum_\mathrm{i} A_\mathrm{i}\, r_\mathrm{i} = 0 \tag{71}$$

that follow from the preceding definitions. The baryon network is also consistent with evolution equations for the net quantum number abundances $Y_\mathcal{X}$:

$$\frac{\mathrm{d}Y_\mathcal{X}}{\mathrm{d}\tau} = r_\mathcal{X}, \qquad Y_\ell = \frac{n_{\ell^-} - n_{\ell^+}}{n}, \qquad Y_\mathcal{S} = \frac{1}{n}\sum_\mathrm{i} S_\mathrm{i}\, n_\mathrm{i}, \tag{72}$$

in which the net quantum number reaction rates $r_\mathcal{X}$ are due only to weak interactions. That these are not independent of the reactions in the baryon network is manifest by the constraints

$$\sum_\ell Y_\ell = \sum_\mathrm{i} Z_\mathrm{i}\, Y_\mathrm{i}, \qquad \sum_\ell r_\ell = \sum_\mathrm{i} Z_\mathrm{i}\, r_\mathrm{i},$$
$$Y_\mathcal{S} = \sum_\mathrm{i} S_\mathrm{i}\, Y_\mathrm{i}, \qquad r_\mathcal{S} = \sum_\mathrm{i} S_\mathrm{i}\, r_\mathrm{i}$$

arising from electric charge neutrality (first row) and the definition of net strangeness (second row).

Under certain conditions there are relevant approximations. For starters, it is common for strangeness $Y_\mathcal{S}$ and the net μ-lepton abundance Y_μ to be neglected altogether. Moreover the number of baryonic species is always limited, usually severely, to those involved in the strongest channels of baryon flow and those contributing the most to energy generation and lepton number exchange. Beyond selection of the most relevant baryonic species there are simplifications associated with increasing degrees of equilibrium. With chemical equilibrium among baryonic species ('nuclear statistical equilibrium' or NSE) established by strong and electromagnetic interactions at sufficiently high density and temperature, the abundances Y_i are associated with chemical potentials μ_i related by the Saha equation, resolved by an equation of state that depends on n, T, $\{Y_\mathcal{X}\}$; this replaces the baryon network of Equation (70), but the net quantum number evolution equations of Equation (72) are essential. At densities where neutrinos are trapped and chemical equilibrium among leptonic species ('weak' or 'beta' equilibrium) is established by weak interactions, the separate charged lepton abundances $\{Y_\ell\}$ may be summed to a single Y_lepton, and even neutrinos characterized by chemical potentials may be brought into the fold, with strangeness abundance determined by equilibrium as well.

The baryon number constraints of Equation (71) are consistent with the alternative expressions

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = -n\,\boldsymbol{\nabla}\cdot\boldsymbol{u}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{N} = 0$$

for baryon number conservation, as the substitution of $n = \sum_\mathrm{i} A_\mathrm{i} n_\mathrm{i} = n\sum_\mathrm{i} A_\mathrm{i} Y_\mathrm{i}$ on the left-hand side of the first of these relations shows. As to the evolution of the individual baryon

species, for the evolution according to comoving observers being given by Equation (70), use of $\boldsymbol{N}_{\mathfrak{i}} = n_{\mathfrak{i}}\,\boldsymbol{U}$ and $n_{\mathfrak{i}} = n\,Y_{\mathfrak{i}}$ results in

$$\boldsymbol{\nabla}\cdot\boldsymbol{N}_{\mathfrak{i}} = n\,r_{\mathfrak{i}},$$

or

$$\frac{1}{\alpha\sqrt{\gamma}}\frac{\partial}{\partial t}\Big(\sqrt{\gamma}\,N_{\mathfrak{i}}\Big) + \frac{1}{\alpha}\,\boldsymbol{D}\cdot\Big(N_{\mathfrak{i}}(\alpha\,\boldsymbol{V}-\boldsymbol{\beta})\Big) = n\,r_{\mathfrak{i}}, \tag{73}$$

in which

$$\boldsymbol{N}_{\mathfrak{i}} = N_{\mathfrak{i}}\,\boldsymbol{n} + N_{\mathfrak{i}}\,\boldsymbol{V}, \qquad\qquad N_{\mathfrak{i}} = \Lambda_{\boldsymbol{V}}\,n_{\mathfrak{i}}$$

are the decomposition relative to and number density measured by fiducial observers. Similarly, Equation (72) yields

$$\frac{1}{\alpha\sqrt{\gamma}}\frac{\partial}{\partial t}\Big(\sqrt{\gamma}\,N_{\mathcal{X}}\Big) + \frac{1}{\alpha}\,\boldsymbol{D}\cdot\Big(N_{\mathcal{X}}(\alpha\,\boldsymbol{V}-\boldsymbol{\beta})\Big) = n\,r_{\mathcal{X}}, \tag{74}$$

which governs the evolution of the net quantum number densities $N_{\mathcal{X}}$ measured by fiducial observers and whose source depends only on weak interactions, implicating neutrinos.

In the case of $B\mathcal{E}$ and $B\mathcal{G}$, the number 4-fluxes $\boldsymbol{N}$ are replaced by number 5-fluxes $\boldsymbol{\mathcal{N}}$, and the 4-velocity $\boldsymbol{U}$ is replaced by the 5-velocity $\mathcal{U}$, and the Levi-Civita connection $\boldsymbol{\nabla}$ is replaced by $\mathcal{D}$. But from the exposition of the number flux of a simple fluid in Sections 3.1 and 3.2, it is clear that the same definitions of densities and abundances and the equations they obey obtained here on $\mathcal{E}$ for a baryon fluid hold as well on $B\mathcal{E}$ and $B\mathcal{G}$ (of course with $\Lambda_{\boldsymbol{V}} \to 1$ on $B\mathcal{G}$).

4.2. Momentum, Energy, and Mass Fluxes

On $\mathcal{E}$ the evolution equations for 3-momentum and total energy, whether relative to fiducial observers in Equations (49) and (50) or comoving observers in Equations (54) and (55), exhibit no explicit dependence on the fluid composition. With n now representing baryon number density, these equations remain unchanged for a baryon fluid, apart from neutrino source terms that arise as follows. Separate the total energy–momentum 4-flux into its contributions $\boldsymbol{T}_F$ from the baryon fluid and $\boldsymbol{T}_R$ from the neutrino radiation:

$$\boldsymbol{T} = \boldsymbol{T}_F + \boldsymbol{T}_R,$$

so that total energy–momentum conservation $\boldsymbol{\nabla}\cdot\boldsymbol{T} = 0$ yields

$$\boldsymbol{\nabla}\cdot\boldsymbol{T}_F = \boldsymbol{Q}_R. \tag{75}$$

On the left-hand side, $\boldsymbol{\nabla}\cdot\boldsymbol{T}_F$ yields precisely the terms in the momentum and energy equations referred to above. On the right-hand side, the 1-form $\boldsymbol{Q}_R = -\boldsymbol{\nabla}\cdot\boldsymbol{T}_R$ is decomposed relative to fiducial or comoving observers as

$$\boldsymbol{Q}_R = \begin{cases} -\,Q_R\,\underline{\boldsymbol{\chi}} + \boldsymbol{A}_R \\ -\,q_R\,\underline{\boldsymbol{\chi}}_{\boldsymbol{U}} + \boldsymbol{a}_R, \end{cases} \tag{76}$$

in which $\boldsymbol{A}_R\cdot\overleftarrow{\boldsymbol{\gamma}} = \boldsymbol{A}_R$ and $\boldsymbol{a}_R\cdot\overleftarrow{\boldsymbol{\gamma}_{\boldsymbol{U}}} = \boldsymbol{a}_R$ are 1-forms tangent to position space as reckoned by fiducial and comoving observers respectively. In the course of solving neutrino transport it is normally the source components q_R and $\boldsymbol{a}_R$ measured by comoving observers that are

most readily computed; the source components Q_R and $\boldsymbol{A}_R$ measured by fiducial observers are obtained in terms of those by contracting both expressions above with $\boldsymbol{n}$ and $\overleftarrow{\gamma}$:

$$\begin{aligned} Q_R &= q_R \, \Lambda_V \, + \, \boldsymbol{a}_R \cdot \boldsymbol{V}, \\ \boldsymbol{A}_R &= \boldsymbol{a}_R \, + \, (\boldsymbol{a}_R \cdot \boldsymbol{V}) \underline{\boldsymbol{\chi}} \, + \, \frac{1}{c^2} \, q_R \, \Lambda_V \, \underline{\boldsymbol{V}}. \end{aligned} \tag{77}$$

In any case, the neutrino radiation source terms resulting from these decompositions add to the right-hand sides of the momentum and energy equations referred to above in the obvious way. For example,

$$\frac{\mathrm{d}\mathfrak{e}}{\mathrm{d}\tau} = \frac{\mathfrak{e} + p}{n} \frac{\mathrm{d}n}{\mathrm{d}\tau} \, + \, q_R \tag{78}$$

generalizes Equation (55) for a simple fluid.

There are implicit dependencies on the fluid composition, of course. For starters, the definition of the 4-velocity $\boldsymbol{U}$, and therefore that of the 3-velocity $\boldsymbol{V}$, depends on the flux of whatever number is conserved—the number of particles in the case of a simple fluid, or the baryon number in the case of a baryon fluid. Moreover, closure of the system requires an equation of state that relates comoving total energy density $\mathfrak{e}$ and pressure p to baryon number density n, temperature T, and the baryon and/or net quantum number abundances $\{Y_{\mathfrak{i}}\}$ and $\{Y_{\mathcal{X}}\}$.

On the other hand, if on $\mathcal{E}$ one decides to explicitly separate rest mass energy from internal energy as in Equation (56), leading to Equation (57) in the case of a simple fluid, the question of composition must more squarely be faced. From the perspective of Poincaré physics the matter is somewhat artificial and even arbitrary, and two possible approaches present themselves; see for instance [20]. One possibility is to declare the rest mass energy density to be $m_B \, c^2 \, n$ for some reasonable constant m_B (such as the atomic mass unit), relegating differences from this due to nuclear binding energy to be part of the internal energy density. Another possibility, more appropriate when one looks ahead to spacetimes $B\mathcal{E}$ and $B\mathcal{G}$, is to respect more carefully the distinction between mass and energy required by the Bargmann approach generally and Galilei physics in particular. This involves definition of a mutable 'mass per baryon' or 'average baryon mass' m, as follows.

The total energy density $\mathfrak{e}$ measured by comoving observers is the sum $\mathfrak{e} = \mathfrak{e}_B + \mathfrak{e}_L + \mathfrak{e}_\gamma$ of contributions from baryons, charged leptons, and photons:

$$\begin{aligned} \mathfrak{e}_B &= \sum_{\mathfrak{i}} m_{\mathfrak{i}} c^2 \, n_{\mathfrak{i}} \, + \, e_B, \\ \mathfrak{e}_L &= \sum_{\ell} m_\ell c^2 \, \left(n_\ell^- + n_\ell^+ \right) \, + \, e_L, \\ \mathfrak{e}_\gamma &= e_\gamma. \end{aligned}$$

Thus the total energy density can be expressed as in Equation (56),

$$\mathfrak{e} = mc^2 \, n + e, \tag{79}$$

with internal energy density $e = e_B + e_L + e_\gamma$ and average baryon mass

$$\begin{aligned} m &= \sum_{\mathfrak{i}} m_{\mathfrak{i}} \, Y_{\mathfrak{i}} + \sum_{\ell} m_\ell \, (Y_{\ell^-} + Y_{\ell^+}) \\ &\approx \sum_{\mathfrak{i}} m_{\mathfrak{i}} \, Y_{\mathfrak{i}}. \end{aligned} \tag{80}$$

(In principle the separate particle and antiparticle charged lepton abundances Y_{ℓ^-} and Y_{ℓ^+} are available as functions of known values of T and the net abundances Y_ℓ, via Fermi–Dirac distributions and the chemical potentials $\mu_{\ell^-} = -\mu_{\ell^+}$ thus implied; but dropping the charged lepton contributions to m is justified by the small electron mass and the tiny muon abundance. It is also worth noting that anytime internal energy is appreciable compared with baryon mass energy, the electrons will be relativistic anyway.) The average baryon mass m is mutable in that

$$\begin{aligned} \frac{\mathrm{d}m}{\mathrm{d}\tau} &= \sum_{\mathrm{i}} m_{\mathrm{i}}\, r_{\mathrm{i}} \\ &= \sum_{\mathrm{i}} (\Delta m)_{\mathrm{i}}\, r_{\mathrm{i}}, \end{aligned} \qquad (\Delta m)_{\mathrm{i}} = m_{\mathrm{i}} - A_{\mathrm{i}}\, m_B.$$

The first line on the left follows from Equation (70). Its right-hand side is subject to large cancellations because of the constraint $\sum_{\mathrm{i}} A_{\mathrm{i}}\, r_{\mathrm{i}} = 0$ from Equation (71). Subtracting this constraint multiplied by a suitably chosen constant m_B (such as the atomic mass unit) results in the second line on the left, showing that it is the mass excess values $(\Delta m)_{\mathrm{i}}$ rather than the absolute masses that drive the rate of change of m. (Again, the contribution from charged leptons could in principle be computed via the chain rule with Y_{ℓ^-} and Y_{ℓ^+} regarded as functions of T and Y_ℓ, but the relative slowness of weak interactions is an additional reason to neglect this.)

Use of this mutable average baryon mass m and its rate of change $\mathrm{d}m/\mathrm{d}\tau$ in Equations (78) and (79) yields

$$\frac{\mathrm{d}e}{\mathrm{d}\tau} = \frac{e+p}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau} + q_m + q_R \qquad (81)$$

with the additional source

$$\begin{aligned} q_m &= -c^2\, n\, \frac{\mathrm{d}m}{\mathrm{d}\tau} \\ &= -n \sum_{\mathrm{i}} b_{\mathrm{i}}\, r_{\mathrm{i}}, \end{aligned} \qquad b_{\mathrm{i}} = (\Delta m)_{\mathrm{i}}\, c^2 \qquad (82)$$

representing the interconversion of mass and internal energy by composition-changing reactions, thus generalizing Equation (57) for a simple fluid. As the binding energies b_{i} of the baryonic species are known to be finite, use of this notation signals an intent to exclude this aspect of the microphysical properties of baryon matter from the $c \to \infty$ limit associated with the passage from Poincaré physics to Galilei physics.

Continuing on $\mathcal{E}$, consider also the balance of kinetic energy from the perspective of fiducial observers. For a simple fluid this is Equation (51). Recall that this is obtained by multiplying particle number conservation, Equation (35) or (36), by constant rest mass energy per particle mc^2—forming, modulo the factor c^2, a mass conservation law expressed as the vanishing divergence of the mass flux $m\,\boldsymbol{N}$—and subtracting from the balance of total energy given by Equation (49). But for a baryon fluid, pulling the mutable average baryon mass m inside the divergence results in the balance equation

$$\boldsymbol{\nabla} \cdot \left(mc^2\, \boldsymbol{N} \right) = -q_m, \qquad (83)$$

leading to

$$\frac{1}{\alpha\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\, E\right) + \frac{1}{\alpha}\boldsymbol{D}\cdot(\alpha\, \boldsymbol{Q} - E\,\boldsymbol{\beta}) = -\overleftarrow{\boldsymbol{S}}\cdot\boldsymbol{D}\phi - \overrightarrow{\boldsymbol{\Sigma}} : \boldsymbol{K} + q_m + Q_R, \qquad (84)$$

in which the neutrino energy source Q_R also appears. Note that the comoving and fiducial neutrino energy sources q_R and Q_R appear in Equations (81) and (84) respectively, while the same energy source q_m due to composition changes appears in both. This is because q_R and Q_R come from different decompositions of the neutrino energy–momentum source $\boldsymbol{Q}_R$, a 1-form, whereas q_m is a scalar source in the balance equation for mass(energy) in Equation (83).

Turn now to the Bargmann spacetime $B\mathcal{E}$. The procedure for obtaining a physical law on $B\mathcal{E}$ from its usual expression on $\mathcal{E}$ is given in Section 6 of Paper IIA. For a simple fluid, this procedure is applied to Equation (48), conservation of energy–momentum, between Equations (121) and (123) of Paper IIA. The same steps can be applied to Equation (75) for a baryon fluid, but allowing for mutable average baryon mass m, resulting in

$$\mathcal{D} \cdot \mathcal{T}_F = 2\, mc^2\, (\boldsymbol{n} \cdot \mathcal{D})\underline{\boldsymbol{\xi}} \,-\, q_m\, \underline{\boldsymbol{\xi}} \,+\, \boldsymbol{Q}_R, \tag{85}$$

which is to be compared with Equation (64) for a simple fluid. In addition to the neutrino radiation source 1-form $\boldsymbol{Q}_R$, composition changes embodied in m give rise to the energy generation source 1-form $-q_m\, \underline{\boldsymbol{\xi}}$, with the energy generation rate q_m again given by Equation (82). (The same neutrino source 1-form $\boldsymbol{Q}_R = -\mathcal{D} \cdot \mathcal{T}_R$ as on $\mathcal{E}$ appears here, with a vanishing action component. For one thing, even though neutrinos have nonvanishing masses, these are tiny and can be ignored with even greater justification than the neglect of charged lepton masses in Equation (80), so that the 'action column' of $\mathcal{T}_R$ containing the neutrino mass flux vanishes. More fundamentally, as discussed in Paper I and Paper IIA and mentioned again in Section 5 herein, covariant tensors (that is, of type $\binom{0}{p}$) on $\mathcal{E}$ acquire no action component when set in $B\mathcal{E}$. Thus, whenever it seems that covariant action components are present in physical laws on $B\mathcal{E}$ reverse-engineered from $\mathcal{E}$, there will always turn out to be compensating cancellations in the resulting decomposed equations, because the reverse engineering procedure cannot of itself generate new information in the action component. Finally, while neutrino transport is beyond the scope of this paper, note that the assumption of massless neutrinos does not preclude the inclusion of the effects of neutrino flavor mixing: even though it is ultimately neutrino mass—specifically, differences in squared mass—that drives flavor mixing, neutrino flavor transformation probabilities or rates are routinely computed with the assumption of ultrarelativistic neutrino kinematics, that is, null neutrino trajectories; see e.g. [21]).

Begin the decomposition of Equation (85) relative to comoving observers by using Equations (27) and (76) to rewrite the right-hand side,

$$\mathcal{D} \cdot \mathcal{T}_F = 2\, m\, n\, c^2 (\mathcal{U} \cdot \mathcal{D})\underline{\boldsymbol{\xi}} \,-\, (q_m + q_R)\, \underline{\boldsymbol{\chi}}_{\mathcal{U}} \,+\, \boldsymbol{a}_R \,-\, \frac{1}{c^2}\, q_m\, \underline{\mathcal{U}},$$

to be compared with Equation (65) for a simple fluid. The decompositions relative to comoving observers of the left-hand side and the first term on the right-hand side proceed as for a simple fluid following Equation (65), except that there is one extra term $n\, (\mathrm{d}m/\mathrm{d}\tau)\, \underline{\mathcal{U}}$ due to mutable m, which cancels the term $-\left(q_m/c^2\right) \underline{\mathcal{U}}$ on the right-hand side just above. Thus the comoving position-space projection on $B\mathcal{E}$ and $B\mathcal{G}$ obtained by contraction with $\overleftarrow{\gamma_{\mathcal{U}}}$ is Equation (67) as for a simple fluid, plus the comoving neutrino momentum source $\boldsymbol{a}_R$. The comoving time projection obtained by contraction with $\mathcal{U}$ is just Equation (81), a version of the first law of thermodynamics obtained earlier for $\mathcal{E}$, valid here also for $B\mathcal{E}$ and $B\mathcal{G}$. The action projection obtained by contraction with $\boldsymbol{\xi}$ vanishes.

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For the decomposition of Equation (85) relative to fiducial observers, return the first term on the right-hand side to the standard form in Equation (62) for a simple fluid, and use Equations (19) and (76) for the new terms associated with the baryon fluid to obtain

$$\mathcal{D} \cdot \mathcal{T}_F = \Big(m \left(\boldsymbol{n} \cdot \boldsymbol{D}\phi \right) - q_m - Q_R \Big) \underline{\boldsymbol{\chi}} \, - \, m \Big(\boldsymbol{n} \cdot \underline{\boldsymbol{\chi}} \Big) \boldsymbol{D}\phi \, + \, \boldsymbol{A}_R \, - \, \frac{1}{c^2} \, q_m \, \underline{\boldsymbol{n}}.$$

The decompositions relative to fiducial observers of the left-hand side and the terms on the right-hand side involving $\boldsymbol{D}\phi$ proceed as for a simple fluid as given in Paper IIA, except that there is one extra term $n \left(\mathrm{d}m / \mathrm{d}\tau \right) \underline{\boldsymbol{n}}$ due to mutable m, which cancels the term $- \left(q_m / c^2 \right) \underline{\boldsymbol{n}}$ on the right-hand side just above. Thus the fiducial position-space projection on $B\mathcal{E}$ and $B\mathcal{G}$ obtained by contraction with $\overleftarrow{\boldsymbol{\gamma}}$ is Equation (50) as for a simple fluid, plus the fiducial neutrino momentum source $\boldsymbol{A}_R$. The fiducial time projection obtained by contraction with $\boldsymbol{n}$ is just Equation (84), the balance of kinetic energy obtained earlier for $\mathcal{E}$, valid here also for $B\mathcal{E}$ and $B\mathcal{G}$. And once again the action projection obtained by contraction with $\boldsymbol{\xi}$ vanishes.

5. Conclusions

The purpose of this paper is to generalize the dynamics of a simple fluid in the Bargmann spacetimes $B\mathcal{E}$ and $B\mathcal{G}$ to the dynamics of a more astrophysically relevant system: a baryon fluid, whose nuclear composition evolves under conditions of high density and temperature. These extended spacetimes $B\mathcal{E}$ and $B\mathcal{G}$ are introduced in order to elucidate a path to a strong-field 'Galilei general relativity' that could serve as a useful approximation to the usual 'Poincaré general relativity' introduced by Einstein. The Bargmann approach facilitates this by allowing a spacetime tensor formalism that includes the conceptual and technical advantages of a pseudo-Riemann manifold—in particular, the presence of a spacetime metric—while respecting the distinction between mass and energy required by Galilei physics.

Unlike the familiar $\boldsymbol{\nabla} \cdot \boldsymbol{T} = 0$ on the usual Einstein spacetime $\mathcal{E}$, the spacetime tensor law for a simple fluid on $B\mathcal{E}$ and $B\mathcal{G}$ obtained in Paper IIA—shown in Equation (1) and reproduced in Equation (62)—has a source term on the right-hand side. This reflects a partial reversion to a Newton-like perspective effected by the Bargmann approach, featuring geodesic fiducial observers and comoving observers subject to an explicit gravitational force rather than the accelerated fiducial observers and geodesic comoving observers encountered in the Einstein perspective. Note the appearance of the particle mass m in this source term. Indeed, the simple fluid assumed in the derivation of Equation (62) is composed of particles of a single type with constant mass m. It is the mutability of the 'mass per baryon' or 'average baryon mass' m of a baryon fluid composed of many interacting baryonic species that provokes the present work.

With the explicit appearance of the 1-form $\underline{\boldsymbol{\chi}}$ normal to the leaves of the spacetime foliation and the potential gradient $\boldsymbol{D}\phi$ tangent to these leaves, the right-hand side of Equation (62) is expressed from the perspective of fiducial observers. This is not particularly unexpected since Galilei physics is associated with absolute time, apparently manifest in the Bargmann approach as a special, even unique, foliation closely associated with the geodesic fiducial observers and an explicit gravitational force. But because the evolving composition of a baryon fluid must first be described by comoving observers, this paper expands beyond the treatment of fluid dynamics according to fiducial observers in Paper IIA by addressing the kinematical (number) and dynamical (energy–momentum–mass) equations governing the fluid as seen by comoving observers. In pursuit of this, still in the context of a simple fluid, the right-hand side of Equation (62) is recast in Equation (64) in a form that, at least on the surface, seems less overtly committed to either fiducial or comoving observers, being given in terms of invariant quantities—the scalar particle rest mass m, the particle

number 5-flux $\mathcal{N}$, and the spacetime gradient $\mathcal{D}\underline{\boldsymbol{\xi}}$ of the (1-form version of the) action vector $\boldsymbol{\xi}$, whose invariance has been noted since Paper I; see also its parallel decompositions in Equations (19) and (27). The link between these two versions of the gravitational force is Equation (21).

Beyond the evolution of composition addressed in Section 4.1, the difference in the dynamical (energy–momentum–mass) spacetime tensor law on $B\mathcal{E}$ and $B\mathcal{G}$ between a simple fluid and a baryon fluid is manifest in the two new source terms on the right-hand side of Equation (85) relative to Equation (64). The neutrino radiation source $\boldsymbol{Q}_R$ is a 1-form that can be decomposed relative to fiducial or comoving observers, yielding different energy and momentum contributions for these two families of observers; see Equations (76) and (77). In contrast, in the energy source term $-\,q_m\,\underline{\boldsymbol{\xi}}$ due to composition changes, the energy generation rate q_m given by Equation (82) is a scalar field with no direct consequences for momentum; therefore its energy contribution is the same for both fiducial and comoving observers, as seen by comparing Equations (81) and (84). On $\mathcal{E}$ this result is obtained somewhat artificially, but on $B\mathcal{E}$ and $B\mathcal{G}$ it is built into the tensor formalism, since the $\underline{\boldsymbol{\xi}}$ that provides the 1-form content of the source term $-\,q_m\,\underline{\boldsymbol{\xi}}$ decomposes in formally parallel ways for fiducial and comoving observers in Equations (19) and (27) respectively. These decompositions of $\underline{\boldsymbol{\xi}}$ also show that the source term $-\,q_m\,\underline{\boldsymbol{\xi}}$ accomplishes two things at once. First, its $\underline{\boldsymbol{\chi}}$ or $\underline{\boldsymbol{\chi}}_{\mathcal{U}}$ component contributes to the energy equation (contraction of Equation (85) with $\boldsymbol{n}$ or $\mathcal{U}$). Second, its $\underline{\boldsymbol{n}}$ or $\underline{\mathcal{U}}$ component contributes to the mass flux equation (contraction of Equation (85) with $\boldsymbol{\xi}$); note Equation (63). In this latter role this source term cancels the variation of m in $\mathcal{D}\cdot(m\,\mathcal{N})$ on the left-hand side of the contraction of Equation (85) with $\boldsymbol{\xi}$, reducing the mass flux equation to redundancy with the conservation of baryon number $\mathcal{D}\cdot\mathcal{N}=0$. (Something like this had to happen, because when $\boldsymbol{\nabla}\cdot\boldsymbol{T}=0$ from $\mathcal{E}$ is set into $B\mathcal{E}$ with vanishing action component, the reverse-engineering into Equation (85) by the procedure of Section 6 of Paper IIA cannot of itself generate new information in the action component. And whatever new information appears in the energy component of Equation (85) is ultimately imported from the baryon evolution worked out in Section 4.1.)

It must be acknowledged that contemplating energy generation by composition changes (and therefore mass changes) is, strictly speaking, incompatible with Galilei physics, which enforces mass conservation (e.g., Paper I). A willingness to be flexible on this point is manifest here in Equation (82) for the energy generation rate q_m, in which the factor c^2 is hidden behind notation for binding energies, known to be finite, thereby shielding this expression from the nonsense that otherwise would result when taking a $c\to\infty$ limit to get from Poincaré physics on $B\mathcal{E}$ to Galilei physics on $B\mathcal{G}$.

Having made this pragmatic allowance, for the sake of an approximation more useful for, e.g., core-collapse supernovae, it is worth extending similar flexibility to all local microscopic considerations reckoned by comoving observers. This possibility is floated near the end of Paper IIA, where such an approach is characterized as 'microscopically Poincaré but macroscopically Galilei'. That is, Poincaré physics is to be allowed in the microscopic description of a comoving fluid element conceived as a local thermodynamic system unto itself, determining for instance the equation of state and allowing pressure and internal energy density (divided by c^2) to meaningfully contribute relative to rest mass density. But Galilei physics is to be enforced at macroscopic scales, manifest in bulk fluid flows that neglect $\gamma(\boldsymbol{V},\boldsymbol{V})/c^2$ and in the differences in spacetime geometry that follow from the differences in the Bargmann metric $\boldsymbol{G}$ in spacetimes $B\mathcal{E}$ and $B\mathcal{G}$. It may be useful to label this general approach $B\mathcal{G}+$. This results in enhanced mass densities—see the definitions of $\rho_{\mathfrak{E}}$ and $\rho_{\mathcal{S}}$ below—but does not otherwise alter the Galilei spacetime geometry. In terms of macroscopic fluid flow, like $B\mathcal{G}$, this extended approach will still lose physical accuracy

to the extent that $\gamma(\boldsymbol{V},\boldsymbol{V})/c^2$ is not negligible, for instance missing the additional inertia provided by $\Lambda_{\boldsymbol{V}} > 1$ that slows the approach of the bulk flow speed toward the speed of light. The absence of gravitational waves on $B\mathcal{G}$ expected in forthcoming Paper IIB would also remain a feature of $B\mathcal{G}+$; gravitational wave signals would have to be post-processed by the usual quadrupole formalism, and their back-reaction on the system would be missing.

In order to make this $B\mathcal{G}+$ approach concrete in the present context, this paper concludes with a summary display of the equations governing the kinematics and dynamics of a baryon fluid on $B\mathcal{E}$ and $B\mathcal{G}+$, decomposed relative to fiducial observers and expressed as partial differential equations, as might be employed in a core-collapse supernova simulation. The fluid kinematics equations describing what the baryon fluid is, and the congruence of worldlines constituting its history, are those governing the total baryon number density N, the number densities $N_{\mathfrak{i}}$ of the baryonic species, and the net lepton number and strangeness densities $N_{\mathcal{X}}$ (see Section 4.1):

$$\frac{\partial}{\partial t}\Big(\sqrt{\gamma}\,N\Big) + \frac{\partial}{\partial x^a}\Big(\sqrt{\gamma}\,N\,(\alpha\,V^a - \beta^a)\Big) = 0,$$
$$\frac{\partial}{\partial t}\Big(\sqrt{\gamma}\,N_{\mathfrak{i}}\Big) + \frac{\partial}{\partial x^a}\Big(\sqrt{\gamma}\,N_{\mathfrak{i}}\,(\alpha\,V^a - \beta^a)\Big) = \alpha\sqrt{\gamma}\,n\,r_{\mathfrak{i}},$$
$$\frac{\partial}{\partial t}\Big(\sqrt{\gamma}\,N_{\mathcal{X}}\Big) + \frac{\partial}{\partial x^a}\Big(\sqrt{\gamma}\,N_{\mathcal{X}}\,(\alpha\,V^a - \beta^a)\Big) = \alpha\sqrt{\gamma}\,n\,r_{\mathcal{X}},$$

with

$$N = \begin{cases} \Lambda_{\boldsymbol{V}}\,n \\ n, \end{cases} \qquad N_{\mathfrak{i}} = \begin{cases} \Lambda_{\boldsymbol{V}}\,n\,Y_{\mathfrak{i}} \\ n\,Y_{\mathfrak{i}}, \end{cases} \qquad N_{\mathcal{X}} = \begin{cases} \Lambda_{\boldsymbol{V}}\,n\,Y_{\mathcal{X}} & (\text{on } B\mathcal{E}) \\ n\,Y_{\mathcal{X}} & (\text{on } B\mathcal{G}+). \end{cases}$$

The fluid dynamics equations governing momentum and kinetic energy are (see Section 4.2)

$$\frac{\partial}{\partial t}\Big(\sqrt{\gamma}\,S_j\Big) + \frac{\partial}{\partial x^a}\Big(\sqrt{\gamma}\,(-\alpha\,\Sigma^a{}_j - \beta^a\,S_j)\Big) = \alpha\sqrt{\gamma}\left(-\rho_{\mathfrak{E}}\,\frac{\partial\phi}{\partial x^j} + \frac{1}{\alpha}\,S_a\,\frac{\partial\beta^a}{\partial x^j} - \frac{1}{2}\,\Sigma^a{}_b\,\gamma^{bc}\,\frac{\partial\gamma_{ca}}{\partial x^j} + (A_R)_j\right)$$

and

$$\frac{\partial}{\partial t}\Big(\sqrt{\gamma}\,E\Big) + \frac{\partial}{\partial x^a}\Big(\sqrt{\gamma}\,(\alpha\,Q^a - E\,\beta^a)\Big) = \alpha\sqrt{\gamma}\left(-S^a\,\frac{\partial\phi}{\partial x^a} - \Sigma^{ab}\,K_{ba} + q_m + Q_R\right),$$

in which

$$S_j = \rho_{\mathcal{S}}\,V_j, \qquad E = \begin{cases} {\Lambda_{\boldsymbol{V}}}^2\,(e+p) - p + m\,n\,c^2\,\Lambda_{\boldsymbol{V}}(\Lambda_{\boldsymbol{V}} - 1) & (\text{on } B\mathcal{E}) \\ e + \dfrac{1}{2}\,m\,n\,\boldsymbol{\gamma}(\boldsymbol{V},\boldsymbol{V}) & (\text{on } B\mathcal{G}+), \end{cases}$$

$$\Sigma^i{}_j = -\Big(V^i\,S_j + p\,\gamma^i{}_j\Big), \qquad Q^i = (E+p)\,V^i,$$

with

$$\rho_{\mathfrak{E}} = \begin{cases} m\,N + \dfrac{1}{c^2}\,E \\ m\,n + \dfrac{1}{c^2}\,e, \end{cases} \qquad \rho_{\mathcal{S}} = \begin{cases} {\Lambda_{\boldsymbol{V}}}^2\left(m\,n + \dfrac{1}{c^2}(e+p)\right) & (\text{on } B\mathcal{E}) \\ m\,n + \dfrac{1}{c^2}(e+p) & (\text{on } B\mathcal{G}+). \end{cases}$$

Like the definition of the binding energies $b_{\mathfrak{i}}$ in Equation (82), use of the notation $\rho_{\mathfrak{E}}$ and $\rho_{\mathfrak{S}}$ for 'mass densities' associated with the total energy and 3-momentum, respectively, signals the exclusion of these expressions from the $c \to \infty$ limit leading to $B\mathcal{G}+$ from $B\mathcal{E}$.

So far, this proposed $B\mathcal{G}+$ approach is physically pragmatic, differing in spirit from a systematic post-Newtonian expansion. In separating the microphysical properties from macrophysical geometry and flows in taking a $c \to \infty$ limit, resulting in enhanced 'mass densities' $\rho_{\mathfrak{E}}$ and $\rho_{\mathfrak{S}}$, it can be regarded as a kind of effective theory partaking of the idea that the concept of 'mass' is less than fundamental, resulting always from unresolved physics at scales not explicitly addressed. Also different from a systematic post-Newtonian expansion is the fact that rather than bootstrapping from a flat background spacetime, from the outset no limits are contemplated here on the strength of gravity manifest in the lapse function α (and the associated nonlinear potential ϕ, see Equation (5)) or the first and second fundamental forms of the position-space leaves of the foliation (3-metric $\boldsymbol{\gamma}$ and extrinsic curvature $\boldsymbol{K}$). The Einstein equation governing these on $B\mathcal{E}$ and $B\mathcal{G}+$ will be worked out in Paper IIB, at which point further consideration will be given to such issues as the mathematical nature and consistency of this physically motivated approximation and the possibility of global conserved quantities on asymptotically flat spacetimes. Only then will it be possible for numerical tests, examples, and comparisons to follow in due course, but, at this point, no serious problems with respect to implementation are foreseen: the handling of nuclear energy generation via an explicit source term, including in connection with shock-capturing schemes, has been used for many years (notably in the FLASH code [22]); and the more bluntly practical and mathematically uncontrolled enhanced-gravity approach in wide current use in core-collapse supernova simulations [23]—keeping higher-order Newton multipole moments, while replacing the Newton monopole with a general relativistic one—has not introduced numerical pathologies.

In summary, beginning from the usual four-dimensional Einstein spacetime $\mathcal{E}$ of 'Poincaré general relativity' (that is, general relativity as traditionally conceived) and passing across the conceptual bridge of five-dimensional Bargmann spacetime $B\mathcal{E}$, the five-dimensional Bargmann spacetime $B\mathcal{G}$ offers the possibility of strong-field, nonlinear gravitation on a curved spacetime consistent with local Galilei physics—a 'Galilei general relativity'. The extension to $B\mathcal{G}+$ proposed here then adds allowance for Poincaré physics on microscopic scales only (the local properties of matter), thereby allowing for energy generation by nuclear reactions, and for local internal energy and stress (but not macroscopic bulk flows) to contribute to the 'mass density' sourcing the gravitational fields. By capturing certain strong-field aspects of traditional general relativity (nonlinear gravitational potentials, enhanced sourcing) while excluding its more highly dynamical complications (gravitational waves, 'Minkowski' bulk flows), $B\mathcal{G}+$ can be expected to provide a convenient and well-motivated approximation to traditional general relativity useful in astrophysical contexts, such as core-collapse supernovae, in which the geometry/gravitation is strong but mostly quasi-static.

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Appendix A. Comoving Observer Decomposition of $\mathcal{D} \cdot \mathcal{T}$

Just as the decomposition of $\boldsymbol{\nabla} \cdot \boldsymbol{T}$ on $\mathcal{E}$ relative to comoving observers appears in Equation (52), a similar decomposition of $\mathcal{D} \cdot \mathcal{T}$ on $B\mathcal{E}$ or $B\mathcal{G}$ is desired. Focusing here on $B\mathcal{E}$, the result turns out to be similar to that on $\mathcal{E}$, though there is some additional complexity along the way. The columns in the decomposition of $\mathcal{T}$ relative to comoving observers as arranged and displayed in Equation (60) can be expressed as three terms,

$$\mathcal{T} \;=\; -(e\,\mathcal{U} - y\,\boldsymbol{\xi}) \otimes \underline{\chi}_{\mathcal{U}} \;+\; p\,\overleftarrow{\gamma_{\mathcal{U}}} \;+\; m\,\mathcal{N} \otimes \underline{\mathcal{U}},$$

each of which will be addressed in turn. The identities involving the action vector field $\boldsymbol{\xi}$ listed in Equation (22) will be employed frequently and without explicit mention.

The first term in the divergence $\mathcal{D} \cdot \mathcal{T}$ yields two parts:

$$\begin{aligned}(\mathcal{D} \cdot \mathcal{T})_{\textcircled{1}} &= -\mathcal{D} \cdot \Big((e\,\mathcal{U} - y\,\boldsymbol{\xi}) \otimes \underline{\chi}_{\mathcal{U}} \Big) \\ &= -\Big(\mathcal{D} \cdot (e\,\mathcal{U} - y\,\boldsymbol{\xi}) \Big)\, \underline{\chi}_{\mathcal{U}} - \Big((e\,\mathcal{U} - y\,\boldsymbol{\xi}) \cdot \mathcal{D} \Big) \underline{\chi}_{\mathcal{U}}.\end{aligned}$$

The first part of the first term is

$$\begin{aligned}(\mathcal{D} \cdot \mathcal{T})_{\textcircled{1a}} &= -\Big(\mathcal{D} \cdot (e\,\mathcal{U} - y\,\boldsymbol{\xi}) \Big)\, \underline{\chi}_{\mathcal{U}} \\ &= -\Big(\mathcal{D} \cdot (e\,\mathcal{U}) \Big)\, \underline{\chi}_{\mathcal{U}} \\ &= \left(-\frac{\mathrm{d}e}{\mathrm{d}\tau} + \frac{e}{n} \frac{\mathrm{d}n}{\mathrm{d}\tau} \right) \underline{\chi}_{\mathcal{U}}\end{aligned}$$

thanks to Equation (43). The second part of the first term is

$$\begin{aligned}(\mathcal{D} \cdot \mathcal{T})_{\textcircled{1b}} &= -\Big((e\,\mathcal{U} - y\,\boldsymbol{\xi}) \cdot \mathcal{D} \Big) \underline{\chi}_{\mathcal{U}} \\ &= -\Big((e\,\mathcal{U} - y\,\boldsymbol{\xi}) \cdot \mathcal{D} \Big) \left(\underline{\boldsymbol{\xi}} - \frac{1}{c^2}\, \underline{\mathcal{U}} \right) \\ &= \frac{e}{c^2} \frac{\mathrm{d}}{\mathrm{d}\tau} \underline{\mathcal{U}} - \left(e + \frac{y}{c^2} \right) (\mathcal{U} \cdot \mathcal{D}) \underline{\boldsymbol{\xi}},\end{aligned}$$

in which Equation (27) has been used to obtain the second line.

The two parts of the second term in $\mathcal{D} \cdot \mathcal{T}$ are

$$\begin{aligned}(\mathcal{D} \cdot \mathcal{T})_{\textcircled{2}} &= \mathcal{D}\big(p\,\overleftarrow{\gamma_{\mathcal{U}}} \big) \\ &= \overrightarrow{\gamma_{\mathcal{U}}} \cdot \mathcal{D} p \;+\; p\,\mathcal{D} \cdot \overleftarrow{\gamma_{\mathcal{U}}}.\end{aligned}$$

The first part of the second term is simply

$$\begin{aligned}(\mathcal{D}\cdot\mathcal{T})_{\textcircled{2a}} &= \overrightarrow{\gamma u}\cdot\mathcal{D}p\\ &= \boldsymbol{\mathcal{D}}p.\end{aligned}$$

The second part is

$$\begin{aligned}(\mathcal{D}\cdot\mathcal{T})_{\textcircled{2b}} &= p\,\mathcal{D}\cdot\overleftarrow{\gamma u}\\ &= -p\,(\mathcal{D}\cdot\mathcal{U})\underline{\chi}_{\mathcal{U}} - p\,(\mathcal{U}\cdot\mathcal{D})\underline{\chi}_{\mathcal{U}} - p\,(\mathcal{U}\cdot\mathcal{D})\,\underline{\xi}\\ &= \left(\frac{p}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau}\right)\underline{\chi}_{\mathcal{U}} + \frac{p}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\underline{\mathcal{U}} - 2\,p\,(\mathcal{U}\cdot\mathcal{D})\underline{\xi},\end{aligned}$$

after once again using Equations (43) and (27) to obtain the last line.

Thanks to Equation (42) and constant m, the third term in $\mathcal{D}\cdot\mathcal{T}$ is simply

$$\begin{aligned}(\mathcal{D}\cdot\mathcal{T})_{\textcircled{3}} &= \mathcal{D}\cdot(m\,\mathcal{N}\otimes\underline{\mathcal{U}})\\ &= m\,n\,\frac{\mathrm{d}}{\mathrm{d}\tau}\underline{\mathcal{U}} - m\,z\,(\mathcal{U}\cdot\mathcal{D})\underline{\xi}\end{aligned}$$

after substitution of Equation (41).

Put all of these contributions to the comoving observer decomposition of $\mathcal{D}\cdot\mathcal{T}$ together and find

$$\begin{aligned}\mathcal{D}\cdot\mathcal{T} = &\left(-\frac{\mathrm{d}e}{\mathrm{d}\tau} + \frac{e+p}{n}\frac{\mathrm{d}n}{\mathrm{d}\tau}\right)\underline{\chi}_{\mathcal{U}} + \left(m\,n + \frac{e+p}{c^2}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\underline{\mathcal{U}} + \boldsymbol{\mathcal{D}}p\\ &- 2\,(e+p)\,(\mathcal{U}\cdot\mathcal{D})\underline{\xi},\end{aligned} \tag{A1}$$

in which the new scalar fields y and z do not appear separately, but have combined to yield the internal energy density e according to Equation (61). The first three terms may be compared with Equation (52), while the last term will readily combine with the right-hand side of Equation (65).

Appendix B. Motion of Fiducial Observers

In order to close the kinematic dictionary between the two observer families, consider the motion of fiducial observers as reckoned by comoving observers. This inverts the usual perspective in which the motion of comoving observers is reckoned by fiducial observers—a perspective which normally (and appropriately) predominates in discussions of the 1 + 3 or 1 + 3 + 1 formulations of physics on $\mathcal{E}$, or on $B\mathcal{E}$ or $B\mathcal{G}$, respectively, because of the fundamental role played by the foliation of spacetime. Focus in particular on the 1 + 3 + 1 formulation on $B\mathcal{E}$.

On the one hand recall from Equation (25)

$$\mathcal{U} = \Lambda_{\boldsymbol{V}}(\boldsymbol{n}+\boldsymbol{V}) - c^2(\Lambda_{\boldsymbol{V}}-1)\,\boldsymbol{\xi} \tag{A2}$$

for the decomposition of the 5-velocity of comoving observers relative to fiducial observers, in which

$$\underline{\boldsymbol{\chi}}\cdot\boldsymbol{V} = 0 = \underline{\boldsymbol{n}}\cdot\boldsymbol{V},$$

with $\boldsymbol{V}$ being the 3-velocity of comoving observers measured by fiducial observers. But it is equally fair to consider

$$\boldsymbol{n} = \Lambda_{V_{\mathcal{U}}}(\mathcal{U} + \boldsymbol{V}_{\mathcal{U}}) - c^2(\Lambda_{V_{\mathcal{U}}} - 1)\,\boldsymbol{\xi} \tag{A3}$$

for the decomposition of the 5-velocity of fiducial observers relative to comoving observers, in which

$$\underline{\boldsymbol{\chi}}_{\mathcal{U}} \cdot \boldsymbol{V}_{\mathcal{U}} = 0 = \underline{\mathcal{U}} \cdot \boldsymbol{V}_{\mathcal{U}},$$

with $\boldsymbol{V}_{\mathcal{U}}$ being the 3-velocity of fiducial observers measured by comoving observers. (As noted in Section 2.2, the action vector $\boldsymbol{\xi}$ is invariant, and therefore the same in both of these decompositions.)

Of present interest are the relations between the new Lorentz factor $\Lambda_{V_{\mathcal{U}}}$ and 3-velocity $\boldsymbol{V}_{\mathcal{U}}$ measured by comoving observers to the usual $\Lambda_{\boldsymbol{V}}$ and $\boldsymbol{V}$ measured by fiducial observers. Equating $\underline{\boldsymbol{n}} \cdot \mathcal{U}$ to $\underline{\mathcal{U}} \cdot \boldsymbol{n}$ and substituting Equations (A2) and (A3), respectively, yields

$$\Lambda_{V_{\mathcal{U}}} = \Lambda_{\boldsymbol{V}}, \tag{A4}$$

because only the action components survive and must be equal. (In addition to $\boldsymbol{V}$ being orthogonal to $\boldsymbol{n}$, and $\boldsymbol{V}_{\mathcal{U}}$ being orthogonal to $\mathcal{U}$, recall that both $\boldsymbol{n}$ and $\mathcal{U}$ are null with respect to $\boldsymbol{G}$ according to Equation (30).) With this equality of Lorentz factors in hand, Equations (A2) and (A3) can be solved for $\boldsymbol{V}_{\mathcal{U}}$ with the result

$$\boldsymbol{V}_{\mathcal{U}} = -\Lambda_{\boldsymbol{V}}(\boldsymbol{V} - \gamma(\boldsymbol{V},\boldsymbol{V})\,\boldsymbol{\chi}), \tag{A5}$$

in which Equation (19) and $c^2\left(\Lambda_{\boldsymbol{V}}{}^2 - 1\right) = \Lambda_{\boldsymbol{V}}{}^2\,\gamma(\boldsymbol{V},\boldsymbol{V})$ have been employed. The expected relations $\underline{\boldsymbol{\chi}}_{\mathcal{U}} \cdot \boldsymbol{V}_{\mathcal{U}} = 0$ and $\underline{\mathcal{U}} \cdot \boldsymbol{V}_{\mathcal{U}} = 0$ and $\underline{\boldsymbol{V}}_{\mathcal{U}} \cdot \boldsymbol{V}_{\mathcal{U}} = \underline{\boldsymbol{V}} \cdot \boldsymbol{V}$ are readily confirmed with this result for $\boldsymbol{V}_{\mathcal{U}}$. Moreover, just as we have

$$\underline{\boldsymbol{\chi}}_{\mathcal{U}} = \Lambda_{\boldsymbol{V}}\left(\underline{\boldsymbol{\chi}} - \frac{1}{c^2}\,\underline{\boldsymbol{V}}\right)$$

in Equation (26), we expect in a reciprocal manner

$$\underline{\boldsymbol{\chi}} = \Lambda_{V_{\mathcal{U}}}\left(\underline{\boldsymbol{\chi}}_{\mathcal{U}} - \frac{1}{c^2}\,\underline{\boldsymbol{V}}_{\mathcal{U}}\right), \tag{A6}$$

which is indeed verified using the equality of Lorentz factors in Equations (A4) and (A5) for $\boldsymbol{V}_{\mathcal{U}}$.

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