

Review

Renormalization Group and Effective Field Theories in Magnetohydrodynamics

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Abstract

We briefly review the recent developments in magnetohydrodynamics, which in particular deal with the evolution of magnetic fields in turbulent plasmas. We especially emphasize (i) the necessity and utility of renormalizing equations of motion in turbulence where velocity and magnetic fields become Hölder singular; (ii) the breakdown of Laplacian determinism of classical physics (spontaneous stochasticity or super chaos) in turbulence; and (iii) the possibility of eliminating the notion of magnetic field lines in magnetized plasmas, using instead magnetic path lines as trajectories of Alfvénic wave packets. These methodologies are then exemplified with their application to the problem of magnetic reconnection—rapid change in magnetic field pattern that accelerates plasma—a ubiquitous phenomenon in astrophysics and laboratory plasmas. Renormalizing rough velocity and magnetic fields on any finite scale l in turbulence inertial range, to remove singularities, implies that magnetohydrodynamic equations should be regarded as effective field theories with running parameters depending upon the scale l . A high wave-number cut-off should also be introduced in fluctuating equations of motion, e.g., Navier–Stokes, which makes them effective, low-wave-number field theories rather than stochastic differential equations.

Keywords: magnetohydrodynamics; turbulence; magnetic reconnection; renormalization group; effective field theories



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1. Introduction

Experiments are the ultimate guide in any physical theory. In assigning an electric charge e to the electron, for instance, we must consider the “measurability” of this charge in the laboratory. It turns out that what we measure for the fundamental charge e indeed depends on the scale of measurement. Energy fluctuations in vacuum, due to the uncertainty principle, may be thought of as creation and annihilation of virtual particles because of mass–energy equivalence. Thus the electron is thought as surrounded by an infinite number of virtual particles, e.g., electrons and positrons, which “screen” the electron’s charge making it a scale-dependent quantity—we measure different values for the electron’s charge as we change the *measurement scale* and get closer to the electron. The measured electric charge is the *renormalized* charge whereas what we initially assign theoretically as electric charge in equations of motion is the *bare* charge. This phenomenon is closely related to the theory of the (Wilsonian) Renormalization Group (RG), which is widely used in quantum field theories and also classical theories. As an example of the latter, in classical theories of liquid diffusion [1], the bare diffusivity can even vanish while the renormalized diffusivity [2] is what satisfies the Stokes–Einstein relation and can be measured experimentally [3]. As another example, the velocity $\mathbf{v}(\mathbf{x}, t)$ and magnetic field $\mathbf{B}(\mathbf{x}, t)$ in plasma

are only mathematical idealizations taken as vector fields at every dimensionless point $\mathbf{x}$ in space, while the physically measurable quantity is, in fact, the average field obtained by coarse-graining the bare field over a finite volume; see Section 2.

What can be measured and the error made in any measurement play even more crucial roles in chaotic and super-chaotic (turbulent) systems where the dynamics are non-linear and extremely sensitive to initial conditions; see Section 3. In fact, in turbulent flows, even the Laplacian determinism of classical physics can be lost at scales much larger than any scale on which any quantum effect might have an effect, see, e.g., [4–6]. The application of these ideas and the crucial difference between chaos and super-chaos in magnetohydrodynamics will be briefly discussed in Section 3.

Another important point is about the utility of the notion of magnetic field lines in magnetized plasmas. Astrophysical flows usually have large Reynolds numbers, which indicates the presence of turbulence [7] where the magnetic field undergoes super-diffusion [8,9] and constantly reconnects everywhere on all scales [5,9–11]. As a main feature of magnetohydrodynamic (MHD) turbulence, this is inconsistent with the idea of magnetic field lines considered as persistent dynamical elements evolving smoothly with time [9,12]. Trajectories of Alfvénic wave packets, or magnetic path lines, provide an alternative, and in some respects a simpler, tool to study the evolution and reconnection of magnetic fields. Magnetic topology change and its possible connection with reconnection can also be approached using path lines in a much simpler way [13], although we will not consider this problem in the present work. Finally, the violation of magnetic flux freezing in the presence of turbulence [4,8,14,15], intimately related to super-chaos and singularity of the magnetic field, will be briefly explored in Sections 3 and 4.

We will exemplify the importance and utility of these concepts in some detail by applying them to the problem of magnetic reconnection, Section 4, in a formalism independent of the notion of field lines, formulated recently [13]. In the problem of magnetic reconnection, one deals with a rapid change in the magnetic field pattern. As previously discussed, the measured field at a point $\mathbf{x}$ is an average field $\mathbf{B}_l$ over a ball of spatial scale $l > 0$ around $\mathbf{x}$, not the “bare” field $\mathbf{B}$. Hence, equations of motion, in this case the induction equation, should be understood as low-wavenumber effective equations on any finite measurement scale $l > 0$. This simple fact becomes crucial in turbulence, where upon averaging, these equations acquire non-linear terms which dominate over other processes in the dynamics. For instance, the non-linear turbulent term in the induction equation *dominates* over all small-scale plasma effects [12,16], rendering non-turbulent reconnection models totally unrealistic in astrophysics [9,11]. In turbulence, both magnetic and velocity fields become Hölder singular with ill-defined spatial derivatives [12,17,18]. Hölder singularity of the velocity (magnetic) field is intimately related to the Richardson diffusion [19] of Lagrangian particles (Alfvénic wave packets), which implies a super-chaotic behavior known as spontaneous stochasticity [20] in which solutions of equations of motion become non-unique [4,6] and initial conditions are completely forgotten! This is totally different from simple *deterministic* chaos (i.e., the butterfly effect) in which solutions diverge exponentially but initial conditions are never forgotten. A renormalization procedure can be applied to such turbulent fields [6,12,14,16,21] to obtain well-defined equations of motion with unique solutions at large (inertial) scales; see also [22] and references therein. This process involves averaging over (or integrating out) the small degrees of freedom below a given scale l . In such a description for systems with strong fluctuations at all length scales, encountered in quantum field theories and turbulent flows, the effective description of the system varies with the resolution length scale l . Instead of seeking some idealized theory valid at vanishingly small length scales, in such an “effective theory”, one attempt

to understand how the description changes as l is varied [18]. This is in fact the base of our modern understanding of quantum field theories.

Despite the fundamental importance of above-mentioned concepts, they seem to remain unappreciated especially in the plasma physics community. For example, both astrophysical observations as well as numerical simulations are in excellent agreement with turbulent reconnection theory, originally proposed by Lazarian and Vishniac [10] and later refined mathematically in a series of works by Eyink and others; e.g., [16,23] based on the methodologies discussed above; see [15,24–28] for non-relativistic regime and, e.g., [29,30] for relativistic regime. Nevertheless, the plasma physics community has essentially neglected all these crucial developments and their implications for decades! Another overlooked phenomenon is the breakdown of magnetic flux freezing [31] in real astrophysical flows due to turbulence as mentioned above [9,11,23,32]. One reason for these important developments to be overlooked might be the unfamiliarity of the community with concepts such as effective field theories and RG analysis, which are nevertheless well-established concepts in other areas such as particle physics, condensed matter physics and statistical physics. Thus to make this short review accessible to a larger audience, we will keep the technical level as simple as possible, focusing on qualitative rather than quantitative aspects, and referring the interested reader to the original literature for more detailed discussions.

In the present review, we will first briefly discuss the concept of finite measurement scales and RG analysis in Section 2. The breakdown of Laplacian determinism in turbulence; the difference between chaos and super-chaos and Richardson diffusion will be discussed in Section 3, where the notion of Alfvénic wave packets as Lagrangian particles is also discussed. Finally, to exemplify the introduced concepts, we briefly revisit the problem of magnetic reconnection in Section 4 using a formalism based on Alfvénic wave packets instead of field lines. This formalism also confirms that astrophysical reconnection is independent of small-scale plasma effects while turbulence plays the main role.

2. Renormalization

Any physical measurement can be performed only on a finite scale $l > 0$ in space, which can be made smaller but it cannot vanish. Although with changing the scale, the value of physical quantities change, but they are all governed by the same physics, i.e., same equations of motion: decreasing the scale l will make measurements “finer”. Mathematically, to obtain an average, coarse-grained, or renormalized version of any field $\mathbf{F}(\mathbf{x}, t)$ on scale l , one may integrate it with a rapidly decaying kernel;

$$\mathbf{F}_l(\mathbf{x}, t) = \int_V \phi\left(\frac{\mathbf{r}}{l}\right) \cdot \mathbf{F}(\mathbf{x} + \mathbf{r}, t) \frac{d^3r}{l^3}, \quad (1)$$

where $\phi(\mathbf{r}) = \phi(r)$ is a smooth and rapidly decaying (scalar) kernel (for mathematical properties required to be satisfied by this kernel, see, e.g., [33], Section 2.1). Hence, $\mathbf{F}_l(\mathbf{x}, t)$ is what an experimentalist can measure for quantity $\mathbf{F}(\mathbf{x}, t)$ in a ball of size l around point $\mathbf{x}$. (The coarse-grained field has been also used to define spatial complexity of vector fields, see e.g., [34].) Let us take Ohm’s law, $\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{\sigma} \mathbf{J} = 0$, as an example, with electric, magnetic and velocity fields $\mathbf{E}, \mathbf{B}, \mathbf{v}$, electrical conductivity σ and current $\mathbf{J} = \nabla \times \mathbf{B}$. The statement is that the physically measurable electric field and current on any given measurement scale $l > 0$, denoted by subscript l , are governed by $\mathbf{E}_l + (\mathbf{v} \times \mathbf{B})_l - \frac{1}{\sigma} \mathbf{J}_l = 0$.

We can of course measure the velocity $\mathbf{v}_l$ as well as magnetic field $\mathbf{B}_l$ on this scale, and write Ohm's law in its original form as

$$\begin{aligned} \mathbf{E}_l + \mathbf{v}_l \times \mathbf{B}_l - \frac{1}{\sigma} \mathbf{J}_l &= \mathbf{v}_l \times \mathbf{B}_l - (\mathbf{v} \times \mathbf{B})_l \\ &:= \mathbf{R}_l, \end{aligned} \quad (2)$$

in terms of the turbulent electric field induced by motions at scales $< l$, represented by the non-linear term $\mathbf{R}_l$, which scales as $|\mathbf{R}_l| \sim |\delta \mathbf{v}(l) \times \delta \mathbf{B}(l)|$ in terms of velocity and magnetic field increments (measured across distance l ; see Equation (9) below). As long as the velocity and magnetic fields are continuous functions with well-defined spatial derivatives, i.e., they are Lipschitz continuous, we have $|\delta \mathbf{v}(l)| \propto l$ and $|\delta \mathbf{B}(l)| \propto l$. Therefore, as we decrease l , the non-linear term $|\mathbf{R}_l| \sim l^2$ decreases [14]. The spatial derivative, e.g., $\nabla \times \mathbf{R}_l$ which enters the renormalized induction equation (obtained by multiplying the bare induction equation by kernel $\phi(\mathbf{r}/l)$ and integrating),

$$\partial_t \mathbf{B}_l = \eta \nabla^2 \mathbf{B}_l + \nabla \times (\mathbf{v}_l \times \mathbf{B}_l - \mathbf{R}_l), \quad (3)$$

with magnetic diffusivity $\eta = 1/\mu_0\sigma$, would scale as $|\nabla \times \mathbf{R}_l| \sim (l \times l)/l = l$ vanishing in the limit of small l . Everything is simple and physically expected; this is the situation in laminar flows. However, this picture changes dramatically in turbulence!

In the following subsection, Section 2, we will see that the Lipschitz continuity of magnetic and velocity fields is lost in a turbulent flow such that $|\delta \mathbf{v}(l)| \propto l^{1/3}$ and $|\delta \mathbf{B}(l)| \propto l^{1/3}$. The derivatives, e.g., of the form $|\nabla \times \mathbf{R}_l| \sim |\mathbf{R}_l|/l \sim (l^{1/3} \times l^{1/3})/l = l^{-1/3}$ will increase as we decrease the scale l ! Unlike laminar flows, spatial derivatives of the “bare” velocity and magnetic fields, e.g., $\nabla \mathbf{v}$ & $\nabla \times \mathbf{B}$ in the Navier–Stokes and induction equations, will blow up. Turbulence is a frequently encountered phenomenon in almost all astrophysical (and laboratory) plasmas where velocity and magnetic fields become non-Lipschitz. The empirical fact indicating the loss of Lipschitz continuity comes from the notion of *anomalous dissipation*.

Anomalous Dissipation

Let us briefly examine how experiments point to the loss of Lipschitz continuity of magnetic and velocity fields in fully turbulent flows mentioned above. In a plasma with viscosity ν , the kinetic energy is viscously dissipated at a rate (per mass) $\epsilon_\nu \equiv \nu |\nabla \mathbf{v}|^2$ while the magnetic energy is dissipated at a rate (per mass) $\epsilon_\eta \equiv \eta |\nabla \mathbf{B}|^2$. One may naively expect that these dissipation rates should vanish in the limit of vanishing viscosity and resistivity, i.e., $\lim_{\nu \rightarrow 0} \nu |\nabla \mathbf{v}|^2 \rightarrow 0$ and $\lim_{\eta \rightarrow 0} \eta |\nabla \mathbf{B}|^2 \rightarrow 0$. However, in a system of integral length L and typical large scale velocity V , these limits correspond to large Reynolds number $Re := LV/\nu$ and large magnetic Reynolds number $Re_m := LV/\eta$. With large Reynolds numbers, the flow is extremely sensitive to perturbations and prone to become turbulent. Experiments and numerical simulations indicate that in turbulence, these energy dissipation rates remain finite even in the limit of vanishing viscosity and resistivity (see [12] and references therein): $\lim_{\nu \rightarrow 0} \nu |\nabla \mathbf{v}|^2 \rightarrow 0$ and $\lim_{\eta \rightarrow 0} \eta |\nabla \mathbf{B}|^2 \rightarrow 0$ (in turbulence). (Enhanced dissipation of kinetic energy is one of the characteristics of turbulent flows. G. I. Taylor was probably the first to suggest that kinetic energy can be dissipated even with infinitesimal viscosity. This idea of non-viscous turbulent dissipation in fully developed turbulence, called anomalous dissipation or the zeroth law of turbulence, was later explored by Kolmogorov and Onsager. For details and references, see [35].) Thus the spatial derivatives of velocity and magnetic fields should blow up to keep these rates finite; in other words, these *dissipative anomalies* imply divergent spatial derivatives for

velocity and magnetic fields: $|\nabla \mathbf{v}| \rightarrow \infty$ and $|\nabla \mathbf{B}| \rightarrow \infty$. We will show presently that increments should scale as

$$\delta v(l) \propto l^{1/3}, \quad (4)$$

and

$$\delta B(l) \propto l^{1/3}. \quad (5)$$

Note that such singular spatial derivatives enter the Navier–Stokes and induction equations making them ill-defined differential equations. In order to avoid such singularities, one can use the Lipschitz continuous renormalized fields, e.g., as defined by Equation (1), at any (inertial) scale $l > 0$. Equations of motion governing these renormalized fields, e.g., the renormalized induction equation given by Equation (3), are thus “effective” field equations valid only on large (inertial range) scales l .

Following Onsager [36], we will obtain the scaling given by (4) for the turbulent velocity field. A similar argument applies to magnetic field, leading to scaling (5). The coarse-grained Navier–Stokes equation is obtained by integrating the (incompressible) bare equation;

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \quad (6)$$

where ρ , ν and p represent, respectively, the mass density, viscosity and kinematic pressure. The resulting renormalized equation reads

$$\partial_t \mathbf{v}_l + \nabla \cdot (\mathbf{v}_l \mathbf{v}_l + \boldsymbol{\tau}_l) = -\nabla p_l + \nu \nabla^2 \mathbf{v}_l, \quad \nabla \cdot \mathbf{v}_l = 0, \quad (7)$$

with renormalized quantities indicated by index l as before, e.g., for the velocity field; $\mathbf{v}_l(\mathbf{x}, t) = \int_V \phi(\frac{\mathbf{r}}{l}) \cdot \mathbf{v}(\mathbf{x} + \mathbf{r}, t) \frac{d^3 r}{l^3}$. The turbulent stress tensor at scale l is defined by $\boldsymbol{\tau}_l = (\mathbf{v}\mathbf{v})_l - \mathbf{v}_l \mathbf{v}_l$; cf. turbulent electromotive force $\mathbf{R}_l$ defined by Equation (2). Only for $\boldsymbol{\tau}_l \equiv \mathbf{0}$, does Equation (7) reduce to the incompressible Euler equation in the naive sense (i.e., not in the weak or renormalized form). Note that spatial derivatives of the renormalized field can be moved to the kernel ϕ , resulting in, e.g., gradient

$$\nabla \mathbf{v}_l(\mathbf{x}, t) = -\frac{1}{l} \int \nabla \phi(\mathbf{r}/l) \mathbf{v}(\mathbf{x} + \mathbf{r}, t) \frac{d^3 r}{l^3} \quad (8)$$

$$= -\frac{1}{l} \int \nabla \phi(\mathbf{r}/l) \delta \mathbf{v}(\mathbf{x}; \mathbf{r}, t) \frac{d^3 r}{l^3} \quad (9)$$

where $\delta \mathbf{v}(\mathbf{r}; \mathbf{x}, t) := \mathbf{v}(\mathbf{x} + \mathbf{r}, t) - \mathbf{v}(\mathbf{x}, t)$ is the velocity increment for separation $\mathbf{r}$. Thus such spatial derivatives, unlike bare derivatives such as $\nabla \mathbf{v}$, are well-defined and finite. It is also crucial to note that the turbulent stress tensor $\boldsymbol{\tau}(\mathbf{v}, \mathbf{v})$ can also be written in another equivalent form in terms of velocity-increments:

$$\boldsymbol{\tau}_l = (\delta \mathbf{v} \delta \mathbf{v})_l - \delta \mathbf{v}_l \delta \mathbf{v}_l. \quad (10)$$

Multiplying the renormalized Navier–Stokes Equation (7) by $\mathbf{v}_l$ and rearranging terms, we find the following balance equation in the inertial range where viscous dissipation is neglected;

$$\partial_t \left(\frac{|\mathbf{v}_l|^2}{2} \right) + \nabla \cdot \left(\mathbf{v}_l \cdot \left[\frac{|\mathbf{v}_l|^2}{2} + p_l \right] + \boldsymbol{\tau}_l \cdot \mathbf{v}_l \right) = -\Pi_l, \quad (11)$$

for the local kinetic energy balance on any inertial length-scale l . Here, the quantity $\Pi_l := -\nabla \mathbf{v}_l : \boldsymbol{\tau}_l$ is the deformation work, non-zero values of which would indicate an energy cascade, i.e., energy flux from resolved scales larger than l to smaller, unresolved scales $< l$. The Lipschitz condition for velocity field reads $|\delta \mathbf{v}(\mathbf{r}; \mathbf{x}, t)| \leq CV \left(\frac{|\mathbf{r}|}{L} \right)^h$ with some dimensionless constant C , integral length L and velocity V . This leads to the scalings

$|\nabla \mathbf{v}_l| \sim l^{h-1}$ and $|\tau_l| \sim l^{2h}$; therefore, $\Pi_l = O(l^{3h-1})$. According to Equation (11), in the limit $\nu \rightarrow 0$, persistent energy cascade from scale l to lower scales can occur only if $\int d^3x \Pi_l(\mathbf{x}, t) \neq 0$. However, the resolution scale l is arbitrary and for any fixed l , we can take viscosity ν sufficiently small for the “ideal” form of Equation (7) to hold, and then further decrease l . However, if the Lipschitz condition holds for any $h > 1/3$, then $\Pi_l = O(l^{3h-1})$ will indicate that $\int d^3x \Pi_l(\mathbf{x}, t) \rightarrow 0$ in the limit $l \rightarrow 0$. This is a contradiction; however, because the rate of energy cascade must be independent of the arbitrarily chosen length-scale l as we decrease l . The conclusion is that somewhere in the turbulent flow, there must appear Hölder singularities corresponding to exponents $h \leq 1/3$ in the limit of vanishing viscosity (or infinite Reynolds number). This prediction can be generalized to higher order scaling exponents for velocity-structure functions within the Parisi-Frisch multifractal model; see [17] and references therein.

3. Super-Diffusion and Super-Chaos

The scalings (4) and (5) have dramatic consequences for determinism in classical physics. From Equation (4); for example, it is obvious that

$$\delta v(l) = dl(t)/dt \propto l^{1/3}, \quad (12)$$

which translates into $dl(t)/dt = \alpha l^{1/3}$ with $\alpha = \text{const.}$, with a solution $l(t) = [l_0^{2/3} + \frac{2}{3}\alpha t]^{3/2}$. Defining the characteristic time for the initial separation $l(0) = l_0$ as $t_0 = \frac{3}{2\alpha} l_0^{2/3}$, for long times, i.e., $t \gg t_0$, it follows that

$$l^2(t) \sim \frac{2}{3}\alpha^3 t^3. \quad (13)$$

As easily seen by setting $l_0 = 0$, which still results in $l^2(t) \sim t^3$ at long times, two particles starting from the *same* point separate to a finite distance. Although it is still to be confirmed by delicate laboratory experiments, this effect has been attested by numerical simulations, see, e.g., [37]. Mathematically, this corresponds to non-uniqueness of particle trajectories, i.e., solutions of $dl(t)/dt = \delta v(l)$. This is indeed a familiar example in the theory of differential equations for non-uniqueness: to guarantee a unique solution for

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (14)$$

the advecting velocity $\mathbf{v}(\mathbf{x}, t)$ should be Lipschitz continuous, i.e., $|\delta \mathbf{v}(l)| \sim l^h$ with $h \geq 1$ (which basically means bounded spatial derivatives). For singular dynamical systems such as Equation (14) with Hölder singular $\mathbf{v}$, there is a continuous infinity of solutions (Kneser’s theorem) and Lyapunov exponents (see Equation (24) in Section 3.3) are not only positive (indicative of rapid divergence of trajectories, i.e., chaos) but they in fact tend to infinity (super-chaos; see below). This is an example of indeterminism in classical physics, but unlike some philosophical considerations, e.g., Norton’s dome thought experiment, it occurs in everyday life, e.g., in Richardson 2-particle diffusion, Equation (13), in turbulent flows.

Equation (12) is the well-known Kolmogorov [38] scaling for turbulent velocity in the inertial range (i.e., range of scales where viscous effects are negligible). Equation (13) is the famous Richardson’s 2-particle diffusion, as it was empirically discovered by Richardson in 1926 [19] in his study of volcanic ash diffusion; see Section 3.4. The lack of uniqueness for solutions of Equation (14) with Hölder continuous velocity $\mathbf{v}(\mathbf{x}, t)$ and the Richardson law are deeply connected to the concept of spontaneous stochasticity, or super-chaos, to be discussed in Section 3.2 below; see also [20]. Spontaneous stochasticity can be understood by considering random advection of fluid particles in a fluid with finite viscosity. Before performing this in Section 3.2; however, let us briefly discuss, in a very qualitative way, how

stochastic equations of motion arise with probability functions written as path integrals and the connection with deterministic chaos.

3.1. Effective Description of Stochastic Systems

Consider a particle moving under the influence of a force in one dimension; $m \frac{dv(t)}{dt} = f$, where the force may depend on velocity and time $f = f(v, t)$. Given the initial conditions $v(t_0) = v_0$ & $x(t_0) = x_0$, one can find the velocity and position of the particle at any later time and this approach is readily generalized to any number of particles as well-known. Formulation of Newtonian mechanics in terms of such initial value problems led Laplace to the conclusion that anyone with the knowledge of “precise” positions and velocities of all particles in the universe at a given time t_0 can in principle “predict” the future, i.e., find all final positions and velocities at any later time. This is Laplacian determinism in classical physics. However, there is no “perfect measurement” in physics even in principle and in fact Newtonian mechanics, and any other theory for that matter, will break down at some scale as we try to make our measurements finer by going to smaller measurement scales. In measuring initial conditions; therefore, we will inevitably make errors Δx_0 and Δv_0 , respectively, in position and velocity. These errors can be represented mathematically by adding random noise terms to initial conditions; $v(t_0) = v_0 + \Delta \tilde{v}_0(t)$ & $x(t_0) = x_0 + \Delta \tilde{x}_0(t)$. Time dependence of the errors (noise terms) simply indicates that they are different in each measurement made at time t : we solve the equation at many different times, $t_1, t_2, \dots, t_N$ each time with slightly different errors, e.g., with velocity errors $\Delta \tilde{v}_0(t_1), \Delta \tilde{v}_0(t_2), \dots$, and then “average” over all the solutions.

How can initial measurement errors affect the final solution? In linear non-chaotic systems, small errors in initial conditions typically lead to small errors in final solution, e.g., for negative Lyapunov exponents, whereas in non-linear chaotic systems, small initial errors may lead to large final errors corresponding to positive Lyapunov exponents (the butterfly effect). This is deterministic chaos; it is deterministic because in principle knowing “precise” values of initial conditions would give us the “precise values” of final solutions. It is chaos because no measurement is perfect and any non-zero error will grow exponentially.

In deterministic chaos, although trajectories (i.e., solutions of equations of motion) diverge exponentially, i.e., initial difference between any pairs of solutions or errors, rapidly grow leading to exponentially large errors in the final solution, but errors in the final solutions are still proportional to the errors in initial conditions; see Equation (24) in Section 3.3. In other words, initial conditions are *never* forgotten. On the other hand, in super-chaos or spontaneous stochasticity, the errors in final solutions *do not* depend on initial errors thus they can diverge even in the limit of vanishing initial measurement errors (the “real” butterfly effect) corresponding to infinite Lyapunov exponents; see Section 3.3. In other words, initial conditions are *completely* forgotten. This implies that two particles starting from the *same* point can separate to finite distances in finite times, violating determinism in classical physics (See also [39] as an interesting a thought experiment on indeterminism in Newtonian physics). An example is the atmosphere: no matter how precise we measure the initial conditions in weather forecasting, the errors after a finite time will diverge. No one can, and will ever be able to, forecast weather for more than about two weeks. (For a non-technical but interesting discussion of these notions, see [40].) Edward Lorenz discovered this “real butterfly effect”, in his seminal 1969 paper [41] where he states:

“... certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small “observational error” will evolve into

two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error."

Back to our discussion on representing measurement errors by adding noise terms to initial conditions, we note that one may equivalently add noise terms to the equations of motion instead turning them to a Langevin equation (in Itô form):

$$\begin{aligned} m \, dv(t) &= \underbrace{f(v, t) \, dt}_{\text{(deterministic part)}} + \underbrace{g(v, t) \tilde{\eta}(t) \, dt}_{\text{(fluctuating part)}} \\ &= f(v, t) \, dt + g(v, t) dW_t, \end{aligned} \quad (15)$$

where $\tilde{\eta}(t)$ may be taken as a Gaussian white noise (thus dW_t as a Wiener process), i.e., a random quantity that has a Gaussian probability with zero mean $\langle \tilde{\eta}(t) \rangle = 0$ and a delta-correlated variance $\langle \tilde{\eta}(t) \tilde{\eta}(t') \rangle = \delta(t - t')$. The function $g(v, t)$ then represents the “strength” of the random noise term. Note that the assumption of zero mean translates into the expected result; $m \langle \frac{dv(t)}{dt} \rangle = \langle f(v, t) \rangle$. The presence of the random noise term in Equation (15), which, in general, can have a Gaussian as well as other probability measure, means that the velocity $v(t)$ is also a random variable. The probability density function for $v(t)$ can be obtained in the form of a path integral using an elementary manipulation of (15); for details see, e.g., Appendix in [4] or Section 3 in [42].

In many body systems, e.g., fluids, thermal noise introduces random fluctuations in particle trajectories. The fluctuation–dissipation Theorem in statistical mechanics indicates that for any process that dissipates energy (e.g., viscous dissipation), a reverse process exists corresponding to fluctuations. Thus any balance equation with a dissipative term, e.g., $\nu \nabla^2 \mathbf{v}$ in the Navier–Stokes Equation (6), should contain a random fluctuating term e.g., representing molecular noise; see Equation (16) below. The “average” equation without noise term then represents an effective description (at large scales) where fluctuations can be neglected. For example, in fluctuating hydrodynamics, to deal with fluctuations at hydrodynamic spatiotemporal scales in thermodynamic equilibrium (originally due to Landau and Lifshitz [43]), one treats the dissipative fluxes of mass, heat and momentum, as stochastic variables [44]. These dissipative fluxes are the macroscopic manifestation of microscopic degrees of freedom. As an example, one has to introduce a heat flux to represent the mechanical (kinetic and potential) energy transfer to internal degrees of freedom (heat). Energy and momentum transfer to random molecular motions, through collisions in a fluid, cause dissipation. Hence, it should not come as a surprise that on larger mesoscopic scales, dissipative fluxes will become random variables, reflecting the random nature of molecular motion. As an example, the fluctuating Navier–Stokes equation is written as

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \nabla \cdot \left(\sqrt{2\nu\rho^{-1}k_B T} \, \tilde{\boldsymbol{\eta}}(\mathbf{x}, t) \right) \quad (16)$$

where temperature T is assumed to be constant, and k_B is Boltzmann’s constant. The white noise symmetric, traceless tensor field $\tilde{\boldsymbol{\eta}}(\mathbf{x}, t)$ represents a thermal fluctuating stress, with mean zero and covariance

$$\begin{aligned} \langle \tilde{\eta}_{ij}(\mathbf{x}, t) \tilde{\eta}_{kl}(\mathbf{x}', t') \rangle &= (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl}) \\ &\quad \times \delta^3(\mathbf{x} - \mathbf{x}') \delta(t - t'). \end{aligned} \quad (17)$$

The coefficient $\sqrt{2\nu\rho^{-1}k_B T}$ is dictated by the fluctuation–dissipation theorem and corresponds to the Gibbs equilibrium distribution for the equal-time velocity statistics, with energy equipartition among wave-number modes; see Appendix A in [45] for a careful discussion.

Finally, note that interpreting equations with noise terms, such as (16), as stochastic differential equations is mathematically problematic due to the presence of delta functions. Physically; however, $\delta^3(\mathbf{x} - \mathbf{x}')$ in these equations contains an implicit high-wave-number cut-off Λ ; $\delta_\Lambda^3(\mathbf{x}) = \frac{1}{V} \sum_{|\mathbf{k}| < \Lambda} e^{i\mathbf{k} \cdot \mathbf{x}}$, reflecting the validity of the underlying theory only above a finite length scale Λ^{-1} . For example, fluid approximation is not valid below the mean-free-path (in a gas) or inter-particle distance (in a liquid). This is another indication that the Navier–Stokes (and induction) equations should be interpreted instead as effective, low-wave-number field theories truncated at some wave-number cutoff Λ which, in the case of the Navier–Stokes equation, is larger than an inverse gradient length and smaller than an inverse microscopic length; see [46].

3.2. Spontaneous Stochasticity and Particle Advection

Now we can return to our discussion of spontaneous stochasticity in terms of stochastic advection of fluid particles in a fluid with finite viscosity ν by a velocity $\mathbf{v}^\nu(\tilde{\mathbf{x}}(t), t)$ which is smooth in the dissipative scales:

$$\frac{d\tilde{\mathbf{x}}(t)}{dt} = \mathbf{v}^\nu(\tilde{\mathbf{x}}(t), t) + \sqrt{2D}\tilde{\boldsymbol{\eta}}(t), \quad (18)$$

where $\tilde{\boldsymbol{\eta}}(t)$ is a Gaussian white noise, and $\sqrt{2D}$ is a constant representing the strength of the noise. Note that the noise term represents the effects of molecular fluctuations on the trajectory $\tilde{\mathbf{x}}$ of the particle moving in the fluid and its role is similar to what we discussed for the simpler case of Equation (15). Analogous mathematical manipulations can be employed to obtain the transition probability for a single fluid particle in a fixed non-random velocity realization $\mathbf{v}^\nu$ as the following path integral (see Appendix in [4]; also [20,47]):

$$P_{\mathbf{v}}^{\nu,D}(\mathbf{x}_f, t_f | \mathbf{x}_0, t_0) = \int_{\mathbf{x}(t_0)=\mathbf{x}_0} \mathcal{D}\mathbf{x} \delta^3[\mathbf{x}_f - \mathbf{x}(t_f)] \times \exp\left(\frac{-1}{4D} \int_{t_0}^t d\tau |\dot{\mathbf{x}}(\tau) - \mathbf{v}^\nu(\mathbf{x}(\tau), \tau)|^2\right). \quad (19)$$

The motivation to consider this problem comes from diffusion of a scalar field $\theta(\mathbf{x}, t)$, e.g., a concentration or temperature field, in a fluid with molecular diffusivity D , with advection-diffusion equation $\partial_t \theta + \mathbf{v}^\nu \cdot \nabla \theta = D \nabla^2 \theta$. (This advection–diffusion equation is in fact the Fokker–Planck equation corresponding to the Langevin Equation (15); see, e.g., [48]. The correspondence between the Onsager–Machlup path integral representation (20) and the diffusion equation in stochastic dynamics is similar to the connection between Feynman’s path integral and the Schrödinger equation in quantum mechanics [49]). The solution can be written using the Feynman–Kac formula

$$\begin{aligned} \theta(\mathbf{x}, t) &= \int d^3x_0 \theta(\mathbf{x}_0, t_0) P_{\mathbf{v}}^{\nu,D}(\mathbf{x}_0, t_0 | \mathbf{x}, t) \\ &= \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a} \theta(\mathbf{a}(t_0), t_0) \\ &\quad \times \exp\left(\frac{-1}{4D} \int_{t_0}^t d\tau |\dot{\mathbf{a}}(\tau) - \mathbf{v}^\nu(\mathbf{a}(\tau), \tau)|^2\right), \end{aligned} \quad (20)$$

for $t_0 < t$. In the limit of vanishing diffusivity, the noise term in Equation (18) vanishes thus the physical expectation may be that in the limit $\nu, D \rightarrow 0$, the solution should become totally deterministic. In other words, the transition probability should collapse to a delta function, i.e.,

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}(t), t), \quad \text{with} \quad P_{\mathbf{v}}^{\nu,D}(\mathbf{x}_f, t_f | \mathbf{x}_0, t_0) \rightarrow \delta^3[\mathbf{x}_f - \mathbf{x}(t_f)]. \quad (21)$$

However, it has been shown [20] that in the joint limit when viscosity and diffusivity both tend to zero, $\nu, D \rightarrow 0$, if the velocity field becomes non-smooth $\mathbf{v}^\nu \rightarrow \mathbf{v}$, the probability may tend to a non-trivial probability density thus Lagrangian trajectories may remain random. The breakdown in the uniqueness of Lagrangian trajectories can appear in different physical limits for turbulent advection; see Section II in [4] and references therein. This phenomenon has been dubbed spontaneous stochasticity because of its similarity with spontaneous symmetry breaking in condensed matter physics and quantum field theory, where, for example, a ferromagnet may retain a non-vanishing magnetization even in the limit of vanishing external magnetic fields. Also, we note that the randomness in spontaneously stochastic (super-chaotic) systems differs from the randomness commonly encountered in turbulence theory, associated with a random ensemble of velocity fields. Instead, the stochasticity in velocity field $\mathbf{v}$ in transition probability $P_{\mathbf{v}}^{\nu,D}$ is for a fixed (non-random) ensemble member $\mathbf{v}$ [4].

The theory of spontaneous stochasticity always starts with dynamics already stochastic, e.g., Landau–Lifschitz fluctuating hydrodynamics. The statement is that even in limit where the evolution equation formally becomes deterministic, the solutions remain random, with universal statistics that are independent of the underlying microscopic source of randomness. It is also important to emphasize that spontaneous stochasticity does not require an infinite Reynolds number, rather a large enough Reynolds number would suffice such that the limiting statistics are obtained.

Spontaneous stochasticity has important implications in magnetohydrodynamics, including magnetic flux freezing. Magnetic flux conservation in turbulent plasmas has been shown neither to hold in the conventional sense nor to be completely violated; instead, it is valid in a statistical sense associated with the spontaneous stochasticity of Lagrangian particle trajectories [4,9,15]. It is common in plasma physics, in particular in magnetic dynamo literature, to appeal to the idea that magnetic flux freezing holds approximately for very large Reynolds numbers (or small viscosity). In reconnection community, it is also common to assume that the magnetic flux freezing is broken by rapid diffusion of magnetic field lines across thin current sheets (where there is strong electric current due to magnetic shear, which leads to reconnection). It turns out that in fact both ideas are correct if understood under the right assumptions and conditions. Magnetic flux through individual material loops which are advected by the flow in the usual sense will not be conserved in the limit of large magnetic Reynolds number due to the development of singular current sheets and vortex sheets. However, magnetic flux in this limit of high Reynolds number should be conserved on average for a random ensemble of loops. In other words, the conservation of magnetic flux holds in a statistical sense if one considers an ensemble of material loops in the flow. This is associated with the spontaneous stochasticity of Lagrangian flows; for details see [4].

3.3. Chaos and Super-Chaos

Spontaneous stochasticity can also be called “super-chaos” which occurs due to the formation of singularities and consequent divergence of Lyapunov exponents to infinity (see below). In such super-chaotic systems, e.g., turbulent flows, vanishingly small random perturbations can be propagated to much larger scales in a finite time. Note that this situation completely differs from simple chaos (the butterfly effect) where trajectories remain unique solutions of equations of motion and despite their explosive divergence, they never forget their initial separation. The fact that in Richardson diffusion particles starting from the *same* point separate to a finite separation at later times resembles the

butterfly effect in chaos theory but in fact it is a completely different phenomenon. As previously discussed, the continuity of the velocity field means that

$$|\mathbf{v}(\mathbf{x}, t) - \mathbf{v}(\mathbf{y}, t)| \leq H|\mathbf{x} - \mathbf{y}|^h \quad (22)$$

for some real $H \geq 0$ and $0 < h \leq 1$ (with $0 < h < 1$ implying Hölder singularity and with $h = 1$ implying Lipschitz continuity). Consider the spatial separation of two arbitrary fluid particles $\mathbf{x}(t)$ and $\mathbf{y}(t)$ at time t , $\delta(t) := |\mathbf{x}(t) - \mathbf{y}(t)|$, which were initially separated by $\delta_0 := |\mathbf{x}(0) - \mathbf{y}(0)|$. Taking the time derivative of $\delta(t)$, we arrive at $d\delta(t)/dt \leq H[\delta(t)]^h$, with simple solution

$$\delta(t) \leq \left[\Delta_0^{1-h} + H(1-h)(t-t_0) \right]^{\frac{1}{1-h}}. \quad (23)$$

In non-turbulent flows, the velocity field is Lipschitz, i.e., $h \rightarrow 1$, thus $\delta(t) \leq \delta_0 e^{H(t-t_0)}$. At long times, assuming a near equality, we get

$$\delta(t) \simeq \delta_0 e^{H(t-t_0)}.$$

Thus even with chaos, for smooth dynamical systems, there is at most exponential divergence of trajectories. The corresponding Lyapunov exponent

$$\lambda := \lim_{t \rightarrow +\infty} \lim_{\delta_0 \rightarrow 0} \frac{1}{t-t_0} \ln \left(\frac{\delta(t)}{\delta_0} \right) \quad (24)$$

is the growth rate with $\lambda > 0$ indicating chaos. Hence, any small change in the initial conditions (e.g., measurement errors) will be exponentially magnified. Therefore, despite the exponential growth, the initial conditions (the initial separations δ_0) are never forgotten. In other words, in the limit when the initial separation vanishes, the final separation also vanishes:

$$\lim_{\delta_0 \rightarrow 0} |\mathbf{x}(t) - \mathbf{y}(t)| \rightarrow 0, \quad (\text{laminar/chaotic flow}). \quad (25)$$

On the other hand, a turbulent velocity field will be non-Lipschitz (or Hölder) with exponent $0 < h < 1$. Hence, in this case, we find $\Delta(t) \simeq [H(1-h)(t-t_0)]^{\frac{1}{1-h}}$, which implies that the information about initial conditions is completely lost. In other words, for long times, no matter how small the initial separation becomes, fluid particles separate super-linearly with time:

$$|\mathbf{x}(t) - \mathbf{y}(t)| \simeq t^{\frac{1}{1-h}}, \quad (\text{turbulent flow}), \quad (26)$$

even in the limit $\delta_0 \rightarrow 0$. This situation corresponds to an infinite Lyapunov exponent $\lambda = +\infty$, where the solutions remain unpredictable at all times even with vanishing measurement errors in the initial conditions.

3.4. Richardson Diffusion

Choosing $h = 1/3$ in Equation (26), corresponding to the Kolmogorov scaling for the velocity field [38], we recover the Richardson law [19] given by Equation (13). It should be mentioned that, historically, Richardson [19] took a different semi-empirical approach based on a probability measure to deduce the two-particle diffusion law $l^2 \sim t^3$. Richardson's probability density for particle separation vector $\mathbf{l} = \mathbf{x}_1 - \mathbf{x}_2$, with a scale-

dependent diffusion coefficient $K(l) \sim K_0 l^{4/3}$, satisfies $\partial_t P(\mathbf{l}, t) = \nabla_{l_i} \left(K(l) \nabla_{l_i} P(\mathbf{l}, t) \right)$ with a similarity solution [4],

$$P(\mathbf{l}, t) = \frac{A}{(K_0 t)^{9/2}} \exp \left(-\frac{9l^{2/3}}{4K_0 t} \right). \quad (27)$$

Using this probability density to average l^2 , one finds $\langle l^2(t) \rangle = (1144/81) K_0^3 t^3$. This is intimately related to Kolmogorov's relation

$$l^2(t) \sim (g_0 \epsilon) t^3, \quad (28)$$

which is, as we have seen before, a solution to the initial value problem $dl(t)/dt = \delta v(l) = (3/2)(g_0 \epsilon l)^{1/3}$, $l(0) = l_0$ for sufficiently long times $t \gg t_0$. Here, g_0 is Richardson–Obukhov constant and ϵ the mean energy dissipation rate.

3.5. Alfvénic Wave Packets

Similarly to the argument following Equation (12), we can use Equation (5) to write $\delta B(l) = \delta V_A(l) = dl(t)/dt \propto l^{1/3}$ or $dl(t)/dt = \beta l^{1/3}$ for some proportionality constant β (where assuming incompressibility, we have absorbed constant density to the definition of magnetic field). Thus,

$$l^2(t) \sim \frac{2}{3} \beta^3 t^3, \quad (29)$$

which implies Richardson two-particle diffusion for magnetic disturbances moving with local Alfvén velocity. Trajectories of these “Alfvénic wave packets” define magnetic path lines. Physically, we can imagine turbulence induces spatial fluctuations of the magnetic field which can be regarded as small disturbances or “Alfvénic wave packets” traveling with the Alfvén velocity $\mathbf{V}_A := \mathbf{B} / \sqrt{4\pi\rho}$ along the local field:

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{V}_A(\mathbf{X}, t), \quad \mathbf{X}(0) = \mathbf{X}_0. \quad (30)$$

Hence, in turbulent flows, magnetic path lines diverge super-linearly with time. As we will see in Section 4, path lines and their divergence in time provide a natural way to study magnetic reconnection without invoking the notion of field lines whose motion in turbulence is a complicated process. Faraday's notion of magnetic field lines has played an extremely important role in electromagnetism and plasma physics. Yet, its utility becomes questionable in turbulent flows where they cannot be defined as smooth curves evolving continuously in time as unique solutions of a differential equation. Their advection and diffusion through a turbulent plasma might deceptively look simple but in fact constitute a complicated problem in magnetohydrodynamics. Path lines provide an alternative tool with simpler dynamics.

Assuming an incompressible flow with constant density ρ , we will absorb the density into the definition of the magnetic field and write $\frac{d\mathbf{X}(t)}{dt} = \mathbf{B}(\mathbf{X}, t)$ as the equation defining path lines. It is easy to see that similar to the velocity field, the anomalous dissipation of the magnetic field, thus its Hölder singularity as indicated by Equation (5), will lead to Richardson diffusion of Alfvénic wave packets, which can be called “magnetic quasi-particles” in the language of modern field theories. Consider two wave packets separated by $|\mathbf{X}(\mathbf{t}) - \mathbf{Y}(\mathbf{t})|$ at time t , initially separated by $\Delta_0 := \Delta(0)$. Using the continuity condition for magnetic field $\mathbf{B}(\mathbf{X}(t), t)$ similar to (22),

$$|\mathbf{B}(\mathbf{X}, t) - \mathbf{B}(\mathbf{Y}, t)| \leq G |\mathbf{X} - \mathbf{Y}|^\xi, \quad (31)$$

with $G \geq 0$ and $0 < g \leq 1$ and repeating the argument which led to Equations (25) and (26), we obtain $\Delta(t) = \Delta_0 e^{Gt}$ implying

$$\lim_{\Delta_0 \rightarrow 0} |\mathbf{X}(t) - \mathbf{Y}(t)| \rightarrow 0, \quad (\text{laminar/chaotic flow})$$

with finite Lyapunov exponents $\lambda = \text{const}$. However, in turbulence, we have

$$\lim_{\Delta_0 \rightarrow 0} |\mathbf{X}(t) - \mathbf{Y}(t)| \simeq t^{\frac{1}{1-g}} \simeq t^{3/2}, \quad (\text{turbulent flow})$$

with infinite Lyapunov exponents $\lambda \rightarrow +\infty$.

An argument analogous to what we advanced for the stochastic scalar diffusion can be applied to magnetic field diffusion. The solution of the advection–diffusion equation for the magnetic field, i.e., the induction equation,

$$\partial_t \mathbf{B} + \mathbf{v} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{v} + \eta \nabla^2 \mathbf{B}, \quad (32)$$

with initial condition $\mathbf{B}(t_0) = \mathbf{B}_0$ can be written as a path integral

$$\begin{aligned} \mathbf{B}(\mathbf{x}, t) &= \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a} \, \mathbf{B}_0[\mathbf{a}(t_0)] \cdot \mathcal{J}(\mathbf{a}, t) \\ &\quad \times \exp\left(\frac{-1}{4\eta} \int_{t_0}^t d\tau |\dot{\mathbf{a}}(\tau) - \mathbf{v}^v(\mathbf{a}(\tau), \tau)|^2\right) \end{aligned}$$

where $\mathcal{J}$ is a 3×3 matrix and $\mathbf{B}$ is interpreted as a row-vector [4]. The condition $\mathbf{a}(t) = \mathbf{x}$ corresponds to solutions of

$$\frac{d\tilde{\mathbf{a}}(\tau)}{d\tau} = \mathbf{v}(\tilde{\mathbf{a}}(\tau), \tau) + \sqrt{2\eta} \tilde{\boldsymbol{\eta}}(\tau), \quad \tilde{\mathbf{a}}(t) = \mathbf{x}, \quad (33)$$

integrated backward in time from $\tau = t$ to $\tau = t_0$. Similarly to the discussion we had for the scalar field diffusion, one may naively apply Laplace asymptotic method to the path integral in the limit of vanishing resistivity and viscosity and expect to recover a single deterministic solution in the limit of $\nu, \eta \rightarrow 0$. However, in this limit, the solution may remain random if the velocity and magnetic fields become non-smooth, which is what is encountered in turbulence. This situation is intimately related to the breakdown of the traditional magnetic flux freezing; for a more detailed discussion see [4].

4. Magnetic Reconnection

Reconnection can be defined as a rapid change, i.e., on time scales much faster than resistive, in magnetic field pattern which accelerates and heats up the plasma [9]. Physically, one expects that such rapid changes in magnetic field direction in a region of plasma should correspond to explosive divergence of Alfvénic wave packets. One way of looking at this phenomenon is to consider the Alfvénic wave packets. As we will see presently, this approach suggests that the failure of Sweet–Parker model [50,51] (also [52]) in astrophysics is due to our assumption that magnetic diffusion is a “normal” diffusion process due to resistivity whereas in real turbulent flows, turbulence causes a super-diffusion (Richardson diffusion) at much larger scales which affects reconnection.

In a fluid of diffusivity D (solvent), the mean square separation (i.e., the mean distance the particle moves at time t relative to a fixed point) of diffusing particles (solute) grows as $\delta_D^2(t) \simeq Dt$ with time. Similarly, resistivity η causes diffusion of Alfvénic wave packets whose mean square separation grows as $\delta_\eta^2(t) \simeq \eta t$. One can understand the classical Sweet–Parker reconnection very easily in terms of the diffusion of these wave packets. In a current

sheet of length Δ and thickness δ , where reconnection proceeds with a typical speed V_R , the conservation of mass leads to $\Delta \times V_R = \delta \times V_A$ with the Alfvén velocity V_A . Magnetic diffusion on the scale of current sheet's thickness δ implies $\delta^2 \simeq \eta t$ where $t \simeq \Delta/V_A$ is the time Alfvénic wave packets require to traverse the current sheet's length Δ . Thus, combining these two basic results, we recover the famous Sweet–Parker reconnection speed

$$V_R \simeq \sqrt{\eta \frac{V_A}{\Delta}}. \quad (34)$$

This speed is of course too slow compared with observations [9]. The reason is that almost all astrophysical flows are turbulent; much faster turbulent diffusion at much larger (inertial) scales, instead of resistive diffusion at much smaller scales, dominates magnetic field evolution and reconnection.

Turbulence is a complicated phenomenon; however, turbulent reconnection is much simpler to understand than many (unrealistic) non-turbulent models which are based on instabilities; for example. As discussed before, in hydrodynamic turbulence, the mean square separation $\delta_v^2(t)$ of pairs of Lagrangian fluid particles grows super-linearly with time, i.e., $\delta_v^2(t) \simeq \epsilon_v t^3$ (Richardson diffusion). In magnetohydrodynamic turbulence, the mean square separation of Alfvénic wave packets grows as

$$\delta_R^2(t) \simeq \epsilon_\eta t^3, \quad (35)$$

corresponding to the Richardson diffusion of wave packets. Using mass conservation $\Delta \times V_R = \delta_R \times V_A$, $t = \Delta/V_A$ and $\epsilon_\eta = u_L^4/V_A L_i$, where L_i is the energy injection scale and u_L is the (isotropic) injection velocity, one obtains the Lazarian–Vishniac reconnection speed:

$$V_R \simeq V_A \sqrt{L/L_i} M_A^2, \quad (36)$$

with Mach number M_A . Thus this important result for fast reconnection can be obtained in a very simple way using magnetic path lines [13] without having to deal with complications such as magnetic field line diffusion [16,23] or stochastic wandering of field lines [10] developed in earlier works. In passing, we also note that the Richardson diffusion of wave packets is related to Richardson diffusion of the magnetic field lines. This new type of dispersion of field lines is sometimes called the Richardson diffusion “in space”, as opposed to the original Richardson dispersion in time [5]. Also, note that Richardson diffusion of wave packets, or magnetic field lines studied in, e.g., [5], is intimately related to the violation of traditional magnetic flux freezing [31] in turbulent flows; for a more detailed discussion see, e.g., [4,5,15].

Almost all non-turbulent reconnection models proposed to enhance the Sweet–Parker rate rely on small scale plasma effects, represented by a non-ideal term $\mathbf{P}$ in the generalized Ohm's law, $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{P}$, which leads to the induction equation

$$\partial_t \mathbf{B}_l = \nabla \times (\mathbf{v}_l \times \mathbf{B}_l - \mathbf{R}_l - \mathbf{P}_l), \quad (37)$$

where we have also included the resistive electric field in $\mathbf{P}$. Reconnection is driven by the non-ideal terms $(\nabla \times \mathbf{R}_l)/B_l$ and $(\nabla \times \mathbf{P}_l)/B_l$ which act as source terms for the differential equations governing magnetic field direction $\hat{\mathbf{B}}_l := \mathbf{B}_l/B_l$ [9,12]. However, $|\nabla \times \mathbf{R}_l|$ grows for l decreasing in the turbulence inertial range, until it becomes comparable to the plasma non-ideal term $|\nabla \times \mathbf{P}_l|$ at the turbulence microscale l_d . For $l \gg l_d$, $|\nabla \times \mathbf{R}_l| \gg |\nabla \times \mathbf{P}_l|$ (see, e.g., [16] and references therein). Hence, in the presence of turbulence, plasma effects essentially play no role in reconnection. This fact, indicated by the early work of Lazarian and Vishniac [10] more than two decades ago, has been confirmed both by simulations and

observations, yet the reconnection community seems to have lagged behind by appealing to instabilities such as tearing modes, while plasma non-ideal effects affect reconnection near electron- and ion-scales, these effects are completely irrelevant to reconnection at scales larger than the ion gyroradius.

5. Final Remarks and Conclusions

The main message of this short review is that magnetic fields especially in astrophysics are usually turbulent, the study of which requires concepts such as regularization/renormalization, stochastic dynamics and effective field theories. Utilizing such tools from other areas of theoretical physics might seem exotic in plasma physics; however, it does not necessarily lead to a more complicated approach. In fact, in some problems such as reconnection, it turns out that notions such as super-chaos and Richardson diffusion [5,8,17] not only are necessary but they can provide a physically more intuitive picture.

Turbulence is a super-chaotic state plagued with indeterminism and singularities, which is totally different from chaos that is inherently deterministic. This is an important matter because the common approach to many problems in plasma physics; for example, dynamo theory and reconnection, is based on either plasma instabilities and non-idealities, e.g., [53,54], or chaos theory, e.g., [55,56] and references therein. However, observations and simulations all point to the ubiquity of turbulence which can be driven by plasma instabilities; magnetic reconnection and/or external sources, e.g., stellar winds and jets in astrophysics (for observational evidence, see, e.g., Section II A in [9] and references therein). In particular, we note that the overall effect of non-ideal plasma mechanisms, e.g., Hall effect, tearing modes instabilities, etc., on large scale astrophysical reconnection is negligible, somehow resembling negligible quantum effects in classical systems.

MHD turbulence and reconnection seem to be interconnected in the sense that one may derive and intensify the other. Turbulent reconnection is not only fast and independent of plasma effects, it should in fact be regarded as an inseparable part of MHD turbulence as it constantly occurs on all inertial scales in a random manner. The original formulation of turbulent reconnection was based on stochastic field line wandering [10], later refined mathematically using the idea of Richardson diffusion of field lines [5,9,11,15,16]. However, as discussed in Section 4, turbulent reconnection can be easily formulated and understood in terms of Alfvénic wave packets moving along the local magnetic field. The latter approach has two advantages: first, it avoids the complication of field lines motion through a turbulent plasma as these curves are not persistent dynamical entities in time and their diffusion has been the source of confusion and misunderstanding in the reconnection literature. The second advantage is the possibility of easily formulating magnetic topology and its rate of change in the context of a dynamical system theory. One implication is that if reconnection is defined as magnetic topology change, it can be fast only in turbulent flows where both reconnection and topology change are driven by spontaneous stochasticity, independent of any plasma effects [13].

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