fluids

MDPI

Article

Lighthill's Theory of Sound Generation in Non-Isothermal and Turbulent Fluids

Swati Routh [1] and Zdzislaw E. Musielak [2,*]

[1] Birla Institute of Technology and Science, Pilani-Dubai, Dubai 345055, United Arab Emirates; swatirouth@dubai.bits-pilani.ac.in
[2] Departmemt of Physics, University of Texas at Arlington, Arlington, TX 76019, USA
* Correspondence: zmusielak@uta.edu

Abstract: Lighthill's theory of sound generation was developed to calculate acoustic radiation from a narrow region of turbulent flow embedded in an infinite homogeneous fluid. The theory is extended to include a simple model of non-isothermal fluid that allows finding analytical solutions. The effects of one specific temperature gradient on the wave generation and propagation are studied. It is shown that the presence of the temperature gradient in the region of wave generation leads to monopole and dipole sources of acoustic emission and that the efficiency of these two sources may be higher than Lighthill's quadrupoles. In addition, the wave propagation far from the source is different than in Lighthill's original work because of the presence of the acoustic cutoff frequency resulting from the temperature gradient.

Keywords: acoustics; sound generation; Lighthill theory; turbulent fluids; fluid mechanics

check for updates

Academic Editor: Ramesh Agarwal

Received: 19 May 2025
Revised: 11 June 2025
Accepted: 12 June 2025
Published: 13 June 2025

Citation: Routh, S.; Musielak, Z.E. Lighthill's Theory of Sound Generation in Non-Isothermal and Turbulent Fluids. *Fluids* **2025**, *10*, 156. https://doi.org/10.3390/fluids10060156

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

A theory of acoustic wave generation by a turbulent jet embedded in an infinite homogeneous fluid was originally developed by Lighthill [1], who showed that Reynolds stresses are sources of quadrupole emission. The theory allows the evaluation of the wave energy flux far away from a finite region of turbulence by assuming that the backreaction of generated waves on the turbulence is negligible. An important result was obtained by Proudman [2], who described the turbulent motions in a jet by the Heisenberg turbulence energy spectrum [3] and derived a general formula for the generated acoustic power output; the well-known Kolmogorov turbulence energy spectrum was proposed earlier [4] but not used by Proudman.

The original Lighthill theory was also extended to include the effects of solid boundaries (e.g., [5,6],) and magnetic fields (e.g., [7,8]). The main prediction of the theory is the now well-known u^8 law of the acoustic power output by the turbulent jet, where u is the jet velocity. Good agreements between this theoretical prediction and the results of several experiments performed for jets of different diameters were reported by Goldstein [9]. The Lighthill–Proudman formula for the acoustic power output was used to evaluate the acoustic wave energy fluxes generated by turbulent motions in the solar (e.g., [10–12]) and stellar (e.g., [13,14],) convection zones and to discuss the role played by acoustic waves in heating the solar and stellar atmospheres.

As mentioned above, the original Lighthill theory concerns only homogeneous media and treats turbulence as isotropic, homogeneous, and decaying in time. A significant extension of Lighthill's theory was carried out by Stein [15], who followed earlier work [16] and included the effects of stratification. Stein's treatment of turbulence was further

improved and modified in [17], and the resulting theory demonstrated that stratification is responsible for monopole and dipole sources of acoustic emission. The modified Lighthill–Stein theory was used to compute the acoustic wave energy spectra for the Sun [17] and late-type stars [18]. Goldreich and Kumar [19] studied the differences between free (decaying) and forced (non-decaying) turbulence, and they demonstrated that the latter is driven by the fluctuating buoyancy force, which leads to dipole emission. However, this resulting dipole emission shows similarities to that obtained by Stein [15].

In all the above applications of the theory of sound generation by turbulent motions, the background fluid was assumed to be isothermal. Therefore, the main aim of this paper is to extend the original Lighthill theory to a non-isothermal fluid. The model of this fluid is assumed to be simple, namely, its density and temperature vary with height and they are related to each other, but pressure remains constant, which means that there is no gravity; such models may give more realistic description of experiments with turbulent jets and also in some laboratory settings. For the considered model, analytical solutions can be obtained, and they can be used to study the effects caused by one specific temperature gradient on the wave generation and propagation. The obtained results show that the gradient leads to monopole and dipole sources of acoustic emission, and that it also is responsible for the acoustic cutoff frequency, which affects the wave propagation.

The paper is organized as follows: a brief description of the original Lighthill theory is given in Section 2; an extension of Lighthill's theory to a non-isothermal fluid is described in Section 3; the acoustic wave energy fluxes computed by using both the original and extended Lighthill theories are presented and compared in Section 4; applications of the obtained theoretical results and their experimental verifications are described in Section 5; finally, our conclusions are given in Section 6.

2. Lighthill Theory of Sound Generation

To describe the sound generation by a turbulent jet, Lighthill [1] derived an inhomogeneous wave equation for a single wave variable by collecting all linear and nonlinear terms on the left-hand side (the propagator) and on the right-hand side (the source function) of the wave equation, respectively, and obtained

$$\hat{L}_{\mathrm{s}}[\rho] = \hat{S}[T_{ij}(u_{\mathrm{t}})]\,, \tag{1}$$

where $\hat{L}_{\mathrm{s}}$ is the acoustic wave propagator given by

$$\hat{L}_{\mathrm{s}} = \frac{\partial^2}{\partial t^2} - c_{\mathrm{s}}^2 \nabla^2\,, \tag{2}$$

ρ represents density perturbations associated with the waves, c_{s} is the speed of sound, u_{t} is the turbulent velocity, and $T_{ij}(u_{\mathrm{t}}) = \rho_0 u_{ti} u_{tj} + p_{ij} - c_{\mathrm{s}}^2 \rho \delta_{ij}$ is Lighthill's turbulence stress tensor, where i and j = 1, 2, and 3 represents its components in different directions, and ρ_0 is the density of the background fluid. Lighthill assumed that the jet was embedded in a uniform atmosphere, which was also at rest, and considered linear (weak) acoustic waves that produce no backreaction on the turbulent flow. He then showed that $T_{ij} \approx \rho_0 u_{ti} u_{tj}$ and that the source function $\hat{S}[T_{ij}(u_{\mathrm{t}})]$ was given by a double divergence of T_{ij}. The physical meaning of the source function is that the stresses produce equal and opposite forces on opposite sides of a fluid element leading to the distortion of its surface without changing the volume (quadrupole emission). In other words, the fluid motions generating acoustic waves behave as a volume distribution of acoustic quadrupoles, so one may write $\hat{S}[T_{ij}(u_{\mathrm{t}})] = S_{\mathrm{quadrupole}}$.

Proudman [2] applied Lighthill's theory to the case when the fluctuating fluid motions are represented by the Heisenberg turbulence energy spectrum [3] and derived a general formula for the generated acoustic power output, P_a. This Lighthill–Proudman formula is usually given in the following form:

$$P_a = \alpha_q \frac{\rho_0 u_t^3}{l_0} M_t^5 \,, \tag{3}$$

where the emissivity coefficient $\alpha_q \approx 38$, l_0 is the characteristic length scale of the turbulence and $M_t = u_t/c_s$ is the turbulent Mach number.

The formula is valid for subsonic turbulence ($M_t << 1$) and it was extensively used in early calculations of acoustic wave energy fluxes generated in the Sun and other stars (e.g., [10–14]). Since the formula does not account for temperature gradients, it was assumed that Equation (3) was satisfied locally in the turbulent region and the total emitted wave energy flux was calculated by performing the integration over the thickness of the wave generation region.

3. Extension of Lighthill's Theory

3.1. Basic Equations

Let us consider a compact region of turbulent flow embedded in a very large volume of an ideal gas and assume that both the turbulent region and the surrounding fluid are non-uniform because of the existence of a temperature gradient. To simplify the problem so that analytical solutions can be obtained, we neglect gravity and consider a model in which the gas pressure p_0 = const; however, the background temperature T_0, density ρ_0, and speed of sound c_s vary with height in the model. The gradients of density and temperature that are related to each other preserve the hydrostatic equilibrium of the background fluid.

To describe the generation and propagation of acoustic waves in this model, we consider a set of hydrodynamic equations and assume that the turbulent flow is subsonic and the waves are linear. In general, the waves propagate in all three spatial directions; however, their propagation only in one specified direction is affected by the gradient. We define $\bar{x} \equiv x_i = (x, y, z)$, with i = 1, 2, and 3, and introduce the velocity in the x_i-direction $u_i(t, \bar{x})$, density $\rho(t, \bar{x})$, and pressure $p(t, \bar{x})$ perturbations. We further assume that the effects of viscosity and heat conduction can be neglected. Based on these assumptions, we linearize the hydrodynamic equations and follow Lighthill [1] to separate the linear and nonlinear terms. This gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho_0 u_i)}{\partial x_i} = -\frac{\partial (\rho u_i)}{\partial x_i} \,, \tag{4}$$

$$\rho_0 \frac{\partial u_i}{\partial t} + \frac{\partial p}{\partial x_i} = -\frac{\partial \rho u_i}{\partial t} - \frac{\partial (\rho_0 u_i u_j)}{\partial x_j} \,, \tag{5}$$

and

$$\frac{\partial p}{\partial t} + \rho_0 c_s^2 \frac{\partial u_i}{\partial x_i} = -u_i \frac{\partial p}{\partial x_i} - c_s^2 \rho \frac{\partial u_i}{\partial x_i} \,, \tag{6}$$

where $c_s = c_s(\bar{x})$ in our non-isothermal model, and the term $\rho_0 u_i u_j$ is the momentum flux tensor or the rate at which the momentum in the x_i-direction crosses the unit surface are in the x_j-direction. Note that any subscript repeated in a single term is to be summed from 1 to 3.

3.2. Wave Equation and Source Function

We derive an inhomogeneous wave equation for the pressure perturbation p associated with the waves by eliminating the other wave variables and obtain

$$\frac{\partial^2 p}{\partial t^2} - \frac{\partial}{\partial x_i}\left[c_s^2(\bar{x})\,\frac{\partial p}{\partial x_i}\right] = S(t, x_i)\,, \tag{7}$$

where the source function $S(t, x_i)$ is given by

$$S(t, x_i) = \frac{\partial}{\partial x_i}\left[c_s^2 \frac{\partial(\rho u_i)}{\partial t}\right] + \frac{\partial}{\partial x_i}\left[c_s^2 \frac{\partial(\rho_0 u_i u_j)}{\partial x_j}\right] - \frac{\partial}{\partial t}\left(u_i \frac{\partial p}{\partial x_i}\right) + c_s^2 \frac{\partial}{\partial t}\left(\rho \frac{\partial u_i}{\partial x_i}\right) \tag{8}$$

It must be noted that, in our model, the inhomogeneous wave equations for the wave velocity u_i and density ρ are of different forms. However, there are relationships that connect all the wave variables and they can be used to derive the wave equations for u_i and ρ once the wave equation for p is known [20].

We again follow Lighthill [1] and treat the source function as being fully determined by a known turbulent flow. To emphasize this point, we label the source function as $S_{\text{turb}}(t, x_i)$. Since the turbulent flow considered in this paper is subsonic, we make a Mach number expansion of the source function and retain only the lowest order terms; the procedure is discussed in great details by Stein [15], and it will not be repeated here. This allows us to write

$$S_{\text{turb}}(t, \bar{x}) \approx \frac{\partial}{\partial x_i}\left[c_s^2 \frac{\partial(\rho_0 u_i u_j)}{\partial x_j}\right]_{turb}, \tag{9}$$

which is consistent with Lighthill's original results if c_s is a const

(see Section 2). Note that in the approach presented in this paper $c_s = c_s(\bar{x})$; however, see Section 3.4).

Hence, the inhomogeneous acoustic wave equation becomes

$$\frac{\partial^2 p}{\partial t^2} - c_s^2 \frac{\partial^2 p}{\partial x_i^2} - 2c_s\left(\frac{\partial c_s}{\partial x_i}\right)\frac{\partial p}{\partial x_i} = S_{\text{turb}}(t, \bar{x})\,, \tag{10}$$

and

$$S_{\text{turb}}(t, \bar{x}) = c_s^2 \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho_0 u_{ti} u_{tj}\right) + 2c_s\left(\frac{\partial c_s}{\partial x_i}\right)\frac{\partial}{\partial x_j}\left(\rho_0 u_{ti} u_{tj}\right)\,, \tag{11}$$

with changes in notation and replacing $[u_i]_{turb}$ in the source function S_{turb} by u_{ti}. It must be noted that Equation (10) reduces to Lighthill's inhomogeneous wave equation (see Equation (1)) where the limit of c_s is a const and with $p = c_s^2 \rho$.

3.3. Transformed Wave Equation

To remove the nonconstant coefficient c_s^2 from the term with the second-order derivative in Equation (10), we introduce the new variable $\partial\tau_i = \partial x_i / c_s$ and obtain

$$\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial \tau_i^2} - \frac{1}{c_s}\left(\frac{\partial c_s}{\partial \tau_i}\right)\frac{\partial p}{\partial \tau_i} = S_{\text{turb}}(t, \tau_i)\,, \tag{12}$$

with

$$S_{\text{turb}}(t, \tau_i) = \frac{\partial^2}{\partial \tau_i \partial \tau_j}\left(\rho_0 u_{ti} u_{tj}\right) + \frac{1}{c_s}\left(\frac{\partial c_s}{\partial \tau_i}\right)\frac{\partial}{\partial \tau_j}\left(\rho_0 u_{ti} u_{tj}\right)\,. \tag{13}$$

As the next step, we remove the first-order derivative from Equation (12) by using the following transformation [21,22]:

$$p(t,\tau_i) = p_1(t,\tau_i)e^{-I_c}\,, \tag{14}$$

where

$$I_c = \frac{1}{2}\int_{\tau_{i0}}^{\tau_i}\frac{1}{c_s}\left(\frac{\partial c_s}{\partial \tilde{\tau}_i}\right)d\tilde{\tau}_i\,. \tag{15}$$

This gives

$$\frac{\partial^2 p_1}{\partial t^2} - \frac{\partial^2 p_1}{\partial \tau_i^2} + \Omega_i^2(\tau_i)p_1 = S_{\rm turb}(t,\tau_i)\,, \tag{16}$$

where

$$\Omega_i^2(\tau_i) = \frac{1}{2}\left[\frac{1}{c_s}\left(\frac{\partial^2 c_s}{\partial \tau_i^2}\right) - \frac{1}{2c_s^2}\left(\frac{\partial c_s}{\partial \tau_i}\right)^2\right], \tag{17}$$

and

$$S_{\rm turb}(t,\tau_i) = \left[\frac{\partial^2}{\partial\tau_i\partial\tau_j}\left(\rho_0 u_{ti}u_{tj}\right) + \frac{1}{c_s}\left(\frac{\partial c_s}{\partial \tau_i}\right)\frac{\partial}{\partial\tau_j}\left(\rho_0 u_{ti}u_{tj}\right)\right]e^{I_c}\,, \tag{18}$$

which shows that the source function is determined by both local ($\partial c_s/\partial x_i$) and global ($I_c$) effects.

3.4. Solution to the Wave Equation and Acoustic Cutoff Frequency

To solve Equation (16), we must specify the temperature gradient in our non-isothermal model. Since the gas pressure $p_0 = RT_0\rho_0/\mu$, where R is the universal gas constant and μ is the mean molecular weight, must be constant in our model, the temperature and density variations with height must be related to each other. Let us consider a model in which both T_0 and ρ_0 vary only in one direction, say, the x_3 (or z) direction and assume that $T_0(\xi) = T_{00}\xi^2$, where $\xi = z/z_0$, with z_0 being a given height and T_{00} is a temperature at this height. For the model to be in hydrostatic equilibrium (p_0 is a const), the density ρ_0 must decrease with ξ as $\rho_0 = \rho_{00}\xi^{-2}$. In this model, the speed of sound c_s is a linear function of ξ and we have $c_s(\xi) = c_{s0}\xi$, where c_{s0} is the speed of sound at z_0.

The non-isothermal model requires that $\tau_1 = x_1/c_s$ and $\tau_2 = x_2/c_s$ but $\tau_3 = ln|\xi\,|/\omega_0$, where $\omega_0 = c_{s0}/z_0$. Hence, we obtain $\xi = e^{\omega_0\tau_3}$ and $c_s(\tau_3) = c_{s0}e^{\omega_0\tau_3}$. We use these results to calculate Ω_i^2 (see Equation (17)), which gives $\Omega_1^2 = 0$, $\Omega_2^2 = 0$, and $\Omega_3^2 = \Omega_0^2$ with $\Omega_0^2 = \omega_0^2/4$ or

$$\Omega_0^2 = \frac{c_{s0}^2}{4z_0^2}\,. \tag{19}$$

An interesting result is that Ω_0 is constant in the τ-space (but not in the z-space), so we can formally make Fourier transforms in time and τ-space. Based on the form of Equation (16) and the fact that Ω_0 = const, we conclude that Ω_0 is the acoustic cutoff frequency introduced by Lamb [23,24] for acoustic waves propagating in a stratified but isothermal atmosphere. The origin and physical meaning of Ω_0 is the same as the cutoff introduced by Lamb. The acoustic cutoff plays important roles in studying Earth's [25,26] and Jupiter's [27] oscillations as well as in helioseismology [28] and asterioseismology [29].

We make Fourier transforms in time and τ-space

$$p_1(t,\tau_i) = \int\int p_2(\omega,k_i)e^{i(\omega t - k_i\tau_i)}d\omega\; d^3k_i\,, \tag{20}$$

where k_i is a wave vector corresponding to τ_i. This gives

$$\int\int[-\omega^2 + k_i^2 + \Omega_i^2]p_2(k_i,\omega)e^{i(\omega t - k_i\tau_i)}\, d\omega\, d^3k_i = S_{\text{turb}}(t,\tau_i)\,, \tag{21}$$

and

$$p_2(\omega,k_i) = \frac{S_{\text{turb}}(\omega,k_i)}{-\omega^2 + k_i^2 + \Omega_i^2}\,, \tag{22}$$

with

$$S_{\text{turb}}(\omega,k_i) = \frac{1}{(2\pi)^4}\int\int S_{\text{turb}}(t,\tau_i)e^{-i(\omega t - k_i\tau_i)}\, dt\, d^3\tau_i\,. \tag{23}$$

Substituting Equation (18) into Equation (23) and integrating by parts results in:

$$S_{\text{turb}}(\omega,k_i) = \frac{-1}{(2\pi)^4}\int(\rho_0 u_i u_j)(\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \frac{c_s'}{c_s}\frac{\partial}{\partial\tau_j})e^{I_c}e^{-i(\omega t - k_i\tau_i)}d^3\tau_i\ \ , \tag{24}$$

where $c_s' = dc_s/d\tau_3 = dc_s/dz$.

After some calculations (see Appendix A.1), $S_{\text{turb}}(t,\tau_i)$ can be written as

$$S_{\text{turb}}(t,\tau_i) = e^{I_c}\rho_0\{k_ik_ju_iu_j - \frac{1}{4}\left(\frac{c_s'}{c_s}\right)^2 u_3u_3 - \frac{c_s''}{2c_s} - \frac{i}{2c_s}[\frac{1}{2}c_s'u_3k_iu_i + \frac{3}{2}c_{sz}'u_3k_ju_j]\} \tag{25}$$

where $c_s'' = dc_s'/d\tau_3 = dc_s'/dz$. The above expression is the final form of the source function for the problem under consideration.

4. Emitted Acoustic Wave Energy Flux

4.1. Mean Acoustic Energy Flux

The mean acoustic energy flux is calculated by

$$F_i = < pu_i > \ . \tag{26}$$

From the momentum conservation and continuity equation, after ignoring gravity and nonlinear terms, the velocity of the fluid (see Equation (54) in Stein [15]) is expressed in terms of the pressure perturbation as

$$u_i = -\frac{1}{\rho_0}(\frac{\partial}{\partial t})^{-1}\frac{1}{c_s}\frac{\partial p}{\partial\tau_i}. \tag{27}$$

Substituting this relation into the energy flux equation leads to the following:

$$F_i = -\ < p\frac{1}{\rho_0 c_s}(\frac{\partial}{\partial t})^{-1}\frac{\partial p}{\partial\tau_i} > \tag{28}$$

and p can be expressed in terms of its Fourier transform as

$$p(t,\tau_i) = \int\frac{S_{\text{turb}}(\omega,k_i)}{-\omega^2 + k_i^2 + \Omega_i^2}e^{i\omega t - ik_i\tau_i}d^3k_i d\omega\,, \tag{29}$$

which gives the following acoustic wave energy flux (see Appendix A.2)

$$F_i(\omega,k_i) = \frac{1}{\rho_0 c_s}\int(\frac{k_i}{\omega})\frac{S_1(\omega,k_i)S_{\text{turb}}^*(\omega,k_i)d^3k_id^3k_i}{\{-\omega^2 + {k_i}^2 + \Omega_i^2\}\{-\omega^2 + {k_i}^2 + \Omega_i^2\}}\,, \tag{30}$$

where the integral is now evaluated.

4.2. Asymptotic Fourier Transform

Lighthill's formula [8] for the asymptotic value of Fourier transforms far from the source is given by

$$\int \frac{F(\vec{k})}{G(\vec{k})} e^{(i\vec{k}\cdot\vec{\tau})} d^3\vec{k} = \frac{4\pi^2}{|\vec{\tau}|} \sum_k \frac{F(\vec{k}) e^{i\vec{k}\cdot\vec{\tau}}}{|\Delta_k G|\sqrt{K}} \tag{31}$$

where the vectors $\vec{k}$ and $\vec{\tau}$ have their components k_i and τ_i, respectively, and the denominator in the integral in (30) G is defined as $G = \vec{k}\cdot\vec{k} - (\omega^2 - \Omega_i^2)$. K is the Gaussian curvature of the slowness surface on which the direction of normal is defined as $|\hat{\tau}| = \frac{\Delta_k G}{|\Delta_k G|} = \frac{(k_1,k_2,k_3)}{\sqrt{(k_1^2+k_2^2+k_3^2)}}$; the slowness surface is the surface in wave number space where the dispersion relation is satisfied. Referring to (29), only those wave numbers and frequencies contribute to the pressure perturbation where the denominator, G, vanishes, i.e., where the dispersion relation (G = 0) is satisfied. The sum is over the set $\vec{k}$ on the slowness surface, G = 0. The cosine of the angle between the sound propagation (group velocity), $\hat{\tau}$, and the vertical is $\cos\theta = \hat{z}\cdot\hat{\tau} = \frac{k_3}{|k|}$. The cosine of the angle between the wave vector, $\vec{k}$, and the vertical is $\cos\theta_k = \frac{\hat{z}\cdot\vec{k}}{|k|} = \frac{k_3}{|k|}$. Hence, in our particular case, the direction of the wave vector is the same as the direction of propagation, that is, $\theta_k = \theta$ (see [14]).

By applying (31) to (30), we obtain the acoustic flux at large distances given by

$$F_i(\omega, k_i) = \lim_{T\to\infty} \frac{8\pi^5}{T} \frac{\hat{\tau}_i}{|\tau_i|^2} \frac{\sqrt{(\omega^2 - \Omega_i^2)}}{\omega\rho_0 c_s} |S_{\text{turb}}(\omega, k_i)|^2 , \tag{32}$$

which requires the source function $S_{\text{turb}}(\omega, k_i)$ to be determined by evaluating spectral efficiency and specifying turbulence and its spectrum.

4.3. Evaluation of Spectral Efficiency

The distribution of energy across different scales of turbulent motions is called the spectral efficiency, and it is typically expressed by a prescribed turbulent energy spectrum as it is now shown. In the absence of a generally accepted model of turbulence, the description of turbulent flow is based on two-point, two-time velocity correlation functions, which are obtained by considering the source at two points in a turbulent fluid at two different times [30,31].

$$S^*_{\text{turb}}(\omega, k_i) = \frac{1}{(2\pi)^4} \int S^*_{\text{turb}}(t'', \tau_i'') e^{-i(\omega t'' - k_i\tau_i'')} d^3\tau_i'' dt'' \tag{33}$$

$$S_{\text{turb}}(\omega, k_i) = \frac{1}{(2\pi)^4} \int S_{\text{turb}}(t', \tau_i') e^{i(\omega t' - k_i\tau_i')} d^3\tau_i' dt' . \tag{34}$$

where primed and double-primed refer to the two turbulent source points.

Averaging the position and time, with $\vec{\tau}_0$ being the vector to the mean position between the two turbulent source points and $t_0 = 0.5(t'' + t')$ being the mean time between t'' and t', the coordinates in the evaluation of $|S(\vec{k}, \omega)|^2$ are transformed as follows:

$$|S_{\text{turb}}(\omega, k_i)|^2 = \frac{1}{(2\pi)^8} \int\int\int\int S^*_{\text{turb}}(t_0 + \frac{t}{2}, \tau_{0i} + \frac{\tau_i}{2})$$
$$S_{\text{turb}}(t_0 - \frac{t}{2}, \tau_{0i} - \frac{\tau_i}{2}) e^{-i\omega t + ik_i\tau_i} d^3\tau_i \; d^3\tau_{0i} \; dt \; dt_0$$

where $\tau_i = \tau_i'' - \tau_i'$ and $t = t'' - t'$ are the space and time intervals between the two points, respectively. Performing the integration over t_0 will result in the time averaging of $|S_{\text{turb}}(\omega, k_i)|^2$ as follows:

$$|S_{\text{turb}}(\omega, k_i)|^2 = \frac{T}{(2\pi)^8} \int\int\int\int < S^*_{\text{turb}}(t_0 + \frac{t}{2}, \tau_{0i} + \frac{\tau_i}{2}) \\ S_{\text{turb}}(t_0 - \frac{t}{2}, \tau_{0i} - \frac{\tau_i}{2}) > e^{-i\omega t + ik_i\tau_i} d^3\tau_i \, d^3\tau_{0i} \, dt \tag{35}$$

Using subscript i,j and l,m for two different locations and two different times, Equation (21) can be employed to calculate $|S^*_{\text{turb}}(t_0 + \frac{t}{2}, \tau_{0i} + \frac{\tau_i}{2}) S_{\text{turb}}(t_0 - \frac{t}{2}, \tau_{0i} - \frac{\tau_i}{2})|$.

$$|S^*_{\text{turb}}(t_0 + \frac{t}{2}, \tau_{0i} + \frac{\tau_i}{2}) S_{\text{turb}}(t_0 - \frac{t}{2}, \tau_{0i} - \frac{\tau_i}{2})| = (\rho_o^2 e^{2I_c})[k_i k_j u_i' u_j' \\ -\frac{1}{4}\frac{(c_{sz}')^2}{c_s^2} u_3' u_3' - \frac{c_{sz}''}{2c_s} u_3{}' u_3{}' + \frac{i}{2c_s}(\frac{3}{2} c_{sz}' u_3' k_j u_j' + \frac{1}{2} c_{sz}' u_3' k_i u_i')][k_l k_m u_l'' u_m'' - \\ -\frac{1}{4}\frac{(c_{sz}')^2}{c_s^2} u_3' u_3' - \frac{c_{sz}''}{2c_s} u_3{}' u_3{}' - \frac{i}{2c_s}(\frac{3}{2} c_{sz}' u_3' k_j u_j' + \frac{1}{2} c_{sz}' u_3' k_i u_i')] \, . \tag{36}$$

Spectral efficiency, $\Upsilon(t, \vec{\tau})$, is obtained by expanding the above equation, ignoring the complex part because, being an odd function of ω, it disappears upon integration over ω. Then, the result is

$$\begin{aligned} \Upsilon(t, \tau_i) = {} & k_i k_j k_l k_m u_i' u_j' u_l'' u_m'' - \frac{1}{2}\left[\frac{1}{2}\left(\frac{c_{sz}'}{c_s}\right)^2 + \frac{c_{sz}''}{c_s}\right] k_i k_j u_i' u_j' u_3'' u_3'' \\ & - \frac{1}{2}\left[\frac{1}{2}\left(\frac{c_{sz}'}{c_s}\right)^2 + \frac{c_{sz}''}{c_s}\right] k_l k_m u_l'' u_m'' u_3' u_3' \\ & + \frac{1}{4}\left[\frac{1}{4}\left(\frac{c_{sz}'}{c_s}\right)^4 + \left(\frac{c_{sz}''}{c_s}\right)^2 + \frac{(c_{sz}')^2 c_{sz}''}{c_s^3}\right] u_3' u_3' u_3'' u_3'' \\ & + \left(\frac{c_{sz}'}{c_s}\right)^2 k_j u_j' k_m u_m'' u_3' u_3'' \, , \end{aligned} \tag{37}$$

which allows us to write the source function as

$$|S_{\text{turb}}(\omega, k_i)|^2 = \frac{T e^{2I_c}}{(2\pi)^8} \int\int\int \rho_0^2 \Upsilon(t, \tau_i) e^{-i\omega t + ik_i\tau_i} d^3\tau_i d^3\tau_{0i} dt \, , \tag{38}$$

and the acoustic wave energy flux becomes

$$\vec{F}(\omega, k_i) = \frac{2\pi\hat{\tau}_i}{|\tau_i|^2} \frac{\sqrt{(\omega^2 - \Omega_i^2)}}{\omega} \int\int\int \frac{\rho_0(\tau_{0i})}{(4\pi^2)^2 c_s(\tau_{0i})} \Upsilon(\vec{\tau}, t) e^{-i\omega t + ik_i\tau_i} d^3\tau_i d^3\tau_{0i} dt \, . \tag{39}$$

Substituting the value of $c_{sz}'/c_s = 2\Omega_0$, Equation (37) simplifies to

$$\Upsilon(t, \tau_i) = < (k_i u_i')^2 (k_i u_i'')^2 > -3\Omega_0^2 \{ < (k_i u_i')^2 u_3'' u_3'' > + < (k_i u_i'')^2 u_3' u_3' > \} \\ +9\Omega_0^4 < u_3' u_3' u_3'' u_3'' > +4\Omega_0^2 < (k_i u_i')(k_i u_i'') u_3' u_3'' > \, . \tag{40}$$

The fourth-order velocity correlation in Equation (40) can be reduced to a second-order velocity correlation [31], and the result is

$$< u_1 u_2 u_3' u_4' > = < u_1 u_2 > < u_3' u_4' > + < u_1 u_3' > < u_2 u_4' > + < u_1 u_4' > < u_2 u_3' > \, , \tag{41}$$

which gives

$$\Upsilon(t,\tau_i) = 2w^4 < v'v'' >^2 + 4\Omega_0^2 w^2 < v'v'' >< u_3'u_3'' > -8\Omega_0^2 w^2 < v'u_3'' >^2 + 18\Omega_0^4 < u_3'u_3'' >^2 , \quad (42)$$

that is derived in Appendix A.3.

4.4. Convolution of Turbulence Energy Spectra

Generally, the Fourier transform of the product of the second-order velocity correlations is expressed as

$$\frac{1}{(2\pi)^4}\int d^3\vec{\tau}\int_{-\infty}^{\infty} e^{-i(\omega t-\vec{k}\cdot\vec{\tau})}dt < u_l'u_m'' >< u_n'u_o'' > = \int\int \lambda_{lm}(\vec{k}-\vec{p},\omega-\sigma)\lambda_{no}(\vec{p},\sigma)d^3\vec{p}\,d\sigma = J_{lmno} , \quad (43)$$

where λ_{ij}, the Fourier transform of the velocity correlation $< u_i'u_j'' >$, is defined as follows:

$$\lambda_{ij} = \frac{1}{(2\pi)^4}\int\int < u_i(\vec{x},t_0)u_j(\vec{x}+\vec{r},t_0+t) > e^{i(\omega t-\vec{k}\cdot\vec{r})}d^3\vec{r}\,dt . \quad (44)$$

From the phenomenological treatment of turbulence, the correlations between the instantaneous velocity components at two different locations in the turbulent region can be evaluated when a turbulent energy spectrum $E(\vec{k},\omega)$ is specified. Assuming the turbulence to be isotropic, homogeneous, and incompressible [30,31], λ_{ij} can be expressed as

$$\lambda_{ij}(\vec{k},\omega) = \frac{E(\vec{k},\omega)}{4\pi k^2}(\delta_{ij}-\frac{k_ik_j}{k^2}) . \quad (45)$$

Even though the fluid is non-isothermal, it can be treated locally as homogeneous and isotropic, hence the application of the above equation. It is further assumed that the turbulence energy spectrum, $E(\vec{k},\omega)$, can be factored into the frequency independent spatial turbulent energy spectrum E(k) and the turbulent frequency factor $\Delta(\omega,k)$, $E(\vec{k},\omega) = E(k)\Delta(\omega,k)$, which in turn can be substituted in Equation (45) to simplify the calculation of Equation (43).

$$\frac{1}{(4\pi)^2}\int\int \frac{E(\vec{k}-\vec{p})}{q^2}\frac{E(\vec{p})}{p^2}\Delta(\omega-\sigma,\vec{k}-\vec{p})\Delta(\sigma,\vec{p}) (\delta_{lm}-\frac{k_lk_m}{q^2})(\delta_{no}-\frac{k_nk_o}{p^2})d^3\vec{p}\,d\sigma = J_{lmno} , \quad (46)$$

where $\vec{q} \equiv \vec{k}-\vec{p}$. The integration of σ, which has the frequency terms only, can be performed separately,

$$g(p,q,\omega) \equiv \int_{-\infty}^{\infty}\Delta(\omega-\sigma,\vec{q})\Delta(\sigma,\vec{p})d\sigma$$

and Equation (46) can be rewritten as

$$\frac{1}{(4\pi)^2}\int \frac{E(\vec{q})}{q^2}\frac{E(\vec{p})}{p^2}g(p,q,\omega)(\delta_{lm}-\frac{k_lk_m}{q^2})(\delta_{no}-\frac{k_nk_o}{p^2})d^3p = J_{lmno} . \quad (47)$$

The integration of $d^3\vec{p}$ is simplified by taking $\vec{k}$ as the axis of the spherical coordinate system, $d^3p = p^2dp\sin\theta d\theta d\phi = p^2dpd\mu d\phi = 2\pi p^2dpd\mu$ with $\mu = \cos\theta_{pk} = \cos\theta$ and $|q| = \sqrt{(k^2+p^2-2kp\mu)}$. The Fourier transforms of velocity correlations appearing in (42) contain four terms, namely, J_{kkkk}, J_{kzkz}, J_{kkzz} and J_{zzzz}. Next we substitute J_{kkkk}, J_{kzkz}, J_{kkzz},

and J_{zzzz} in (42), take $(\frac{1}{8\pi})\int_0^\infty dp\int_{-1}^{+1} d\mu \frac{E(q)E(p)}{q^2} g(p,q,\omega)$ as common, and define the remaining equation as $f(\omega,\theta,p,q,\mu)$.

$$f(\omega,\theta,p,q,\mu) = f_q + f_d + f_m$$

$$f_q = 2(w)^4 \frac{p^2}{q^2}(1-\mu^2)^2,$$

$$f_d = -8w^2\Omega_0^2\left\{\mu^2\cos^2\theta_k(1-\frac{p^2}{q^2}(1-\mu^2)) + (\frac{p^2}{q^2}\mu^2 - \frac{pk\mu}{q^2})\frac{1}{2}(1-\mu^2)\sin^2\theta_k\right\}$$
$$+4w^2\Omega_0^2\left\{\frac{p^2}{q^2}(1-\mu^2)\{1-\mu^2\cos^2\theta_k - \frac{1}{2}(1-\mu^2)\sin^2\theta_k\}\right\},$$

and

$$f_m = 18\Omega_0^4\left\{1-\mu^2\cos^2\theta_k - \frac{1}{2}(1-\mu^2)\sin^2\theta_k + \frac{k^2}{q^2}\{-\cos^2\theta_k + \mu^2\cos^4\theta_k\right.$$
$$+\frac{1}{2}(1-\mu^2)\cos^2\theta_k\sin^2\theta_k\} + \frac{p^2}{q^2}\{-\mu^2\cos^2\theta_k + \mu^4\cos^4\theta_k$$
$$+3\mu^2(1-\mu^2)\cos^2\theta_k\sin^2\theta_k - \frac{1}{2}(1-\mu^2)\sin^2\theta_k + \frac{3}{8}(1-\mu^2)^2\sin^4\theta_k\}$$
$$\left.+\frac{2pk\mu}{q^2}\{\cos^2\theta_k - \mu^2\cos^4\theta_k - \frac{3}{2}(1-\mu^2)\cos^2\theta_k\sin^2\theta_k\}\right\}$$

The presented explicit forms of $f(\omega,\theta,p,q,\mu)$ allow us to calculate and give the final expression for the acoustic wave energy flux generated by turbulent motions.

4.5. Acoustic Wave Energy Flux and Its Discussion

The emitted acoustic energy flux for a given frequency, calculated in τ space, is given by

$$\vec{F}(t,\tau_i) = \frac{1}{16}\frac{\hat{\tau}_i}{|\tau_i|^2}\frac{(\omega^2-\Omega_i^2)^{1/2}}{\omega}e^{2I_c}\int\frac{\rho_0(\tau_{0i})}{c_s(\tau_{0i})}d^3\tau_{0i}$$
$$\int_0^\infty dp\int_{-1}^{+1} d\mu\frac{E(q)E(p)}{q^2}g(p,q,\omega)f(\omega,k,p,q,\mu)\,, \tag{48}$$

and the expression is valid for a fluid with temperature and density gradients related to each other, and with constant pressure, which means that there is no gravity. The flux was derived under the assumption of isotropic and homogeneous turbulence, whose spectrum and frequency factor must be specified.

As the above results show, the generated acoustic wave energy flux is calculated by making the multipole expansion of the source function given by Equation (36). This allows the identification of contributions from different wave sources and writing the source function in the following form: $S_a[p_o,u_t] = S_{quadrupole} + S_{dipole} + S_{monopole}$, where $S_{quadrupole} \sim \omega^4$, $S_{dipole} \sim \omega^2\Omega_0^2$, and $S_{monopole} \sim \Omega_0^4$. The dipole and monopole source terms describe the conversion of kinetic energy into acoustic energy resulting from forcing the mass and momentum in a fixed region of space to fluctuate, respectively, and are produced by the density gradient.

In the original approach presented by Stein [15], see also [17], the external gravitational force is responsible for the density gradient and the stratification of the background solar atmosphere, which is assumed to be isothermal. Studies by Goldreich and Kumar [19]

revealed that the forced turbulence occurring in the solar convection zone is driven by the fluctuating buoyancy force, which represents the coupling of gravity to density fluctuations associated with the turbulent field, and it leads to dipole emission that may dominate over the quadrupole source. There are similarities between the dipole terms obtained in these two papers, which shows that the fluctuating buoyancy force can be accounted for by either the method proposed in [19] or by the multipole expansion considered in [15].

In all previous approaches to acoustic wave generation by isotropic and homogeneous turbulence [1,2,7,15,17,19], the background fluid was assumed to be isothermal. The results presented in this paper demonstrate that the temperature gradient leads to monopole and dipole emissions, which depend directly on the acoustic cutoff frequency. The model considered in this paper is simple; nevertheless, its results are important because temperature gradients exist in realistic planetary, solar, and stellar atmospheres, and they affect the hydrostatic equilibrium of such atmospheres by changing their densities and pressures. Moreover, temperature gradients also make the wave speed to be a function of atmospheric height, which modifies the wave cutoff frequencies and wave propagation conditions.

The acoustic power output, P_s, obtained in this paper can be written in the following form

$$P_s = \frac{\rho_o u_t^3}{l_o}\left(\alpha_q M_t^5 + \alpha_d M_t^3 + \alpha_m M_t\right) \tag{49}$$

where α_q, α_d, and α_m represent the emissivity coefficients of quadrupole, dipole, and monopole sources, respectively, and $M_t = u_t/c_s$ is the turbulent Mach number. The previous results (e.g., [1,2,15]) showed that quadrupole emission dominates in the acoustic wave energy spectrum, and it is a sensitive function of the wave frequency, the turbulent energy spectrum, the turbulent frequency factor, and the physical parameters in the region of wave generation [17]. However, there are some special astrophysical conditions when contributions from the dipole and monopole sources may become important (e.g., [14,19]). They may also become significant in some realistic jet experiments and laboratory settings where the acoustic cutoff frequency plays a role because of the presence of temperature gradients.

The results presented in this paper can be used to include temperature gradients in Stein's approach [15] that considered only stratification. Such an extension would make the theory of sound generation to be applicable to various background media with different gradients in physical parameters, including planetary, solar, and stellar atmospheres. However, based on the theory developed in this paper, it seems unlikely that it can be done analytically; instead, it would require numerical simulations (e.g., [32]), similar to those described in [33] or recently performed in [34].

It must be also pointed out that the presence of the acoustic cutoff frequency in the developed theory of sound generation restricts frequencies of the generated waves to those that are higher than the cutoff at the wave source. Moreover, the wave propagation away from the source is affected by the cuttoff and may lead to wave reflection that may limit the wave energy transfer to the layers located above the wave source. Both effects make the presented theory more realistic than the original Lighthill theory of acoustic wave generation.

5. Possible Numerical Applications and Experimental Verifications

The general formula for the acoustic wave energy flux given by Equation (48) is an exact analytical expression obtained for a specific temperature gradient. Thus, by selecting the turbulent energy spectrum and the turbulent frequency factor and setting a turbulent and non-isothermal layer with the required temperature gradient, the formula can be used to test the accuracy of the numerical codes developed to perform sound generation by such layers.

The acoustic analogy predictions based on the original Lighthill theory of sound generation were tested against direct numerical simulation (DNS), and good agreements were reported (e.g., [35,36]). The hybrid direct numerical simulations performed in [35] combined DNS with the Lighthill acoustic analogy and found the acoustic power in the simulations was quadrupole. Similarly, a good agreement was found in simulations of acoustic noise generation during the transition of isothermal and non-isothermal mixing layers [36]. However, with the DNS being only restricted to moderate and small Reynolds numbers, other ways of testing the theory are required for high Reynolds numbers that are observed in some turbulent jets (e.g., [37]). Moreover, a correlation analysis of flow and sound in non-isothermal subsonic jets that used large eddy simulations demonstrated that the acoustic correlations cancel each other in hot jets, but they enhance each other in cold jets [38].

The main prediction of the Lighthill theory that the sound generation obeys the u^8 law of the acoustic power output by the turbulent jet, where u is the jet velocity, was found to be in good agreements with the results of several experiments performed for jets of different diameters (e.g., [9]). A computational aeroacoustics prediction based on Lighthill's theory to calculate noise from subsonic turbulent jets was shown to be in a good agreement with narrowband measurements for isothermal and hot jets with an acoustic Mach number ranging from 0.5 to 1.0 [39]. At high Reynolds numbers, a unique data set of measurements in a large wind tunnel was obtained and analyzed in [40], and it was demonstrated that the Lighthill main result is a good approximation, which provides additional support to Lilley's approach described in [37]. Studies of high-speed airstream flows through pipeline valves showed the origin of acoustic resonances as well as stochastic noise generation that can be accurately described by the method of Lighthill acoustic analogy [41].

The above numerical and experimental verifications of the original Lighthill theory of sound generation can also be used to verify the results presented in this paper. Specifically, numerical simulations of non-isothermal jets can be performed, and the obtained numerical results can be compared to the theoretical predictions resulting from Equation (48); however, performing such numerical computations is out of the scope of this paper and will be reported elsewhere. Similarly, the effects of temperature gradients in turbulent jets can be investigated using the developed theory, whose predictions can be tested by performing experiments with non-isothermal and turbulent jets.

6. Conclusions

The theory of generation of acoustic waves by a region of isotropic and homogeneous turbulence embedded in a fluid with gradients in temperature and density, but constant pressure, is developed. The theory extends the original Lighthill theory of sound generation by accounting for the effects of the gradients on the wave source and on the wave propagation outside the source. It is shown that the gradients lead to monopole and dipole wave emission, whose efficiency may exceed Lighthill's quadrupole emission depending on the values of the gradients. The wave propagation away from the source is also affected by the gradients, which give the origin to the acoustic cutoff frequency. It is the latter that uniquely determines the efficiency of monopole and dipole emissions, and it also sets up the wave propagation conditions in the fluid.

The main result of this paper is the general formula for the generated acoustic wave energy flux. To guarantee analytical solutions, the formula is obtained for a simple model of non-isothermal fluid, in which the speed of sound varies linearly with height, which causes variations in density; however, pressure remains constant as there is no gravity in this model. The presented theory and its results are compared to the original Lighthill work [1] as well as to the extended version of Lighthill's theory that was developed by Stein [15],

who showed that atmospheric stratification is responsible for the origin of monopole and dipole wave emissions. The results of this paper demonstrate that temperature gradients are also responsible for both monopole and dipole emissions because they modify the equilibrium at the background fluid; the effect would be even more prominent in planetary, solar, and stellar atmospheres, where temperature gradients can be strong and they will significantly influence the hydrostatic equilibrium in these atmospheres.

Finally, previously performed numerical simulations of jets can be extended to non-isothermal jets and the obtained numerical results can be compared to those resulting from the theory developed in this paper. It is suggested that a direct comparison between the predictions of the theory and experimental results obtained for non-isothermal and turbulent jets can also be made.

Author Contributions: Conceptualization, S.R. and Z.E.M.; Methodology, S.R. and Z.E.M.; Formal analysis, S.R. and Z.E.M.; Writing—original draft, S.R. and Z.E.M. All authors have read and agreed to the published version of the manuscript.

Funding: S.R. research was partially funded by Birla Institute of Technology and Science, Pilani-Dubai Campus.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: We are indebted to the three anonymous reviewers for their constructive comments and suggestions that allowed us to significantly improve the original version of this paper.

Conflicts of Interest: The authors declare that they have no conflicts of interest regarding this manuscript.

Appendix A

Appendix A.1. Derivation of the Source Function

The general form of the source function is given by Equation (24), and it can be written in the following form

$$\begin{aligned} S_{\text{turb}}(\omega, k_i) &= \frac{-1}{(2\pi)^4} \int (\rho_0 u_i u_j)\left(\frac{\partial^2}{\partial \tau_i \partial \tau_j} + c'_{si}\frac{\partial}{\partial \tau_j}\right) e^{I_c} e^{-i(\omega t - k_i \tau_i)} d^3\tau_i \, dt \\ &= \frac{-1}{(2\pi)^4} \int (\rho_0 u_i u_j) f(\tau_i, t) d^3\tau_i \, dt \,, \end{aligned} \tag{A1}$$

where

$$\begin{aligned} f(t, \tau_i) &= \left(\frac{\partial^2}{\partial \tau_i \partial \tau_j} + c'_{si}\frac{\partial}{\partial \tau_j}\right) e^{I_c} e^{-i(\omega t - k_i \tau_i)} \\ &= \left[-k_i k_j + i\frac{1}{2}k_j c'_{si} + i\frac{1}{2}k_i c'_{sj} + \frac{1}{4}c'_{si}c'_{sj} + c'_{si} i k_j + \frac{1}{2}c'_{si}c'_{sj}\right] e^{I_c} e^{-i(\omega t - k_i \tau_i)} \\ &= \left[-k_i k_j + \frac{3}{4}c'_{si}c'_{sj} + i\left(\frac{1}{2}k_i c'_{sj} + \frac{3}{2}k_j c'_{si}\right)\right] e^{I_c} e^{-i(\omega t - k_i \tau_i)} \,. \end{aligned} \tag{A2}$$

Substituting Equation (A2) into Equation (A1) gives

$$\begin{aligned} S_{\text{turb}}(\omega, k_i) = \frac{1}{(2\pi)^4} \int \left\{k_i k_j - \frac{3}{4}c'_{si}c'_{sj} - i\left[\frac{1}{2}c'_{sj}k_i + \frac{3}{2}c'_{si}k_j\right]\right\} (e^{I_c}\rho_0 u_i u_j) \\ e^{-i(\omega t - k_i \tau_i)} d^3\tau_i \, dt \,, \end{aligned} \tag{A3}$$

which when compared to

$$S_{\text{turb}}(\omega, k_i) = \frac{1}{(2\pi)^4} \int S_{\text{turb}}(t, \tau_i) e^{-i(\omega t - k_i \tau_i)} d^3\tau_i \, dt \,, \tag{A4}$$

results in

$$S_{\text{turb}}(t, \tau_i) = e^{I_c} \rho_0 \{ k_i k_j u_i u_j - \frac{1}{4} \left(\frac{c_s'}{c_s} \right)^2 u_3 u_3 - \frac{c_s''}{2c_s} - \frac{i}{2c_s} [\frac{1}{2} c_s' u_3 k_i u_i + \frac{3}{2} c_{sz}' u_3 k_j u_j] \} \tag{A5}$$

that is Equation (25) in the main text.

Appendix A.2. Calculation of the Acoustic Flux

Time averaging the flux requires the frequency of both factors p and u_i to be the same. Hence $k_i' = k_i'' = k_i$. Substituting (29) into (28) and performing the required algebra, the flux turns out to be

$$F_i(k_i) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \frac{1}{\rho_0 c_s} \int \frac{k_i}{\omega''} \frac{S_{\text{turb}}(\omega', k_i) S_{\text{turb}}^*(\omega'', k_i) e^{i(\omega' - \omega'')t} \, d^6 k_i d\omega' d\omega''}{\{ -\omega'^2 + {k_i}^2 + \Omega_i^2 \} \{ -\omega''^2 + {k_i}^2 + \Omega_i^2 \}} \,, \tag{A6}$$

where the "*" denotes a conjugate. Using $\int_{-\infty}^{\infty} e^{i(\omega' - \omega'')t} dt = 2\pi\delta(\omega' - \omega'')$ to integrate by time, the equation is reduced to

$$F_i(k_i) = \frac{1}{\rho_0 c_s} \int (\frac{k_i}{\omega''}) \frac{S_{\text{turb}}(\omega', k_i) S_{\text{turb}}^*(\omega'', k_i) \delta(\omega' - \omega'') d^6 k_i \, d\omega' d\omega''}{\{ -\omega'^2 + {k_i}^2 + \Omega_i^2 \} \{ -\omega''^2 + {k_i}^2 + \Omega_0^2 \}} \,. \tag{A7}$$

Integrating by $d\omega''$ and using $F_i(k_i) = \int F_i(\omega', k_i) d\omega'$, the expression for $F_i(\tau_i)$ is evaluated as

$$F_i(\omega, k_i) = \frac{1}{\rho_0 c_s} \int (\frac{k_i}{\omega}) \frac{S_{\text{turb}}(\omega, k_i) S_{\text{turb}}^*(\omega, k_i) d^3 k_i d^3 k_i}{\{ -\omega^2 + {k_i}^2 + \Omega_0^2 \} \{ -\omega^2 + {k_i}^2 + \Omega_i^2 \}} \,, \tag{A8}$$

which is Equation (30) in the main text.

Appendix A.3. Fourth-Order Turbulent Correlations

The fourth-order velocity correlation in Equation (40) can be reduced to a second-order velocity correlation [30], which gives

$$\begin{aligned} \Upsilon(\tau_i, t) = w^4 < v'^2 v''^2 > -3\Omega_0^2 w^2 < v'^2 u_3''^2 > -3\Omega_0^2 w^2 < v''^2 u_3'^2 > \\ +9\Omega_0^4 < u_3'^2 u_3''^2 > +16\Omega_0^2 w^2 < v' v'' u_3' u_3'' > \,. \end{aligned} \tag{A9}$$

Each term in the above equation can be simplified by using

$$\begin{aligned} < v' v' v'' v'' > &= 2 < v' v'' >^2 \\ < v' v' u_3'' u_3'' > &= 2 < v' u_3'' >^2 \\ < v'' v'' u_3' u_3' > &= 2 < v'' u_3' >^2 \\ < u_3' u_3' u_3'' u_3'' > &= 2 < u_3' u_3'' >^2 \\ < v' v'' u_3' u_3'' > &= < v' v'' >< u_3' u_3'' > + < v' u_3'' >^2 \end{aligned}$$

The resulting spectral efficiency turns out to be (42) in the main text after taking $< v' u_3'' >^2 = < v'' u_3' >^2$.

References

1. Lighthill, M.J. On sound generated aerodynamically I. General theory. *Proc. R. Soc. Lond. A* **1952**, *211*, 564.
2. Proudman, I. The generation of noise by isotropic turbulence. *Proc. R. Soc. Lond. A* **1952**, *214*, 119.
3. Heisenberg, W. On the theory of statistical and isotropic turbulence. *Proc. R. Soc. Lond. A* **1947**, *189*, 456.
4. Kolmogorov, A.N. Doklady Akademy Nauk. *SSSR* **1941**, *30*, 301.
5. Curle, N. The influence of solid boundaries upon aerodynamic sound. *Proc. R. Soc. Lond. A* **1955**, *231*, 505–514.
6. Ffowcs Willaims, J.E.; Hall, L.H. Aerodynamic sound generation by turbulent flow in the vicinity of a scattering half plane. *J. Fluid Mech.* **1970**, *40*, 657. [CrossRef]
7. Kulsrud, R.M. Effect of Magnetic Field on Generation of Noise by Isotropic Turbulence. *Astrophys. J.* **1955**, *121*, 461. [CrossRef]
8. Lighthill, M.J. Studies on Magneto-Hydrodynamic Waves and other Anisotropic Wave Motions. *Phil. Trans. R. Soc. Lond. A* **1960**, *252*, 397.
9. Goldstein, M.E. *Aeroacoustic*; Mc-Graw-Hill International Book Co.: New York, NY, USA, 1976.
10. Unno, W.; Kawabata, K. The effect of short-periodic oscillations in the photosphere on the spectral line profile. *Publ. Astron. Soc. Jpn.* **1955**, *7*, 21.
11. de Jager, C.; Kuperus, M. The acoustic energy flux of the sun and the formation of the corona. *Bull. Astron. Inst. Neth.* **1961**, *16*, 71.
12. Kuperus, M. The Transfer of Mechanical Energy in the Sun and the Heating of the Corona. *Rech. Astron. Obs. Utrecht* **1965**, *17*, 1.
13. Renzini, A.; Cacciari, C.; Ulmschneider, P.; Schmitz, F. Theoretical chromospheres of late type stars. I - Acoustic energy generation. *Astron. Astrophys.* **1977**, *61*, 39.
14. Bohn, H.U. Generation of acoustic energy from convection zones of late type stars. *Astron. Astrophys.* **1984**, *136*, 338.
15. Stein, R.F. Generation of acoustic and gravity waves by turbulence in an isothermal stratified atmosphere. *Sol. Phys.* **1967**, *2*, 385. [CrossRef]
16. Unno, W.; Kato, S. On the generation of acoustic noise from the turbulent atmosphere. *Publ. Astron. Soc. Jpn.* **1962**, *14*, 417.
17. Musielak, Z.E.; Rosner, R.; Stein, R.F.; Ulmschneider, P. On sound generation by turbulent convection: A new look at old results. *Astrophys. J. Part* **1994**, *423*, 474. [CrossRef]
18. Ulmschneider, P.; Theurer, J.; Musielak, Z.E. Acoustic wave energy fluxes for late-type stars. *Astron. Astrophys.* **1994**, *315*, 212.
19. Goldreich, P.; Kumar, P. The interaction of acoustic radiation with turbulence. *Astrophys. J.* **1988**, *326*, 462. [CrossRef]
20. Musielak, Z.E.; Musielak, D.E.; Mobashi, H. Method to determine acoustic cutoff frequencies in nonisothermal atmosphere. *Phys. Rev. E* **2006**, *73*, 036612-1. [CrossRef]
21. Murphy, G.M. *Ordinary Differential Equations and Their Solutions*; D. Van Nostrand Company: New York, NY, USA, 1960
22. Kahn, P.B. *Mathematical Methods for Scientists and Engineers*; John Wiley: New York, NY, USA, 1960
23. Lamb, H. On the theory of waves propagated vertically in the atmosphere. *Proc. Lond. Math. Soc. Ser.* **1908**, *2*, 122.
24. Lamb, H. *The Dynamical Theory of Sound*; Dover Publications Inc.: Mineola, NY, USA, 1960.
25. Suda, N.; Nawa, K.; Fukao, Y. Earth's background oscillations. *Science* **1998**, *279*, 2089. [CrossRef]
26. Rhie, J.; Romanowicz, B. Excitation of Earth's continuous free oscillations by atmospheric-ocean-seafloor coupling. *Nature* **2004**, *431*, 552. [CrossRef] [PubMed]
27. Lee, U. Acoustic oscillations of Jupiter. *Astrophys. J.* **1993**, *405*, 359. [CrossRef]
28. Basu, S. Global seismology of the Sun. *Living Rev. Sol. Phys.* **2016**, *13*, 2. [CrossRef]
29. Garcia, R.; Ballot, J. Asteroseismology of solar-type stars. *Living Rev. Sol. Phys.* **2019**, *16*, 4. [CrossRef]
30. Bachelor, G.K. *The Theory of Homogeneous Turbulence*; Cambridge University Press: Cambridge, UK, 1953.
31. Hinze, J.O. *Turbulence*; McGraw Hill: New York, NY, USA, 1975.
32. Candel, S.M. A Review of Numerical Methods in Acoustic Wave Propagation. In *Recent Advances in Aeroacoustics*; Krothapalli, A., Smith, C.A., Eds.; Springer: New York, NY, USA, 1986.
33. Ben-Ami, Y.; Manela, A. Acoustic wave propagation at nonadiabatic conditions: The continuum limit of a thin acoustic laye. *Phys. Rev. Fluids* **2020**, *5*, 033401. [CrossRef]
34. Kuźma, B.; Kadowaki, L.H.S.; Murawski, K.; Musielak, Z.E.; Poedts, S.; Yuan, D.; Feng, X. Magnetoacoustic cutoff effect in numerical simulations of the partially ionized solar atmosphere. *Phil. Trans. R. Soc. A* **2024**, *382*, 20230218. [CrossRef]
35. Sarkar, S.; Hussaini, M.Y. *Computation of the Sound Generated by Isotropic Turbulence*; ICASE Report 93-74; NASA: Washington, DC, USA, 1993.
36. Fortuné, V.; Lamballais, E.; Gervais, Y. Noise radiated by a non-isothermal, temporal mixing layer. Part I: Direct computation and prediction using compressible DNS. *Theor. Comput. Fluid Dyn.* **2004**, *18*, 61–81. [CrossRef]
37. Lilley, G.F. The radiated noise from isotropic turbulence. *Theor. Comput. Fluid Dyn.* **1994**, *6*, 281–301. [CrossRef]
38. Liu, Q.; Lai, H. Correlation analysis of flow and sound in non-isothermal subsonic jets based on large eddy simulations. *Phys. Fluids* **2022**, *34*, 045125. [CrossRef]
39. Ilário, C.R.S.; Azarpeyvand, M.; Rosa, V.; Self, R.H.; Meneghini, J.R. Prediction of jet mixing noise with Lighthill's acoustic analogy and geometrical acoustics. *J. Acoust. Soc. Am.* **2017**, *141*, 1203. [CrossRef] [PubMed]

40. Zhou, Y.; Praskovsky, A.; Oncley, S. On the Lighthill relationship and sound generation from isotropic turbulence. *Theor. Comput. Fluid Dyn.* **1995**, *7*, 355. [CrossRef]
41. Bai, C.; Chen, T.; Yu, W. Numerical and experimental research on single-valve and multi-valve resonant systems. *Front. Mater.* **2021**, *8*, 756158. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.