fluids

MDPI

Article

Line Thermals from Finite Sources in Neutral and Stable Environments, with Application to Thermal Plumes from Submarines

Jason H. Middleton

School of Aviation, The University of New South Wales, Sydney, NSW 2052, Australia; j.middleton@unsw.edu.au

Abstract

The first application of theory of the rise of line thermals was to understand the rise of turbulent smoke plumes emitted from smoke stacks into a cross-wind. Initial solutions required numerical calculations. In this article analytical solutions are found, and these are used here to explore solutions for the rise of buoyant line wakes from submarine vehicles. Solutions cater for wakes in both neutral and stable environments, and for sources which have either negative or positive initial buoyancy. Account is also taken of sources with differing size and initial momentum. Practical examples of submarine thermal wake flows are given using neutral and typical stably stratified upper ocean conditions and a range of source conditions. A key result is that small-diameter submarine wakes with high temperatures produced in weakly stratified ocean waters will have a large height of rise, and may easily reach the surface. By contrast, large-source-diameter wakes, with temperatures close to ambient and emitted into strongly stratified oceans, will have very small heights of rise.

Keywords: plumes; submarines; wakes

1. Introduction

A line thermal is essentially a cylindrical segment of fluid oriented horizontally, or almost so, and with a density different to that of the local environment. By virtue of the density difference, the thermal will be driven to rise or fall in the local environment. The interior of such a flow is almost always turbulent, and the turbulence acts to entrain ambient local fluid, increasing the total mass. In a neutral environment with constant density, the total buoyancy is conserved, but as the thermal rises in a stratified environment the total buoyancy changes. Buoyant plumes, jets and thermals were considered in [1], using dimensional analysis methods for the scaling. Analytical solutions were found for asymptotic flows from point sources in stably and neutrally stratified environments. These concepts and theories are used here for the analogous flow of the line plume.

The earliest experiments on line thermals were conducted by [2]. Early theories of line plumes were developed by [3], and by [4]. The analytical theory for bent-over plumes in a stable environment was further developed by [5] using the entrainment hypothesis originally proposed by [1] whereby entrainment is proportional to the product of the surface area of the buoyant element, and to its mean vertical velocity.

Using a transformation of independent variable from height to time, analytical solutions are found here for the line thermal flows which are of finite source size and have positive initial momentum, as described in [5,6]. Basic assumptions include the Boussinesq

Check for updates

Academic Editor: Ziad Saghir

Received: 30 December 2025
Revised: 13 February 2026
Accepted: 22 February 2026
Published: 26 February 2026

Copyright: © 2026 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

approximation (that density effects are negligible and only appear in buoyancy terms), that the flow remains of self-similar cylindrical shape, and that entrainment of ambient fluid is proportional to both mean vertical velocity and surface area of the cylinder.

For plumes, there has long been a recognition that the vertical momentum at the source (so-called forced plumes) may play a role in the flow development, as shown by [7,8]. Such effects are included in the theory presented here. Flows in an unstable local environment were explored briefly by [4], and while these may be used here in an analogous way to that developed by [9], they are not further developed here. The present solutions have not been previously published, so provide a unique opportunity for understanding the basic physical processes.

An important application of the line plume solutions described here is to model the wake emitted by a submarine vehicle which is moving in a submerged state. The thermal wake is a direct result of the power generation required in a crewed vehicle. A number of recent studies of wakes caused by submarine vehicles are listed in the comprehensive review [10], confirming that at times wakes may surface, creating a thermal signature which would be visible in satellite or airborne thermal infra-red imagery. A number of references therein focus on the creation of surface and internal wave wakes, but none appear to focus on the rates of dilution and rates and extent of rise of buoyant wakes in either mixed or stratified waters. The present study, where the theory is applied to submarine vehicle wakes, appears to be unique in this aspect.

In terms of submarine wakes, there is no attempt here to model actual wakes, as submarine wake design is highly confidential. However, these solutions, which are applicable for idealised wakes, may be used to gain an appreciation of core flow outcomes, such as heights of rise from the source, and dilutions. Highly buoyant sources in oceans with weak stratification have large heights of rise, potentially leading to line wake thermals reaching the surface. By contrast, larger sources with small density differences have small heights of rise, and would be constrained in the stratified deeper ocean below the surface layers.

2. The Line Thermal in a Neutral or Stably Stratified Environment

Line thermals resulting from chimney plumes in a cross-flow can either be a diffuse arrangement of circulation approximately cylindrical in shape or can consist of two bifurcated lines of concentrated vorticity [11], and here only line thermals of approximately circular cross-section are considered.

A schematic diagram of a segment of a line thermal is shown in Figure 1, where R is the radius, W is the mean vertical velocity, Z is the height, and the thermal density is ρ.

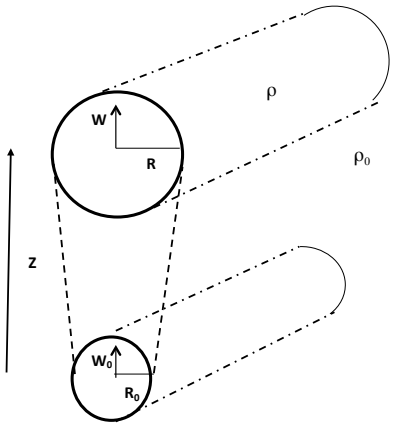


Figure 1. Showing the rise of line thermal with radius R, height Z and mean vertical velocity W, with the zero subscript indicating source values. Thermal and local environmental densities are ρ and ρ_0.

2.1. The Equations

The bulk conservation equations are described in [5] and represent quantities per unit length of a buoyant and turbulent line thermal which is emitted from a source with average horizontal velocity no different from the receiving ambient fluid. The development of the analysis closely follows the methodology used by [9], with this flow being essentially a two-dimensional analogy of the flow described by [9]. The environmental stratification N and the buoyancy force per unit mass B strongly impact the flow, and the basic equations which express changes in time of volume, momentum and buoyancy in a cylindrical segment of unit length of buoyant flow are given by (1), (2) and (3) below.

$$\frac{d\pi R^2}{dt} = 2\pi\alpha RW \tag{1}$$

$$\frac{d\pi R^2 W}{dt} = \pi R^2 B \tag{2}$$

$$\frac{d\pi R^2 B}{dt} = -\pi R^2 W N^2. \tag{3}$$

Increase in volume occurs through turbulent entrainment into the thermal, which is assumed proportional to the surface area, and to the average thermal vertical velocity W, with the proportionality defined by the entrainment constant α. Increase in momentum is driven by the buoyancy force per unit mass, which is

$$B = \frac{g(\rho_0 - \rho)}{\rho_1}. \tag{4}$$

Here ρ_1 is the reference density, ρ_0 is the environmental density at thermal height, ρ is the thermal density and g is the gravity constant.

Buoyancy changes are due to the entrainment of fluid with local density, which itself may have a vertical gradient, represented here by the buoyancy frequency N defined by

$$N^2 = -\frac{g}{\rho_1}\frac{d\rho_0}{dz}. \tag{5}$$

A uniform environment has $N = 0$ and a stable environment $N^2 > 0$. Note that the only adjustable parameter in these equations is the entrainment constant α.

Simplifying the equations is readily done by defining bulk quantities per unit length of momentum M and buoyancy F as

$$M = R^2 W \tag{6}$$

$$F = R^2 B \tag{7}$$

Equations (1)–(4) now appear as

$$\frac{dR^2}{dt} = \frac{2\alpha M}{R} \tag{8}$$

$$\frac{dM}{dt} = F \tag{9}$$

$$\frac{dF}{dt} = -N^2 M \tag{10}$$

Solutions to (9) and (10) are sinusoidal in structure (and of analogous format to the equations for spherical thermals found by [9]), and with source values indicated by subscript zero; these solutions are

$$M = M_0 \csc\delta \sin\theta \tag{11}$$

$$F = F_0 \sec\delta \cos\theta \tag{12}$$

The angle $\theta = Nt + \delta$ is a variable which increases with time, and the constant δ is a phase angle representing a balance of source values through

$$\tan\delta = NM_0/F_0. \tag{13}$$

Angle δ is a measure of the relative influences of initial momentum. In analogy to the solutions of [9], the time $t = -N^{-1}\delta^{-1}$ is a virtual origin for a virtual plume where the initial buoyancy is $F = F_0 \sec\delta$. The momentum equation can now be written in terms of the source buoyancy as

$$M = F_0 N^{-1} \sec\delta \sin\theta. \tag{14}$$

The line thermal grows linearly with Z as defined by $R = R_0 + \alpha(Z - Z_0)$, as may be shown from (1), which can be derived from $dR/dt = \alpha W$, and the definition $W = dZ/dt$. Integrating (8) gives the radius cubed R^3 as

$$R^3 = R_0^3 + 3\alpha \int_0^t M dt \tag{15}$$

Using (10) gives an analytical expression for radius R as a function of buoyancy F:

$$R^3 = R_0^3 + 3\alpha N^{-2}(F_0 - F). \tag{16}$$

Now define the dimensionless buoyancy f as

$$f = \frac{F}{F_0} = \sec\delta \cos\theta \tag{17}$$

and the dimensionless constant Γ as

$$\Gamma = \frac{R_0^3 N^2}{3\alpha F_0}. \tag{18}$$

The parameter Γ is the source radius (to the third power) made dimensionless using stratification N and source buoyancy F_0. $\Gamma = 0$ represents an infinitely thin line source. The thermal radius is now given as a function of f, and is

$$R^3 = 3\alpha F_0 N^{-2}(\Gamma + 1 - f). \tag{19}$$

while the height Z is found from $Z - Z_0 = \alpha^{-1}(R - R_0)$, and is

$$Z = 3^{1/3}(F_0 N^{-2}\alpha^{-2})^{1/3}((\Gamma + 1 - f)^{1/3} - \Gamma^{1/3}). \tag{20}$$

Equation (6) is used to find the vertical velocity $W = MR^{-2}$.

The analytical solutions obtained here may be used to predict flows rising from both positively buoyant ($F_0 > 0$) and negatively buoyant ($F_0 < 0$) sources. The background environment may be stable ($N > 0$) or neutral ($N = 0$), and the initial momentum (assumed positive) may be non-zero with $M_0 > 0$, a forced line thermal.

In stratified cases, flows with $F_0 > 0$ which are emitted upwards will have source balance angle δ in the range $0 < \delta < \pi/2$, and $\theta = Nt + \delta$ grows with time until $\theta = \pi/2$ when the buoyancy drops to zero (neutral buoyancy), although the plume is still rising then. Finally, when $\theta = \pi$, the vertical momentum dies out, but the buoyancy is now negative so the flow will then descend, continuing to entrain. The theory does not cater for this further descent.

For flows from small sources where $\Gamma << 1$ and $\delta << 1$, asymptotic values of height for neutral buoyancy Z_n and maximum height Z_m are ${Z_n}^3 = 3F_0N^{-2}\alpha^{-2}$ and ${Z_m}^3 = 6F_0N^{-2}\alpha^{-2}$.

For flows with initial negative buoyancy, $F_0 < 0$, and the source balance angle lies within $\pi/2 < \delta < \pi$. The thermal momentum (assumed initially positive) is continually reduced by the negative buoyancy until the maximum height is reached, when $\theta = \pi$ and the vertical momentum $M = 0$.

For thermals emitted with no initial vertical velocity and with finite F_0 but from a small origin, consideration of (20) with negligible Γ gives

$$Z^3 = 3F_0N^{-2}\alpha^{-2}(1 - f). \tag{21}$$

For small times, $1 - f \approx \theta^2/2 = N^2t^2/2$, and the radius is given by $R^3 = 3/2\alpha F_0t^2$, and $Z^3 = 3/2\alpha^{-2}F_0t^2$.

For unstratified flows $N = 0$, the total buoyancy becomes constant $F = F_0$, and solutions to the basic equations are

$$M = M_0 + F_0t \tag{22}$$

$$R^3 = R_0^3 + 3\alpha M_0t + 3/2\alpha F_0t^2 \tag{23}$$

$$Z^3 = \alpha^{-3}R^3 \tag{24}$$

2.2. Calculating the Solutions

In a stratified environment, dimensionless solutions can be found using dimensional values of F_0 and N. To cater for solutions with negative buoyancy the sign variable sgn is used, noting $sgn = -1$ for $F_0 < 0$ and $sgn = +1$ for $F_0 > 0$. Thus $F_0 = sgn(|F_0|)$.

The scaled dimensionless solutions for momentum and buoyancy are

$$MN/|F_0| = m = sgn \sec\delta \sin\theta \tag{25}$$

$$F/|F_0| = f = sgn \sec\delta \cos\theta. \tag{26}$$

The radius is

$$R(N^2/(\alpha|F_0|)^{1/3} = r = 3^{1/3}(\Gamma + sgn - f)^{1/3} \tag{27}$$

and the height of the thermal above the source is given by

$$Z(N^2\alpha^2/(|F_0|)^{1/3} = z = 3^{1/3}((\Gamma + sgn - f)^{1/3}) - \Gamma^{1/3}). \tag{28}$$

The vertical velocity is found from the relationship $w = mr^{-2}$ and has scaling $(|F_0|N/9\alpha^2)^{1/3}$.

The rate of dilution formula for thermals in stratified fluids has exactly the same form as for spherical thermals [9], and is $D = ((\Gamma + 1 - f)/\Gamma)^{2/3}$, with values of $D = (1 + 1/\Gamma)^{2/3}$ at $f = 0$ and $D = (1 + 2/\Gamma)^{2/3}$ at $f = -1$ (top of rise). Dilution at top of rise is larger from smaller sources, which are characterised by smaller Γ.

2.3. Describing the Solutions

Utilising F_0 and N as dimensional scaling variables allows more explicit solutions to be found for source radius (through values of Γ) and source momentum (through values of δ).

Before proceeding with the discussion, it is important to understand the structure of the equations. The source buoyancy flux F_0 and the environmental stratification N are used to create the dimensional scaling of the equations. The numerical results therefore cannot explicitly depict the way in which the flows might change as a consequence of changes in either of those parameters. Instead, the dependence on the source characteristics (volume and momentum) can be readily determined with this approach.

Figure 2 depicts the dimensionless solutions as a function of time. The top left panel is for a positively buoyant source in a stable environment. Buoyancy f drops from its initial maximum positive value to $f = 0$ as the plume reaches the height of neutral buoyancy (when $\theta = \pi/2$) at which time momentum m is at a maximum. Further rise to the maximum height occurs when $\theta = \pi$ and $f = -1$.

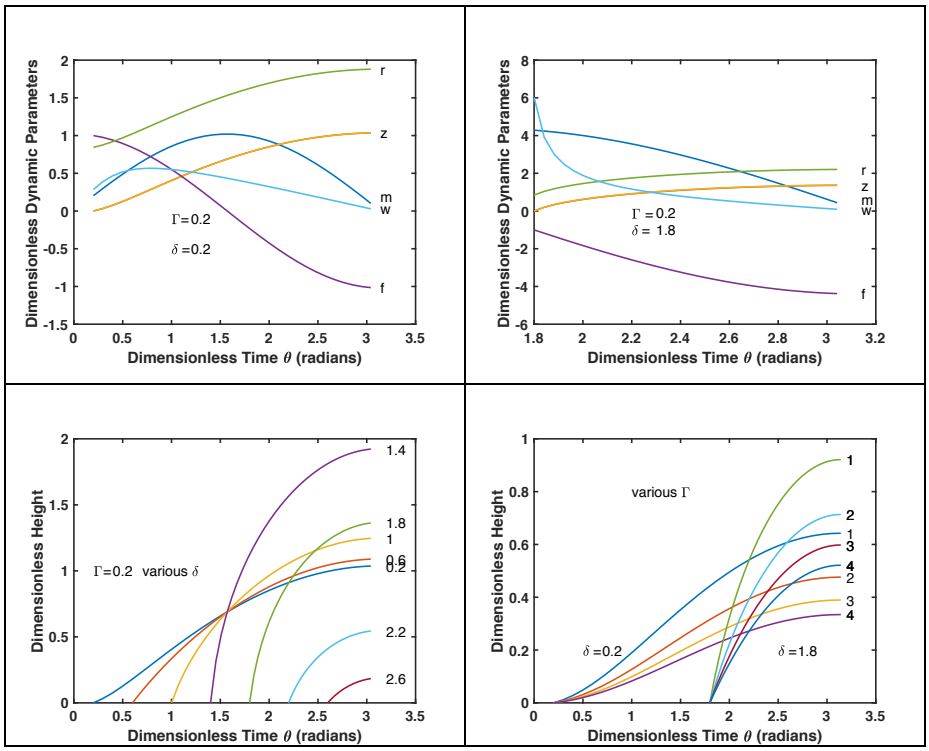


Figure 2. Showing the dimensionless values of momentum m, buoyancy f, radius r, vertical velocity w and height of rise z for selected values of parameters Γ and δ for both positively and negatively buoyant line thermals in a stably stratified environment.

The vertical velocity w initially accelerates under buoyancy forces, then gradually reduces to zero at the top of rise. Height and radius have the same shape as expected from their linear relationship.

The top right panel shows parameter values for negative buoyancy, where the negative buoyancy grows in magnitude from its initial negative value to the value of $f = -\sec\delta$, and momentum and vertical velocity (both initially positive) decay to zero.

The bottom left panel shows how the flow parameters change with differing values of δ (initial momentum) for $\Gamma = 0.2$. For $\delta < \pi/2$ the source buoyancy is positive and higher values of delta give higher heights of rise due the greater source momentum. But for

$\delta > \pi/2$ the source buoyancy is negative and greater values of δ actually reduce the vertical velocity and thus the height of rise, as is evident from (13).

The bottom right panel shows the dependency on Γ, showing clearly that larger values result in lower heights of rise. This is due to the larger surface area for entrainment resulting in larger entrainment with higher volume growth.

Similar results hold for spherical thermals [9], but not for forced plumes [7], where higher values of source momentum result in the plume (actually initially a jet if the source momentum is large) initially spreading at a more rapid rate.

Consideration of (12) and (14) shows that the quantity

$$Q^2 = M^2 + N^{-2}F^2 = N^{-2}F_0^2 \sec^2 \delta \tag{29}$$

is conserved at all times for line thermals in stratified environments. This is also true for spherical thermals [9]. Q is effectively a conservation of vertical momentum, being a product of terms of volume (mass) per unit length and vertical velocity.

Solutions for unstable environments may also be developed in an analogous way to that implemented by [9], but these are not presented here.

2.4. Some Practical Examples of Submarine Wake Calculations

The upper ocean is usually composed of a surface mixed layer of uniform density overlying colder stratified waters below. This surface layer can be quite deep (to 200 m or more) when there are very strong winds and large surface waves which have a strong mechanical mixing function. However the surface mixed layer may be almost non-existent in waters which have had a recent history of high solar heating and low winds, such as occurs in equatorial oceans. Below the surface mixed layer are layers where density increases with depth, usually due to lower temperatures. These have varying density gradients, with higher values being typically $N = 10^{-2}$ s^{-1}, and lower values perhaps $N = 10^{-3}$ s^{-1}.

It is assumed in these examples that the thermal wake from a submarine (the source) will be of essentially cylindrical shape, approximately horizontal, and have minimal initial vertical velocity. The cylindrical thermal wake is produced by the mixing of the engine cooling effluent with the physical wake of the submarine hull and propeller. It is also assumed that the wake water will be essentially the same seawater as that of the surrounding environment at that same depth but have higher temperatures. For practical calculations it is necessary to know how the difference in temperature affects the ocean density. The full equation of state of seawater is complicated and depends on temperature, salinity and pressure. The practical formula for quick calculations of source buoyancy in seawater is $F_0 = gR_0^2\beta\Delta T$, where $\beta = \rho^{-1}d\rho/dT$ is the thermal expansion coefficient (approximately 2.10^{-4} s^{-1} at temperature 15 °C and salinity 35 psu [12]).

It is also essential to specify the entrainment coefficient value and here we choose $\alpha = 0.35$ ([4]). We also choose the value for gravity as $g = 9.81$ m s^{-2}.

There is little unclassified open-source data available about submarine plumes, so the following discussion is aimed at building a general understanding.

For the case where the ocean is unstratified (the surface mixed layer), the plume rise continues until the ocean surface is reached, or until the line plume is torn apart by the mixed layer turbulence. As an example, consider a plume of source radius 3 m, and a temperature excess of $\Delta T = 1$ °C with zero initial momentum. Then $F_0 = 0.0177$ m^{3}s^{-2}, and Equations (23) and (24) may be used to find the rise as a function of time. Over 10 min, the plume will rise to approximately 34 m, and have a temperature excess of 0.04 °C. If the source radius is 3 m and the excess source temperature is 2 °C, then the plume will rise to $Z_n = 45$ m in 10 min, and will then have a residual temperature excess of 0.05 °C.

For stratified flows, examples are as follows, and to proceed, the parameter Γ must be determined. Assume the ocean stratification is $N = 10^{-2}$ s^{-1}, the temperature difference of the wake is $\Delta T = 1$ °C and the plume source radius is $R_0 = 3$ m. The source buoyancy $F_0 = gR_0^2\beta\Delta T = 0.0177$ m^{3}s^{-2} and $\Gamma = 0.15$. From the perspective of the source physics balance the source is small, and the asymptotic value of the height of rise to neutral buoyancy is $Z_n^3 = 3F_0N^{-2}\alpha^{-2}$ resulting in $Z_n = 16$ m. Note that the maximum height is double that, but the plume at its maximum height will be negatively buoyant and sink again.

As a second example consider the case of a source with the same temperature difference emitted into a weakly stratified ocean. Assume the ocean stratification is $N = 10^{-3}$ s^{-1}, the temperature difference is $\Delta T = 1$ °C and the plume source radius is $R_0 = 3$ m. The source buoyancy $F_0 = gR_0^2\beta\Delta T = 0.0177$ m^{3}s^{-2} and $\Gamma = 0.001$, and the height of rise to neutral buoyancy is $Z_n = 75$ m.

To enhance interpretation it is perhaps useful to see the joint effects of source buoyancy F_0 and ocean stratification N. For presentation purposes here we will use the stability period $2\pi N^{-1}$ instead of N. Figure 3 shows the contours of maximum height of rise with the horizontal axis being that for F_0 and the vertical axis the buoyancy stability period in minutes.

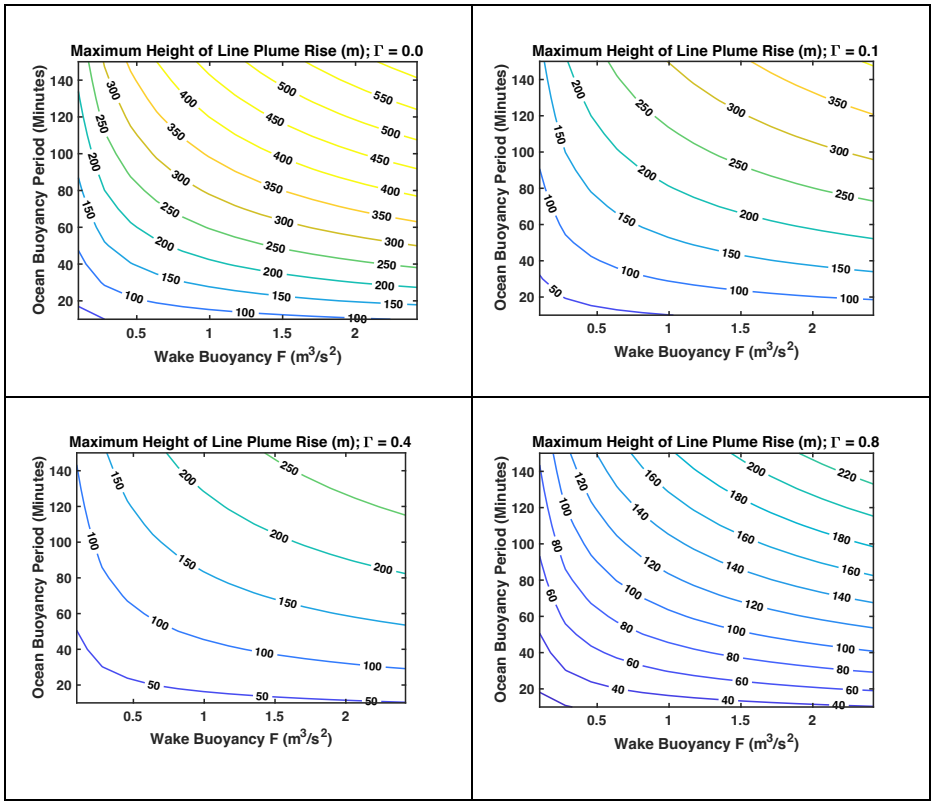


Figure 3. Showing a contoured diagram of the height of rise (m) as a function of ocean stability period (minutes) and source buoyancy F_0 for $\Gamma = 0,\ 0.1,\ 0.4$ and 0.8.

It is clear that strong stratification (short buoyancy periods), in conjunction with weakly buoyant sources (low F_0 values), will heavily restrict wake rise. On the other hand the combination of a weakly stratified ocean and strong positive source buoyancy will drive wakes to rise considerably over several hundred metres. Should a wake be generated in the surface mixed layer, or break into it from below, then it may well reach the ocean surface if the mixed layer turbulence is low or the mixed layer shallow.

Conclusions are clear. Thermal wakes from submarines operating submerged in shallow continental shelf waters are likely to be observable at the sea surface to airborne or satellite high-resolution infra-red thermal imagers. One effective mitigation, as used in offshore submerged ocean wastewater plumes, is to enhance design-effluent wakes into propeller wakes to enhance the mixing of wake effluent with ocean waters as much as possible initially so as to dilute the plume with deeper, usually colder waters, which will minimise temperature differences and buoyant rise.

3. General Discussion

Turbulence and entrainment in buoyant flows are complex processes and the literature is vast. There is no attempt here at trying to refine the solutions to account for complex effects of entrainment, or for line thermals of non-cylindrical or changing shape. Also the effects of rotation due to the thermal wake being possibly emitted into the propeller wake of a submarine have not been addressed.

The parameter Γ characterises the diameter of the line source and has a strong impact on the height of rise with significant decreases as the source size becomes larger. The dimensionless ratio NM_0/F_0 is the dimensionless momentum and so is a direct measure of the source strength, and is characterised here by the parameter $\tan\delta = NM_0/F_0$. Higher values of δ are associated with larger initial momentum M_0, and result in significantly greater heights of rise.

The total specific momentum Q defined in Equation (29) is determined at the source and retains its value as the line thermal rises, and is a conserved quantity defined first in the literature on spherical thermal rise in stratified flows by [9]. The internal contributions to Q reflect the change in balance between buoyancy and upward momentum. Upward momentum increases when the thermal has positive buoyancy, but decreases when the buoyancy is negative. Conceptually, buoyant sources with zero momentum $\delta = 0$ may be considered as virtual sources for more realistic thermal sources having some initial momentum.

It is appropriate to review the role of the entrainment parameter α, which in this theoretical model is the only adjustable parameter. The value of α for line jets and plumes is now described with some certainty [13]. However, the entrainment parameter value for line thermals (from a single source as modeled here) for both neutrally and stably stratified surrounding fluids has not yet been determined with the same level of definition by experiments. The earliest work of Csanady [4] where the observed value was $\alpha = 0.35$ was followed up by the more complex calculations of [14], where their theory comprises a number of length scales and empirically defined parameters, and has become the standard for engineering applications. However, reference to their Figure 9.27 (p374) shows the ratio of b/z in the far field to be approximately $8/44 = 0.41$, only slightly larger than the earlier value. For this reason, the earlier observed value was chosen for the quantitative examples.

For the plumes modeled here, possible horizontal shear at the boundary between receiving fluid and the plume may play a role in enhancing turbulence (and entrainment), while in stratified flows, the energy required to mix fluids of differing density at the bottom and top of the plume might reduce entrainment in those areas. So while the earlier observations are relevant in a broad sense, observations of plumes with characteristics typical of submarine vehicle wakes would appear to be the best way of determining α for this specific application. The present analytical solution with the single adjustable parameter α is readily applicable and has the benefit to the reader of simple calculations, as opposed to the more complex assessments of length scales and adjustable parameters as used in the coastal engineering literature. The mathematical structure of the buoyancy and momentum equations for spherical and line thermals is analogous, being of sinusoidal form, although the scaling is different for each due to the different geometry.

In summary, assumptions made by [9] for spherical thermals have been used here for line thermals emitted from finite-size sources having initial vertical momentum. The analytical solutions for momentum and buoyancy have a sinusoidal structure in time for flows in stable environments, and the flows are straight-sided. There are many criticisms that may be directed at this theory, which is at best approximate. Yet the principal value of this paper is perhaps that the analytical expressions allow for a clear understanding of the key effects of buoyancy. In addition, this appears to be the only publication on this topic.

Small-diameter submarine wakes with high temperatures emitted in weakly stratified ocean waters will have a large height of rise, and may easily reach the surface. By contrast, large-source-diameter wakes, with temperatures close to ambient emitted into strongly stratified oceans, will have very small heights of rise. Submarines submerged but operating in shallow continental shelf waters where the ocean is well mixed will almost certainly create a thermal wake which reaches the surface and is visible in infra-red thermal imagery.

Background turbulence in the environment is assumed here to be negligible, and ref. [4] notes its importance in ultimately destroying the structure of the buoyant element. This is likely to be important in the oceanic mixed layer where background turbulence is not damped by stratification, and is instead enhanced by a number of processes including wind stress, wind–wave breaking and Langmuir circulation. Another complexity arises, which is known in the literature on sewage plume rise in the ocean. Should an element of thermal or plume reach the surface under momentum alone, but with negative relative buoyancy at that level, then it may subsequently descend below the surface again, and be only visible momentarily. This occurs occasionally with sewer outfalls which emit from the seabed in coastal waters.

The theory is easily applicable for a single layer, or for several layers. For oceans with a stratified layer (or layers) below an unstratified surface mixed layer, the calculations can be undertaken within each layer. For example estimates would be made of the rise of the line thermal that is emitted from the source within the first stratified layer to the top of that layer, then those values provided as source input values for line thermal rise in the next layer above, whether that be stratified or not.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Morton, B.R.; Taylor, G.I.; Turner, J.S.T. Turbulent gravitational convection from maintained and instantaneous sources. *Proc. Roy. Soc.* **1956**, *A234*, 1–23. [CrossRef]
2. Richards, J.M. Experiments on the motions of isolated cylindrical thermals through unstratified surroundings. *Int. J. Air Water Poll.* **1963**, *7*, 17–34.
3. Slawson, P.R.; Csanady, G.T. On the mean path of buoyant bent-over chimney plumes. *J. Fluid Mech.* **1967**, *28*, 311–322. [CrossRef]
4. Csanady, G.T. *Turbulent Diffusion in the Environment*; D. Riedel: Dordrecht, The Netherlands, 1980; 249p.
5. Middleton, J.H. The rise of forced plumes in a stably stratified crossflow. *Bound. Layer Met.* **1986**, *36*, 187–198. [CrossRef]
6. Middleton, J.H. Times of rise for turbulent forced plumes. *Tellus* **1979**, *31*, 82–88. [CrossRef]
7. Morton, B.R. Forced Plumes. *J. Fluid Mech.* **1959**, *5*, 151–163. [CrossRef]
8. Morton, B.R.; Middleton, J.H. Scale diagrams for forced plumes. *J. Fluid Mech.* **1973**, *58*, 165–176. [CrossRef]
9. Middleton, J.H.; Mumford, P.J. Thermals from finite sources in stable and unstable environments. *J. Fluid Mech.* **2023**, *960*, A41. [CrossRef]

10. Huang, F.; Meng, Q.; Cao, L.; Wan, D. Wakes and free surface signatures of a generic submarine in the homogeneous and linearly stratified fluid. *Ocean Eng.* **2022**, *250*, 111062. [CrossRef]
11. Turner, J.S.T. *Turbulent Diffusion in the Environment*; Cambridge University Press: Cambridge, UK, 1973; 367p.
12. Gill, A.E. *Atmosphere-Ocean Dynamics*; Academic Press: Cambridge, MA, USA, 1982; 662p.
13. Richardson, J.; Hunt G.R. What is the entrainment coefficient of a pure turbulent line plume? *J. Fluid Mech.* **2022**, *934*, A11. [CrossRef]
14. Fischer, H.B.; List, E.J.; Koh, R.C.Y.; Imberger, J.; Brooks, N.H. *Mixing in Inland and Coastal Waters*; Academic Press: Cambridge, MA, USA, 1979; 483p.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.