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Review

The Universal Presence of the Reynolds Number

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Abstract: The Reynolds number is a fundamental parameter in fluid dynamics, initially introduced by O. Reynolds in 1883 to characterize the transition between laminar and turbulent flow in fluids and necessary in the scaling of viscous resistance. Over time, its application has expanded significantly, becoming essential for studying a vast range of fluid phenomena—from microscopic scales such as cellular motion to macroscopic scales like turbulent flows and even intergalactic dynamics. The article highlights the universal relevance of the Reynolds number across various fields, including its adaptation to non-Newtonian fluids and granular flows. It emphasizes how the Reynolds number has evolved from a simple dimensionless group to a critical tool for understanding complex physical processes across different scales and environments.

Keywords: Reynolds number; similarity; scaling

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1. Introduction

Osborne Reynolds (1842–1912) was a British engineer, physicist, and educator renowned for his pioneering contributions to fluid mechanics and hydraulics. Born in Belfast and educated at Queens' College, Cambridge, he became the first professor of engineering at Owens College, today the University of Manchester, a position he held for nearly four decades. Reynolds' early work spanned magnetism, electricity, and heat transfer, but he soon focused on fluid mechanics, making significant advancements in turbine pumps, lubrication theory (1886), and heat transfer. His 1883 paper on the law of resistance in parallel channels remains a classic, as does his 1895 work, which presents the criterion for flow stability—a seminal contribution to fluid dynamics. The "Reynolds number" and "Reynolds stress" are named after him due to his foundational work in turbulence and fluid flow modeling. A Fellow of the Royal Society since 1877, Reynolds contributed to developing the Whitworth laboratories at Owens College and received numerous accolades, including the Royal Society's gold medal in 1888.

It is interesting to note that Reynolds' 1895 article is not mentioned in the obituaries written after his death. These obituaries highlight that "he wrote more than 70 papers on mechanics and physics, including 'The Law of Resistance in Parallel Channels', 'Theory of Lubrication', and 'Flow of Gases', all of which are familiar to engineers". However, his work *Sub-Mechanics of the Universe* is praised as "one of the most remarkable scientific works of the 19th century" [1,2]. *Sub-Mechanics* represents a more speculative exploration of fundamental concepts, aiming to develop a unified theory for various physical phenomena. Unlike his other seminal papers, it lacks the experimental rigor and mathematical precision that defined his contributions to mechanics and physics. In this work, Reynolds [3,4] hypothesized the existence of a foundational element of the universe, which he called the "purely mechanical medium" (comparable to the ether), to explain the mechanics of

fluid motion, the behavior of forces, and the interaction of matter at a fundamental level. However, this line of reasoning was soon overshadowed by developments in physics, notably the Michelson–Morley experiment (1887) [5] and Einstein's theory of relativity (1905) [6], both of which eliminated the need for an ether in explaining the behavior of light and gravity.

Rott presents a vivid history of the Reynolds number [7]. He mentions that Arnold Sommerfeld introduced the term "Reynolds number" in hydrodynamics in 1908 at the Fourth International Congress of Mathematicians held in Rome [8], using "R" to represent the dimensionless quantity $\rho UH/\mu$, which became widely adopted across fluid mechanics. U is the flow velocity, H is the pipe diameter, and ρ and μ are the density and dynamic viscosity of the fluid, respectively. However, this term was often mistakenly attributed to Blasius, who used it in his 1912 and 1913 publications [9,10]. Prandtl also played a significant role in the history of the Reynolds number, introducing it in his 1910 work on hydrodynamic similarity, where it was denoted by ξ [11]. Lord Rayleigh's 1892 work first referenced Reynolds' similarity law, which became fundamental in resolving the sphere drag problem [12]. Using dimensional analysis, he concluded that the pressure gradient in a pipe is a power function of the dimensionless parameter (cw/ν), i.e., the (dimensionless) pressure drop should be proportional to $(cw/\nu)^n$, where c is the pipe diameter, w is the velocity, and ν is the kinematic viscosity, respectively. Experimental data indicated that for large cw/ν values, n must be equal to 2, after which Raleigh shows that the pressure gradient is independent of the viscosity. In a note of less than one page published in 1913 [13], Rayleigh discusses results obtained by Eiffel regarding the resistance and fall velocity of spheres. He mentions (without providing references) that, according to the analyses by Stokes and Reynolds, the drag coefficient must depend on a single parameter (ν/VL), where V is the fall velocity, and L is a linear dimension proportional to $S^{1/2}$, with S being the "diametral surface" of the sphere. According to Rott [7], Rayleigh refers to Stokes' 1850 work [14], where, in studying the similarity of pendulum motion in a viscous medium, Stokes concludes that for systems to be similar, in addition to geometric similarity, they must be dynamically similar if they satisfy the proportion $a^2/T \propto \mu'$, where a corresponds to a line, and T is the time it takes the pendulum to travel along the line. μ' is defined as μ/ρ, which Stokes refers to as the "index of friction of the fluid" and endorses its use, noting that its dimensions are "the square of a line divided by a time, whence it will be easy to adapt the numerical value of μ' to a new unit of length or of time". In this way, Stokes would be the first to establish the importance of the Reynolds number (without naming it as such) in preserving the dynamic similarity of systems where viscosity is the cause of the resistive force.

Reynolds' 1883 pioneering work on flow visualization and critical Reynolds numbers was crucial in understanding turbulent flow. His experiments demonstrated that turbulent flow occurs above a critical dimensionless number, which varies with factors like pipe diameter and fluid properties [15]. In his paper, using dimensional analysis and taking advantage of Stokes' observation that μ/ρ or μ' "is a quantity of the nature of the product of a distance and a velocity", he established that the "birth of eddies [depends] on some definite value of $\rho\ c\ U/\mu$", where c is a "single linear parameter, say the radius of the tube", and U is a "single velocity parameter, say the mean velocity along a tube". To determine the cause of the "eddies", he used a result obtained by Stokes in 1843, which states that "under certain circumstances, the steady motion becomes unstable, so that an indefinitely small disturbance may lead to a change to sinuous motion". To establish the instability condition, Reynolds did not use the dimensionless parameter he derived. As was customary at the time, viscosity was not directly used in the computations. Instead, he used the Poiseuille relation that connects relative viscosity $P = \nu/\nu_0$ with temperature

(ν_0 is the viscosity of water at 0 °C), expressing numerically the condition of instability not in terms of the parameter $\rho\ c\ U/\mu$, as he had initially proposed in his paper, but as $P/(UD) = B_s$, with B_s in s/m^2. Reynolds found two extreme values for B_s. The first, corresponding to "at which steady motion broke down", is 43.73 s/m^2, and the second, "at which previously existing eddies would die out", is 278 s/m^2. In terms of the Reynolds number $Re = UD/\nu = 1/(\nu_0 B_s)$, using $\nu_0 = 1.793 \times 10^{-6}$ m^2/s [16], these threshold values correspond to Re equal to 12,736 and 2006, respectively. Reynolds later refined these ideas in his 1895 paper, using the Reynolds number explicitly to explore the transition from laminar to turbulent flow, establishing that the flow is stable for $Re < 1900$ and unstable for $Re > 2000$ [17]. Reynolds' application of dimensional analysis and power laws laid the foundation for modern turbulence theory, with his findings influencing both theoretical and practical aspects of fluid mechanics. The Reynolds number became crucial in understanding flow behavior and pressure losses in pipes, with significant contributions from Prandtl and von Kármán in later years [7]. By 1910, the Reynolds number had gained broader acceptance in engineering, particularly in aerodynamics and hydraulics, as seen in Blasius' influential work on pressure loss in pipes and turbulence theory [10]. At the 50th General Assembly of the Association of German Engineers in Mainz in 1909, Prandtl mentioned, referring to the scaling of airplanes and airships for testing in wind tunnels, that "in the absence of gravity effects, another similarity rule can be specified (it was probably first specified by O. Reynolds), to which the friction processes also strictly conform" [18]. At the beginning of the second decade of the 20th century, the concept of similarity, scaling, and the use of dimensionless parameters in fluid mechanics and, particularly, in hydraulics, was already well established, with notable works by von Mises (1914) [19], Forchheimer (1914) [20], Weber (1918) [21], among others. In the 1918 Annual General Meeting of the Shipbuilding Technical Society, held in Berlin, Weber presented an extensive paper (119 pages) in which he definitively laid the foundations of the laws of similarity and their use in physical models, "with special consideration for applications in shipbuilding". In it, Weber states that to ensure the dominance of gravitational forces, the Froude number, which he defines as $\varphi = V^2/(L\ g)$, must be preserved in the model and prototype. V and L correspond to characteristic velocity and length, respectively, and g is the acceleration due to gravity. To preserve the effect of viscous frictional forces, the Reynolds number must be the same in both the model and the prototype, denoted as $\psi = VL/\nu$. The preservation of capillary effects is achieved through the parameter $\xi = V^2L/\omega$, where ω is a "measure of dynamic capillarity", defined as $\omega = S/\rho$, with S being the surface tension and ρ the density. Weber refers to the dimensionless parameter ξ as "the unnamed number". It is now known as the Weber number. Additionally, in his work, Weber analyzes the deformation of an elastic solid, determining the Cauchy number (which he denotes as χ), and the case of phenomena subject to "general gravity". The dimensionless parameter he introduces is denoted η and corresponds to the ratio of centrifugal inertial force to the resultant force of Newton's law of universal gravitation. Sommerfeld's notation for the Reynolds number had also become popular in turbulence studies, at least among German authors. As an example, R is used by Noether (1921) [22], Schiller (1922) [23], and Heisenberg (1924) [24], among other authors.

The reader can find information about Osborne Reynolds' biography, his works, and the history of the dimensionless parameter named after him in the articles by Rott (1990), Jackson (1995 and 2004), and Jackson and Launder (2007) [7,25–27]. The purpose of this article is not to recount the history of the number of Reynolds but to examine the concept that the Reynolds number has evolved beyond its origins in fluid dynamics, demonstrating its diverse applications in systems involving motion. Initially defined as the ratio of inertial to viscous forces in a fluid, the Reynolds number has undergone significant

reinterpretation. Today, it is a crucial analytical tool for understanding phenomena across various scales—from cellular motion at the microscopic level to turbulent flows and even intergalactic dynamics. This article highlights the Reynolds number's universal relevance across multiple disciplines, including its innovative adaptations for non-Newtonian fluids and granular flows.

Recently, a review article titled "The Reynolds number: A journey from its origin to modern applications" by Saldana et al. [28] was published in *Fluids*. Although it addresses the same topic as the present article, both covering the Reynolds number from its origin to more recent applications across a wide variety of fields, there are notable differences between the two. The article by Saldana et al. includes a bibliometric analysis and is more reference-oriented, featuring an extensive list of citations. It is more focused on the practical history of the Reynolds number, with a strong emphasis on its role in engineering and the development of numerical and experimental techniques. Although the present manuscript also provides a broad exploration of the Reynolds number applications across various disciplines, its central research focus lies in examining whether the Reynolds number, as a concept, has evolved beyond its original definition in fluid dynamics. Specifically, we investigate how its reinterpretation and adaptation have enabled its application to diverse systems, such as non-Newtonian fluids, granular flows, plasma physics, and relativistic fluid dynamics. This work hypothesizes that the Reynolds number's universal relevance stems from its ability to adapt to different physical contexts while retaining its fundamental meaning as a ratio of inertial to viscous forces. By addressing this hypothesis, the article aims to demonstrate that the Reynolds number is not merely a historical artifact of fluid mechanics but a versatile analytical tool with far-reaching implications.

2. Derivation of the Reynolds Number

2.1. The Reynolds Number as the Ratio Between Inertia and Viscous Forces

The derivation of the Reynolds number as a dimensionless parameter that relates inertia to viscous forces can be done in several ways. Probably the simplest is through dimensional analysis, as Reynolds did [15]. Generating dimensionless parameters is quite simple. The tricky part is the ability to recognize the variables that genuinely govern the physical process being studied and a deep understanding of the phenomenon. In his derivation, Reynolds acknowledges the effect of viscosity on the flow characteristics ("*the more viscous a fluid is, the less prone it is to eddying or sinuous motion*", in cursive in Reynolds' paper), as well as the usefulness of characterizing this effect through μ', the "index of friction of fluid" defined by Stokes [14] and now called kinematic viscosity, v, which is "a quantity of the nature of the product of a distance and a velocity". Assuming the motion depends "on a single velocity parameter" $\mathcal{U}$, and "on a single linear parameter" $\mathcal{L}$, we obtain the following:

$$Re = \frac{\mathcal{U}\,\mathcal{L}}{\nu} \tag{1}$$

Another way to obtain the Reynolds number is by taking the ratio between the inertia and the viscous force in the flow. Considering, following Reynolds [15], a unique length and velocity scales, $\mathcal{L}$ and $\mathcal{U}$, and no external temporal forcing acting on the flow, so that the only possible time scale is $\mathcal{T} = \mathcal{L}/\mathcal{U}$, the inertia of the flow, I, scales as follows:

$$I = \frac{d}{dt}(m\,\mathcal{U}) \;\sim\; \frac{\rho\,\mathcal{V}\,\mathcal{U}}{\mathcal{T}} \;\sim\; \frac{\rho\,\mathcal{L}^3\,\mathcal{U}}{\mathcal{L}/\mathcal{U}} = \rho\,\mathcal{L}^2\,\mathcal{U}^2 \tag{2}$$

In the previous relation, m is the mass, ρ is the density, and $\mathcal{V}$ is the volume of the fluid element. The resistive force of viscous origin scales with the flow characteristics in the following way:

$$F_V = \tau\, A_w \;\sim\; \mu\, \dot{\gamma}\, A_w \;\sim\; \mu\, \frac{\mathcal{U}}{\mathcal{L}}\, \mathcal{L}^2 = \mu\, \mathcal{U}\, \mathcal{L} \tag{3}$$

where $\dot{\gamma}$ is the shear rate, and A_w is the area element where the viscous stress acts. Defining the Reynolds number as the ratio of inertia to viscous force, $Re = I/F_V$, we obtain $Re = \mathcal{U}\, \mathcal{L}/\nu$.

The Reynolds number naturally arises as one of the control parameters governing the Navier–Stokes equations, which for body forces resulting from a gravitational field ($\vec{g}$) is written as follows:

$$\rho \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) = -\nabla p + \mu\, \nabla^2 \vec{V} + \rho\, \vec{g} \tag{4}$$

Always assuming only one velocity and one characteristic length of the flow, denoting dimensionless variables with an asterisk, the velocity $\vec{V}$ and the coordinate vector $\vec{x}$ are $\vec{V} = \mathcal{U}\, \vec{V}^*$ and $\vec{x} = \mathcal{L}\, \vec{x}^*$. Note that the scaling of time, pressure, and gravity are not given; they result from the prescribed length $\mathcal{L}$ and velocity $\mathcal{U}$. Thus, the non-dimensional form of the Navier–Stokes equation becomes

$$\rho \left(\mathcal{U}\, \frac{\partial \vec{V}^*}{\partial t} + \frac{\mathcal{U}^2}{\mathcal{L}}\, \vec{V}^* \cdot \nabla^* \vec{V}^* \right) = -\frac{1}{\mathcal{L}}\, \nabla^* p + \mu \frac{\mathcal{U}}{\mathcal{L}^2}\, \nabla^{*2} \vec{V}^* + \rho\, g\, \hat{e}_g \tag{5}$$

$\hat{e}_g$ is the unit vector of $\vec{g}$.

$$\rho \frac{\mathcal{U}^2}{\mathcal{L}} \left(\frac{\mathcal{L}}{\mathcal{U}}\, \frac{\partial \vec{V}^*}{\partial t} + \vec{V}^* \cdot \nabla^* \vec{V}^* \right) = -\frac{1}{\mathcal{L}} \nabla^* p + \mu \frac{\mathcal{U}}{\mathcal{L}^2}\, \nabla^{*2} \vec{V}^* + \rho\, g\, \hat{e}_g \tag{6}$$

Thus, the time scale arises naturally as $\mathcal{T} = \mathcal{L}/\mathcal{U}$. The dimensionless time and pressure are $t^* = t\, \mathcal{U}/\mathcal{L}$ and $p^* = p/\left(\rho \mathcal{U}^2\right)$, and Equation (6) is written as follows:

$$\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \nabla^* \vec{V}^* = -\nabla^* p^* + \frac{\mu}{\rho \mathcal{U} \mathcal{L}}\, \nabla^{*2} \vec{V}^* + \frac{g\, \mathcal{L}}{\mathcal{U}^2}\, \hat{e}_g \tag{7}$$

In this non-dimensional equation, two dimensionless control parameters appear: the inverse of the Reynolds number and the inverse of the Froude number, as defined by Weber [21]:

$$Fr = \frac{\mathcal{U}^2}{g \mathcal{L}} \tag{8}$$

The dimensionless Navier–Stokes equation becomes

$$\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \nabla^* \vec{V}^* = -\nabla^* p^* + \frac{1}{Re} \nabla^{*2} \vec{V}^* + \frac{1}{Fr} \hat{e}_g \tag{9}$$

It is customary to join pressure and gravity in one term. Defining h as a vertical coordinate, positive upwards, $g\, \hat{e}_g = \nabla\, g\, h$. Defining a modified pressure as $\hat{p} = p + \rho\, g\, h$ [29], Equation (9) is reduced to

$$\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \nabla^* \vec{V}^* = -\nabla^* \hat{p}^* + \frac{1}{Re} \nabla^{*2} \vec{V}^* \tag{10}$$

As $\hat{p} = p + \rho\, g\, h = p + \rho\, g\, \mathcal{L}\, h^*$ and $\nabla h = \nabla^* h^*$, the Froude number is included in $\nabla^* \hat{p}^*$.

2.2. The Reynolds Number from the Point of View of Similarity

In his articles related to the law of similarity of flows with friction, Blasius [9,10] scales the fluid and flow variables in system 1 for those in system 2, such that he defines $f_l = l_2/l_1$ as the ratio of lengths, $f_u = u_2/u_1$ as the ratio of velocities, $f_\nu = \nu_2/\nu_1$ as the ratio of kinematic viscosities, and other similar ratios. Instead of working with pressure, he works with the piezometric head, $h = p/\gamma$, where γ is the specific weight of the fluid. Considering that the inertial, pressure, and viscous friction terms are of the following form

$$\frac{\gamma}{g}\, u \frac{\partial u}{\partial x}, \quad \gamma \frac{\partial h}{\partial x}, \quad \frac{\gamma}{g}\, \nu \frac{\partial^2 u}{\partial x^2}, \tag{11}$$

their scales must be preserved in both systems, resulting in the following [20]:

$$\frac{f_\gamma\, f_u^2}{f_g} = \frac{f_\gamma\, f_h}{f_l} = \frac{f_\gamma\, f_\nu\, f_u}{f_g\, f_l^2} \tag{12}$$

from which it follows:

$$\frac{f_l\, f_u}{f_\nu} = 1, \tag{13a}$$

$$f_h = \frac{f_u^2}{f_g} \tag{13b}$$

From the Equation (13a), it is deduced that the processes are similar only if

$$\frac{u_1 l_1}{v_1} = \frac{u_2 l_2}{v_2} \tag{14}$$

that is, the Reynolds number in system 1 must be equal to the Reynolds number in system 2, $Re_1 = Re_2$. From the second scaling condition (Equation (13b)), the dimensionless parameter gh/u^2 is generated, corresponding to the Euler number in terms of the pressure head.

2.3. The Reynolds Number as the Ratio of Characteristic Times

The Navier–Stokes equation (Equation (4)) represents the transport of momentum in the flow. When presented divided by the density, it is easy to see that the kinematic viscosity ν gives the momentum diffusion coefficient. Thus, two-time scales are distinguished: one associated with the flow, corresponding to $\mathcal{T} = \mathcal{L}/\mathcal{U}$, and another associated with the time scale related to momentum transport by viscous diffusion over the length scale $\mathcal{L}$, denoted as $\mathcal{T}_V$. By dimensional analysis, $\mathcal{T}_V = \mathcal{L}^2/\nu$. Hence, the Reynolds number can also be interpreted as the ratio between the time scale associated with viscous diffusion and the time scale of the flow:

$$Re = \frac{\mathcal{T}_V}{\mathcal{T}} = \frac{\mathcal{U}\, \mathcal{L}}{\nu} \tag{15}$$

2.4. Generalization of the Reynolds Number for Non-Cylindrical Ducts

Equation (4) can be written as follows:

$$\rho \frac{d\vec{V}}{dt} = -\nabla \hat{p} + \nabla \cdot \tau_{ij} \tag{16}$$

where the stress tensor for a Newtonian fluid is given by $\tau_{ij} = \mu\, \dot{\gamma}_{ij}$, with $\dot{\gamma}_{ij}$ the shear rate given by $\dot{\gamma}_{ij} = \left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)$. Calling $\left|\dot{\gamma}\right|$ to the norm of $\dot{\gamma}_{ij}$, a scale of the shear stress

tensor is $\mu\,|\dot{\gamma}|$. From the definition of $\dot{\gamma}_{ij}$, the time scale associated with the shear rate is $\mathcal{T}_{shear} \sim 1/|\dot{\gamma}|$. Thus, the dimensionless stress tensor can be written as $\tau_{ij} = (\mu/\mathcal{T}_{shear})\,\tau^*_{ij}$. Replacing the scaled variables in Equation (16), it becomes

$$\rho\,\frac{\mathcal{U}}{\mathcal{T}}\,\frac{d\vec{V}^*}{dt^*} = -\nabla\hat{p} + \frac{\mu}{\mathcal{L}\,\mathcal{T}_{shear}}\,\nabla^*\cdot\tau^*_{ij} \tag{17}$$

or,

$$\frac{d\vec{V}^*}{dt^*} = -\frac{\mathcal{T}}{\rho\,\mathcal{L}\,\mathcal{U}}\nabla\hat{p} + \frac{\mu}{\rho\,\mathcal{L}\,\mathcal{U}}\frac{\mathcal{T}}{\mathcal{T}_{shear}}\nabla^*\cdot\tau^*_{ij} \tag{18}$$

To recover Equation (10), the time scale associated with the flow must match the time scale associated with the shear rate, i.e., $\mathcal{T}_{shear} = \mathcal{T}$.

So far, only one length scale has been considered in the analysis, which is appropriate for the flow in a cylindrical tube, where $\mathcal{L}$ can be taken as the diameter, as Reynolds did. The question is as follows: What is the proper scale $\mathcal{L}$ when the geometry involves more than one longitude? Equations (2) and (3) allow us to consider a duct of arbitrary cross-section. Let P_w the "wetted perimeter", i.e., the perimeter of the duct wall where the shear stress is acting, and A the cross-section area of the tube, and Δx the length of a portion of the duct in the direction of the main flow. The fluid volume in Δx is $\mathcal{V} = A\,\Delta x$, and the area where the shear stress acts is $A_w = P_w\,\Delta x$. Thus, inertia and viscous force are given by

$$I = \frac{d}{dt}(m\,\mathcal{U}) \;\sim\; \frac{\rho\,\mathcal{V}\,\mathcal{U}}{\mathcal{T}} \;\sim\; \frac{\rho\,A\,\Delta x\,\mathcal{U}}{\mathcal{T}} \tag{19}$$

$$F_V = \tau\,A_w \;\sim\; \mu\,|\dot{\gamma}|\,A_w \;\sim\; \frac{\mu}{\mathcal{T}_{shear}}\,P_w\,\Delta x \tag{20}$$

Using the result that the flow and shear rate time scales are equal, the Reynolds number, defined as the ratio between inertia and viscous forces, becomes

$$Re = \frac{I}{F_V} = \frac{\mathcal{U}}{\nu}\frac{A}{P_w} \tag{21}$$

The hydraulic radius is defined as $R_H = A/P_w$ so that an appropriate length scale for a tube of arbitrary cross section is $\mathcal{L} = R_H$, and the proper Reynolds number is as follows:

$$Re_{R_H} = \frac{\mathcal{U}R_H}{\nu} \tag{22}$$

The hydraulic radius for the flow in a cylindrical tube of diameter D is $R_H = D/4$, for a rectangular duct of sides a and b is $R_H = ab/(2a+2b)$, and for the flow between two concentrical cylinders of diameters D_{outer} and D_{inner} is $R_H = (D_{outer} - D_{inner})/4$. For a flow with a free surface in a rectangular channel, with width b and flow depth H, the hydraulic radius is $R_H = bH/(b + 2\mathrm{H})$.

The standard values of *Re* that define flow regime are as follows: for *Re* (Re_{R_H}) < 2000 (500), the regime is laminar; for *Re* (Re_{R_H}) > 4000 (1000), the regime is turbulent. Between those limits, the regime is named the laminar-turbulent transition.

2.5. The Reynolds Number as a Control Parameter in the Mathematical Analysis of the Navier–Stokes Equations

In the dimensionless form of the Navier–Stokes equation (Equation (10)), the Reynolds number is its only control parameter, and mathematical analysis and the derivation of analytical solutions may be possible by considering limiting values of *Re*. The most obvious cases are the extremes. One of them is when $Re \to \infty$, that is, a fluid with zero viscosity in which the second-order differential terms of the equations disappear (reducing the spatial

boundary conditions), and the Euler equation is recovered. The other extreme case is when $Re \rightarrow 0$, a situation in which the flow's inertia is negligible, simplifying the analysis of the Navier–Stokes equation since, in this case, the nonlinear terms disappear (Stokes regime). The book by Drazin and Riley [30] presents a wide variety of problems whose analytical solutions feature the Reynolds number as a control parameter. Analytical solutions based on asymptotic expansions or through perturbation theory are parameterized in terms of the Reynolds number. The solutions are expressed as series, which are usually expanded in terms of powers (integer or fractional) of Re or $1/Re$, or functions of $\log(Re)$ [31,32]. The Reynolds number also naturally appears as a parameter that defines the stability of flows described by the Navier–Stokes equations [33–35].

3. The Universality of the Reynolds Number

As seen, the physical meaning of the Reynolds number is a quantification of the relevance of fluid inertia compared to the viscous force opposing the motion. At high Re values, inertia dominates, while at low values, viscous forces prevail. Thus, the Reynolds number predicts the upper limit of laminar flow and the lower boundary of turbulent flow. Furthermore, it is a key parameter in scaling resistance to motion. Although its application began with pressurized flows of Newtonian fluids in pipes, its use quickly extended to other fluid mechanics phenomena involving a length scale, a velocity variable, and a resistive force. Without aiming to present an exhaustive compilation, this document outlines various fields of fluid mechanics where a dimensionless parameter has been defined and identified as a Reynolds number. It must be noted that what specific authors call the Reynolds number, or the modified Reynolds number, others provide a different name.

3.1. Reynolds Number and Biofluid Mechanics

Broadly, biofluid mechanics refers to fluid flows influenced by the surfaces of living beings. These flows can be external or internal. Examples of the former include the flow around a bird or insect in flight or around bacteria, fish, or swimmers moving in a liquid medium. Internal biological flows pertain to phenomena such as blood flow and respiration, always associated with mass and heat transfer. A characteristic of biological flows is that the motion is typically affected by highly flexible and structured surfaces [36]. The relationship between fluid mechanics and biology is not new. It is known that Poiseuille's interest in pipe flow arose from his studies on blood flow in the circulatory system [37,38]. However, this association dates even further back to the Greeks Herophilus (before 300 B.C.) and Erasistratus (310–250 B.C.) of the Alexandria medical school, who were likely the first to study blood flow and liken the heart to a pump [39].

The motion of living organisms in a fluid medium spans a vast range of Reynolds numbers. In these cases, the characteristic length is associated with a body dimension, and the velocity corresponds to its displacement velocity in a stationary fluid or the relative velocity in a current. The medium is typically water or air. Water is considered the fluid medium for bacteria: $\mathcal{L} \sim 10$ µm and $\mathcal{U} \sim 10$ µm/s yielding $Re \sim 10^{-4}$. On the other extreme of the length scales, for a whale $\mathcal{L} \sim 10$ m and $\mathcal{U} \sim 10$ m/s, resulting in $Re \sim 10^{7}$ [40]. Diverse dominant forces in motion can be expected in this wide range of Reynolds numbers. For example, at very low Reynolds numbers, the fluid behaves as if it has no inertia (Stokes flow), and Equation (10) reduces to

$$0 = -\nabla^{*}\hat{p}^{**} + \nabla^{*2}\vec{V}^{*} \tag{23}$$

In this very low Re limit, the appropriate scaling for the pressure is the viscous component rather than inertia, so the dimensionless pressure in this case is

$$\hat{p}^{**} = \frac{\hat{p}\,\mathcal{L}}{\mu\,\mathcal{U}} \tag{24}$$

Written with dimensions, Equation (23) is $\nabla\hat{p} = \mu\nabla^2\vec{V}$, indicating a balance between the driving force (due to the pressure gradient) and the resistive viscous force. The linearity of this equation, along with the continuity equation and the boundary conditions, imply flow reversibility. Organisms with motion in low Reynolds number regimes have developed unique locomotion mechanisms distinct from those outside the Stokes range. This mechanism is associated with the existence of flagella, thin organelles whose periodic motion in a fluid environment enables movement [41].

Some organisms have developed locomotion based on undulatory and oscillatory body motion [42]. In this case, a temporal forcing of motion arises, associated with a certain frequency ω. Thus, the time scale associated with the flow is no longer $\mathcal{L}/\mathcal{U}$ but $\mathcal{T} = 1/\omega$, i.e., $t = t^{**}/\omega$, so the dimensionless Navier–Stokes equation is rewritten as follows:

$$Re_\omega \frac{\partial\vec{V}^*}{\partial t^{**}} + Re\,\vec{V}^*\cdot\nabla\vec{V}^* = -\nabla^*\hat{p}^{**} + \nabla^{*2}\vec{V}^* \tag{25}$$

A new Reynolds number emerges:

$$Re_\omega = \frac{\mathcal{L}^2\,\omega}{\nu} \tag{26}$$

Re_ω is named the oscillatory Reynolds number [40]. Taylor presented it in his pioneering article on the swimming of microscopic organisms [43]. Its velocity scale is given by the characteristic velocity of the fluid induced by the oscillatory motion, $\mathcal{L}\omega$. When a body vibrates, inertial stresses are generated by the interaction between the vibrating surface and the surrounding fluid. Re and Re_ω do not necessarily have the same order of magnitude. For example, an insect hovering in the air in a fixed position has a small Re but a very large Re_ω due to the high-frequency movement of its wings [40]. The oscillatory Reynolds number is frequently expressed in terms of another dimensionless parameter, the Strouhal number, defined as $St = \mathcal{L}\,\omega/\mathcal{U}$, such that $Re_\omega = St\,Re$.

To explain how a propulsive tail can move a body through a viscous fluid without relying on the reaction caused by the fluid's inertia, Taylor [43] simplified the three-dimensional problem of flagellar propulsion in spermatozoa and large arrays of cilia by modeling as transversely wavy sheets. If k is the wavenumber, ω is the oscillation frequency, the wave propagation velocity is ω/k, and pressure scales with $\mu\,\omega$. It is possible to define a new Reynolds number $Re_{\omega k} = \omega/\left(k^2\,\nu\right)$ considering the length scale as $1/k$. The effect of $Re_{\omega k}$ on the displacement velocity of the transversely wavy sheets is of second order [40].

The locomotion of microorganisms possessing dense arrays of motile cilia has been modeled as a spherical squirmer of radius a that moves through small, axisymmetric deformations of its surface. The radial and tangential velocities on its surface in a co-moving frame of reference are expressed in terms of series of Legendre and associated Legendre polynomial [44]. B_1 is the amplitude of the first coefficient of the series describing the surface tangential velocity, which is associated with the velocity of a squirmer in potential flow, $u_{spf} = 2B_1/3$. The only length scale for a squirmer in an unbounded Newtonian fluid is the radius a. Thus, considering $\mathcal{L} \sim a$ and $\mathcal{U} \sim u_{spf}$, Chisholm et al. define the Reynolds number associated with the motion of the squirmer as $Re = 2B_1\,a/(3\nu)$ [44].

A simplified model for analyzing experimental data regarding fish locomotion was presented by Gurka et al. [45]. It considers that the flow around the fish can be assimilated to the motion of a harmonic oscillatory motion parallel to a plate (Stokes second problem). Taking the frequency of the oscillatory motion of the fish's body as the frequency of the oscillating plate ω, the length scale as the depth penetration of the viscous wave $\mathcal{L} = 5\sqrt{2\,\nu/\omega}$, and the fish speed U as the velocity scale $\mathcal{U}$, a Reynolds number is formed: $Re = 10\,U/\sqrt{2\,\omega\,\nu}$. Gurka et al. call this Reynolds number "periodic swimming number" and denote it as P [45].

3.2. Reynolds Number in Nano and Microfluidics

Among all the dimensionless numbers, the Reynolds number is the most referenced in the context of microfluidics. Interestingly, it may also be one of the least significant for microfluidics. In nearly all cases, the fluids used in microfluidic devices have Reynolds numbers low enough that inertial effects become negligible [46]. The usual forms of the Reynolds number are those presented in Equations (1) or (2), with $\mathcal{U}$ being the flow average velocity and $\mathcal{L}$ equal to the diameter of the micro pipe or, in the case of rectangular ducts, using the corresponding hydraulic radius. It is also common to find a Reynolds number based on the flow rate Q and the diameter D [47,48]:

$$Re = \frac{Q}{\nu D} \tag{27}$$

Studies in microfluidics have shown that a slip velocity exists on the boundary. Although the boundary condition considering a slip velocity goes back to Navier, it was shadowed by Stokes' experiments on the non-slip condition [49]. Although non-slip is a valid condition in common large-scale flows, it cannot be overlooked in nano and microfluidics. Consider a plane Couette flow between two plates, driven by the motion of the upper plate, which does not allow slip, but in the lower wall, a slip condition exists. If λ_s is the slip (or penetration) length, Niavarani and Priezjev proposed the following Reynolds number [50]:

$$Re = \frac{U_{upp}\,h}{\nu}\left(1 + \frac{\lambda_s}{h}\right) \tag{28}$$

where U_{upp} is the velocity of the upper wall, and h the plates' separation. The previous result is obtained directly from $Re \sim I/F_V$. The shear stress remains constant across the fluid in the case of a two-dimensional pure Couette flow and from the Navier condition: $\tau_{wall} = \mu u_s/\lambda_s$, where u_s is the slip velocity at the wall. The velocity distribution indicates $U_{upp}/u_s = (h + \lambda_s)/\lambda_s$. Additionally, $\mathcal{V} = h\Delta x$ and $A = \Delta x$, where Δx is a length element of the wall in the direction of flow. Considering $\mathcal{L} = h$ and $\mathcal{U} = U_{upp}$, we have $I \sim \rho U_{upp}^2 \Delta x$. and $F_V = 2\,\tau_{wall}\,\Delta x$, which follows that the ratio I/F_V corresponds to the Reynolds number defined in Equation (28).

3.3. Reynolds Number in Boundary Layers

For simplicity, two-dimensional or axisymmetric flow will be considered next. The key of Prandtl's boundary layer concept introduced in 1904 [51] is the existence of two length scales, one in the flow direction, say $\mathcal{L}$, and another normal to it, the boundary layer thickness δ, such that $\delta \ll \mathcal{L}$. Navier–Stokes equations are greatly simplified with this assumption, and the so-called Prandtl equations are obtained [52,53]. The flow is rotational within the boundary layer, and the potential flow equations are valid outside it. Usually, $\mathcal{L}$

is associated with a coordinate attached to the body or the main flow direction, x. Thus, two Reynolds numbers arise, Re_x and Re_δ:

$$Re_x = \frac{U_0 \ x}{\nu}, \tag{29a}$$

$$Re_\delta = \frac{U_0 \ \delta}{\nu} \tag{29b}$$

where U_0 is the velocity of the irrotational flow (outside the boundary layer); generally, it depends on the x coordinate. For a boundary layer over a plate or solid body, Prandtl defined δ as the distance from the wall to the location where the flow velocity is $0.99U_0$. Other definitions of boundary layer thickness exist, like displacement thickness, momentum thickness, and energy thickness [52,53], which originate from Reynolds numbers based upon them. In Pohlhausen's method for determining the boundary layer separation point, the parameter $\Lambda_p = -\left(\delta^2/(\mu \ U_0)\right)(dp/dx)$ appears. The pressure gradient can be related to the velocity gradient in the irrotational region, where Bernoulli's equation holds. This gradient expresses the previous term as $\Lambda = \left(\delta^2/\nu\right)(dU_0/dx)$. The parameter Λ_p can be interpreted as the ratio between the force associated with the pressure gradient and the viscous forces [54]. However, when the pressure gradient is related to the velocity gradient, the dimensionless Λ can be interpreted as the ratio between the momentum change due to the acceleration of the velocity in space and the viscous forces. Thus, Λ represents an "accelerating Reynolds number". If the momentum thickness is used instead of the boundary layer thickness δ, the accelerating Reynolds number is denoted as λ [52,53,55].

The no-slip condition and viscosity govern the flow near a flat disk rotating with a constant angular velocity Ω about an axis perpendicular to its plane in a fluid initially at rest. As a result, the fluid layer in direct contact with the disk is dragged along and pushed outward by centrifugal force. New fluid particles are continuously drawn toward the disk in the axial direction and then ejected centrifugally, creating a fully three-dimensional flow that behaves like a pump. The thickness δ of the layer "carried" with the disk is given by $\delta \ \sim \ \sqrt{\nu/\Omega}$. For a circular disk of radius R, the Reynolds number is defined as follows [52,53]:

$$Re = \frac{R^2 \ \Omega}{\nu} \tag{30}$$

Equation (26) is similar to Equation (30), but there is a difference. In the latter, Ω is the disk's angular velocity, and in Equation (26), ω is the surface's oscillation frequency.

3.4. Reynolds Number in Rotating Flows and Hydraulic Machines

A glance at a rotating flow was given in the previous subsection. Another kind of rotating flow is generated in the gap between two concentric cylinders when one is rotating with an angular velocity Ω. A Reynolds number used for this flow is the following [56]:

$$Re = \frac{r_o \ \Omega \ d}{\nu} \tag{31}$$

where r_o is the radial distance to the outer cylinder, $d = D_o - D_i$ is the annular gap, with D_0 and D_i the diameter of the outer and inner cylinders, respectively. Singh and Rajvanshi [57], in their study with eccentric cylinders of inner and outer radii R_i and R_o rotating with angular velocities Ω_i and Ω_o worked with a Reynolds number defined as follows:

$$Re = \frac{R_o\sqrt{R_i^2\Omega_i^2 + R_o^2\Omega_o^2}}{v} \tag{32}$$

Ludwieg (cited in reference [58]) analyzed his data obtained from concentric cylinder experiments with the Reynolds number given by

$$Re = \frac{\Omega_o \, (R_o - R_i)^2}{\nu} \tag{33}$$

Joseph and Munson [58] studied the stability of the flow between two concentric cylinders with the inner moving along its axis with velocity U_c. They defined a generalized Reynolds number as

$$Re = \frac{\sqrt{R_i^2 \, (R_o - R_i)^2 + U_c^2} \, (R_o - R_i)}{\nu} \tag{34}$$

Bishop et al. [59], in their research concerning the characteristics of a natural convective oscillatory flow in horizontal, cylindrical annuli, defined three Reynolds numbers that they named amplitude Reynolds number Re_A, wavelength Reynolds number Re_W, and period Reynolds number Re_T:

$$Re_A = \frac{R_o \, A_o (R_o - R_i)}{\nu \, T} \tag{35}$$

$$Re_W = \frac{R_i \, A_i \, W}{\nu \, T} \tag{36}$$

$$Re_T = \frac{R_i \, A_i \, (R_o - R_i)}{\nu \, T} \tag{37}$$

In the above equations, A_o and A_i represent the amplitude of the oscillatory flow at the inner and outer cylinder radius, respectively, and W and T are the wavelength and the period of the oscillatory flow, respectively. Weidman [60] investigated experimentally the spin-up and spin-down of a fluid in the gap between two concentric cylinders, where the inner cylinder either accelerated to a constant angular velocity or decelerated until stopped. He defined a Reynolds number based on the fluid's local geostrophic velocity, $V_g = |u - \Omega \, r|$, and the Eckman layer thickness, $\delta = \sqrt{\nu / \Omega}$, i.e., $Re = V_g \, \delta / \nu$, resulting in the following:

$$Re = \frac{|u - \Omega \, r|}{\sqrt{\nu \, \Omega}} \tag{38}$$

It is easy to conclude that the appropriate Reynolds number for a sphere of diameter D rotating with angular velocity Ω in an infinite medium with kinematic viscosity ν is as follows [61]:

$$Re = \frac{D^2 \, \Omega}{\nu} \tag{39}$$

In the analysis of the effect of a wall located a distance d from the center of a sphere with radius R, the following modified Reynolds number appears [62]:

$$Re = \frac{R^2 \, \Omega}{\nu} \left(\frac{d}{R} - 1 \right) \tag{40}$$

In general, the flow of a fluid located in the space between two concentric spheres rotating around an axis does not include the gap thickness or the difference in the radii of the spheres in the definition of the Reynolds number, as is the case for flow between two cylinders. A relationship of the following type is usually used:

$$Re = \frac{\Omega_{ref} \, R_{ref}^2}{\nu} \tag{41}$$

where R_{ref} and Ω_{ref} are the reference variables used for the radius and angular velocity, which depend on the author. Typically, the analysis is accompanied by other dimensionless

parameters to account for the gap and the difference in angular velocities of the spheres, such as $(\Omega_o - \Omega_i)/\Omega_i$ and $(R_o - R_i)/R_i$ [63], or the angular velocity ratio Ω_o/Ω_i and the radius ratio R_o/R_i [64,65]. $R_{ref} = R_o$ is used in references [64–68], among others. In contrast, $R_{ref} = R_i$ is used in [63,69–71]. The angular velocity of the outer sphere, $\Omega_{ref} = \Omega_o$, is employed in references [64,66,67], while $\Omega_{ref} = \Omega_i$ is used in [63,71]. Gulwadi and Elkouh [72] use two Reynolds numbers in their analysis: one associated with R_o and Ω_o and another with R_i. Nakabayashi et al. [70] also use two Reynolds numbers, with Ω_i or Ω_o as the reference angular velocity, but keeping $R_{ref} = R_i$ fixed. Bartels uses $\Omega_{ref} = 800\nu/R_o^2$ [68].

In turbomachines, the Reynolds number is presented in terms of the rotational speed N and the characteristic diameter D, such that

$$Re = \frac{\rho\, N\, D^2}{\mu} \tag{42}$$

Additionally, other Reynolds numbers are defined based on some characteristic of the blades, $Re_i = u_i \ell_i/\nu$. For example, u_i can be the impeller tip speed, and ℓ_i the impeller tip width at the impeller exit; or u_i can be the inlet relative velocity at mid-span and ℓ_i the chord length (the length of the flow path) [73]. In gas turbines, the Reynolds number also depends on the exhaust pressure, decreasing with decreasing exhaust pressures as indicated by the definition of the Reynolds number (e.g., peripheral Reynolds number) [74]:

$$Re = \frac{\rho\, u\, D}{\mu} = \frac{u\, D\, p_s}{\mu_s\, R\, T_s} \tag{43}$$

where u is the rotor tip speed, p is the pressure, μ is the dynamic viscosity, T is the temperature, and R is the gas constant. The sub-index s denotes the turbine's exit section or exhaust.

In performance simulations of radial compressors, instead of the actual Reynolds number, the Reynolds Number Index RNI is used, which is the ratio of the actual Reynolds number Re, and a reference Reynolds number Re_{ref}, for the same Mach number [75]:

$$RNI = \frac{Re}{Re_{ref}} \tag{44}$$

Use $\rho_s = p_s/(R\, T_s)$ and the fact that commonly there is no change in length scale, and preserve the Mach number in the actual and reference turbine ($Ma = Ma_{ref}$, i.e., $u/\sqrt{\gamma\, R\, T_s} = u_{ref}/\sqrt{\gamma_{ref}\, R_{ref}\, T_{s,ref}}$), where γ is the exponent of the isentropic process. Considering that $\gamma \approx \gamma_{ref}$, Equation (44) becomes

$$RNI = \frac{p_s}{p_{ref}} \sqrt{\frac{R_{ref}\, T_{ref}}{R\, T}}\, \frac{\mu_{ref}}{\mu} \tag{45}$$

3.5. Reynolds Number in Turbulence

Since its origin, the Reynolds number and turbulence have been inseparably linked, and it is impossible to refer to turbulence without referencing some version of the Reynolds number. As seen in the Introduction, this dimensionless parameter allows us to establish when the flow transitions from laminar to turbulent. We can say that the Reynolds number is at the origin of turbulence, not only because of its importance in experimentally discerning when "the steady motion becomes unstable" and "changes to sinuous motion". As Sommerfeld emphasized analytically, the Reynolds number is a key control parameter in the theoretical approach for determining when an infinitesimal disturbance in the flow

grows exponentially, making it unstable. The equation that describes the behavior of disturbances is called the Orr–Sommerfeld equation [52,53], and it allows the generation of a neutral stability curve from which critical values for the Reynolds number, velocity, and wave number of the disturbance can be determined, beyond which any flow disturbance is amplified. For boundary layers over a flat plate, such curves can be found in the books by Schlichting and Gersten [52] and White [53].

In its origin, the length scale $\mathcal{L}$ was associated with the diameter D of the pipe, and the velocity scale $\mathcal{U}$ with the mean flow velocity U. In other words, they correspond to "large" length and velocity scales. The concept of eddies in turbulent flow had already been developed before Reynolds' publications [49]. The size of these eddies is limited by the size of the installation or scenario where the flow occurs, L. Considering that the energy dissipation rate ϵ is due to viscous friction, a dimensional analysis allows us to obtain the size of the smallest vortices in the flow, as well as their velocity and associated times, resulting in

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \quad \mathcal{v} = (\nu\,\epsilon)^{1/4}, \quad t_K = \left(\frac{\nu}{\epsilon}\right)^{1/2} \tag{46}$$

The above length, velocity, and time scales are known as Kolmogorov scales, and the Reynolds number formed from them is 1 [76]:

$$Re_K = \frac{\mathcal{v}\,\eta}{\nu} = 1 \tag{47}$$

The energy from the largest vortices passes through the smaller ones until it is dissipated in the smallest vortices in a cascade process. Suppose ℓ corresponds to the size of the largest vortices in the flow, which is associated with a velocity $\mathcal{u}$, and the kinetic energy per unit mass of these vortices is proportional to $\mathcal{u}^2$, the rate of kinetic energy transfer per unit mass to the smaller vortices is $\mathcal{u}^3/\ell$. This energy is entirely dissipated at the Kolmogorov scale, so $\mathcal{u}^3/\ell \sim \epsilon$. Using Equation (46), the scale ratio is

$$\frac{\eta}{\ell} = Re_\ell^{-3/4}, \quad Re_\ell = \frac{\mathcal{u}\;\ell}{\nu} \tag{48}$$

The scale ℓ is related to the integral scales of turbulence, which are determined from the velocity correlation functions [76]. The Reynolds number from Equation (48) can be expressed in terms of turbulent kinetic energy k and the dissipation rate ϵ. In fact, $k \sim \mathcal{u}^2$ and $\ell \sim \mathcal{u}^3/\epsilon$, resulting in

$$Re_\ell = \frac{k^2}{\nu\,\epsilon} \tag{49}$$

Between the integral length scale ℓ and the Kolmogorov scale η, another important scale in turbulence, known as the Taylor microscale λ, is defined by the turbulent fluctuations of the strain rate. For isotropic turbulence, the following holds:

$$\overline{\left(\frac{\partial u'}{\partial x}\right)^2} \equiv \frac{\overline{u'^2}}{\lambda^2} = \frac{\mathcal{u}^2}{\lambda^2} \tag{50}$$

In Equation (50), u' is the turbulent component along the x-direction of the velocity vector, resulting from the Reynolds decomposition, where the instantaneous velocity is separated into a time-averaged component and a fluctuation around the mean value: $\vec{V} = \overline{\vec{V}} + \vec{V}'$. In homogeneous turbulence, it holds that $\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$, so it is possible to consider $\mathcal{u}^2 = \overline{u'^2}$ in Equation (50) unambiguously. The Taylor microscale measures the scale of the turbulence at which velocity fluctuations are highly correlated over small distances. In simpler terms, it is the scale at which the flow begins to exhibit significant

local velocity gradients, but before it reaches the Kolmogorov scales where dissipation occurs. The Taylor microscale is a key intermediate scale in the cascade process, often seen as a boundary between energy-containing and dissipative eddies.

The Reynolds number based on the Taylor microscale is defined as

$$Re_\lambda = \frac{u\ \lambda}{\nu} \tag{51}$$

The ratio of the Taylor microscale and integral length scale is as follows [76]:

$$\frac{\lambda}{\ell} = \left(\frac{15}{A}\right)^{1/2} Re_\ell^{-1/2} \tag{52}$$

where A is a constant of the order of 1. It is easy to see that $Re_K \ll Re_\lambda \ll Re_\ell$ from the above relationships, which implies $\eta \ll \lambda \ll \ell$.

Near absolute zero, quantum effects significantly influence fluid properties, leading to behaviors so distinct from ordinary fluids that they are termed superfluids. A superfluid behaves as a mixture of two components: a frictionless superfluid and a normal viscous fluid. These components move independently without momentum transfer. Superfluid flow exhibits no viscosity, no heat transfer, and is always potential flow, while normal flow exerts drag forces. This dual-flow concept explains the macroscopic dynamical properties of superfluid and was proposed by Landau in 1941, as mentioned in reference [77]. The Euler equation governs the superfluid component, but it includes an additional term corresponding to the gradient of the chemical potential.

Additionally, a parameter q, which governs mutual friction and dissipation, appears to multiply the term involving the cross product of the superfluid vorticity and velocity [78]. Since the superfluid component has no viscosity, the term $1/q$ plays a similar role in the generation of turbulence. However, this parameter alone is not sufficient to describe the characteristics of turbulence. A circulation coefficient κ, which characterizes the quantized circulation around a quantum vortex, also arises. Based on this, a Reynolds number associated with the superfluid component is defined as $Re_s = \mathcal{U}\mathcal{L}/\kappa$. For fully developed turbulence, $q \ll 1$ and $Re_s \gg 1$ must be satisfied simultaneously [78]. It should be noted that Landau's model disagrees with more recent experimental observations, which indicate that energy dissipation in superfluids occurs not through viscosity, as in ordinary fluids, but through the emission of acoustic waves [79]. A theoretical explanation emerges when a term accounting for the quantum stress tensor $\overline{\overline{\Sigma}}$ is added to the Euler equation. This term is given by $\nabla \cdot \overline{\overline{\Sigma}}$, with $\overline{\overline{\Sigma}} = (\hbar/(2m))^2 \rho_s \nabla \nabla \rho_s$, with $\hbar$ being the reduced Planck's constant, m the mass of one bosom, and ρ_s the superfluid density. Noting the similitude between $\nabla \cdot \overline{\overline{\Sigma}}$ and the stress tensor gradient $\nabla \cdot \overline{\overline{\tau}}$ in the Navier–Stokes equation, the quantum circulation is defined as $\kappa = h/m$, where is h the Planck's constant. For helium II, $\kappa \sim 10^{-3}$ cm^2/s [79].

3.6. Reynolds Number in Porous Media

The porous medium consists of a matrix with interconnected void spaces, typically exhibiting irregular characteristics, through which a liquid or gas flows driven by a pressure gradient. The quantification of these spaces is often simplified by defining a characteristic length and a measure of the porosity or void fraction in the solid matrix. Porous media are frequently heterogeneous and anisotropic, and the flow within the pores can be either laminar or turbulent. The transition between these two regimes, typically characterized by a Reynolds number, is difficult to predict. Additionally, interactions between the fluid and the pore surfaces, such as the liquid's viscosity and capillary forces, can influence the flow

in complex ways, particularly in microscopic pores, where intermolecular and capillary forces play a significant role.

There are numerous definitions of the Reynolds number in the context of flow through porous media. Discrepancies observed in the analysis and conclusions of some experimental data have been attributed to differences in the definitions of this dimensionless parameter [80]. The characteristic velocity $\mathcal{U}$ and the length scale $\mathcal{L}$ are challenging to determine, contributing to the variety of Reynolds number definitions used in this field.

In their work on mass transfer between particles and fluid in fluidized beds, Dwivedi and Upadhyay [81] use the variable G (superficial mass flow rate) to define four Reynolds numbers. In three of them, the length scale used is the diameter of the equivalent particle forming the porous medium, $\mathcal{L} = D_p$, and in the fourth, $\mathcal{L} = \sqrt{A_p}$, where A_p is the geometric surface area of the particle. In terms of the Darcy velocity U_D (defined as the flow rate per unit total surface area, including both the solid phase and the pores), instead of G and denoting ε as the void fraction, they are

$$Re_p = \frac{\rho\, U_D\, D_p}{\mu} \tag{53}$$

$$Re_{p,\varepsilon 1} = \frac{\rho\, U_D\, D_p}{\mu\, \varepsilon} \tag{54}$$

$$Re_{p,\varepsilon 2} = \frac{\rho\, U_D\, D_p}{\mu\, (1-\varepsilon)} \tag{55}$$

$$Re_{A_p} = \frac{\rho\, U_D\, \sqrt{A_p}}{\mu} \tag{56}$$

Ziołkowska and Ziołkowski [80] add a Reynolds number based on the specific surface area a as the characteristic length:

$$Re_a = \frac{4\, U_D}{\nu\, a} \tag{57}$$

The specific surface area is computed from

$$a = \frac{6\,(1-\varepsilon)}{\psi\, D_z} \tag{58}$$

$D_z = (6V/\pi)^{1/3}$ is the grain (or pellet) equivalent diameter, with V the grain (or pellet) volume. $\psi = F/\left(\pi D_z^2\right)$ is the shape factor of bed element, with F the grain (or pellet) surface area. Similar to Equations (53) and (55), in reference [80], Reynolds numbers based on D_z instead of D_p are also used.

Prieur du Plessis and Woudberg [82] use the following Reynolds number:

$$Re_{D_h} = \frac{\rho U_D D_h}{\mu} \tag{59}$$

where D_h is the hydraulic diameter, which for porous media can be related to the solid surface A_p as $D_h = 6V/A_p$.

Boleve et al. [83] studied electrical field's controlling parameters associated with water seepage through a porous material. When fluid flows through a porous medium, the shear forces on the fluid near the surface of the pores cause ions in the fluid to move, leading to a separation of charge. This separation creates an electrical potential across the material. In their definition of the Reynolds number, they used the seepage velocity, U, and $\mathcal{L} = R$, where R is the radius of the capillaries of the porous medium or $\mathcal{L} = \Lambda$, with Λ a characteristic flow length. For a granular medium formed by a unimodal particle size distribution, $\Lambda = d_0/(2m(F-1))$. d_0 is the mean particle diameter, and $F = \varepsilon^{-m}$ is the

electrical formation factor. The exponent m is often called the cementation exponent. The Reynolds number neglecting electroosmotic effects is obtained from

$$Re_\Lambda = \frac{\rho^2 g\, K\, \Lambda}{\mu^2\,(1 + Re_\Lambda)} \frac{h}{L} \tag{60}$$

where K is the permeability, g is the acceleration due to gravity, h is the hydraulic head, and L is the length of the porous pack. From Equation (60), a quadratic equation for Re_Λ is obtained, from which the positive root must be considered. Note that in this equation, the dependence on velocity has been replaced by the hydraulic gradient, h/L, which is often easier to determine.

Some authors [84,85] have used a characteristic length based on the permeability K of the porous media, $\mathcal{L} = \sqrt{K}$, such as

$$Re_K = \frac{U_D\,\sqrt{K}}{\nu} \tag{61}$$

K can be estimated using the Carman–Kozeny equation, $K = D_p^2\,\varepsilon^3 / \left(180\,(1-\varepsilon)^2\right)$.

Jin and Kuznetsov [85] in their study on turbulent flows in porous media defined a Reynolds number based on the macroscopic strain tensor, $s_{Dij} = \left(\partial u_{Di}/\partial x_j + \partial u_{Dj}/\partial x_i\right)/2$, such that $\mathcal{U} = \sqrt{K}\,\left|s_{Dij}\right|$ with $\left|s_{Dij}\right| = \sqrt{2 s_{Dij} s_{Dij}}$, and $\mathcal{L} = \sqrt{K}$, resulting in:

$$Re_{s_{Dij}} = \frac{K\,\left|s_{Dij}\right|}{\nu} \tag{62}$$

Versions of the previous relations have been used for two-phase flows, such that a Reynolds number is defined for the solid phase and another for the liquid phase [86,87].

3.7. Reynolds Number in Multiphase Flows

Multiphase flows are those in which two or more phases of different nature (liquid, gas, or solid) coexist and move simultaneously through a medium. These phases can be found in dispersed or stratified mixtures, and the flows can have various configurations depending on how the phases are distributed. The phases can interact with each other through their interfaces, which influences the overall flow dynamics. These interactions include mass, heat, and momentum transfer processes. The Reynolds number is crucial for characterizing multiphase flows because it determines their interactions in addition to defining the flow regime in each phase. The most important interaction between the phases is the drag force, which depends on the Reynolds number and the concentration of the dispersed phase in the carrier phase.

The drag force $\vec{F}_D$ for an isolated particle in an infinite fluid is given by

$$\vec{F}_D = \frac{1}{2}\,\rho\, C_D\, A_p \left|\vec{V}_r\right| \vec{V}_r \tag{63}$$

where ρ is the fluid density, A_p is the particle's cross-sectional area, and $\vec{V}_r = \vec{V}_p - \vec{V}_f$ is the relative velocity of the particle ($\vec{V}_p$) to the fluid velocity ($\vec{V}_f$). C_D is the drag coefficient, which depends on the particle's Reynolds number, geometry, and orientation relative to the approaching flow. The previous relationship is valid for solid particles, drops, or bubbles. However, in the latter two cases, their interaction with the fluid can deform them, altering

their surface area and drag coefficient as they move. The Reynolds number of a spherical particle (or bubble or drop) is given by

$$Re_P = \frac{\left|\vec{V}_r\right| D_P}{\nu} \tag{64}$$

If the particle is not spherical, a suitable characteristic length is used instead of D_P.

When the particle is not isolated, but there is suspension, the particle's concentration affects the drag coefficient, and the definition of the Reynolds number needs to consider the concentration of the particles in the mixture, c. For dilute suspensions ($c \lesssim 0.05 - 0.1$), the following Reynolds number can be used [88]:

$$Re_s = \frac{(1-c)\left|\vec{V}_r\right| D_P}{\nu} \tag{65}$$

The above expression considers the kinematic viscosity of the fluid phase. For higher concentrations, the particle "feels" a higher drag effect that can be considered using a dynamic viscosity associated with the suspension or mixture, μ_m, used to define the Reynolds number Re_m [89,90]:

$$Re_m = \frac{\rho\left|\vec{V}_r\right| D_P}{\mu_m} \tag{66}$$

The viscosity μ_m can be computed from the following [89–91]:

$$\mu_m = \mu\left(1 - \frac{c}{c_{pm}}\right)^{-2.5 c_{pm} \mu^*} \tag{67}$$

Equations (66) and (67) are valid for solid particles, drops, or bubbles. For solid particles, $c_{pm} \approx 0.62$ is the concentration for maximum packing, and $\mu^* = 1$. For bubbles and drops, it depends on the viscosity of the carrier phase, μ, and the viscosity of the fluid within the bubble or drop, μ_p, according to

$$\mu^* = \frac{\mu_p + 0.4\,\mu}{\mu_p + \mu} \tag{68}$$

Many other expressions for μ_m can be found in the literature [92].

In the design of spray systems, the flow characteristics at the nozzle exit are crucial for liquid atomization and the generation of fluid particles with a relatively homogeneous size distribution. One of the atomization methods involves tangentially injecting the liquid into the nozzle, creating a strong swirling motion that forms an air core at the center of the nozzle. This results in the liquid exiting as thin, axisymmetric sheet, then breaking into tiny fluid particles. The formation of the air core depends on a Reynolds number associated with the liquid flow rate and the geometry of the tangential entry ports, Re_t. The discharge coefficient depends on a Reynolds number, $Re_{\Delta p}$, which is based on the pressure gradient between the nozzle inlet and outlet, defined as

$$Re_t = \frac{\rho\, Q\, D_{tp}}{\mu\, A_{tp}}\,, \tag{69a}$$

$$Re_{\Delta p} = \frac{\sqrt{\rho\,\Delta p}\, D_0}{\mu} \tag{69b}$$

where Q is the discharge, D_{tp} is the diameter of the tangential entry ports of the nozzle, A_{tp} is the total area of entry ports, and D_0 is the diameter of outlet's nozzle [93]. The final size of the droplets depends on other variables like surface tension, gas properties (density and viscosity), and gas-to-liquid momentum ratio.

The mathematical modeling of multiphase flows involves coupling the continuity, momentum, and energy equations for the different phases involved. One of the most significant challenges is the interaction between phases, where one component exerts a drag on the others. Modeling can be quite complex depending on the conditions and the phases involved. For example, in gas–liquid mixtures within a closed pipe, a liquid film may form along the perimeter of the pipe, with a gas core in the center, which also drags liquid drops. Without delving into the various modeling approaches developed in this paper, Reynolds numbers—local or global—are frequently encountered in these models to calculate the pressure losses associated with the different phases or to determine the interphase momentum transfer. Below, as a sample, we present some of the Reynolds numbers found in multiphase flow modeling.

Brennen [94] considers a Reynolds number for the effective gas flow given by

$$Re_{eff} = \frac{U_C\, L_C}{\nu_C}\,(1+\xi) \tag{70}$$

where the subindex C refers to the "continuous" (carrier) phase, and ξ is the loading parameter, defined as follows:

$$\xi = \frac{c_D\, \rho_D}{c_C\, \rho_C} \tag{71}$$

The subindex D designs the "discrete" or disperse phase. The density of the mixture is $\rho = c_D\, \rho_D + c_C\, \rho_C$.

It has been observed in laminar flow that when small particles move through a pipe, they migrate away from the pipe wall and toward the center of the flow (Segre and Silberberg effect). This migration contrasts with the intuitive expectation that particles should follow the fluid's motion and tend to settle. This phenomenon occurs because the particles experience shear forces in the fluid and move through a region with a velocity gradient. Thus, a shear Reynolds number ($Re_{\dot{\gamma}}$) is generated [93], defined as follows:

$$Re_{\dot{\gamma}} = \frac{|\dot{\gamma}|\, D_p^2}{\nu} = \left|\frac{\partial u}{\partial r}\right| \frac{D_p^2}{\nu} \tag{72}$$

where u is the axial velocity, and r is the radial coordinate. It must be noted that $Re_{\dot{\gamma}}$ is a local Reynolds number as it is a function of r.

In the study of the role of the ambient fluid in gravitational granular flow dynamics [95], the authors derived two Reynolds numbers while nondimensionalizing the equations for the solid and liquid phases. One Reynolds number is associated with grain collisions (Re_{gc}), and the other with the fluid (Re_f). They are

$$Re_{gc} = \frac{c_{LP}\, \rho_s\, h_0\, \sqrt{g' h_0}}{\mu_{sc}}, \tag{73a}$$

$$Re_f = \frac{(1-c)\, \rho\, h_0\, \sqrt{g'\, h_0}}{\mu + \mu_T} \tag{73b}$$

c_{LP} is the loose packing concentration of the solids, ρ_s is the solid density, h_0 is the length scale of the liquid–solid mixture, $g' = g(\rho_s - \rho)/\rho_s$ is a reduced gravity, and μ_{sc} is the grain collisional viscosity, which depends on the diameter and density of the grains, solid

concentration, elastic restitution coefficient, and the granular temperature, computed from the grain velocity fluctuations. The turbulent viscosity μ_T was determined using a *k*-epsilon model of the turbulence.

3.8. Reynolds Number in Fluvial Mechanics

Like the flow in pipes and channels, the Reynolds number plays a fundamental role in river mechanics. In addition to the Reynolds number from Equation (22), which determines whether the flow will be laminar or turbulent, other dimensionless numbers related to flow resistance arise. If turbulent flow develops over a smooth wall, a region in contact with the wall is generated, called the viscous sublayer, where viscous forces dominate over turbulent forces. The thickness of this sublayer (δ_V) is unknown a priori, but the variables it scales with can be determined through dimensional analysis. In this region, a shear stress acts, which can be characterized as the one acting on the wall, τ_0. The relevant fluid properties are its density ρ and viscosity μ. Thus, it is possible to generate a dimensionless parameter, $\Pi = (\delta_V\, \rho/\mu)\, \sqrt{\tau_0/\rho}$. The value of Π is determined experimentally, yielding $\Pi \approx 5$. The term $\sqrt{\tau_0/\rho}$ has the dimensions of velocity, is denoted u_*, and is called friction velocity. Therefore, $\Pi = \delta_V u_* /\nu$ corresponds to a Reynolds number whose length scale is the thickness of the viscous sublayer $\mathcal{L} = \delta_V$, and the velocity scale is the friction velocity $\mathcal{U} = u_*$, so we have the following:

$$Re_{\delta_V} = \frac{\delta_V\, u_*}{\nu} \tag{74}$$

The notation $Re_{\delta_V} = \delta_V^+$ is also used. Equation (74) indicates the presence of another length scale, ℓ_V, associated with the relevant variables near the wall: τ_0, ρ and μ or, equivalently, u_* and ν. These variables are called wall variables or inner variables and $\ell_V = \nu/u_*$. Then, Re_{δ_V} corresponds to the ratio of two lengths, the thickness of the viscous sublayer, and the inner length: $Re_{\delta_V} = \delta_V/\ell_V$.

We have referred to a smooth wall without specifying what exactly defines a smooth wall. Strictly speaking, every surface has roughness; however, from a hydrodynamic perspective, the wall is defined as hydrodynamically smooth if the roughness is entirely within the viscous sublayer. If the size of the roughness is k_s, this means $k_s < \delta_V$. Thus, a Reynolds number naturally arises by comparing k_s with ℓ_V, $Re_{k_s} = k_s\, /\ell_V$, named the roughness Reynolds number:

$$Re_{k_s} = \frac{k_s\, u_*}{\nu} \tag{75}$$

The notation $Re_{k_s} = k_s^+$ is also used. Thus, a wall is called hydrodynamically smooth if $Re_{k_s} = k_s^+ < 5$. If the size of the roughness is such that it completely inhibits the development of a viscous sublayer, the wall is called hydrodynamically rough; this occurs if $Re_{k_s} = k_s^+ > 70$. For Re_{k_s}'s intermediate values, the wall is named a smooth-to-rough transition.

The type of wall conditions the resistance to flow. The shear stress at the bottom (τ_0) is related to the average flow velocity (U), for example, through the Darcy friction factor (f) using the following relationship:

$$\tau_0 = \frac{1}{8}\, f\, U^2, \quad \frac{u_*}{U} = \sqrt{\frac{f}{8}} \tag{76}$$

In general, f is a function of the Reynolds number and the relative roughness, i.e., $f = f(Re_{R_H},\ k_s/R_H)$. For a laminar regime, it depends only on the Reynolds number, $f = 16/Re_{R_H}$. There are relationships for turbulent flows in open channels based on the semi-analytical Prandtl–Nikuradse expression obtained for pipes where the numeri-

cal constants have been adjusted [96]. In natural, non-cohesive channels, the roughness k_s is determined by a representative size of the sediment that forms the bed.

When the material that forms the bed can move due to the flow, a complex and continuous relationship exists between bed deformation, flow pattern alteration, and sediment movement. This deformation requires determining the condition under which the flow can move the sediment (incipient motion condition) and the amount of sediment transported (solid discharge). Shields [97] determined that the critical shear stress at which non-cohesive particle movement occurs depends on what he called the particle Reynolds number, Re_*:

$$Re_* = \frac{u_* \, D_s}{\nu} \tag{77}$$

where D_s is the characteristic size of the bed's sediment. As first noticed by Vanoni [98], the Shield diagram (i.e., the dimensionless critical shear stress, $\tau_* = u_*^2/(g \, R \, D_s)$, as a function of Re_*, where $R = \rho_s/\rho - 1$, and ρ_s and ρ are the sediment and water densities) is not practical because in order to find the critical shear stress for incipient motion, one must know the critical shear velocity u_*. The relationship can be cast in an explicit form if another Reynolds number exists: $Re_p = \sqrt{g \, R \, D_s^3}/\upsilon$, such that $Re_* = \tau_*^{1/2} \, Re_p$ [99]. Other relationships based on the critical shear stress have been developed by various authors [100].

When the particles of the bed are set in motion, change the bed configuration forming bedforms that depend on the flow characteristics and sediment properties. Thus, for example, ripples are present when a viscous sublayer exists [100,101] and antidunes in supercritical flows [100]. The flow resistance when there are bedforms has two components: one associated with the friction induced by the grains (roughness) and the other with the geometry. The first is computed using a friction coefficient, as mentioned above. The second demands ad hoc modeling.

Vortex sand ripples are bedforms created by oscillatory flow and are commonly generated beneath waves [102,103]. The behavior of vortex ripples in an oscillatory flow is controlled by the wave Reynolds number, which is defined $Re = A^2 \, \omega/\nu$, where A is the wave semi-orbital excursion, ω is the radian frequency of the wave, given by $\omega = 2\pi/T$, T is the flow oscillation's period, and ν is the kinematic viscosity of the water [103,104].

The sediment discharge in rivers is divided into bedload and suspension load. Einstein's approach to determining the bedload depends on the ratio $Re_{k_s}/Re_{\delta_V} = k_s/\delta_V$ [100,105]. Other bed-load formulas require computing a critical shear stress, which can depend on Re_*.

Bagnold's approach to the sediment transport problem must be mentioned [106]. He defines a Reynolds number for a layer formed by solids sheared in a fluid and denoted by G:

$$G = \frac{D_s}{\mu} \sqrt{\frac{\rho_s}{\lambda} \, \tau_0} \tag{78}$$

where λ is the linear concentration. The above equation can be written as

$$G = \frac{D_s}{\mu/\rho_s \sqrt{\lambda}} \sqrt{\frac{\tau_0}{\rho_s}} \tag{79}$$

Bagnold expresses that G is closely analogous to the fluid Reynolds number $(D_s/\nu) \, \sqrt{\tau_0/\rho} = D_s \, u_*/\nu$. At the bed, the linear concentration remains constant at 14, the limiting concentration at which a mixture of water and a sheared array of solid grains was found to cease to behave as a Newtonian fluid and begin to behave as a rheological "paste" [106].

3.9. Reynolds Number in Granular Flows

The distinction between granular and multiphase flows is somewhat arbitrary, and some references from one subsection could easily belong to the other or both. Granular flows refer to "dry" flows of solid particles, meaning there is no liquid between the particles, only air, which remains stationary. Due to the significant difference between the density of air and that of liquids, it is common to assume that air does not play an important role in the dynamics of dry granular flow. However, in the reference [95], it is shown that while buoyancy can be neglected, the same does not apply to the effect of the viscosity of interstitial air (the kinematic viscosity of air is an order of magnitude higher than that of water), and the movement of the solid phase induces air movement.

Granular flows are classified into three regimes: quasi-static, with slow deformations and frictional particle interactions; gaseous, with rapid, dilute flow where particles interact via collisions; and liquid-like, where the material is dense but flows, with both collisional and frictional interactions [107].

The initial definition of the Reynolds number as the ratio of inertia to viscous forces can be extended to granular flows if it is considered as the ratio between inertia and the dissipative term in the equation of motion. In granular flows, the dissipative term is associated with frictional (tangential) stresses. For a dense material flow (liquid-like), Schaeffer [108] deduces that the following equation gives the velocity:

$$\rho_s \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) = -k \nabla \left(T_\tau \mid \dot{\gamma} \mid^{-1} \dot{\gamma}_{ij} \right) - \nabla T_\tau + \rho \vec{g} \tag{80}$$

where $T_\tau = tr(\tau_{ij})/3$ is the trace of the stress tensor divided by 3, and k is a material-dependent constant related to the internal friction angle. Thus, the inertia-related term scales as $\rho_s \mathcal{U}^2/\mathcal{L}$, while the dissipative term $-k \nabla \left(\tau \mid \dot{\gamma} \mid^{-1} \dot{\gamma}_{ij} \right)$ scales as $k T/\mathcal{L}$, thus $inertia/dissipative\ term \sim \rho_s \mathcal{U}^2/(k T)$. If, for example, a silo is considered, $T \sim \rho_s g \mathcal{L}$, then

$$Re_{gran} = \frac{\mathcal{U}^2}{kg\mathcal{L}} \tag{81}$$

Some authors define an effective viscosity for the granular medium as [108,109]

$$\mu_{eff} = \frac{\eta\, p_s}{|\dot{\gamma}|} \tag{82}$$

where η is the effective friction coefficient given by

$$\eta = \eta_1 + \frac{\eta_2 - \eta_1}{I_0/I + 1} \tag{83}$$

Here, η_1, η_2, and I_0 are constants with typical $\tan 21^\circ$, $\tan 33^\circ$, and 0.3, values, respectively. I is the inertial number defined as

$$I = \frac{\dot{\gamma}\, D_s}{\sqrt{p_s/\rho_s}} \tag{84}$$

where D_s is the particle diameter, ρ_s its density, p_s is the pressure, and $\dot{\gamma}$ is the second invariant of the shear rate. As can be seen, this definition of viscosity is not an intrinsic property of the material but depends on local flow characteristics.

For dilute granular flows, expressions for effective viscosity have been derived based on the kinetic theory of gases, for example [110].

$$\mu_{eff} = \frac{5\pi}{96}\rho_s \, D_p \, \sqrt{T_g} \tag{85}$$

T_g is the so-called granular temperature, corresponding to the root mean square value of grain velocity fluctuations.

3.10. Reynolds Number in Non-Newtonian Fluid Flows

Non-Newtonian fluids are those in which the relationship between stress and strain rate is not linear. In these cases, no single definition exists for viscosity since it depends on the strain rate. Essentially, two types of viscosity can be defined: apparent viscosity (μ_A) and local (or tangent) viscosity (μ_L). They are usually determined from a rheogram, which represents the relationship between shear stress and strain rate, obtained experimentally for each specific fluid. The measurement results in a relationship between τ_{ij} and $\dot{\gamma}_{ij}$, or, when using a rotational rheometer, $\tau_{r\theta}$ and $\dot{\gamma}_{r\theta}$. For simplicity, in the following discussion, stress and strain rate will be denoted simply as τ and $\dot{\gamma}$. The apparent and local viscosities are defined as follows:

$$\mu_A = \frac{\tau}{\dot{\gamma}}, \qquad \mu_L = \frac{d\tau}{d\dot{\gamma}} \tag{86}$$

For an Ostwald–de Waele fluid, defined as $\tau = K\,\dot{\gamma}^{\,n}$, the two viscosities are $\mu_A = K\,\dot{\gamma}^{\,n-1}$ and $\mu_L = n\,K\,\dot{\gamma}^{\,n-1}$.

One way to define the Reynolds number is to express the shear stress on the wall of a pipe in terms of the friction factor ($\tau_0 = \frac{1}{8}\,\rho\,f\,U^2$) and ensure that the friction factor in the laminar regime in a pipe for a non-Newtonian fluid has the same form as that of a Newtonian fluid, that is, $f = 64/Re$. The equivalent Reynolds number Re_E is frequently written as $F\,Re_{eq}$, where F is a function that depends on the yield stress, τ_y. The way to obtain Re_{eq} is as follows: in a cylindrical pipe with a radius R_0, the discharge is given by $Q = \pi\,R_0^2\,U = 2\pi\,\int_0^{R_0} u\,r\,dr$. We can write the strain rate $\dot{\gamma}$ as a function of τ, $\dot{\gamma} = g(\tau)$. As $\tau/\tau_0 = r/R_0$ give the shear stress distribution, changing variables from u to τ, the equation for the discharge can be integrated in terms of the shear stress and relate the velocity with the shear stress on the wall τ_0. The following relation is obtained: $2U/D = -\left(1/\tau_0^3\right)\,\int_0^{\tau_0}\tau^2 g(\tau)\,d\tau$. Thus, a relation among U, τ_0, and the parameters of the specific rheological model can be used to determine $f = 8\tau_0/\left(\rho\,U^2\right)$ and obtain $Re_{eq.}$ The results of applying of the aforementioned procedure to three common types of rheology are presented in Table 1.

Table 1. Equivalent Reynolds for common rheological models. $\xi = \tau_y/\tau_0$.

MODEL	F	Re_{eq}
Ostwald–de Waele $\tau = K\,\dot{\gamma}^{\,n}$	1	$8\left(\frac{n}{3n+1}\right)^n \frac{\rho\,U^2}{K\,(2U/D)^n}$
Bingham's plastic $\tau = \tau_y + \mu_B\,\dot{\gamma}$	$1 - \frac{4}{3}\,\xi + \frac{1}{3}\,\xi^4$	$\frac{\rho\,U\,D}{\mu_B}$
Herschel and Bulkley $\tau = \tau_y + K\,\dot{\gamma}^{\,n}$	$\left[(3n+1)^n(1-\xi)^{n+1}\right]\times$ $\left[\frac{(1-\xi)^2}{3n+1} + \frac{2\xi(1-\xi)}{2n+1} + \frac{\xi^2}{n+1}\right]^n$	$8\left(\frac{n}{3n+1}\right)^n \frac{\rho\,U^2}{K\,(2U/D)^n}$

The equivalent Reynolds number for Ostwald–de Waele or Herschel and Bukley models is often referred as "Metzner and Reed Reynolds number", denoted as Re_{MR}, was

introduced in the analysis performed by those authors in their study on power law fluid flows [111]. As with Newtonian fluids, when inertia overcomes the stabilizing effect of viscous forces, non-Newtonian fluid flow becomes unstable and turbulence arises. Metzner and Reed studied the limits of laminar, transitional, and turbulent-flow regions for power-law (Ostwald–de Waele) fluids and their associated friction factors. Hanks determined the criterion for the laminar-turbulent transition in the flow of fluids presenting a yield stress, modeled as Bingham plastics or Powell–Eyring fluids. In these cases, the critical Reynolds number depends on the Hedström number, defined for a Bingham plastic fluid as $He = \rho \tau_y D^2/\mu_B^2$. For the Powell–Eyring model (given by $\tau = \mu_\infty \dot{\gamma} + B^{-1}\sinh^{-1}(\dot{\gamma}/A)$) the Reynolds and Hedström numbers are defined as $Re = \rho\ U\ D/\mu_\infty$ and $He = \rho D^2/(\mu_B^2 B)$, respectively [112,113]. A review by Yusufi et al. on modeling the flow of Herschel–Bulkley fluids in pipes [114] presents an analysis of the laminar-turbulent transition in these fluids. They use a Reynolds number defined as $Re_w = \rho U D/\mu_w$, where $\mu_w = K^{1/n}\tau_0/(\tau_0 - \tau_y)^{1/n}$. Haldenwang et al. studied the impact of different Reynolds number definitions on the pressure drop prediction in sludge flows in pipes [115]. The laminar-turbulent transition of non-Newtonian fluids in open channel flows has been found to depend on both the Reynolds and Froude numbers [116].

3.11. Reynolds Number in Plasma and Magnetohydrodynamics

A plasma is an ionized gas in which a significant fraction of its atoms or molecules have lost or gained electrons, forming ions and free electrons. It has unique properties, such as conducting electricity and responding to electromagnetic fields. The equations of Vlasov–Maxwell govern plasma. The Vlasov equation describes the time evolution of the distribution function of a collisionless plasma consisting of charged particles with long-range interaction. Maxwell's equations are a set of equations that describe entirely electromagnetic phenomena. On the other hand, magnetohydrodynamics (MHD) is a macroscopic approximation in which the plasma is treated as a continuous conducting fluid without resolving the individual dynamics of electrons and ions. Assuming quasi-neutral equilibrium and frequent collisions, MHD is described by the following:

(i) The Navier–Stokes equations, including the electromagnetic Lorentz force $\vec{J} \times \vec{B}$, where $\vec{J}$ is the current density and $\vec{B}$ the magnetic field

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\ \vec{V}\right) = -\nabla\,\hat{p} + \mu\ \nabla^2\ \vec{V} + \vec{J} \times \vec{B} \tag{87}$$

(ii) The following induction equation

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \nabla \times \vec{B} + \frac{1}{\sigma\ \mu_0}\nabla^2 \vec{B} \tag{88}$$

where σ is the electrical conductivity, and μ_0 is the permeability of free space. The product $\sigma\ \mu_0$ has dimensions of $(\text{lengh}^2/\text{time})^{-1}$. Thus, the magnetic diffusivity or magnetic viscosity is defined as $\nu_m = 1/(\sigma\ \mu_0)$.

(iii) The following Maxwell's equations simplified for MHD

$$\nabla\cdot\vec{B} = 0, \tag{89a}$$

$$\nabla \times \vec{B} = \mu_0\ \vec{J} \tag{89b}$$

Making dimensionless the equation of momentum and induction with a characteristic fluid density ρ_0, magnetic field value, B_0, and using Equation (89b) ($J_0 = B_0/(\mu_0\,\mathcal{L})$), Equation (87) becomes

$$\rho^* \left(\frac{\partial \overrightarrow{V}^*}{\partial t^*} + \overrightarrow{V}^* \cdot \nabla^* \overrightarrow{V}^* \right) = -\nabla^* \hat{p}^* + \frac{\mu}{\rho_0\,\mathcal{U}\,\mathcal{L}} \nabla^{*2} \overrightarrow{V}^* + \frac{B_0^2}{\mu_0\,\rho_0\,\mathcal{U}^2} \overrightarrow{J}^* \times \overrightarrow{B}^* \tag{90}$$

Another dimensionless parameter appears in addition to the Reynolds number. It is named Alfvén number, $Al = B_0^2/\left(\mu_0 \rho_0 \mathcal{U}^2\right)$.

(iv) Similarly, the induction equation becomes

$$\frac{\partial \overrightarrow{B}^*}{\partial t^*} = \nabla^* \times \nabla^* \times \overrightarrow{B}^* + \frac{\nu_m}{\mathcal{U}\,\mathcal{L}} \nabla^{*2} \overrightarrow{B}^* \tag{91}$$

The new dimensionless parameter in front of $\nabla^{*2}\overrightarrow{B}^*$ is the inverse of a magnetic Reynolds number [117,118]:

$$Re_m = \frac{\mathcal{U}\,\mathcal{L}}{\nu_m} = \mathcal{U}\,\mathcal{L}\,\sigma\,\mu_0 \tag{92}$$

The magnetic Reynolds number is the ratio of advection of the magnetic field to magnetic diffusion. It compares the rate of macroscopic motion of the fluid with the rate of magnetic diffusion. If $Re_m >> 1$, magnetic induction dominates, and the magnetic field is "frozen" into the fluid, i.e., the hydrodynamic and electromagnetic fields are coupled rigidly. If $Re_m << 1$, magnetic diffusion dominates, and the magnetic field decouples from the fluid motion. In this case, the electromagnetic field's diffusion rate through the hydrodynamic field is appreciable and there is leakage flow or free decay [118,119].

The magnetic Reynolds number can be interpreted in terms of time scales. A characteristic time for free decay of the magnetic field is $\mathcal{T}_{mDecay} \sim \mathcal{L}^2/\nu_m$. A characteristic time of the flow field is $\mathcal{T} = \mathcal{L}/\mathcal{U}$. Thus, $Re_m = \mathcal{T}_{mDecay}/\mathcal{T}$.

In plasmas, the magnetic Reynolds number (Re_m) is often more important than the hydrodynamic Reynolds number (Re) because plasmas are highly conducting, and magnetic effects dominate. Examples are the solar corona and interstellar medium. Depending on the specific problem, the hydrodynamic Reynolds number (Re) and the magnetic Reynolds number (Re_m) are important in MHD. Examples of MHD are found in industrial applications (liquid metals) and geophysics (Earth core).

The matter of the initial stages of the universe is a conducting plasma, and the need of a primordial turbulence in the formation of galaxies and stars has been emphasized since the works of von Weizsäcker [120] and Gamow [121]. In the early universe, in addition to the Re_m, the so-called kinetic Reynolds number Re_{kin} is also important [122], where the coefficient of thermal diffusivity replaces the kinematic viscosity, ν_{th}, as they are directly related according to the kinetic theory of gases [123]. Thermal diffusivity can be estimated as $\nu_{th} \sim (\alpha_{em} T)^{-1}$, where α_{em} is the electromagnetic (or fine-structure) coupling constant [124], and T is the temperature. In both Re_m and Re_{kin}, the velocity scale $\mathcal{U}$ is a function of the electron and ion velocities, and the length scale $\mathcal{L}$ is considered to be on the order of the Hubble radius (the distance at which the universe's expansion rate equals the speed of light) at the corresponding epoch. As the early universe evolves, both the kinetic and magnetic Reynolds numbers decrease, with Re_m being approximately twenty orders of magnitude larger than Re_{kin} which is, in turn, smaller than one.

3.12. Reynolds Number in Relativistic Fluid Mechanics

Navier–Stokes theory is well established for describing dissipative fluid dynamics in non-relativistic settings. However, extending it to relativistic systems is problematic because the naive relativistic generalization leads to acausal and unstable behavior. This issue arises because dissipative currents in Navier–Stokes theory respond instantaneously to dissipative forces. This behavior is unphysical, according to this theory. Efforts have been made to address this by introducing a finite response time for dissipative currents, even in non-relativistic fluid dynamics. Israel and Stewart were among the first to address this issue in their 1979 article by developing a second-order theory of relativistic dissipative fluid dynamics [125]. Their approach introduces a finite relaxation time for dissipative currents, ensuring they do not respond instantaneously to dissipative forces. The term "second-order" includes terms proportional to the square of the Knudsen number. These terms are necessary to describe this relaxation process [126].

The fluid-dynamical limit is typically assessed using the Knudsen number ($Kn = \ell_{micro}/\mathcal{L}_{macro}$), defined as the ratio of microscopic to macroscopic length or time scales. Macroscopic scales are estimated from fluid-dynamical gradients, while microscopic scales correspond to the mean-free path or collision time. When $Kn \ll 1$, a clear separation exists between microscopic and macroscopic scales. This separation allows the system's dynamics to be described using only a few macroscopic fields as microscopic details can be effectively integrated out [127].

Fluid dynamics is expected to be valid near local thermal equilibrium. Deviations from equilibrium can be quantified using macroscopic variables by defining a set of ratios of dissipative quantities to equilibrium pressure or density, which serve as generalized forms of the inverse Reynolds number [127].

Denicol et al. proposed three inverse Reynolds numbers in their 2012 paper. By 2024, this set had expanded to six [126]. Following the notation of Denicol et al. [127], they are

$$Re_{\Pi}^{-1} = \frac{|\Pi|}{P_0}, \quad Re_{n}^{-1} = \frac{|n^{\mu}|}{n_0}, \quad Re_{\pi}^{-1} = \frac{|\pi^{\mu\nu}|}{P_0} \tag{93}$$

where Π is the bulk viscous pressure, n^{μ} is the particle-diffusion current, and $\pi^{\mu\nu}$ is the shear stress tensor. Additionally, P_0 and n_0 represent the pressure and particle density in local equilibrium, respectively. The term "inverse Reynolds number" comes from Re_{π}^{-1} as it relates to the usual Reynolds number under non-relativistic conditions.

4. Reynolds Number Presence: From Nanostructures to the Cosmic Horizon

As shown in the previous section, when there is fluid flow or the movement of a body within a fluid, it is possible to find Reynolds numbers associated with the phenomenon across virtually the entire range of measurable length scales. These scales can range from the order of a micrometer (as in the case of the size of *E. coli* [41] or, if the characteristic length is considered to be the diameter of the flagellum, on the order of 100 Å (~0.01 μm) [41]) to the size of the observable universe or cosmic horizon [122]. Similarly, the Reynolds number spans a range from approximately 10^{-5} for the movement of bacteria [41] to 10^{8} for a typhoon [128]. For stars, using Equation (48), Canuto and Christensen-Dalsgaard estimate that $Re > 10^{10}$ [129]. Since its conception and definition in the 19th century, the meaning of the Reynolds number has remained unchanged: it is the ratio of inertial forces to viscous forces in a flow, regardless of how it is derived. The velocity or length scales must be chosen appropriately depending on the application. Similarly, viscosity—originally considered a fluid property—can sometimes also depend on the flow, as seen in non-Newtonian fluids. In contrast, in others, such as plasmas, it remains primarily a fluid property.

By analogy, dimensionless parameters have been named "Reynolds numbers" in various fields, such as the magnetic Reynolds number defined by Elsasser in 1954 [117], inverse Reynolds numbers in relativistic fluid mechanics [126], the kinetic Reynolds number in cosmology [122], and Reynolds numbers in superfluids [78,79]. In all cases, viscosity is replaced by a coefficient that represents a resistive effect on the flow or arises from a term in the equation of motion that plays a role analogous to the stress tensor gradient in the Navier–Stokes equation.

As has been shown, the Reynolds number originated as a criterion for determining the transition of flow from laminar to turbulent [15,17] and as a criterion of similarity [10,18–21]. It has also been extensively used in the search for analytical solutions to the Navier–Stokes equations as it constitutes a control parameter for the equations, defining two extreme cases: $Re \ll 1$ (Stokes regime) and $Re \to \infty$ (potential flow). However, the Reynolds number alone is insufficient to represent all possible flow dynamics or maintain similarity in physical models. For example, the following are two situations where this occurs: laminar flow at very low Reynolds numbers and the scaling of flow dynamics in wind tunnels to full-scale flow.

4.1. When the Number of Reynolds Is Not Enough: Flows at Milli, Micro, and Nano Length Scales

Millifluidics refers to a field that deals with fluids in channels with dimensions on the scale of 1 mm to 10 mm. It is not simply a flow or a flow regime but rather a field of study and application focused on manipulating and understanding fluid behavior at this specific scale. The flows within millifluidic systems are typically laminar due to their low Reynolds number, which is usually in the range of $Re = 10$–10^2. When the characteristic dimensions of the channels are on the order of 100 nm to 1 mm, the associated Reynolds numbers span from 10^{-6} to 10. Nanofluidics, on the other hand, involves length scales smaller than 100 nm, with $Re < 10^{-6}$ [130].

Fluid bulk properties dominate millifluidics and microfluidics. In addition to inertia and viscous force, those derived from surface and interfacial tension are relevant [131]. Capillary force can drive fluids forward or lift them against gravity when confined within a narrow tube with a hydrophilic surface. This phenomenon enables passive fluid pumping in microfluidic devices [130]. Here, in addition to the Reynolds number, other dimensionless parameters are relevant as indicated below: Capillary number, corresponding to the ratio between viscous and surface tension forces, given by $Ca = \mu\mathcal{U}/\sigma$ (σ is the surface tension), and Péclet number, relating convection to diffusion, $Pe = \mathcal{U}\mathcal{L}/D$. D is the molecular diffusion coefficient of a specific component, and Pe determines the channel length required for that component to diffuse across the channel width [46]. When polymers are present in a solvent, two other important dimensionless parameters arise, both associated with the polymer's relaxation time T_R, which stem from the viscoelastic behavior of the solution. The first is the Deborah number, which measures the polymer's relaxation time relative to the deformation process' characteristic time, or the flow time scale, T_P. It is defined as $De = T_R/T_P$ [46,132]. The value of T_P depends on the process. For example, in a flow with a characteristic velocity scale U_0 in a channel that contracts over a length L_0, the geometric time scale is given by $T_P = L_0/U_0$, which is the time required for a polymer to traverse the channel. Similarly, in an oscillatory flow with frequency ω, the oscillation time scale is $T_P = 1/\omega$. The second dimensionless number arising in viscoelastic fluids is the Weissenberg number, given by $Wi = T_R\dot{\gamma}$, where $\dot{\gamma}$ is the shear rate. It can be shown that Wi represents the ratio of elastic to viscous forces [132,133].

In nanofluidics, channel dimensions are smaller than 100 nm, comparable to several molecules' size. As a result, intermolecular forces such as electrostatic forces, van der Waals attractions, hydrogen bonding, and steric repulsion become significant [130]. For

instance, when a surface is charged, it attracts counterions (ions with a charge opposite to that of the charged surface, they are attracted to the surface due to electrostatic forces) and repels coions (ions with a charge of the same sign as the charged surface, they are repelled from the surface due to electrostatic forces) in the fluid, creating a characteristic length known as the Debye length (λ_D). Within this length, the electric potential decreases exponentially to zero within this length as the distance from the charged surface increases. The Debye length typically ranges from 1 to 100 nm. If a is the characteristic size of the particle, the Debye number is defined as $Db = a/\lambda_D$. This parameter indicates whether electrostatic interactions are long range ($Db << 1$) or short range ($Db >> 1$). In millifluidics or microfluidics, where channel dimensions are much larger than the Debye length, surface charges are generally shielded by counterions and can be considered negligible. However, in nanofluidics, where channel dimensions are comparable to or smaller than the Debye length, the electric field generated by the charged surface can penetrate the nanochannels and influence phenomena such as ion movement.

Overall, interactions in nanofluidics are highly complex and involve a combination of various intermolecular forces like those resulting in van der Waals attraction, hydrogen bonding, steric repulsion, electrostatic force, etc. The van der Waals force is related to the Hamaker constant (A), a material-dependent constant that quantifies the strength of van der Waals forces between particles, surfaces, or molecules. A is typically on the order of 10^{-20} to 10^{-19} joules for many common materials [134]. The Hamaker number is defined as $Ha = A/(k_B T)$, where k_B is the Boltzmann constant and T the temperature.

The Derjaguin number arises from the comparison between the van der Waals force to viscous force in a system: $De = F_{vdW}/(6\pi\mu RU)$, where F_{vdW} is the van der Waals force, μ the fluid dynamic viscosity, R is the particle effective radius, and U is the relative velocity. The force F_{vdW} depends on the geometry of the particles and their separation (h). For two spheres with radii R_1 and R_2, $F_{vdW} = AR_{eff}/(6h^2)$, where $R_{eff} = R_1R_2/(R_1 + R_2)$. For two flat parallel surfaces of area S, $F_{vdW} = AS/(6\pi h^3)$ [135,136].

There is not a specific name for the dimensionless parameter involving hydrogen bonding. However, the ratio of hydrogen bond energy (E_{HB}) to thermal energy: $Hy = E_{HB}/(k_B T)$ is commonly used in studies of hydrogen bonding to determine whether thermal fluctuations can disrupt hydrogen bonds. This ratio is often discussed in the context of molecular interactions.

Coulomb's law governs electrostatic interactions. A standard dimensionless parameter for electrostatic forces is the Debye–Hückel number or the ratio of electrostatic energy to thermal energy: $DH = q^2/(4\pi\epsilon k_B T)$, where q is the charge of the particle, ϵ is the permittivity of the medium, and $k_B T$ is the thermal energy. This parameter determines the strength of electrostatic interactions relative to thermal fluctuations.

As can be seen, for flows with $Re \ll 1$, the Reynolds number is insufficient to characterize the flow adequately, and other relevant phenomena begin to emerge, described by sets of new dimensionless parameters. Notably, the small dimensions that suppress inertial nonlinearity bring attention to other physical phenomena that are less familiar at the macroscale [41,46]. The article by Squires and Quake [46] provides a comprehensive list of applications within the range of very low Reynolds numbers for Newtonian fluid flows (specifically, laminar regime flows), which are now commonly referred to as milli-, micro-, and nanofluidics.

It should also be noted that the laminar flow condition for $Re \ll 1$ does not apply to all types of rheologies. A surprising phenomenon occurs in flows with low Reynolds numbers of viscoelastic fluids (typically polymeric solutions) as they can exhibit turbulence-like behavior despite their negligible inertia. This behavior is similar to the turbulence found in Newtonian fluid flows that exceed the critical Reynolds number [137]. This phenomenon,

known as "elastic turbulence", arises from the elastic properties of the solution, which stem from the elastic characteristics of viscoelastic fluids, even at very low Reynolds numbers ($Re \ll 1$). While traditional turbulence is driven by inertia, elastic turbulence arises from the elastic stresses caused by the stretching and deformation of polymers in the flow. The Weissenberg number characterizes the onset of elastic turbulence. Schiamberg et al. reported measurements in torsional, parallel-plate flow of a polymer solution that showed elastic turbulence in experiments with Reynolds number $Re = R^2\Omega/\nu$ (Equation (30)) in the range 0.023–0.59 [138]. Weissenberg numbers were in the range $Wi = 0.30$–4.97. It must be emphasized that elastic turbulence is not the same as hydrodynamic turbulence as the destabilizing mechanisms are fundamentally different.

Even though the Reynolds number is a crucial dimensionless parameter for characterizing fluid flow and enabling scaling between different systems, it has inherent limitations. Scaling issues are not limited to flow conditions where $Re \ll 1$, and it is not an exaggeration to say that they occur across the entire range of Reynolds numbers. Preserving Re alone does not guarantee perfect similarity as it only accounts for the ratio of inertial to viscous forces. Other dimensionless numbers, such as the Froude number (for gravitational effects), the Capillary or Weber numbers (for surface tension effects), the Weissenberg number (for viscoelastic effects), and the Mach number (for compressibility effects), may also be significant depending on the flow conditions. Neglecting these other parameters can lead to inaccurate predictions, especially when dealing with complex fluids, high-speed flows, or systems where surface tension or gravitational forces play a significant role. Therefore, while matching Reynolds number is a valuable starting point, a comprehensive scaling analysis requires consideration of all relevant dimensionless groups to capture the underlying physics of the flow accurately.

4.2. When the Number of Reynolds Is Not Enough: Limitations of the Reynolds Number in Flow Scaling

While the concept of the Reynolds number has not changed since its origin, the range of values it takes in practical applications has significantly expanded, as shown in the review by Saldana et al. [28]. These authors present a wide array of technical applications in which the range of Reynolds number values continues to grow. These applications include the design of aircraft, automobiles, ships, and submarines, as well as industrial processes such as heat exchangers, mixers, and more—alongside the classical applications involving flow in pipes and channels. As noted in Section 2, the Reynolds number is fundamental for the proper scaling of flows with friction and represents a key advancement in the development of studies using reduced-scale models. Wind tunnel experiments have been crucial to the advancement of aviation. It is worth noting that wind tunnels were already being used for studies on aerial locomotion long before the scaling laws were fully established. Although the first wind tunnel was built by Francis H. Wenham in 1871 to study the lift and drag of flat plates, and in 1901 the Wright brothers constructed a wind tunnel to investigate the lift of wing models (their first flight took place in 1903) [139], it was not until 1908, with Prandtl's wind tunnel in Göttingen, that these facilities began to be fully exploited. It was then that the application of Reynolds similarity relations in experiments and the analysis of results became standard practice [18].

It is interesting to note that, in the study of the aerodynamic characteristics of vehicles, the current challenges—such as shape optimization to reduce drag, stability, control effectiveness, leading-edge separation, pitching, and others ([28] and references therein)—are virtually the same as those outlined by Prandtl in his presentation at the 50th General Assembly of the Association of German Engineers [18]. Prandtl emphasized the importance of stability, controllability, and accurate scaling in aerodynamic testing, highlighting the need for Reynolds number similarity between models and full-scale aircraft. He pointed

out the scaling limitations due to the inverse relationship between velocity and length, which can lead to impractical test conditions. While using water as a working fluid offers lower kinematic viscosity, its high density makes it unsuitable due to excessive power demands and forces on the tested model. Pressurized air emerged as a viable alternative as its dynamic viscosity remains nearly constant with pressure while density increases linearly. In 1926, Munk and Miller reported the pressurized wind tunnel developed by NACA (now NASA), capable of reaching 20 atmospheres, achieving a Reynolds number of 2.7×10^6 for a Fokker D.VII model, compared to 1.35×10^5 at atmospheric pressure [140].

It is common to express the Reynolds number of a wind tunnel as Re per unit length ($Re/\mathcal{L}$) since the characteristic length corresponds to that of the tested object (for example, for an aircraft wing, it is usually the chord length). The maximum $Re/\mathcal{L}$ values reached by the early wind tunnels were 6.5×10^5 m^{-1} (Göttingen, 1908), 1.4×10^6 m^{-1} (Eiffel, 1909), 2.6×10^6 m^{-1} (Eiffel, 1914), 3.6×10^6 m^{-1} (Göttingen, 1917), etc., (values calculated from Reference [141]). A significant leap in the maximum $Re/\mathcal{L}$ value came with the pressurized wind tunnel of the NACA, which increased the values of earlier wind tunnels by an order of magnitude, reaching 3×10^7 m^{-1} [140].

As Prandtl pointed out [18], higher flow velocities make air compressibility increasingly important, requiring that pressure waves be considered in the scaling process. These waves must be scaled to preserve the Mach number between the model and the full-scale prototype. The Mach number is defined as $Ma = U/c$, where U is the flow velocity, and c is the speed of sound in air. As the Reynolds number increases, the Mach number also increases. The flow is considered subsonic for $Ma < 0.8$, but mild compressibility corrections are required when Ma approaches 0.8 locally. When $0.8 < Ma < 1.3$, the flow is transonic. It should be noted that even if an aircraft is flying, for example, at $Ma = 0.85$, the flow over the wings can exceed $Ma = 1$, and shock waves may be generated. For $1.2 < Ma < 3$, the flow is called supersonic. For $3 < Ma < 5$, the flow is high supersonic. In this range, the temperature at stagnation points can exceed the melting point of certain metals, and near $Ma = 5$, air temperature can reach combustion levels. The flow is called hypersonic for $Ma > 5$, and it is in this regime that spacecrafts reentering the atmosphere operate [142].

Pressurized wind tunnels must be used to physically model flows in the transonic or supersonic regime. However, due to the high pressures required to achieve the working velocities, there is an increase in air temperature, which affects the flow properties, dynamic similarity, and the integrity of materials. To address this, cryogenic wind tunnels have been designed and built. These tunnels operate at ambient pressure but use air (or another gas, such as gaseous nitrogen) at low temperatures to achieve high Reynolds numbers, compatible with the condition imposed by the Mach number [143]. This approach decouples aerodynamic effects due to Reynolds number, Mach number, and aeroelasticity. The largest high-pressure, cryogenic, transonic wind tunnels are the National Transonic Facility from NASA ($0.1 < Ma < 1.2$, $Re/\mathcal{L}$ up to 4.4×10^8 m^{-1}, working pressures between 1 to 8 atm, temperature range: -157 °C to 66 °C) and the European Transonic Wind Tunnel ($0.13 < Ma < 1.3$, $Re/\mathcal{L}$ up to 2.3×10^8 m^{-1}, working pressures between 1.1 to 4.4 atm, temperature range -160 °C to 40 °C) [144,145].

Low-temperature wind tunnels are not restricted to aerodynamic studies as they have also been used to study the effects of the Reynolds number on high-speed trains [146].

The requirements that resulted from the development of aviation after World War II, as well as in aerospace engineering, have demanded the need to design and build supersonic and hypersonic wind tunnels. In these, the Mach number is large enough that the compression and temperature increase in the air (or gas used) must be taken into account, along with the resulting changes in density and viscosity, complicating scaling [147,148]. In high-speed flows, the transition from laminar flow to turbulent flow

is important because turbulent boundary layers significantly increase the heat transfer to the surface, which is a major concern for thermal protection systems [149]. Hypersonic air plasma wind tunnels, such as the SCIROCCO arc jet facility (Italian Aerospace Research Center), are designed to simulate the extreme conditions experienced by spacecraft during atmospheric re-entry. These facilities generate high-enthalpy flows, which include high temperatures and pressures, to replicate the thermal and aerodynamic loads on spacecraft materials and components. The gas used in the wind tunnel must faithfully replicate the atmospheric composition relevant to the mission. For re-entry to Earth, this usually means air (a mixture of nitrogen and oxygen). For other planetary missions, such as Mars, the gas composition may need to simulate atmospheres rich in CO_2. The Reynolds numbers achieved in arc-jet facilities are often much lower than those encountered during actual re-entry. These high Reynolds numbers are due to the plasma flow's low density and high viscosity [150–152].

High Reynolds water tunnels are used frequently in cavitation studies like those from the U.S. Navy Large Cavitation Channel (LCC) or the Garfield Thomas Water Tunnel. The LCC is the largest water tunnel in the world reaching a Reynolds number based on the momentum thickness θ ($Re_\theta = \mathcal{U}\theta/\nu$) of about 10^5 (corresponding to a Reynolds number based on the boundary layer thickness (δ) of $Re_\delta = \mathcal{U}\delta/\nu \sim 10^6$) for turbulent boundary layers. The withdrawal of LCC is its high operating costs. Less expensive to operate are water tunnels with lower maximum Reynolds numbers, like that of Oklahoma State University, with maximum Reynolds number of $Re_\theta \sim 10^4$ ($Re_\delta \sim 10^5$) [153,154].

In conclusion, while the Reynolds number remains a cornerstone of fluid dynamics and scaling analysis, its limitations necessitate the consideration of additional dimensionless parameters to fully capture the complexities of fluid behavior. The evolution of experimental facilities, from early wind tunnels to modern cryogenic and hypersonic wind tunnels, reflects the ongoing effort to address these challenges. These advancements have enabled more accurate modeling of high-speed, compressible, and extreme flow conditions, critical for aerospace engineering applications to cavitation studies. However, achieving perfect scaling remains an elusive goal due to the interplay of various forces and the practical constraints of experimental setups. Developing specialized facilities, such as high-pressure and cryogenic wind tunnels, has provided valuable tools for decoupling the effects of Reynolds number, Mach number, and aeroelasticity.

5. Discussion and Conclusions

The Reynolds number, introduced initially to describe the transition between laminar and turbulent flows and as a scaling parameter, has evolved into a fundamental dimensionless parameter applicable across various disciplines. This paper has demonstrated its universal relevance by highlighting its use in various physical systems, including classical fluid mechanics, biofluid dynamics, nano- and microfluidics, boundary layer theory, rotating flows, multiphase systems, porous media, and even relativistic fluid mechanics. The adaptability of the Reynolds number underscores its role as a unifying principle in the study of fluid motion and transport phenomena.

One key takeaway from this study is that while the traditional definition of the Reynolds number is based on the ratio of inertial to viscous forces, its interpretation varies significantly depending on the specific system under consideration. For example, in microfluidics, where inertial effects are negligible, the Reynolds number primarily confirms the dominance of viscous forces. In turbulence studies, it governs the energy cascade process from large to small eddies, while in biofluid mechanics, it dictates locomotion strategies at different scales, from bacteria to whales. In relativistic fluid dynamics, inverse

Reynolds numbers are used to quantify deviations from equilibrium, demonstrating that even in extreme conditions, the concept remains indispensable.

Furthermore, this study has shown that modifications and generalizations of the Reynolds number naturally arise when applied to complex systems. For non-Newtonian fluids, effective viscosity-dependent Reynolds numbers better describe flow behavior. In rotating flows, alternative formulations account for centrifugal effects, in magnetohydrodynamics, the magnetic Reynolds number governs the interaction between fluid motion and magnetic fields, and the kinetic Reynolds number appears as an important parameter in the initial stages of the Universe. These variations highlight the flexibility of the Reynolds number as an analytical tool capable of adapting to different governing forces and constraints.

A notable observation is that different scientific communities sometimes define Reynolds-like numbers differently based on their needs. While this flexibility is beneficial, it also underscores the importance of clearly specifying definitions when comparing results across fields. Standardizing certain modifications could improve interdisciplinary communication and facilitate a broader understanding of flow phenomena.

The Reynolds number remains fundamentally defined as the ratio of inertial to viscous forces, a concept without significant changes in its physical interpretation. Its core meaning—capturing the balance between these forces—continues to be universally applicable across fluid mechanics. Attempts to redefine or extend *Re* beyond this fundamental ratio often result in parameters that no longer represent the original concept. Thus, while the physical meaning of *Re* has remained consistent, its application has evolved to address increasingly complex systems.

By analogy, "Reynolds numbers" have been defined in different fields, corresponding to dimensionless parameters that compare the effect of inertia with some resistive force to motion. The analogous Reynolds number has emerged as a critical ability for describing the interaction between fluid motion and magnetic fields, highlighting the adaptability of *Re* to new governing forces. Similarly, inverse Reynolds numbers are used in relativistic fluid dynamics to quantify deviations from equilibrium in extreme conditions, such as those in astrophysical or high-energy systems. These developments demonstrate that while the concept of *Re* has been adapted to analogous situations, it has expanded into new and challenging domains, offering valuable insights into diverse physical phenomena.

In conclusion, the Reynolds number remains a cornerstone of fluid mechanics and beyond. Its widespread applicability across multiple disciplines demonstrates its fundamental importance in understanding and predicting flow behavior in diverse physical systems. Future research may continue to refine and expand its use, spanning from the smallest to the largest scales of the universe. The Reynolds number solidifies its status as an essential tool for scientific and engineering advancements by continually adapting to new challenges.

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Abbreviations

The following abbreviations are used in this manuscript:

LCC	U.S. Navy Large Cavitation Channel
MHD	Magnetohydrodynamics
NACA	U.S. National Advisory Committee for Aeronautics (1915–1958)
NASA	U.S. National Aeronautics and Space Administration (is the successor to NACA)

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