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Magnetohydrodynamic Turbulence in a Spherical Shell: Galerkin Models, Boundary Conditions, and the Dynamo Problem

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Abstract: The ‘dynamo problem’ requires that the origin of the primarily dipole geomagnetic field be found. The source of the geomagnetic field lies within the outer core of the Earth, which contains a turbulent magnetofluid whose motion is described by the equations of magnetohydrodynamics (MHD). A mathematical model can be based on the approximate but essential features of the problem, i.e., a rotating spherical shell containing an incompressible turbulent magnetofluid that is either ideal or real but maintained in an equilibrium state. Galerkin methods use orthogonal function expansions to represent dynamical fields, with each orthogonal function individually satisfying imposed boundary conditions. These Galerkin methods transform the problem from a few partial differential equations in physical space into a huge number of coupled, non-linear ordinary differential equations in the phase space of expansion coefficients, creating a dynamical system. In the ideal case, using Dirichlet boundary conditions, equilibrium statistical mechanics has provided a solution to the problem. As has been presented elsewhere, the solution also has relevance to the non-ideal case. Here, we examine and compare Galerkin methods imposing Neumann or mixed boundary conditions, in addition to Dirichlet conditions. Any of these Galerkin methods produce a dynamical system representing MHD turbulence and the application of equilibrium statistical mechanics in the ideal case gives solutions of the dynamo problem that differ only slightly in their individual sets of wavenumbers. One set of boundary conditions, Neumann on the outer and Dirichlet on the inner surface, might seem appropriate for modeling the outer core as it allows for a non-zero radial component of the internal, turbulent magnetic field to emerge and form the geomagnetic field. However, this does not provide the necessary transition of a turbulent MHD energy spectrum to match that of the surface geomagnetic field. Instead, we conclude that the model with Dirichlet conditions on both the outer and the inner surfaces is the most appropriate because it provides for a correct transition of the magnetic field, through an electrically conducting mantle, from the Earth’s outer core to its surface, solving the dynamo problem. In addition, we consider how a Galerkin model velocity field can satisfy no-slip conditions on solid boundaries and conclude that some slight, kinetically driven compressibility must exist, and we show how this can be accomplished.



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1. Introduction

The quest to understand the nature of the Earth’s mostly dipole magnetic field stretches far back in time and has a fascinating history [1]. Just over a hundred years ago, it was recognized that global geomagnetism was likely due to dynamo action caused by magnetofluid dynamics within the Earth’s liquid metal outer core [2]. The study of magnetofluid motion

was eventually termed ‘magnetohydrodynamics’ (MHD) [3], and a quest began to prove that MHD was relevant and could solve the ‘dynamo problem’ [4,5].

In the last century, there were many unsuccessful attempts to solve the dynamo problem, e.g., kinetic dynamo theories [6–15] (magnetic field is not dynamic) and mean-field electrodynamics [16–19] (essentially a circular argument). Although it was not even certain that MHD theory was pertinent, this was finally proven through numerical simulation when a computer model based on the MHD equations produced a magnetic field that mimicked the geomagnetic field, including a reversal of the dominant dipole component [20,21]. Thus, MHD was relevant, although a theoretical solution to the dynamo problem using the basic MHD equations was still lacking.

On a parallel track, research in homogeneous MHD turbulence was moving forward [22–25]. This research was based on Fourier expansions of the interacting velocity and magnetic fields that produce MHD turbulence in a periodic box, i.e., a 3-torus, which has no boundaries and thus no need for boundary conditions (b.c.s). Computationally, this approach showed that a dominant, largest-scale, quasi-steady magnetic mode always seemed to arise dynamically [26] due to ‘broken ergodicity’, defined [27] as occurring when, ‘In a system that is non-ergodic on physical timescales the phase point is effectively confined in one subregion or component of phase space’. Broken ergodicity is expressed as broken symmetry [28] in that the system’s largest-scale modes, which, in the long term, are predicted to have zero mean value, in fact develop large mean values (not all equal) with very small standard deviation rather than having a symmetric path winding around the origin of the phase space [29,30].

The cause for this broken ergodicity was found by applying the principles of equilibrium statistical mechanics [31–34] to the dynamical system of Fourier coefficients that represent ideal homogeneous MHD turbulence in a periodic box. Specifically, minimizing the entropy functional of the system produced the expectation value of the magnetic energy, which led to the three largest-scale modes dominating the energy spectrum, having expectation values that were $O(M)$ larger than other modes (where M is the total number of Fourier modes). The largest-scale modes (the ‘dipole’) were large enough to disassociate themselves from the dynamics of the turbulent system and become quasi-steady. This provided a theoretical explanation the broken ergodicity in the ideal MHD turbulence [29,35], a theory which is reviewed and extended in reference [30]. Furthermore, numerical simulations of real (by which we mean that the model system is forced and dissipative) homogeneous MHD turbulence in quasi-equilibrium revealed that the ideal statistical results applied to the largest scales of real MHD turbulence [36–38]. Statistical theory and numerical results for both the ideal and real cases are expanded on greatly in the references cited, and the reader is directed to these for further details.

The interior layers of the Earth are not yet well understood [39] and appear to be relatively complicated [40]. These layers have boundaries that are thought to be close to those of oblate spheroids, which, however, differ from spheres by only 1 part in 400 [41]. For mathematical and computational purposes, these details are typically avoided by modeling the Earth’s interior as having spherical boundaries with the mantle and outer core being approximated as spherical shells and the inner core as a sphere. The successful simulations of geomagnetic field reversal also had enough detail to allow for the mantle and the outer and inner cores to rotate with slightly different axes and rotation rates, but coupled dynamically through magnetic and viscous torque [20,21]. These simulations represented velocity, magnetic, and temperature variations using expansions in terms of Chebyshev functions in the radial and spherical harmonic functions of longitude and latitude, and they marched forward in time using spectral transform methods [42,43]. These expansions were well-suited to their task but were not Galerkin expansions.

Galerkin expansions are defined to be series of orthogonal functions, in which each individual term satisfies the imposed b.c.s [44]. In the case of homogeneous turbulence [45], Fourier series in a cartesian coordinate system are used since the expansion functions appropriate for a periodic box are sines and cosines, or equivalently, $\exp(i\mathbf{k} \cdot \mathbf{x})$, where $\mathbf{k}$ is a wavevector and $\mathbf{x}$ is a physical space position vector. For motion inside a sphere, using a spherical polar coordinate system, the Galerkin expansion functions are spherical Bessel functions in radial coupled with spherical harmonics [46,47], while for a spherical shell, a combination of spherical Bessel and Neumann functions in the radius are used [48]. These Galerkin expansions were based on earlier work on ‘force-free’ magnetic fields [49,50], and all required that the normal component of the magnetic field vanish at the boundaries. The statistical mechanics that led to a solution to the dynamo problem in the Fourier case [35] was later seen to apply to ideal MHD turbulence in a spherical shell with perfectly conducting boundaries [48]. Here, we will show that the statistical solution is applicable to a wide variety of b.c.s, including those that might appear to be, but in the end are not, more appropriate than the perfectly conducting boundaries assumed previously.

In regard to boundary conditions, at solid physical boundaries that are not moving with respect to a rotating coordinate system, in ideal flow, the normal component of velocity is zero, and in real flow, *all* components of fluid velocity must go to zero, while the *normal* component of the magnetic field must be continuous (and not necessarily zero). The tangential magnetic field will also be continuous at the boundaries between a magnetofluid and solid material if both have relative magnetic permeability of unity (as it is for the outer core, bounded by inner core and mantle), unless there is a surface electrical current on the boundary [51]. Since the nature of the outer core’s boundaries are not well known, the imposition of conditions is somewhat problematic. Thus, the precise b.c.s given above are often not completely enforced in theoretical or computational models, with the implied or explicit assumption that this has negligible effects on at least the qualitative results that these models produce. For example, in another successful simulation of geomagnetic activity, using finite difference methods in radius [52] rather than Chebyshev polynomials [20,21], the tangential components of velocity obeyed stress-free conditions at the boundaries, rather than going to zero. Here, we will show that a Galerkin expansion of the velocity field can satisfy no-flow and no-slip b.c.s if a slight amount of kinetically driven compressibility is introduced.

Specifying the value of a function on a boundary is called a Dirichlet b.c., while specifying the value of its normal derivative is called a Neumann b.c. [51]. Using Dirichlet b.c.s to study MHD activity within spherical boundaries is clearly appropriate for the velocity, and, again, has also been applied to the magnetic field by modeling the magnetofluid as enclosed by perfectly conducting material [46,47,53]. However, this concept seems to prevent the magnetic field from emerging to form the geomagnetic field because a perfect conductor develops a surface current that blocks the magnetic field from entering the surrounding media. If the surrounding material is not perfectly conducting, then the magnetic field may be expected to emerge, the more so as the electrical conductivity of the material decreases. The outer core is expected to have high electrical conductivity ($\approx 2 \times 10^6$ S), with 13–20% higher conductivity in the inner core [54,55]. While the electrical conductivity of the mantle near the core-mantle boundary (CMB) is not known, it is in all probability larger than the conductivity of ~ 4 S/m estimated for the mantle at about 1000 km above the CMB [56,57]. Supporting the existence of large electrical conductivity near the CMB, realistic dynamo simulations needed thin layers of high conductivity near outer core boundaries to ensure sufficient coupling between the mantle and the inner and outer cores in order to produce Earth-like magnetic dipole reversal and other observed geophysical phenomena [20,21].

On the other hand, if we accept that MHD turbulence relaxes to an equilibrium with mostly a tangential magnetic field at the boundaries due to entrainment by the velocity field in a highly conducting magnetofluid, as required by Alfvén’s theorem [58], then the concept of perfectly conducting b.c.s need not be evoked. In this case, Dirichlet b.c.s may be used and the purely tangential magnetic field at the CMB may be continued into the Earth’s electrically conducting mantle and thence into the Earth’s global magnetic field [59]. Alternatively, each term in a Galerkin expansion of the magnetic field can be required to satisfy Neumann b.c.s, or some form of mixed Dirichlet and Neumann b.c.s, as these also allow for the emergence of a non-zero normal component of the outer core magnetic field, connecting it to the geomagnetic field; however, these overlook the electrical conductivity of the mantle. These alternative approaches involving Neumann conditions do not appear to have been investigated, and that is what we present here: an examination of Galerkin methods using Neumann, as well as mixed Dirichlet-Neumann b.c.s, and a comparison of these with those using strictly Dirichlet b.c.s.

It will be seen that the principal effect of varying the b.c.s imposed on the outer core in Galerkin models of MHD turbulence is that the wavenumbers associated with Neumann or mixed b.c.s are slightly shifted from those arrived at with purely Dirichlet b.c.s. The result is that the statistical analysis leading to a solution to the dynamo problem is essentially unaffected for MHD turbulence in a state of equilibrium. The effect of imposing a Neumann b.c. on the outer boundary is, again, that it allows the normal (radial) component of the magnetic field interior to the spherical shell to emerge and be directly connected to the radial component to the exterior geomagnetic field. However, in imposing a Neumann b.c. on the CMB, one can ignore the effect of mantle electrical conductivity but, in doing so, one identifies a poor connection between the observed geomagnetic energy spectrum and that which is expected from a turbulent magnetofluid. On the other hand, including mantle electrical conductivity, a model using Dirichlet b.c.s, led to a greatly improved correlation of the geomagnetic energy spectrum with a magnetic energy spectrum expected to arise in MHD turbulence [59]. Thus, this earlier model appears optimal, and we will demonstrate this conclusion by examining models using alternative b.c.s.

2. Outline

As an outline of what appears next and where it leads, we list the following:

Section 3 *Mathematical Model*. The basic equations of incompressible MHD turbulence are presented, along with a brief description of the spherical shell containing the turbulent magnetofluid.

Section 4 *Boundary Conditions*. A definition of Dirichlet and Neumann b.c.s is given, along with an introduction to the expansion functions that will be used to represent the velocity and magnetic fields.

Section 5 *Galerkin Expansions for a Spherical Shell*. here, we give details about the orthogonal functions and the ‘poloidal–toroidal’ expansions that incorporate them to represent the velocity and magnetic fields.

Section 6 *Helical Representation and Statistical Theory*. Since magnetic helicity is essential to understanding MHD turbulence, particularly its statistical mechanics, we show how the basic Galerkin expansion functions can be transformed into those where each has a definite helicity, for both the velocity and magnetic fields; a brief introduction to the statistical theory of ideal MHD turbulence is also given.

Section 7 *Boundary Conditions and Orthogonality*. The exact form of the radial part of the expansion functions needed to satisfy Dirichlet or Neumann b.c.s, or some combination of these, is given, as well as the wavenumbers associated with each set of b.c.s.

Section 8 *The Velocity Field*. We show how the Galerkin expansion of the velocity field

can be extended to meet no-slip b.c.s, an extension that requires adding a compressible component to the velocity field.

Section 9 *The Magnetic Field*. The mathematical representation of the magnetic field inside and outside the spherical shell is described, an a distinction is made as to the efficacy of Dirichlet versus Neumann b.c.s with regard to mapping the geomagnetic field from the Earth's surface down to the CMB.

Section 10 *Discussion*. Here, we review the various topics covered in this paper and show how they connect together to essentially solve the dynamo problem (albeit in a simplified model system).

Section 11 *Conclusion*. We emphasize the importance of the results contained in this paper and the viability of the encompassing theory that, though approximate, leads to a solution to the dynamo problem.

3. Mathematical Model

The Earth is composed of four basic layers: crust, mantle, outer core, and inner core. The outer core contains liquid iron with some nickel and trace elements and is therefore an electrically conducting fluid, i.e., a *magnetofluid* in which magnetic fields arise and are maintained dynamically. The inner core consists of solid iron with some nickel but fewer trace elements and thereby has, again, an electrical conductivity 13–20% higher than the $\approx 2 \times 10^6$ S of the outer core [54,55]. For the purposes of analysis and computation, the outer core containing this magnetofluid is approximated most simply as an electrically conducting magnetofluid within a rotating spherical shell. Since the outer core has density variations of only $\pm 10\%$ of mean value [60], it can be treated as incompressible, as it often is in geodynamo simulations, which nevertheless produce Earth-like results [20,21,52].

Here, we also model the outer core as containing an incompressible magnetofluid. We use spherical polar coordinates r, θ, φ , where $r_1 \leq r \leq r_2$, and assume sharp boundaries at the bottom ($r = r_1$) and top ($r = r_2$) of the outer core. The physical values of these radii are reasonably well known [60] and we use an inner core boundary (ICB) radius of $R_{ICB} = 1220$ km as a characteristic length. Then, in non-dimensional units, $r_1 = 1$ and $r_2 = r_o = 2.85$, while the Earth's surface is at $r_e = 5.22$.

The motion of an incompressible, turbulent magnetofluid is governed by the non-relativistic MHD equations [61–64]. As appropriate for our purpose, the non-dimensional form of the MHD equations, in a rotating frame of reference with constant angular velocity Ω_o and no mean magnetic field, is

$$\frac{\partial \omega}{\partial t} = \nabla \times [\mathbf{u} \times (\omega + 2\Omega_o) + \mathbf{j} \times \mathbf{b}] + \nu \nabla^2 \omega, \quad (1)$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \eta \nabla^2 \mathbf{b}. \quad (2)$$

Here, the velocity $\mathbf{u}$ and magnetic field $\mathbf{b}$ satisfy $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0$, and the associated vorticity ω and electric current $\mathbf{j}$ are $\omega = \nabla \times \mathbf{u}$ and $\mathbf{j} = \nabla \times \mathbf{b}$. Mass density does not appear because it equals unity and because the flow is non-relativistic; charge separation effects are negligible, as is well known [65]. The kinematic viscosity ν and the magnetic diffusivity η generally vary with position but are usually assumed constant in studies of MHD turbulence, while in geodynamo simulations, 'hyperviscosity' and 'hyperdiffusivity' are employed, in which $\nu \nabla^2 \omega \rightarrow (-1)^{n-1} \nu_n \nabla^{2n} \omega$ in (1) and $\eta \nabla^2 \mathbf{b} \rightarrow (-1)^{n-1} \eta_n \nabla^{2n} \mathbf{b}$ in (2), $n = 1, 2, 3, \dots$. The reason for using hyperviscosity and hyperdiffusivity is to bring the dissipation length scales within the resolution of numerical simulations [20,21].

Here, we assume that ν and η are constants (possibly zero). Then in the non-dimensional Equations (1) and (2), $\nu^{-1} = \text{Re}$ and $\eta^{-1} = \text{Rm}$, where Re and Rm are the

kinetic and magnetic Reynolds numbers, respectively. In regard to the outer core, typical values for these are estimated ([60], Table 5) to be $Re \approx 10^6$ and $Rm \approx 10^3$. In order to apply equilibrium statistical mechanics, we assume that Re and Rm are large enough so that we can consider the ideal case, where $\nu = \eta = 0$. In this case, the energy E [32] and magnetic helicity H_M [66] are invariants for a turbulent magnetofluid in a state of equilibrium, i.e., when there is no net transfer of E or H_M across boundaries. Cross helicity would also be an invariant if Ω_o were negligible [67], but this is not generally true for planets or stars.

In a numerical simulation based on (1) and (2), the magnetofluid makes a transition from some prescribed initial conditions into an equilibrium state, in either the ideal (no viscosity or resistivity) or real (driven and dissipative) case. In a simple (but not too simple) model, complicating thermal and compositional effects can be excluded such that only the velocity and magnetic fields need be considered. As stated earlier, we can apply the principles of equilibrium statistical mechanics [31–34] to ideal, incompressible MHD turbulence. In the case where periodic or Dirichlet conditions are applied to boundaries, a solution to the ‘dynamo problem’ has ensued [35,48]. Here, we examine the effects that varying the b.c.s can have on the mathematical model and on our previous conclusions.

To reiterate what was stated in the Introduction, mathematical and computational models often use orthogonal function expansions to represent velocity $\mathbf{u}$ and magnetic field $\mathbf{b}$, as well as density and temperature if these are included, as they are in Boussinesq models [20,21,68]. Early examples are found in the use of Fourier expansions to study homogeneous MHD turbulence in a periodic box [32,69]. Orthogonal functions typically used for a spherical shell consist of spherical harmonics $Y_{lm}(\theta, \varphi)$ paired with functions of radius. These radial functions may be Chebyshev polynomials for efficient computation [20,21], or a combination of spherical Bessel and Neumann functions used in Galerkin method modeling and numerical simulation [46–48,53]. An important feature of Galerkin methods is that they transform (1) and (2) into a high-dimensional dynamical system, where the dynamical variables are the independent coefficients of the Galerkin expansions representing $\mathbf{u}$ and $\mathbf{b}$; this, in turn, allows for statistical analysis, as was initially applied to Fourier models of homogeneous, ideal fluid turbulence [70] and magnetofluid turbulence [22], and later significantly extended to explain the broken ergodicity seen in numerical simulations [25,26,29,35].

4. Boundary Conditions

In a spherical shell, a necessary b.c. for the velocity field inside a sphere surrounded by solid material is that its radial component is zero at a spherical boundary, i.e., a Dirichlet b.c. This is sufficient for ideal flow but for real flow may require an additional no-slip [20,21] Dirichlet or no-stress [52] Neumann tangential b.c. Here, as already mentioned, we will consider the effect of choosing Dirichlet or Neumann or mixed b.c.s on Galerkin models of turbulent magnetofluids in a spherical shell. These Galerkin expansions will be based on orthogonal functions related to so-called Chandrasekhar–Kendall (C-K) functions [50].

C-K functions [call them $\mathbf{H}_{lmn}(r, \theta, \varphi)$] are chosen so that each $\mathbf{H}_{lmn}$ satisfies $\nabla \times \mathbf{H}_{lmn} = \gamma_{lmn} \mathbf{H}_{lmn}$; i.e., each C-K function is an eigenfunction of the curl operator [71,72]. If $\gamma_{lmn} > 0$, the function $\mathbf{H}_{lmn}$ has positive helicity, while if $\gamma_{lmn} < 0$, it has negative helicity. Of course, the total velocity or magnetic field is a combination of positive and negative helicity terms and is not itself generally an eigenfunction of the curl. The spherical shell’s inner and outer boundaries are, again, at $r_1 = 1$ and $r_2 = r_o = 2.85$, respectively. If we require, as ref. [50] does, that $\hat{\mathbf{r}} \cdot \mathbf{H}_{lmn} \sim g_l(k_{ln} r_{1,2}) Y_{lm}(\theta, \varphi) = 0$, then $g_l(k_{ln} r_1) = g_l(k_{ln} r_2) = 0$ are Dirichlet b.c.s; this ensures that the normal component of velocity goes to zero on the boundaries. The $g_l(k_{ln} r)$ combine spherical Bessel and Neumann functions, the exact form depending on the b.c.s.

We could also require Neumann b.c.s, $dg_l(k_{ln}r)/dr = 0$, at the boundaries $r = 1$ and r_o , or Dirichlet b.c.s on one boundary and Neumann b.c.s on the other. Neumann b.c.s are not usually appropriate for the velocity (unless some compressibility is needed to enforce certain b.c.s, as will be seen) but can be applied to the magnetic field. In fact, it will be shown that there are infinitely many possibilities of mixed Dirichlet–Neumann b.c.s on the magnetic field. However, we will also see that choices are constrained by the assumed properties of the material outside and inside the spherical shell, and that some choices may be more viable than others.

C-K eigenfunctions with Dirichlet b.c.s have been used for analysis of MHD turbulence in a cylinder [53], as well as for low-resolution numerical simulations of MHD turbulence inside a sphere [46,47]. In these early papers, the radial components of both velocity and magnetic field at the boundaries were zero. The requirement that the normal component be zero on a boundary originated in paper on ‘force-free magnetic fields’ [50], where the magnetic field took the form of separate, non-intersecting force-free flux tubes, where the α in $\nabla \times \mathbf{H} = \alpha \mathbf{H}$ was constant for a given flux tube but changed discontinuously when going to another flux tube. Each flux tube completely contained its interior magnetic field and this required $\hat{\mathbf{n}} \cdot \mathbf{H} = 0$ on the flux tube boundary, where $\hat{\mathbf{n}}$ is a unit normal vector on the surface. Later, this requirement was kept because the interest was in studying fusion plasmas which are typically held inside a metal container [53]. When C-K functions were then applied to the computation of MHD turbulence inside a sphere, the requirement that the normal component was zero on a boundary was justified for the magnetic field by assuming that the bounding surfaces were perfectly conducting [46,47]. However, in the case of Earth’s outer core, even though the mantle has some conductivity, it cannot be assumed to be perfectly conducting and thereby to completely contain the interior magnetic field simply because the geomagnetic field emerges out of the CMB and is observable. At the inner core boundary (ICB), of course, perfect conductivity may be assumed in a model of ideal MHD turbulence as the inner core has high electrical conductivity.

In numerical simulations of real MHD using a Chebyshev–tau method [20,21], in fully-developed flow, the magnetic field emerges at the CMB (with a kink [20,21,73], implying the presence of surface electrical current on the CMB) to become the geomagnetic field, while at the ICB, it enters the electrically conducting inner core only in a shallow layer. In these simulations, the mantle was given finite electrical conductivity in a thin layer just above the CMB equal to that of the outer core. Having these layers of electrical current both above the CMB and below the ICB was found to be necessary for the simulation to produce realistic results, e.g., in the frequency and duration of geomagnetic dipole reversals. The *ex post facto* justification for assuming electrical conductivity in the mantle follows from observational data and subsequent analysis that indicates that the mantle is, indeed, electrically conducting at least down to 1000 km above the CMB [56,57]. Presumably, the lower mantle is also electrically conducting, though exactly how much is unknown, but probably less than in the inner and outer cores due to differences in composition [54,55].

Nevertheless, in ideal MHD (and approximately in high magnetic Reynolds number dissipative flow), according to Alfvén’s theorem [58], magnetic flux tubes are entrained with velocity and thus may be considered as having a negligible normal component near the boundaries, which emerges tangentially into an electrically conducting mantle. While mantle electrical conductivity has been estimated, as mentioned, down to about 1000 km above the CMB [56,57], below that level, it can be modeled as increasing exponentially to a finite value at the CMB [59,74]. This exponentially increasing mantle conductivity can then be integrated radially, creating a model where mantle conductivity is concentrated on a spherical surface interior to the mantle, with corresponding surface electrical current. Doing so alters the way the geomagnetic field at the Earth’s surface is mapped down to the

CMB, with the result that the predicted magnetic spectrum at the CMB is much closer to what a turbulent MHD magnetic spectrum would look like than not doing so. This altered mapping is considered explicitly in Section 7.1.

Using C-K functions with Dirichlet b.c.s also allows for a statistical mechanical analysis of the set of dynamic coefficients of the corresponding Galerkin expansions of the velocity and magnetic fields [48]. The statistical theory that results is essentially identical to that of a Fourier periodic-box model, especially in its prediction of a dominant dipole magnetic field; furthermore, the Fourier method (periodic box) results are validated through many numerical simulations [29,75,76]. In regard to numerical simulations using spherical Galerkin methods with Dirichlet b.c.s [46,47], these are still too low-resolution and will remain so until a fast Bessel transformation (FBT) analogous to the FFT can be found. In the meantime, higher-resolution Fourier method numerical simulations serve as a computational surrogate for the spherical shell case. It is this surrogacy that provided numerical insight and helped lead to a solution to the ‘dynamo problem’ [30].

This ideal result is important but it would seem worthwhile to investigate whether the standard C-K functions [50], which satisfy Dirichlet b.c.s, could be modified to satisfy Neumann (or mixed) b.c.s, for which the normal component of the magnetic field is not zero at a boundary but, instead, its radial derivative is zero. This would allow for the passage of the normal component of the internal magnetic field through the CMB to connect to the externally observed geomagnetic field, with or without having to go through an electrically conducting mantle. This seems important for theoretical analysis and particularly for the Galerkin-method numerical simulation of MHD processes during the transition to an equilibrium turbulent state (once higher-resolution numerical simulations using FBTs are available for the spherical case).

Our model assumptions are that, for a chosen set of b.c.s, equilibrium has been reached and has constant or quasi-steady values of energy and magnetic helicity in either ideal or forced-dissipative real MHD turbulence (‘forced’ means introducing energy into the magnetofluid at some intermediate length scale [36–38]). In equilibrium, the net transfer of magnetic energy and magnetic helicity across the boundaries becomes negligible by definition. This allows for a statistical mechanical treatment of ideal MHD turbulence, resulting in a theory that differs for each choice of b.c.s only in the sets of wave numbers involved, which is an important though minor detail. The essential structure and predictions of the statistical theory [30] are unchanged, and any combination of Dirichlet and Neumann b.c.s lead to a solution to the dynamo problem.

The essential realization here is that Galerkin mathematical models need not require the perfectly conducting boundaries associated with Dirichlet conditions but need only have equilibrium MHD turbulence within a spherical shell in either the ideal or real case (i.e., viscous and resistive but forced so as to achieve quasi-equilibrium). In these models, Neumann or mixed Dirichlet–Neumann conditions, rather than strictly Dirichlet conditions, can be applied to one or both boundaries. Net transfer of energy and magnetic helicity across these boundaries becomes essentially zero at equilibrium such that energy and magnetic helicity within spherical shell maintain relatively constant values. In the ideal case, there is no dissipation or need to input energy and magnetic helicity into the system. In the real case, if we require in our model system that energy and magnetic helicity are input at the inner core boundary at the same rate that they are dissipated in the outer core or transferred to the mantle, then MHD turbulence can be assumed to have entered a quasi-equilibrium state after a period of transition. In Fourier models, it has been found through numerical simulation that ideal statistical results are relevant for predicting large-scale behavior in real MHD turbulence in quasi-equilibrium [36–38]. The statistical theory based

on Fourier methods has, in turn, been shown to be essentially the same as that based on a spherical Galerkin method with Dirichlet b.c.s [48].

Here, we move beyond Dirichlet b.c.s to determine the effect that alternative b.c.s have on the statistical mechanics of ideal MHD turbulence. Elements of the discussion given above will become clearer as we delve into the mathematical structure of spherical shell Galerkin methods and b.c.s. First, we will look at how the orthogonal functions used by Galerkin methods are defined.

5. Galerkin Expansions for a Spherical Shell

The magnetic field $\mathbf{b}$ and velocity field $\mathbf{u}$, as well as other divergenceless vector fields in a spherical geometry, can be represented by a ‘poloidal–toroidal decomposition’ [77]:

$$\mathbf{b}(\mathbf{x}, t) = \nabla \times [\mathbf{r}B(r, \theta, \varphi, t)] + \nabla \times (\nabla \times [\mathbf{r}A(r, \theta, \varphi, t)]) \quad (3)$$

$$\mathbf{u}(\mathbf{x}, t) = \nabla \times [\mathbf{r}U(r, \theta, \varphi, t)] + \nabla \times (\nabla \times [\mathbf{r}W(r, \theta, \varphi, t)]) \quad (4)$$

We use non-dimensional coordinates r , θ , and φ , where $1 \leq r \leq r_o$, $0 \leq \theta \leq \pi$, and $0 \leq \varphi \leq 2\pi$; again, for the Earth, $r_o = 2.85$.

We wish to expand B and A in (3), as well as U and W in (4), in terms of a complete set of orthogonal functions, so we begin with

$$B(r, \theta, \varphi, t) = \sum_{l,m,n} b_{lmn}(t) g_l(k_{ln}r) Y_{lm}(\theta, \varphi), \quad A(r, \theta, \varphi, t) = \sum_{l,m,n} a_{lmn}(t) g_l(k_{ln}r) Y_{lm}(\theta, \varphi) \quad (5)$$

$$U(r, \theta, \varphi, t) = \sum_{l,m,n} u_{lmn}(t) f_l(q_{ln}r) Y_{lm}(\theta, \varphi), \quad W(r, \theta, \varphi, t) = \sum_{l,m,n} w_{lmn}(t) f_l(q_{ln}r) Y_{lm}(\theta, \varphi). \quad (6)$$

Above, the $Y_{lm}(\theta, \varphi)$ are spherical harmonics and the $g_l(k_{ln}r)$ and $f_l(q_{ln}r)$ are linear combinations of the spherical Bessel functions of the first kind, $j_l(z)$, and of the second kind, $n_l(z)$ (called spherical Neumann functions), where $z = k_{ln}r$ or $q_{ln}r$. The exact form of the $g_l(k_{ln}r)$ and $f_l(q_{ln}r)$ will be given below, while the wavenumbers k_{ln} and q_{ln} will be determined by the b.c.s, which can be different for $\mathbf{b}$ and $\mathbf{u}$. An important property of the functions in (5) and (6) is that they satisfy Helmholtz’s equation:

$$\nabla^2 g_l(k_{ln}r) Y_{lm}(\theta, \varphi) = -k_{ln}^2 g_l(k_{ln}r) Y_{lm}(\theta, \varphi), \quad \nabla^2 f_l(q_{ln}r) Y_{lm}(\theta, \varphi) = -q_{ln}^2 f_l(q_{ln}r) Y_{lm}(\theta, \varphi). \quad (7)$$

The Galerkin method requires that when (5) is put into (3) and (6) is put into (4), the individual terms in the resulting expansions each satisfy the b.c.s. These vector expansions can be transformed into a helical representation, as will be undertaken below, where each term must be a solution to the Helmholtz equation, that is, an eigenfunction of the Laplacian operator in spherical coordinates [78]; this requires that the individual terms in (5) and (6) also be eigenfunctions of the Helmholtz equation; i.e., they must satisfy (7).

If we place (5) into (3) and (6) into (4), we obtain the Galerkin poloidal–toroidal expansions:

$$\mathbf{b}(\mathbf{x}, t) = \sum_{l,m,n} [b_{lmn}(t) \mathbf{T}_{lmn}(\mathbf{x}) + a_{lmn}(t) \mathbf{P}_{lmn}(\mathbf{x})], \quad (8)$$

$$\mathbf{u}(\mathbf{x}, t) = \sum_{l,m,n} [u_{lmn}(t) \mathbf{T}_{lmn}(\mathbf{x}) + w_{lmn}(t) \mathbf{P}_{lmn}(\mathbf{x})], \quad (9)$$

$$\mathbf{T}_{lmn}(\mathbf{x}) = c_{lmn} g_l(k_{ln}r) \mathbf{r} \times \nabla Y_{lm}(\theta, \varphi), \quad (10)$$

$$\nabla \times \mathbf{T}_{lmn}(\mathbf{x}) = \mathbf{P}_{lmn}(\mathbf{x}), \quad \nabla \times \mathbf{P}_{lmn}(\mathbf{x}) = k_{ln}^2 \mathbf{T}_{lmn}(\mathbf{x}), \quad (11)$$

$$b_{l,-m,n}(t) = (-1)^m b_{lmn}^*(t), \quad a_{l,-m,n}(t) = (-1)^m a_{lmn}^*(t). \quad (12)$$

In (10), c_{lmn} is a normalizing constant, to be defined presently, and the overall minus sign, which occurred because $\nabla \times \mathbf{r}Y_{lm} = -\mathbf{r} \times \nabla Y_{lm}$, has been dropped; also, $*$ denotes complex conjugation. Also, for (9), we replace $g_l(k_{ln}r)$ by $f_l(q_{ln}r)$ in (10) and the wavenumbers k_{ln} with q_{ln} in (11) since the radial functions and wavenumbers depend on b.c.s and will generally be different for the $\mathbf{b}$ and $\mathbf{u}$ expansion (8) and (9).

Both (8) and (9) have zero divergence since $\nabla \cdot \mathbf{T}_{lmn}(\mathbf{x}) = \nabla \cdot \mathbf{P}_{lmn}(\mathbf{x}) = 0$, while (12) is required by the realness of $\mathbf{b}$ [and similarly for u_{lmn} and w_{lmn} in (9)]. The definition of $\mathbf{T}_{lmn}(\mathbf{x})$ and $\mathbf{P}_{lmn}(\mathbf{x})$ in terms of vector spherical harmonics will be given below, where it will be seen that the $\mathbf{T}_{lmn}(\mathbf{x})$ and $\mathbf{P}_{lmn}(\mathbf{x})$ form an orthogonal set for $1 \leq r \leq r_o$, each member of which will be seen to satisfy the b.c.s (so that we have a Galerkin expansion), while the wavenumbers k_{ln} and q_{ln} are numerical constants with summation indices l, m, n ranging over prescribed, finite sets of integer values. Thus, we have two sets of complex coefficients; b_{lmn} and a_{lmn} to represent the magnetic field, and u_{lmn} and w_{lmn} to represent the velocity field.

'Uncurling' $\mathbf{b} \equiv \nabla \times \mathbf{a}$ produces the associated vector potential $\mathbf{a}$; using the relations given above:

$$\mathbf{a}(\mathbf{x}, t) = \sum_{l,m,n} \left[a_{lmn}(t) \mathbf{T}_{lmn}(\mathbf{x}) + k_{ln}^{-2} b_{lmn}(t) \mathbf{P}_{lmn}(\mathbf{x}) \right]. \quad (13)$$

The vector potential is important because it is used to define the magnetic helicity which, along with energy, is an ideal invariant for rotating MHD turbulence [66,67].

Using k_{ln} as a generic wavenumber, the vector functions $\mathbf{T}_{lmn}(\mathbf{x})$ and $\mathbf{P}_{lmn}(\mathbf{x})$ are defined as follows:

$$\mathbf{T}_{lmn}(\mathbf{x}) = \hat{g}_l(k_{ln}r) \hat{\Phi}_{lm}(\theta, \varphi), \quad 1 \leq l \leq L, \quad -l \leq m \leq l, \quad 1 \leq n \leq N, \quad (14)$$

$$\mathbf{P}_{lmn}(\mathbf{x}) = -\frac{1}{r} \left[\sqrt{l(l+1)} \hat{g}_l(k_{ln}r) \hat{\mathbf{Y}}_{lm}(\theta, \varphi) + \frac{d}{dr} [r \hat{g}_l(k_{ln}r)] \hat{\Psi}_{lm}(\theta, \varphi) \right]. \quad (15)$$

The orthonormal vector functions $\hat{\mathbf{Y}}_{lm}(\theta, \varphi)$, $\hat{\Psi}_{lm}(\theta, \varphi)$, and $\hat{\Phi}_{lm}(\theta, \varphi)$ are the (normalized) vector spherical harmonics of [79] and are based on the spherical harmonics $Y_{l,m}(\theta, \varphi)$ described in [51]:

$$Y_{l,-m}(\theta, \varphi) = (-1)^m Y_{lm}^*(\theta, \varphi), \quad (16)$$

$$\hat{\mathbf{Y}}_{lm}(\theta, \varphi) = \hat{\mathbf{r}} Y_{lm}(\theta, \varphi), \quad (17)$$

$$\hat{\Psi}_{lm}(\theta, \varphi) = \frac{r}{\sqrt{l(l+1)}} \nabla Y_{lm}(\theta, \varphi), \quad (18)$$

$$\hat{\Phi}_{lm}(\theta, \varphi) = \hat{\mathbf{r}} \times \hat{\Psi}_{lm}(\theta, \varphi). \quad (19)$$

The radial functions $\hat{g}_l(k_{ln}r)$ in (14) and (15) form an orthonormal set for $n = 1, 2, \dots, N$ at each value of l over the range $1 \leq r \leq r_o$, and are linear combinations of the spherical Bessel and Neumann functions, as will be detailed in the next section.

Here, let us define integrals over radius r as $\langle F \rangle_r$ and over solid angle σ as $\langle G \rangle_\sigma$, where

$$\langle F \rangle_r = \int_1^{r_o} F(r) r^2 dr, \quad \langle G \rangle_\sigma = \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} G(\theta, \varphi) \sin \theta d\theta d\varphi. \quad (20)$$

Thus, the integral over the volume of the spherical shell interior is

$$\langle F(r)G(\theta, \varphi) \rangle = \int_1^{r_o} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} F(r)G(\theta, \varphi) r^2 \sin \theta dr d\theta d\varphi = \langle F \rangle_r \langle G \rangle_\sigma. \quad (21)$$

The orthonormality properties required of the Galerkin basis functions are

$$\langle \hat{g}_l(k_{ln}r) \hat{g}_l(k_{lk}r) \rangle_r = \delta_{nk}, \quad \langle \hat{\mathbf{V}}_{lm}^{(i)} \cdot \hat{\mathbf{V}}_{pq}^{(j)*} \rangle_\sigma = \delta_{ij} \delta_{lp} \delta_{mq}, \quad (22)$$

$$\hat{\mathbf{V}}_{lm}^{(1)} = \hat{\mathbf{Y}}_{lm}(\theta, \varphi), \quad \hat{\mathbf{V}}_{lm}^{(2)} = \hat{\mathbf{\Psi}}_{lm}(\theta, \varphi), \quad \hat{\mathbf{V}}_{lm}^{(3)} = \hat{\mathbf{\Phi}}_{lm}(\theta, \varphi). \quad (23)$$

The radial functions $\hat{g}_l(k_{ln}r)$ will be discussed in Section 7.

Using these and the properties given in (14) and (15), we see that the integrals of the inner products of $\mathbf{T}_{lmn}(\mathbf{x})$ and $\mathbf{P}_{lmn}(\mathbf{x})$ over the volume enclosed by the spherical boundaries are

$$\langle \hat{\mathbf{T}}_{pqs} \cdot \hat{\mathbf{T}}_{lmn}^* \rangle = \langle \hat{g}_p(k_{ps}r) \hat{g}_l(k_{ln}r) \rangle_r \langle \hat{\mathbf{\Phi}}_{pqs} \cdot \hat{\mathbf{\Phi}}_{lmn}^* \rangle_\sigma = \delta_{pl} \delta_{qm} \delta_{sn}, \quad (24)$$

$$\langle \hat{\mathbf{P}}_{pqs} \cdot \hat{\mathbf{P}}_{lmn}^* \rangle = k_{ln}^2 \langle \hat{\mathbf{T}}_{pqs} \cdot \hat{\mathbf{T}}_{lmn}^* \rangle, \quad (25)$$

$$\langle \hat{\mathbf{T}}_{pqs} \cdot \hat{\mathbf{P}}_{lmn}^* \rangle = 0. \quad (26)$$

While we use a model with an inner core, others consider a model with no inner core [46,47]. The results we present do not depend on whether there is an inner core or not. The only adjustment necessary is that the radial expansion functions (39) become $g_l(k_{ln}r) = j_l(k_{ln}r)$ when there is no inner core. In this case, the numerical values of the k_{ln} in Tables 1–4 (to be discussed in the next section) change slightly, but this does not affect our analysis, which is independent of the exact values of the k_{ln} , as these are always distinct and well-ordered solutions of $g_l(k_{ln}) = 0$, as discussed by [80].

Table 1. Case DD: values for k_{ln} , $g_l(k_{lnr_o}) = g_l(k_{ln}) = 0$, for $l, n = 1, \dots, 10$.

$n =$	1	2	3	4	5	6	7	8	9	10
$l = 1$	1.8638	3.4929	5.1612	6.8434	8.5316	10.2231	11.9165	13.6110	15.3063	17.0022
2	2.1497	3.6788	5.2927	6.9440	8.6129	10.2912	11.9750	13.6623	15.3520	17.0433
3	2.5042	3.9411	5.4851	7.0931	8.7339	10.3928	12.0625	13.7391	15.4204	17.1050
4	2.8910	4.2636	5.7330	7.2884	8.8935	10.5273	12.1785	13.8410	15.5112	17.1869
5	3.2898	4.6294	6.0300	7.5271	9.0903	10.6938	12.3225	13.9678	15.6244	17.2890
6	3.6911	5.0227	6.3684	7.8062	9.3227	10.8915	12.4940	14.1190	15.7595	17.4111
7	4.0909	5.4310	6.7396	8.1218	9.5890	11.1194	12.6923	14.2942	15.9162	17.5528
8	4.4882	5.8457	7.1347	8.4695	9.8870	11.3762	12.9167	14.4929	16.0943	17.7140
9	4.8828	6.2618	7.5452	8.8441	10.2145	11.6608	13.1664	14.7145	16.2932	17.8943
10	5.2749	6.6767	7.9642	9.2399	10.5685	11.9718	13.4405	14.9586	16.5127	18.0935

Table 2. Case DN: values for $k_{ln}, g_l(k_{lnr_o}) = g'_l(k_{ln}) = 0$, for $l, n = 1, \dots, 10$.

$n =$	1	2	3	4	5	6	7	8	9	10
$l = 1$	1.5204	2.8820	4.4533	6.0930	7.7581	9.4352	11.1187	12.8061	14.4961	16.1877
2	1.9558	3.1616	4.6242	6.2136	7.8512	9.5110	11.1827	12.8615	14.5449	16.2314
3	2.4134	3.5504	4.8776	6.3932	7.9901	9.6243	11.2784	12.9444	14.6179	16.2967
4	2.8542	4.0064	5.2102	6.6309	8.1740	9.7745	11.4054	13.0544	14.7151	16.3836
5	3.2765	4.4829	5.6144	6.9265	8.4024	9.9610	11.5633	13.1914	14.8360	16.4919
6	3.6866	4.9498	6.0711	7.2799	8.6753	10.1835	11.7516	13.3549	14.9805	16.6214
7	4.0895	5.3984	6.5513	7.6888	8.9933	10.4418	11.9701	13.5446	15.1483	16.7718
8	4.4878	5.8323	7.0289	8.1427	9.3577	10.7365	12.2186	13.7602	15.3390	16.9429
9	4.8827	6.2566	7.4916	8.6220	9.7680	11.0685	12.4973	14.0016	15.5526	17.1346
10	5.2749	6.6748	7.9391	9.1043	10.2184	11.4394	12.8067	14.2689	15.7887	17.3465

Table 3. Case ND: values for $k_{ln}, g'_l(k_{lnr_o}) = g_l(k_{ln}) = 0$, for $l, n = 1, \dots, 10$.

$n =$	1	2	3	4	5	6	7	8	9	10
$l = 1$	0.8237	2.5981	4.2804	5.9696	7.6623	9.3568	11.0525	12.7487	14.4454	16.1424
2	1.1993	2.8337	4.4368	6.0845	7.7526	9.4312	11.1156	12.8035	14.4938	16.1858
3	1.5916	3.1494	4.6626	6.2537	7.8868	9.5419	11.2098	12.8854	14.5663	16.2507
4	1.9834	3.5162	4.9482	6.4741	8.0632	9.6883	11.3346	12.9941	14.6625	16.3369
5	2.3713	3.9114	5.2826	6.7412	8.2801	9.8693	11.4894	13.1292	14.7822	16.4444
6	2.7549	4.3193	5.6539	7.0503	8.5350	10.0837	11.6735	13.2902	14.9250	16.5727
7	3.1351	4.7311	6.0502	7.3954	8.8256	10.3300	11.8860	13.4765	15.0907	16.7218
8	3.5124	5.1422	6.4615	7.7699	9.1487	10.6070	12.1260	13.6876	15.2787	16.8911
9	3.8875	5.5511	6.8799	8.1665	9.5008	10.9127	12.3927	13.9228	15.4887	17.0805
10	4.2608	5.9572	7.3006	8.5782	9.8777	11.2454	12.6849	14.1815	15.7201	17.2895

Table 4. Case NN: values for $k_{ln}, g'_l(k_{lnr_o}) = g'_l(k_{ln}) = 0$, for $l, n = 1, \dots, 10$.

$n =$	1	2	3	4	5	6	7	8	9	10
$l = 1$	0.6913	2.0849	3.6042	5.2332	6.8964	8.5737	10.2580	11.9463	13.6370	15.3294
2	1.1554	2.4672	3.8232	5.3761	7.0021	8.6576	10.3276	12.0057	13.6890	15.3755
3	1.5780	2.9277	4.1425	5.5886	7.1595	8.7828	10.4316	12.0947	13.7667	15.4446
4	1.9795	3.4019	4.5473	5.8695	7.3680	8.9486	10.5695	12.2128	13.8700	15.5364
5	2.3702	3.8599	5.0085	6.2176	7.6271	9.1546	10.7409	12.3597	13.9986	15.6508
6	2.7546	4.2985	5.4897	6.6278	7.9371	9.4004	10.9453	12.5351	14.1522	15.7875
7	3.1350	4.7232	5.9634	7.0857	8.2986	9.6864	11.1825	12.7385	14.3305	15.9463
8	3.5124	5.1394	6.4201	7.5671	8.7103	10.0133	11.4526	12.9697	14.5332	16.1269
9	3.8875	5.5501	6.8617	8.0479	9.1642	10.3826	11.7560	13.2287	14.7601	16.3291
10	4.2608	5.9569	7.2931	8.5155	9.6432	10.7944	12.0940	13.5157	15.0110	16.5526

6. Helical Representation and Statistical Theory

Since the statistical theory of ideal MHD turbulence is referred to often in this paper, here, we make a brief digression from our discussion about Galerkin expansions.

We can write (8) and (13) in terms of an explicitly helical set of vector basis functions:

$$\mathbf{H}_{lmn}^{\pm}(\mathbf{x}) = \frac{1}{\sqrt{2}} [\pm k_{ln} \mathbf{T}_{lmn}(\mathbf{x}) + \mathbf{P}_{lmn}(\mathbf{x})], \quad \nabla \times \mathbf{H}_{lmn}^{\pm}(\mathbf{x}) = \pm k_{ln} \mathbf{H}_{lmn}^{\pm}(\mathbf{x}). \quad (27)$$

The $\mathbf{H}_{lmn}^{\pm}(\mathbf{x})$ are Chandrasekhar–Kendall functions [50] and are orthonormal eigenfunctions of the curl operator [46,47]. In terms of (27), the magnetic field (8), vector potential (13), and current $\mathbf{j} = \nabla \times \mathbf{b}$ become

$$\mathbf{b}(\mathbf{x}, t) = \sum_{l,m,n} [h_{lmn}^+(t) \mathbf{H}_{lmn}^+(\mathbf{x}) + h_{lmn}^-(t) \mathbf{H}_{lmn}^-(\mathbf{x})], \quad (28)$$

$$\mathbf{a}(\mathbf{x}, t) = \sum_{l,m,n} k_{ln}^{-1} [h_{lmn}^+(t) \mathbf{H}_{lmn}^+(\mathbf{x}) - h_{lmn}^-(t) \mathbf{H}_{lmn}^-(\mathbf{x})], \quad (29)$$

$$\mathbf{j}(\mathbf{x}, t) = \sum_{l,m,n} k_{ln} [h_{lmn}^+(t) \mathbf{H}_{lmn}^+(\mathbf{x}) - h_{lmn}^-(t) \mathbf{H}_{lmn}^-(\mathbf{x})]. \quad (30)$$

The helical coefficients h_{lmn}^+ and h_{lmn}^- are related to a_{lmn} and b_{lmn} by the transformations

$$h_{lmn}^+ = \frac{1}{\sqrt{2}} (a_{lmn} + k_{ln}^{-1} b_{lmn}), \quad h_{lmn}^- = \frac{1}{\sqrt{2}} (a_{lmn} - k_{ln}^{-1} b_{lmn}). \quad (31)$$

The velocity (4) can also be expressed in a helical representation, using

$$v_{lmn}^+ = \frac{1}{\sqrt{2}} (w_{lmn} + k_{ln}^{-1} u_{lmn}), \quad v_{lmn}^- = \frac{1}{\sqrt{2}} (w_{lmn} - k_{ln}^{-1} u_{lmn}) \quad (32)$$

so that, for the velocity $\mathbf{u}$ and vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, we have

$$\mathbf{u}(\mathbf{x}, t) = \sum_{l,m,n} [v_{lmn}^+(t) \mathbf{H}_{lmn}^+(\mathbf{x}) + v_{lmn}^-(t) \mathbf{H}_{lmn}^-(\mathbf{x})], \quad (33)$$

$$\boldsymbol{\omega}(\mathbf{x}, t) = \sum_{l,m,n} k_{ln} [v_{lmn}^+(t) \mathbf{H}_{lmn}^+(\mathbf{x}) - v_{lmn}^-(t) \mathbf{H}_{lmn}^-(\mathbf{x})]. \quad (34)$$

If (28)–(34) are put into (1) and (2), then these few partial differential equations in x -space become many ordinary differentials in phase space:

$$\frac{dv_{lmn}^\pm}{dt} = K_{lmn}^\pm - \nu k_{ln}^2 v_{lmn}^\pm, \quad \frac{dh_{lmn}^\pm}{dt} = J_{lmn}^\pm - \eta k_{ln}^2 h_{lmn}^\pm. \quad (35)$$

Here, $K_{lmn}^\pm$ and $J_{lmn}^\pm$ are transformations of the non-linear terms in (1) and (2) into phase space achieved using the orthonormality properties of the helical set of vector basis functions $\mathbf{H}_{lmn}^\pm(\mathbf{x})$ defined in (27), where the orthonormality properties follow from those of the $\mathbf{T}_{lmn}(\mathbf{x})$ and $\mathbf{P}_{lmn}(\mathbf{x})$ given in Section 5. The Equation (35) represent a dynamical system and an ideal one if $\nu = \eta = 0$. Numerically, the Equation (35) have been solved by [46,47] for the spherical rather than the spherical shell case, using a spectral method but not a spectral transform method, which greatly limited grid size and dynamical resolution. Also, helical representations of Fourier expansions using $\exp(i\mathbf{k} \cdot \mathbf{x})$, along with the statistical mechanics of ideal MHD turbulence, are discussed in depth elsewhere (for example, in [30]).

This helical representation facilitates the development of the statistical mechanics of MHD turbulence, as the probability density function for rotating, ideal MHD turbulence is $D = Z^{-1} \exp(-\alpha E - \gamma H_M)$, where Z is the partition function and the ‘inverse temperatures’, α and β , can be expressed as functions of the initially unknown expectation value of the magnetic energy. In a helical representation with suitable b.c.s, energy E and magnetic helicity H_M take the following form:

$$E = \frac{1}{2} \sum_{lmn} (|v_{lmn}^+|^2 + |v_{lmn}^-|^2 + |h_{lmn}^+|^2 + |h_{lmn}^-|^2), \quad (36)$$

$$H_M = \frac{1}{2} \sum_{lmn} k_{ln}^{-1} (|h_{lmn}^+|^2 - |h_{lmn}^-|^2). \quad (37)$$

Using these results, we see that the probability density function for rotating, ideal MHD turbulence takes the following form:

$$D = \prod_{lmn}^{m \geq 0} Z_{lmn}^{-1} \exp c_m \left[\alpha \left(|v_{lmn}^+|^2 + |v_{lmn}^-|^2 \right) + \left(\alpha + \frac{\gamma}{k_{ln}} \right) |h_{lmn}^+|^2 + \left(\alpha - \frac{\gamma}{k_{ln}} \right) |h_{lmn}^-|^2 \right]. \quad (38)$$

Here, $c_0 = \frac{1}{2}$ and $c_m = 1$ for $m \neq 0$, while the Z_{lmn} are modal partition functions.

Minimizing the entropy functional leads to $\alpha \approx |\gamma|/k_{11}$ with $\alpha - |\gamma|/k_{11} > 0$, $m = 0, \pm 1$. Thus, either the h_{1m1}^+ or the h_{1m1}^- have a great deal more energy than any other coefficients, and this is either the positive or the negative magnetic helicity dipole mode with the smallest wave number $k_{11} < k_{ln}$, $l > 1$, i.e., the largest associated length-scale. Extensive detail is given elsewhere [30].

One important point to note is that this theory applies to a turbulent magnetofluid contained within a simple geometry, such as periodic box, sphere, spherical shell, or cylinder (ρ, ϕ, z) with periodic b.c.s in the z -direction, etc., where the velocity and magnetic fields can be expressed as Galerkin expansions. What we discuss here, focusing on the spherical shell, is that its radial b.c.s can be some combination of Dirichlet and Neumann conditions. What we wish to determine is which combination is the best one.

7. Boundary Conditions and Orthogonality

Let us now consider the radial integral in (20) and the radial orthogonality requirement in (22). The properties of spherical Bessel and spherical Neumann functions, $j_l(z)$ and $n_l(z) = (-1)^{l+1} j_{-l-1}(z)$, respectively, which are solutions to the radial differential equation that emerges from the Helmholtz equation when written in spherical polar coordinates, are well known (e.g., see [78]). In a spherical shell, the radial function $g_l(k_{ln})$ is a combination of these spherical Bessel and Neumann functions:

$$g_l(k_{ln}r) = \alpha_{ln} j_l(k_{ln}r) + \beta_{ln} n_l(k_{ln}r), \quad 1 \leq l \leq L, \quad 1 \leq n \leq N. \quad (39)$$

The coefficients α_{ln} , β_{ln} and the wavenumbers k_{ln} in (39) are determined by the choice of b.c.s, as will be discussed shortly. The $g_l(k_{ln}r)$ are normalized to become $\hat{g}_l(k_{ln}r)$ appearing in (14), where

$$\hat{g}_l(k_{ln}r) = N_{ln}^{-1} g_l(k_{ln}r). \quad (40)$$

The normalization constant N_{ln} depends on the b.c.s chosen, and its various forms will also be discussed presently. Orthogonality of the $g_l(k_{ln}r)$ will be achieved by choosing α_{ln} and β_{ln} in (39) so that the b.c. at $r = r_0$ is satisfied automatically, while the b.c. at $r = 1$ will be used to determine the wavenumbers k_{ln} .

Applying the orthogonality integral (22) to the set of functions (39) yields, using known results [78],

$$I_{np}^l = \left\langle g_l(k_{ln}r) g_l(k_{lp}r) \right\rangle_r = K_{np}^l(r_0) - K_{np}^l(1) = N_{ln}^2 \delta_{np}, \quad (41)$$

$$K_{np}^l(r) = r^2 \frac{k_{lp} g_l(k_{ln}r) g_l'(k_{lp}r) - k_{ln} g_l(k_{lp}r) g_l'(k_{ln}r)}{k_{ln}^2 - k_{lp}^2}. \quad (42)$$

The normalization constant N_{ln} that appears in (40) can be found by taking the limit $\lim_{n \rightarrow p} I_{np}^l = N_{ln}^2$, which requires applying L'Hôpital's rule to (42).

Since $I_{np}^l = I_{pn}^l$, for each l , there are $\frac{1}{2}N(N+1)$ Equation (41) for N variables k_{ln} , which is an over-determined system for $N > 1$, and has no general solution. However,

a special solution can be found by observing [81], as we have, where C_o^l and D_o^l are arbitrary constants:

$$(a) K_{np}^l(r_o) = 0 \quad \text{if} \quad (b) C_o^l g_l(k_{ln} r_o) + D_o^l k_{ln} g_l'(k_{ln} r_o) = 0. \quad (43)$$

Using the b.c. (43b), the coefficients α_{ln} , β_{ln} in (39) become

$$\alpha_{ln} = C_o^l n_l(k_{ln} r_o) + D_o^l k_{ln} n_l'(k_{ln} r_o), \quad (44)$$

$$\beta_{ln} = -[C_o^l j_l(k_{ln} r_o) + D_o^l k_{ln} j_l'(k_{ln} r_o)]. \quad (45)$$

To completely define the functions $g_l(k_{ln} r)$ that satisfy the orthogonality condition (41), we first choose values for C_o^l and D_o^l and then pick a completely independent b.c. at $r = 1$:

$$K_{np}^l(1) = 0 \rightarrow C_1^l g_l(k_{ln}) + D_1^l k_{ln} g_l'(k_{ln}) = 0 \rightarrow \{k_{ln} | 1 \leq l \leq L, 1 \leq l \leq N\}. \quad (46)$$

Here, C_1^l and D_1^l are again arbitrary constants, possibly different for each value of l . The set of wavenumbers k_{ln} , associated with the parameters C_o^l , D_o^l , C_1^l , and D_1^l , can then be found numerically, for example, by using a bisection method [82]. Thus, we have an infinite number of mixed Dirichlet and Neumann b.c.s at our disposal.

Although we have an infinite number of choices, in this paper, we will only consider four cases:

$$\text{Case DD: } C_o^l = 1, D_o^l = 0, C_1^l = 1, D_1^l = 0; \quad g_l(k_{ln} r_o) = g_l(k_{ln}) = 0; \quad (47)$$

$$\text{Case DN: } C_o^l = 1, D_o^l = 0, C_1^l = 0, D_1^l = 1; \quad g_l(k_{ln} r_o) = g_l'(k_{ln}) = 0; \quad (48)$$

$$\text{Case ND: } C_o^l = 0, D_o^l = 1, C_1^l = 1, D_1^l = 0; \quad g_l'(k_{ln} r_o) = g_l(k_{ln}) = 0; \quad (49)$$

$$\text{Case NN: } C_o^l = 0, D_o^l = 1, C_1^l = 0, D_1^l = 1; \quad g_l'(k_{ln} r_o) = g_l'(k_{ln}) = 0. \quad (50)$$

On the spherical shell, Case DD has Dirichlet conditions on both boundaries; while Case NN has Neumann conditions on both boundaries; Case DN imposes Dirichlet b.c.s at $r = r_o$ with Neumann b.c.s at $r = 1$; and Case ND imposes Neumann b.c.s at $r = r_o$ with Dirichlet b.c.s at $r = 1$. We could build mixed b.c.s by choosing different Cases listed above to apply to different ranges of the index l ; however, this treats the multipoles (identified by the value of l) differently at the boundaries and there seems no compelling reason to do this.

We now examine these cases in more detail, and we discuss how the magnetic field internal to the spherical shell connects to those external to the shell.

7.1. Case DD: Dirichlet Conditions on Both Boundaries

The parameters (47) lead to the Dirichlet conditions $\widehat{g}_l(k_{ln} r) = 0$ at both $r = 1$ and $r = r_o$. Using these parameters, we find, from (39), (41), (42), (44), and (45),

$$g_l(k_{ln} r) = n_l(k_{ln} r_o) j_l(k_{ln} r) - j_l(k_{ln} r_o) n_l(k_{ln} r), \quad \widehat{g}_l(k_{ln} r) = N_{ln}^{-1} g_l(k_{ln} r) \quad (51)$$

$$N_{ln}^2 = \frac{1}{2k_{ln}^4 r_o} \left(\frac{1}{r_o} - \left[\frac{n_l(k_{ln} r_o)}{n_l(k_{ln})} \right]^2 \right), \quad (52)$$

$$g_l(k_{ln}) = 0, \quad n = 1, 2, \dots, N; \quad l = 1, 2, \dots \quad (53)$$

Therefore, $g_l(k_{ln} r_o) \equiv 0$, requiring that (53) holds determines the k_{ln} , which must satisfy

$$g_l(k_{ln}) = n_l(k_{ln} r_o) j_l(k_{ln}) - j_l(k_{ln} r_o) n_l(k_{ln}) = 0. \quad (54)$$

Using the results above, along with (18) and (19), the normalizing constant in (10) is seen to be $c_{lmn} = (\sqrt{l(l+1)}N_{ln})^{-1}$. The k_{ln} in (54) clearly depends on the ratio r_o . Again, we use $r_o \equiv 2.85$, which is appropriate for the Earth's outer core. The k_{ln} are found numerically [82] and for $l, n = 1, \dots, 10$, they appear in Table 1. We can use the asymptotic properties of spherical Bessel and Neumann functions [78] to show that when $k_{ln} \rightarrow \infty$, (51) becomes

$$\lim_{k_{ln} \rightarrow \infty} g_l(k_{ln}r) = \frac{1}{k_{ln}^2 r_o r} \sin[k_{ln}(r_o - r)]. \quad (55)$$

Thus, as $k_{ln} \rightarrow \infty$, (54) becomes

$$\lim_{k_{ln} \rightarrow \infty} g_l(k_{ln}) = \frac{1}{k_{ln}^2 r_o} \sin[k_{ln}(r_o - 1)] = 0 \Rightarrow k_{ln} \approx \frac{n\pi}{r_o - 1}, \quad (56)$$

$$\Rightarrow \lim_{k_{ln} \rightarrow \infty} \hat{g}_l(k_{ln}r) = \frac{1}{r} \sqrt{\frac{2}{r_o - 1}} \sin[k_{ln}(r_o - r)]. \quad (57)$$

If we compare these approximate values for k_{ln} with the largest values in Table 1, we see that there is, in fact, a convergence of the approximate and computed values.

Although **b** is zero on the boundaries for Case DD, it can be connected to the geomagnetic field by using observations that the mantle has a slight electrical conductivity [56,57]. The connection is achieved using a model in which the effect of electrical conductivity in the volume of the mantle is replaced by an electrically conducting spherical surface within the mantle, located between the Earth's surface and the CMB [59]. The result is that when the geomagnetic field at the Earth's surface, standardized as the International Geomagnetic Reference Field (IGRF) [83], is mapped down to the CMB, the spectrum matches much more closely to what might be expected of MHD turbulence than otherwise [59], as will be detailed and illustrated in Section 9.2. This result gives some—at least qualitative—validity to the method just described for approximating the effect of mantle electrical conductivity on the magnetic connection between surface and CMB. A similar approach could be applied to the inner boundary, thereby taking into account the electrical conductivity of the inner core and defining the inner core magnetic field in terms of the outer core's dynamic magnetic field.

7.2. Case DN: Dirichlet Conditions on the Outer and Neumann Conditions on the Inner Boundary

The parameters (48) lead to the Dirichlet conditions $\hat{g}_l(k_{ln}r) = 0$ at $r = r_o$ and $\hat{g}'_l(k_{ln}) = 0$ at $r = 1$. Using these parameters, we find, from (39), (41), (42), (44), and (45),

$$g_l(k_{ln}r) = n_l(k_{ln}r_o)j_l(k_{ln}r) - j_l(k_{ln}r_o)n_l(k_{ln}r), \quad (58)$$

$$N_{ln}^2 = \frac{1}{2k_{ln}^4 r_o} \left(1 - r_o \left[\frac{n_l(k_{ln}r_o)}{n'_l(k_{ln})} \right]^2 \left[1 - \frac{l(l+1)}{k_{ln}^2} \right] \right), \quad (59)$$

$$g'_l(k_{ln}) = 0, \quad n = 1, 2, \dots, N; \quad l = 1, 2, \dots \quad (60)$$

Therefore, $\hat{g}_l(k_{ln}r_o) \equiv 0$ and requiring that (60) holds determines the k_{ln} , which must satisfy

$$g'_l(k_{ln}) = n_l(k_{ln}r_o)j'_l(k_{ln}) - j_l(k_{ln}r_o)n'_l(k_{ln}) = 0. \quad (61)$$

Using the results above, along with (18) and (19), the normalizing constant in (10) is seen to be $c_{lmn} = (\sqrt{l(l+1)}N_{ln})^{-1}$. The k_{ln} in (61) clearly depend on the ratio r_o ; again, we use $r_o \equiv 2.85$, which is appropriate for the Earth's outer core. Again, the k_{ln} are found

numerically [82] and for $l, n = 1, \dots, 10$, they appear in Table 2. When $k_{ln} \rightarrow \infty$, (58) becomes the same as (55), i.e.,

$$\lim_{k_{ln} \rightarrow \infty} g_l(k_{ln}r) = \frac{1}{k_{ln}^2 r_o r} \sin[k_{ln}(r_o - r)] \Rightarrow \hat{g}_l(k_{ln}r) = \frac{1}{r} \sqrt{\frac{2}{r_o - 1}} \sin[k_{ln}(r_o - r)]. \quad (62)$$

Thus, as $k_{ln} \rightarrow \infty$, (61) is found by taking the derivative of both sides of (62) with respect to $k_{ln}r$ and dropping higher-order terms:

$$\lim_{k_{ln} \rightarrow \infty} g'_l(k_{ln}) = -\frac{1}{k_{ln}^2 r_o} \cos[k_{ln}(r_o - 1)] = 0 \Rightarrow k_{ln} \approx \frac{(2n - 1)\pi}{2(r_o - 1)}, \quad (63)$$

$$\Rightarrow \hat{g}_l(k_{ln}r) = \frac{1}{r} \sqrt{\frac{2}{r_o - 1}} \sin[k_{ln}(r_o - r)]. \quad (64)$$

Comparing these approximate values for k_{ln} with the largest values in Table 2, we again see a convergence of the approximate and computed values.

In Case DN, the magnetic field at the CMB, due to MHD turbulence in the outer core, can be connected to the geomagnetic field in the same manner as described above for Case DD. At the ICB, a Neumann condition provides a non-zero normal component that can be connected to an inner core magnetic field; it also requires an induced surface current. The diffusion time for the inner core is estimated to be an order of magnitude longer than advection time in the outer core [54]. For the purpose of analysis, this can be construed as requiring that the magnetic field existing in the outer core have a fixed value at the ICB. The simplest fixed—and perhaps most reasonable—value, at least on average, seems to be $\mathbf{b} = 0$ at the ICB, and we are back at Case DD. However, Glatzmaier and Roberts discovered that numerical simulations require an inner core magnetic field to exist and serve as an anchor for the dynamic outer core magnetic field; otherwise, the numerical simulations did not mimic the palaeomagnetic record, lacking sufficient stability and reversing much too often. For details, see references [20,21].

7.3. Case ND: Neumann Conditions on the Outer and Dirichlet Conditions on the Inner Boundary

The parameters (49) lead to the Dirichlet conditions $\hat{g}'_l(k_{ln}r) = 0$ at $r = r_o$ and $\hat{g}_l(k_{ln}) = 0$ at $r = 1$. In this Case, we have

$$g_l(k_{ln}r) = n'_l(k_{ln}r_o)j_l(k_{ln}r) - j'_l(k_{ln}r_o)n_l(k_{ln}r), \quad (65)$$

$$N_{ln}^2 = \frac{1}{2k_{ln}^4 r_o} \left(1 - \frac{l(l+1)}{k_{ln}^2 r_o^2} - r_o \left[\frac{n'_l(k_{ln}r_o)}{n_l(k_{ln})} \right]^2 \right), \quad (66)$$

$$g_l(k_{ln}) = 0, \quad n = 1, 2, \dots, N; \quad l = 1, 2, \dots, L. \quad (67)$$

Therefore, $\hat{g}'_l(k_{ln}r_o) \equiv 0$ and requiring that (67) holds determines the k_{ln} , which must satisfy

$$g_l(k_{ln}) = n'_l(k_{ln}r_o)j_l(k_{ln}) - j'_l(k_{ln}r_o)n_l(k_{ln}) = 0. \quad (68)$$

Using the results above, along with (18) and (19), the normalizing constant in (10) is seen to be $c_{lmn} = (\sqrt{l(l+1)}N_{ln})^{-1}$. The k_{ln} in (68) clearly depend on the ratio r_o ; again, we use $r_o \equiv 2.85$, which is appropriate for the Earth's outer core. The k_{ln} are found numerically [82]

and for $l, n = 1, \dots, 10$, they appear in Table 3. Now, we take the derivative of (55) with respect to $k_{ln}r_o$, dropping higher-order terms, to show that when $k_{ln} \rightarrow \infty$, (65) becomes

$$\lim_{k_{ln} \rightarrow \infty} g_l(k_{ln}r) = \frac{1}{k_{ln}^2 r_o r} \cos[k_{ln}(r_o - r)]. \quad (69)$$

Thus, as $k_{ln} \rightarrow \infty$, (68) becomes

$$\lim_{k_{ln} \rightarrow \infty} g_l(k_{ln}) = \frac{1}{k_{ln}^2 r_o} \cos[k_{ln}(r_o - 1)] = 0 \Rightarrow k_{ln} \approx \frac{(2n-1)\pi}{2(r_o - 1)}, \quad (70)$$

$$\Rightarrow \hat{g}_l(k_{ln}r) = \frac{1}{r} \sqrt{\frac{2}{r_o - 1}} \cos[k_{ln}(r_o - r)]. \quad (71)$$

Comparing these approximate values for k_{ln} with the largest values in Table 3, again, we see a convergence of the approximate and computed values.

In case ND, $\mathbf{b} = 0$ at the ICB, and we could use the method described in the section on Case DD, if it were desired, to model the inner core magnetic field. In terms of a statistical analysis of MHD turbulence in the outer core, knowledge of the inner core's magnetic field is not required, so it can be assumed to be zero unless there was a good reason not to do so.

7.4. Case NN: Neumann Conditions on Both Boundaries

The parameters (50) lead to the Dirichlet conditions $\hat{g}'_l(k_{ln}r) = 0$ at $r = r_o$ and $\hat{g}'_l(k_{ln}) = 0$ at $r = 1$. Thus, in Case NN, we obtain

$$g_l(k_{ln}r) = n'_l(k_{ln}r_o)j_l(k_{ln}r) - j'_l(k_{ln}r_o)n_l(k_{ln}r), \quad (72)$$

$$N_{ln}^2 = \frac{1}{2k_{ln}^4 r_o} \left(1 - \frac{l(l+1)}{k_{ln}^2 r_o^2} - r_o \left[\frac{n'_l(k_{ln}r_o)}{n'_l(k_{ln})} \right]^2 \left[1 - \frac{l(l+1)}{k_{ln}^2} \right] \right), \quad (73)$$

$$g'_l(k_{ln}) = 0, \quad n = 1, 2, \dots, N; \quad l = 1, 2, \dots, L. \quad (74)$$

Therefore, $\hat{g}'_l(k_{ln}r_o) \equiv 0$, requiring that (74) holds determines the k_{ln} , which must satisfy

$$g'_l(k_{ln}) = n'_l(k_{ln}r_o)j'_l(k_{ln}) - j'_l(k_{ln}r_o)n'_l(k_{ln}) = 0. \quad (75)$$

The functions (72) have also been studied by [84], and similar results were obtained. Using the results above, along with (18) and (19), the normalizing constant in (10) is seen to be $c_{lmn} = (\sqrt{l(l+1)}N_{ln})^{-1}$. The k_{ln} in (75) clearly depends on the ratio r_o ; again, we use $r_o \equiv 2.85$, which is appropriate for the Earth's outer core. Yet again, the k_{ln} are found numerically [82], and for $l, n = 1, \dots, 10$, they appear in Table 4. Finally, we take the derivative of (69) with respect to $k_{ln}r$, again dropping higher-order terms, to show that when $k_{ln} \rightarrow \infty$, (72) becomes

$$\lim_{k_{ln} \rightarrow \infty} g_l(k_{ln}r) = \frac{1}{k_{ln}^2 r_o r} \cos[k_{ln}(r_o - r)]. \quad (76)$$

Thus, as $k_{ln} \rightarrow \infty$, (75) becomes

$$\lim_{k_{ln} \rightarrow \infty} g'_l(k_{ln}) = \frac{1}{k_{ln}^2 r_o} \sin[k_{ln}(r_o - 1)] = 0 \Rightarrow k_{ln} \approx \frac{(n-1)\pi}{r_o - 1}, \quad (77)$$

$$\Rightarrow \hat{g}_l(k_{ln}r) = \frac{1}{r} \sqrt{\frac{2}{r_o - 1}} \cos[k_{ln}(r_o - r)]. \quad (78)$$

Comparing these approximate values for k_{ln} with the largest values in Table 4, yet again, we see a convergence. Note that the asymptotic form of the wavenumbers k_{ln} in (77) for Case NN differ from those in (56) for Case DD in order to match the computed values in Tables 1 and 4, respectively.

8. The Velocity Field

If we take the Galerkin expansion for velocity (9), we can find $\hat{\mathbf{r}} \cdot \mathbf{u}(\mathbf{x}, t)$ and $\hat{\mathbf{r}} \times \mathbf{u}(\mathbf{x}, t)$ using (14), (15), and (17)–(19),

$$\hat{\mathbf{r}} \cdot \mathbf{u}(\mathbf{x}, t) = \sum_{l,m,n} w_{lmn}(t) \hat{\mathbf{r}} \cdot \mathbf{P}_{lmn}(\mathbf{x}) = - \sum_{l,m,n} w_{lmn}(t) \frac{1}{r} \sqrt{l(l+1)} \hat{g}_l(k_{ln}r) Y_{lm}(\theta, \varphi), \quad (79)$$

$$\begin{aligned} \hat{\mathbf{r}} \times \mathbf{u}(\mathbf{x}, t) &= \sum_{l,m,n} [u_{lmn}(t) \hat{\mathbf{r}} \times \mathbf{T}_{lmn}(\mathbf{x}) + w_{lmn}(t) \hat{\mathbf{r}} \times \mathbf{P}_{lmn}(\mathbf{x})] \\ &= - \sum_{l,m,n} \left[w_{lmn}(t) \hat{g}_l(k_{ln}r) \hat{\mathbf{T}}_{lm}(\theta, \varphi) + u_{lmn}(t) \frac{1}{r} \frac{d}{dr} [r \hat{g}_l(k_{ln}r)] \hat{\mathbf{\Phi}}_{lm}(\theta, \varphi) \right]. \end{aligned} \quad (80)$$

If Dirichlet conditions (Case DD) are applied on the both outer and inner boundaries, i.e., $r = r_o$ and 1, respectively, as described in Section 7.1, then we have $\hat{\mathbf{r}} \cdot \mathbf{u}(\mathbf{x}, t)|_{r_o,1} = 0$ because $\hat{g}_l(k_{ln}r)|_{r_o,1} = 0$. Thus, the normal component of velocity is zero on the boundaries as required. However, on the boundaries, (80) becomes

$$\hat{\mathbf{r}} \times \mathbf{u}(\mathbf{x}, t)|_{r_o,1} = - \sum_{l,m,n} u_{lmn}(t) k_{ln} \hat{g}'_l(k_{ln}r)|_{r_o,1} \hat{\mathbf{\Phi}}_{lm}(\theta, \varphi) \neq 0. \quad (81)$$

We see that the no-slip condition is not obeyed because $\hat{g}'_l(k_{ln}r)|_{r_o,1} \neq 0$, which is admissible for ideal MHD but not strictly so for real MHD.

In the case of real flow, we can enforce a no-slip condition to supplement $\hat{\mathbf{r}} \cdot \mathbf{u}(\mathbf{x}, t)|_{r_o,1} = 0$ by introducing compressibility through the addition of new terms to (9):

$$\mathbf{u}(\mathbf{x}, t) = \sum_{l,m,n} \left[v_{lmn}(t) \nabla \left\{ \hat{f}(\kappa_{ln}r) Y_{lm}(\theta, \varphi) \right\} + u_{lmn}(t) \mathbf{T}_{lmn}(\mathbf{x}) + w_{lmn}(t) \mathbf{P}_{lmn}(\mathbf{x}) \right] \quad (82)$$

Here, $\hat{f}(\kappa_{ln}r)$ is equivalent to the Case NN function $\hat{g}_l(k_{ln}r)$ described in Section 7.4, which satisfies $\hat{f}'(\kappa_{ln}r)|_{r_o,1} = 0$; the κ_{ln} are the same as the k_{ln} that appear in Table 4. Now, we require

$$\hat{\mathbf{r}} \times \mathbf{u}(\mathbf{x}, t)|_{r_o,1} = \sum_{l,m,n} \left[v_{lmn}(t) \frac{\sqrt{l(l+1)}}{r} \hat{f}(\kappa_{ln}r) - u_{lmn}(t) k_{ln} \hat{g}'_l(k_{ln}r) \right] \hat{\mathbf{\Phi}}_{lm}(\theta, \varphi) \neq 0 \quad (83)$$

Due to the independence of the $\hat{\mathbf{\Phi}}_{lm}(\theta, \varphi)$ for different l and m , this leads to the requirement

$$\sum_n \left[v_{lmn}(t) \frac{\sqrt{l(l+1)}}{r} \hat{f}(\kappa_{ln}r) - u_{lmn}(t) k_{ln} \hat{g}'_l(k_{ln}r) \right]_{r_o,1} = 0. \quad (84)$$

The set of coefficients $\{v_{lmn}(t)\}$ can be chosen such that for $n > 2$, $v_{lmn}(t) = 0$; then, (84) becomes

$$v_{lm1}(t) \hat{f}(\kappa_{l1}r_o) + v_{lm2}(t) \hat{f}(\kappa_{l2}r_o) = U_{lm}(r_o) = \frac{r_o}{\sqrt{l(l+1)}} \sum_{n=1}^N u_{lmn}(t) k_{ln} \hat{g}'_l(k_{ln}r_o), \quad (85)$$

$$v_{lm1}(t) \hat{f}(\kappa_{l1}) + v_{lm2}(t) \hat{f}(\kappa_{l2}) = U_{lm}(1) = \frac{1}{\sqrt{l(l+1)}} \sum_{n=1}^N u_{lmn}(t) k_{ln} \hat{g}'_l(k_{ln}). \quad (86)$$

These two linear equations are easily solved to find the coefficients $v_{lm1}(t)$ and $v_{lm2}(t)$, and this solution ensures that the no-slip condition is satisfied, as well as the no-flow condition $\hat{\mathbf{r}} \cdot \mathbf{u}(\mathbf{x}, t)|_{r_o,1} = 0$.

The velocity expansion (82) now satisfies realistic b.c.s, and the assumption is that the magnetofluid is almost incompressible, requiring only a slight compressibility to satisfy the requirement that $\mathbf{u}|_{r_o,1} = 0$, as long as $\nabla \cdot \mathbf{u} \approx 0$. The incompressible MHD Equations (1) and (2) are still used, except that the velocity that drives the evolution is (82), but with no separate dynamical equation for the $v_{lmn}(t)$, which are determined by the $u_{lmn}(t)$ and $w_{lmn}(t)$ as described above. This model can be tested numerically by those who have developed computer codes based on C-K eigenfunctions [46,47]; we do not have such codes ourselves.

9. The Magnetic Field

The magnetic field has been modeled as satisfying Dirichlet b.c.s within a sphere using spherical Bessel functions [46,47] and within a spherical shell using both spherical Bessel and Neumann functions [48,59]. These Dirichlet b.c.s were used following earlier work on force-free magnetic fields [49,50] and because they allowed for straightforward proofs of the ideal invariance of energy E and magnetic helicity H_M in a rotating system (and cross helicity H_C in a non-rotating system) [48,66,67]. These proofs of ideal invariance involved the vanishing of surface integrals connected to transfer of either E or H_M or H_C into or out of the magnetofluid through the bounding surfaces; these surface integrals vanished in Fourier models due to periodicity and in the spherical Bessel–Neumann models because the normal components of both velocity and magnetic field were zero on the boundaries. Here, in this paper, we have kept the requirement that $\mathbf{u}$ must vanish on a boundary, as discussed in the previous section. For the magnetic field, however, we have considered other possibilities, i.e., Neumann or mixed Dirichlet–Neumann b.c.s, as presented in Section 5.

In regard to the Earth, as has been discussed, the mantle has relatively low electrical conductivity compared to the outer core, except perhaps in a thin shell just above the CMB, such as was needed for realistic numerical simulations of the dynamic geomagnetic field [20,21]. In this context, Dirichlet–Dirichlet or Dirichlet–Neumann b.c.s and their associated Galerkin expansions of Sections 7.1 or 7.2 might be appropriate. D-D or D-N b.c.s could be useful because, if the normal component of the magnetic field internal to the outer core is zero at the CMB, as required by Dirichlet conditions, then the only way to transport it to the surface is to have a conducting mantle as described below. Otherwise, continuity of the normal component, being zero at the CMB without a conducting mantle, forces the external magnetic field to be zero. The alternative is that the normal component is discontinuous, which does not follow from Maxwell’s equations ([51] pp. 352–353).

Alternatively, the effect of mantle conductivity might be assumed to be negligible, with Neumann conditions imposed on the CMB and perhaps the ICB. In these scenarios, either Neumann–Dirichlet b.c.s or Neumann–Neumann b.c.s and the associated Galerkin methods of Sections 7.3 or 7.4 would seem to be most appropriate. These alternatives, however, do not resolve the flatness problem, while imposing either D-D or D-N conditions does, as will be seen below.

9.1. Magnetic Field and Electric Current Inside the Spherical Shell

The magnetic field $\mathbf{b}$ is given by (8). Taking the curl of $\mathbf{b}$ gives us the electric current $\mathbf{j}$:

$$\mathbf{j}(\mathbf{x}, t) = \nabla \times \mathbf{b}(\mathbf{x}, t) = \sum_{l,m,n} \left[k_{ln}^2 a_{lmn}(t) \mathbf{T}_{lmn}(\mathbf{x}) + b_{lmn}(t) \mathbf{P}_{lmn}(\mathbf{x}) \right]. \quad (87)$$

The general continuity equation for electric charge ρ_e and current $\mathbf{j}$ is

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (88)$$

In MHD [65], the first term on the left side is negligible in the bulk of the fluid such that $\nabla \cdot \mathbf{j} = 0$ there. However, on the boundary $r = r_o$ in Case ND (or on the Neumann boundaries in Cases DN and NN), it cannot be neglected. In Case ND, for example, the surface charge ρ_s has a defining equation (found by integrating over an infinitesimal volume surrounding a surface element):

$$\frac{\partial \rho_s}{\partial t} = -\hat{\mathbf{r}} \cdot \mathbf{j}|_{r_o} = -\sum_{l,m,n} b_{lmn}(t) \hat{\mathbf{r}} \cdot \mathbf{P}_{lmn}(\mathbf{x})|_{r_o}. \quad (89)$$

Now, using (15) and (17), on the boundary $r = r_o$, we have

$$\frac{ds_{lm}(t)}{dt} = \frac{\sqrt{l(l+1)}}{r_o} \sum_n b_{lmn}(t) \hat{g}_l(k_{ln}r_o), \quad \rho_s = \sum_{l,m} s_{lm}(t) Y_{lm}(\theta, \varphi). \quad (90)$$

In addition, at $r = r_o$, the normal component on the spherical shell is

$$\hat{\mathbf{r}} \cdot \mathbf{b}|_{r_o} = \sum_{l,m,n} a_{lmn}(t) \hat{\mathbf{r}} \cdot \mathbf{P}_{lmn}(\mathbf{x})|_{r_o} = -\sum_{l,m,n} \frac{\sqrt{l(l+1)}}{r_o} a_{lmn}(t) \hat{g}_l(k_{ln}r_o) Y_{lm}(\theta, \varphi). \quad (91)$$

In Cases ND or NN, $\hat{g}_l(k_{ln}r_o) \neq 0$ such that when the current $\mathbf{j}$ hits the boundary $r = r_o$ it creates a time-dependent, fluctuating layer of surface charge ρ_s ; we also have $\hat{\mathbf{r}} \cdot \mathbf{b} \neq 0$ at $r = r_o$ in these cases. If there was an electrically conducting mantle above $r = r_o$ and the conductivity was high enough, it would be necessary to model its variation with position and to determine mantle current flow and the resultant total magnetic field. This greatly complicates the analysis unless mantle conductivity is assumed to be negligible with respect to outer core conductivity; in this case, (90) can be used to determine the fluctuating electric charge at the CMB, while the normal component of $\mathbf{b}$ at $r = r_o$ can be matched to the geomagnetic potential field mapped down to $r = r_o$.

However, if it is assumed that the magnetic field $\mathbf{b}$ within the outer core is entrained with the fluid flow, then $\hat{\mathbf{r}} \cdot \mathbf{b} = 0$ at $r = r_o$ and we have a Dirichlet condition and $\hat{g}_l(k_{ln}r_o) \neq 0$ at $r = r_o$ and there is also no surface charge because $\hat{\mathbf{r}} \cdot \mathbf{j} = 0$ at $r = r_o$. (Similar results follow for N or D conditions at $r = 1$ and are easily generated.) In Cases DD or DN, we can introduce mantle conductivity, even if it is small compared to outer core electrical conductivity; doing so leads to useful results, as we will see.

9.2. Magnetic Field Outside the Spherical Shell

At the surface of the Earth, the magnet potential field may be written as follows, where $r_e = 5.22$ is the mean radius of the Earth:

$$\mathbf{B} = -\nabla r_e \sum_{l,m} B_{lm}(t) \left(\frac{r_e}{r}\right)^{l+1} Y_{lm}(\theta, \varphi). \quad (92)$$

The coefficients B_{lm} are linearly related to the Schmidt quasi-normalized [85] Gauss coefficients g_{lm} and h_{lm} of the International Geomagnetic Reference Field (IGRF) [83]. The normal component of $\mathbf{B}$ at $r = r_o$ is

$$\hat{\mathbf{r}} \cdot \mathbf{B}|_{r_o} = \sum_{l,m} (l+1) B_{lm}^o(t) Y_{lm}(\theta, \varphi), \quad B_{lm}^o(t) = B_{lm}(t) \left(\frac{r_e}{r_o}\right)^{l+2}, \quad B_{lm} = \sqrt{\frac{4\pi c_m}{2l+1}} (g_{lm} - ih_{lm}). \quad (93)$$

Here, $g_{lm} = g_{l,-m}$, $h_{lm} = -h_{l,-m}$, and $c_m = \frac{1}{2}$ for $m \neq 0$, while $c_0 = 1$.

The Mauersberger–Lowes spectrum $R_l(r_e)$ of the IGRF [83,85] is defined by

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi |\mathbf{B}|^2 \sin \theta d\theta d\phi = \sum_{l=1}^L R_l(r), \quad (94)$$

$$R_l(r) = \left(\frac{r_e}{r}\right)^{2l+4} R_l(r_e) \quad R_l(r_e) = (l+1) \sum_{m=0}^l \left[(g_{lm})^2 + (h_{lm})^2 \right]. \quad (95)$$

Mapping this directly down to the CMB, we obtain $R_l(r_o) = (r_e/r_o)^{2l+4} R_l(r_e)$, a magnetic power spectrum at $r = r_o$ that, as we will see, is much too flat.

9.2.1. Neumann Conditions

If (93) is directly projected down to the CMB ($r = r_o$), we see that the B_{lm} are increased by a factor of $(r_e/r_o)^{l+2}$ over their Earth surface values to become the B_{lm}^o at the CMB. This leads to the flatness problem seen in Figure 1, which indicates that a Neumann condition on the magnetic field at $r = r_o$ is perhaps not appropriate.

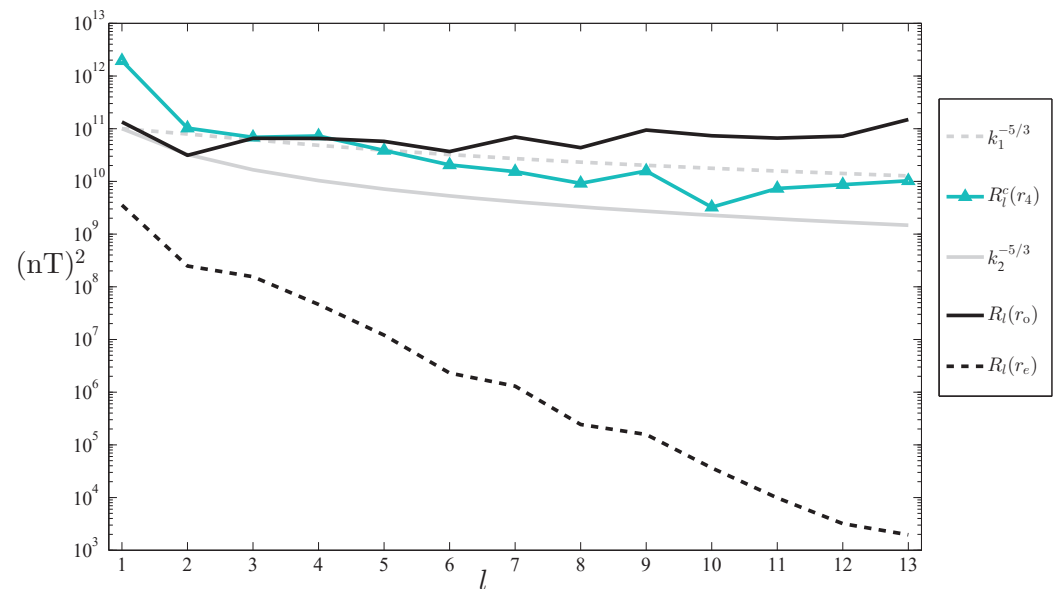


Figure 1. (Colour online) A comparison: International Geomagnetic Reference Field (IGRF) [83] spectra $R_n(r)$ for the year 2020 at the Earth’s surface, $r = r_e$, and projected down to the CMB, $r = r_o$, with a spectrum that takes into account mantle electrical conductivity, i.e., $R_n^o(r_4)$ from reference [59]. Also shown is the region where Kolmogorov turbulent spectra $k^{-5/3}$ might be expected to appear, the region being bounded by using $k_1 = k_{l1}$ and $k_2 = k_{l2}$, where the first ten values of each are shown in Table 1. For more details concerning $R_n^o(r_4)$, please see Section 9.2.

9.2.2. Dirichlet Conditions

However, the mantle does have electrical conductivity $\sigma(r)$, which can be modeled as exponentially increasing from $\sigma_s = 4$ S/m at $r_s = 5371.2$ km (1000 km deep) towards the CMB. This was achieved in reference [59], where $\sigma(r)$ is integrated over the range $r_o \leq r \leq r_s$ to yield an effective conductance σ_c (S) residing on a spherical surface at $r = r_c$, where $r_o < r_c < r_s$:

$$\sigma(r) = \sigma_s \exp[\alpha(r_s - r)], \quad r_o \leq r \leq r_s, \quad \alpha = \frac{\log(\sigma_o/\sigma_s)}{r_s - r_o}, \quad \sigma_o = \sigma(r_o), \quad (96)$$

$$\sigma_c = \int_{r_o}^{r_s} \sigma(r) dr = \frac{\sigma_o - \sigma_s}{\alpha}, \quad r_c = \int_{r_o}^{r_s} r \sigma(r) dr = r_o + \frac{1}{\alpha} \left[1 - \frac{\sigma_s}{\sigma_c} (r_s - r_o) \right]. \quad (97)$$

Between the CMB and the surface current at $r = r_c$, there is a magnetic potential field, but of the following form:

$$\boldsymbol{\beta} = -\nabla r_e \sum_{l,m} f_l(r) H_{lm}(t) Y_{lm}(\theta, \varphi), \quad (98)$$

$$f_l(r) = \left(\frac{r_e}{r} \right)^{l+1} + \frac{l+1}{l} \left(\frac{r_e}{r_o} \right)^{2l+1} \left(\frac{r}{r_e} \right)^l. \quad (99)$$

The form of $f_l(r)$ follows from the Case DD requirement that $\hat{\mathbf{r}} \cdot \boldsymbol{\beta} = 0$ at $r = r_o$. Next, using (92) and (98) and requiring that $\hat{\mathbf{r}} \cdot (\mathbf{B} - \boldsymbol{\beta}) = 0$ at $r = r_c$ gives

$$B_{lm} = \left[1 - \left(\frac{r_c}{r_o} \right)^{2l+1} \right] H_{lm}. \quad (100)$$

Using this, the magnetic power spectrum at the CMB becomes

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi |\boldsymbol{\beta}|^2 \sin \theta d\theta d\phi = \sum_{l=1}^L R_l^c(r_o), \quad (101)$$

$$R_l^c(r_o) = \left(\frac{r_e}{r_o} \right)^{2l+4} F_l(r_c) R_l(r_e) \quad F_l(r_c) = \frac{2l+1}{l} \left[\left(\frac{r_c}{r_o} \right)^{2l+1} - 1 \right]^{-2}. \quad (102)$$

If we compare (95) with (102), we see that taking mantle conductivity into account, we gain a factor $F_l(r_c)$ that fixes the flatness problem. This is seen in Figure 1, where the representative values $\sigma_o = 10^3$ S/m, $\sigma_c = 341$ kS, and $r_c = 3.13$ (or dimensional $R_c = 3815$ km), with $r_o = 2.85$, have been taken from [59].

In Figure 1, there also appear variations of Kolmogorov spectra that are expected to arise in a turbulent fluid or magnetofluid [86–88]. These are inertial range energy spectra that go as $k^{-5/3}$, where k is the associated wavenumber. The challenge is in comparing these Kolmogorov spectra to geomagnetic spectra, which are given in terms of spherical harmonic order l , which are associated with many different wavenumbers, as Table 1, for example, shows. Here, as representative wavenumbers for each order, we use $l = 1, 2, \dots, 13$, $k_1 = k_{l1}$, and $k_2 = k_{l2}$, drawn from Table 1. The positioning of these is arbitrary in Figure 1 but placed so as to provide an easy comparison. The qualitative importance of Figure 1 is that the IGRF spectra map better corresponds to the CMB if mantle electrical conductivity is taken into account [59] than if it is not.

10. Discussion

In this paper, we examined a mathematical model that had been used to solve to the dynamo problem [30]. The model has a turbulent magnetofluid contained within a rotating spherical shell whose motion is determined by basic equations of incompressible MHD; i.e., thermal effects and density variations are excluded and the only dynamical quantities are the velocity and the magnetic field. The rotating spherical shell represents the Earth's outer core, and its outer and inner boundaries represent the CMB and ICB. A primary focus of this paper was to see how different boundary conditions (b.c.s) affected the solution to the dynamo problem.

We assumed no-flow conditions on the velocity and examined the possibility of also ensuring no-slip conditions on the velocity at the boundary. One result of our work is

that we were able to find a procedure for imposing no-slip conditions, although this required introducing some compressibility into the erstwhile incompressible magnetofluid. In regard to the magnetic field, there were many options to consider with respect to its b.c.s. Investigation of possible b.c.s was based on poloidal–toroidal decompositions involving Galerkin expansion of both velocity and magnetic fields.

The form of the orthogonal functions serving as the basis of the Galerkin expansions relegated considerations regarding how to define and meet the various b.c.s with respect to the radial part of these functions, $\hat{g}_l(k_{ln}r)$. Using r_b to represent either the outer or inner boundary, the choices were Dirichlet (D) conditions, where $\hat{g}_l(k_{ln}r_b) = 0$, or Neumann (N) conditions, where $\hat{g}'_l(k_{ln}r_b) = 0$, or one of an infinite number of combinations of these. A manageable number of these combinations, with D or N on either boundary, gave us four cases in total, namely, DD, DN, ND, and NN, where the first letter denotes the outer boundary and the second letter denotes the inner boundary.

For the velocity within an incompressible magnetofluid, only the DD option was appropriate, which satisfied a no-flow condition, but not a no-slip condition, at a boundary. We found that a no-slip condition could be met by allowing some compressibility into the model system. This involved adding a velocity potential term to the existing divergenceless poloidal–toroidal decomposition of the velocity field. This procedure awaits testing by those who possess the necessary computer codes (and desire) to do so.

In regard to the magnetic field, we considered the four options mentioned: DD, DN, ND, and NN. For the magnetic field, a D b.c. forced the magnetic field to obey $\hat{\mathbf{r}} \cdot \mathbf{b} = 0$ on a boundary, while an N b.c. allowed $\hat{\mathbf{r}} \cdot \mathbf{b} \neq 0$ on a boundary. Imposing an N condition might seem the most appropriate approach, but several factors, including Alfvén’s theorem and numerical results, advocated for the D option, at least on the outer boundary. The final choice of a D condition on the outer boundary was made because when the electrical conductivity of Earth’s mantle was modeled, imposing a D condition on the outer boundary removed the ‘flatness problem’ of the geomagnetic field when mapped down to the outer boundary, the CMB.

Our solution to the dynamo problem is based on the statistical mechanics of ideal MHD turbulence [30]. This theory had been alluded to several times in this paper, so we included a brief digression to present an overview of it. The statistical theory is founded on the invariance of energy and magnetic helicity in an equilibrium, turbulent magnetofluid. Since magnetic helicity is essential to understanding MHD turbulence, particularly its statistical mechanics, we showed how the basic Galerkin expansions could be transformed into those where each basis function has a definite helicity, for both the velocity and magnetic fields. This allowed for a brief introduction to the statistical theory of ideal MHD turbulence, a theory which is essential to solving the dynamo problem.

Boundary conditions affect how orthogonality is ensured for the Galerkin basis functions and, in particular, how the radial parts of the basis functions are designed. The radial part of the Galerkin functions are, as mentioned, the functions $\hat{g}_l(k_{ln}r)$, which are a linear combination of spherical Bessel and Neumann functions. The forms that the $\hat{g}_l(k_{ln}r)$ take to satisfy any of the b.c. options, i.e., DD, DN, ND and NN, and in particular, the set of wavenumbers k_{ln} associated with each of these options, are unique and must be calculated numerically.

As already mentioned, we have shown in this paper how the Galerkin expansion of the velocity field can be extended to meet no-slip b.c.s. In regard to the mathematical representation of the magnetic field inside and outside the spherical shell, after the various options are examined in detail, a distinction is made as to the efficacy of D versus N b.c.s with regard to mapping the geomagnetic field from the Earth’s surface down to the CMB. As seen in Figure 1, mapping the Gauss coefficients of IGRF [83] down to the CMB

for a non-electrically conducting mantle produces an approximately flat magnetic power spectrum $R_l(r_o)$, while a magnetic energy spectrum originating from MHD turbulence in the outer core is expected to fall off as $k_{ln}^{-5/3}$.

The cause of this mismatch seems to be a neglect of mantle electrical conductivity, along with the assumption that an N b.c. on the outer boundary was the appropriate one. Another factor vitiating the imposition of N b.c.s is, as mentioned before, Alfvén's theorem, which says that magnetic field lines are entrained with the flow of a magnetofluid with large enough Reynolds numbers. If mantle electrical conductivity is taken into account and the b.c.s of Case DD are used [59], then mapping the Gauss coefficients down to the CMB produces, as seen, a magnetic spectrum that closely resembles what might be expected from a turbulent magnetofluid [59].

We could use b.c.s from any of the Cases (DD, DN, ND or NN), with a spherical shell model containing a turbulent magnetofluid that is in equilibrium, along with statistical mechanics to produce a solution to the dynamo problem. However, as an approximation that has been employed in many applications, and one that is the simplest and produces viable results, Case DD b.c.s is still appropriate for the reasons listed above. Case DN is an alternative set of b.c.s—and, in fact, any could be used—but N conditions are not as fully developed, which would seem to complicate the model; thus, its further development is required.

11. Conclusions

Incompressible MHD and the model systems that were considered here are all able to represent some critical features of the Earth and its magnetism, while other features remain unknown and perhaps unknowable. The 'real' Earth does not have exactly spherical surfaces separating its mantle, outer core, and inner core, nor are these layers spherically symmetric, incompressible, or homogeneous in their material properties. However, idealized models of the Earth are necessary for theoretical and computational analysis to begin to make progress in identifying mechanisms essential to understanding geomagnetic and other geophysical processes. From there, we can move forward, adding more variables and complications.

To understand 'reality', we build models that we know can never fully represent all the details inherent in the physical world around us. The initial task is to capture the qualitative features of the physical processes that are observed and call for explanation. The 'dynamo problem' of determining the origin of the Earth's quasi-steady, energetic, mostly dipole magnetic field is one example. The model, with various possible boundary conditions, which seems the simplest and yet contains the essential features, is the one discussed here: A rotating spherical shell fill with a turbulent, incompressible magnetofluid in equilibrium. We do not allow ourselves to be diverted by questions of how the system arrives at this state, nor of how it is maintained, nor of how complicating details such as density and temperature variations might affect the results of our analysis, nor of any of the many other questions that might be considered. Those are important questions, whose answers will sharpen the quantitative details, but they are secondary consideration when the important qualitative features of a solution need to be found.

What has been found in our previous examinations of the rotating spherical shell model with Dirichlet b.c.s is that the statistical mechanics of ideal MHD turbulence, along with an analysis of the associated dynamical system, solves the dynamo problem in a qualitative sense (though the associated mathematics are precise and quantitative). The solution follows the assumption that the turbulent magnetofluid is in equilibrium, with constant energy and magnetic helicity. We mention the statistical theory here but do not develop it as this is available in detail elsewhere [30]. Here, we have focused on examining

optional b.c.s and concluded that DD b.c.s are sufficient to solve the dynamo problem and that a D condition on the outer boundary, along with a conducting mantle, are necessary to remove the flatness problem [59].

These are our conclusions and the interested reader is invited to examine the references cited herein for further information and illumination.

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References

1. Turner, G. *North Pole, South Pole: The Epic Quest to Solve the Great Mystery of Earth's Magnetism*; The Experiment: New York, NY, USA, 2011.
2. Larmor, J. How could a rotating body such as the sun become a magnet? *Rep. Brit. Assoc. Adv. Sci.* **1919**, *87*, 159–160.
3. Alfvén, H. On the Existence of Electromagnetic-Hydrodynamic Waves. *Ark. Mat. Astron. Och Fys.* **1943**, *29*, 1–7. [\[CrossRef\]](#)
4. Elsässer, W.M. Hydromagnetic dynamo theory. *Rev. Mod. Phys.* **1956**, *28*, 135–163. [\[CrossRef\]](#)
5. Weiss, N.O. The Dynamo Problem. *Q. J. Roy. Astron. Soc.* **1971**, *12*, 432–446.
6. Roberts, P.H. Kinematic Dynamo Models. *Philos. Trans. R. Soc. Lond. A* **1972**, *272*, 663–698.
7. Gubbins, D. Numerical Solutions of the Kinematic Dynamo Problem. *Philos. Trans. R. Soc. Lond. A* **1973**, *274*, 493–521.
8. Kumar, S.; Roberts, P.H. A Three-Dimensional Kinematic Dynamo. *Philos. Trans. R. Soc. Lond. A* **1975**, *344*, 235–258.
9. Zel'dovich, Y.B.; Ruzmaikin, A.A.; Molchanov, S.A.; Sokoloff, D.D. Kinematic dynamo problem in a linear velocity field. *J. Fluid Mech.* **1984**, *144*, 1–11. [\[CrossRef\]](#)
10. Archontis, V.; Dorci, S.B.F.; Nordlund, A. Numerical simulations of kinematic dynamo action. *Astron. Astrophys.* **2003**, *397*, 393–399. [\[CrossRef\]](#)
11. Chen, L.; Herreman, W.; Li, K.; Livermore, P.W.; Luo, J.W.; Jackson, A. The optimal kinematic dynamo driven by steady flows in a sphere. *J. Fluid Mech.* **2018**, *839*, 1–32. [\[CrossRef\]](#)
12. Dudley, M.L.; James, R.W. Time-Dependent Kinematic Dynamos with Stationary Flows. *Philos. Trans. R. Soc. Lond. A* **1989**, *425*, 407–429.
13. Schekochihin, A.; Cowley, S.; Maron, J.; Malyskin, L. Structure of small-scale magnetic fields in the kinematic dynamo theory. *Phys. Rev. E* **2001**, *65*, 016305. [\[CrossRef\]](#)
14. Pongkitiwanichakul, P.; Nigro, G.; Cattaneo, F.; Tobias, S.M. Shear-Driven Dynamo Waves in the Fully Nonlinear Regime. *Astrophys. J.* **2016**, *825*, 23. [\[CrossRef\]](#)
15. Nigro, G.; Pongkitiwanichakul, P.; Cattaneo, F.; Tobias, S.M. What is a large-scale dynamo? *Mon. Not. Roy. Astro. Soc.* **2017**, *464*, L119. [\[CrossRef\]](#)
16. Moffatt, H.K. *Magnetic Field Generation in Electrically Conducting Fluids*; Cambridge U.P.: New York, NY, USA, 1978.
17. Krause, F.; Rädler, K.-H. *Mean-Field Magnetohydrodynamics and Dynamo Theory*; Oxford U.P.: Oxford, UK, 1980.
18. Rädler, K.-H. Mean-Field Dynamo Theory: Early Ideas and Today's Problems. In *Magnetohydrodynamics—Historical Evolution and Trends*; Molokov, S., Moreau, R., Moffatt, H.K., Eds.; Springer: Dordrecht, The Netherlands, 2007; pp. 55–72.
19. Hughes, D.W. Mean field electrodynamics: Triumphs and tribulations. *J. Plasma Phys.* **2018**, *84*, 735840407. [\[CrossRef\]](#)
20. Glatzmaier, G.A.; Roberts, P.H. A three-dimensional self-consistent computer simulation of a geomagnetic field reversal. *Nature* **1995**, *377*, 203–209. [\[CrossRef\]](#)
21. Glatzmaier, G.A.; Roberts, P.H. A three-dimensional convective dynamo solution with rotating and finitely conducting inner core and mantle. *Phys. Earth Planet. Int.* **1995**, *91*, 63–75. [\[CrossRef\]](#)
22. Frisch, U.; Pouquet, A.; Leorat, J.; Mazure, A. Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence. *J. Fluid Mech.* **1975**, *68*, 769–778. [\[CrossRef\]](#)
23. Fyfe, D.; Montgomery, D. High beta turbulence in two-dimensional magneto-hydrodynamics. *J. Plasma Phys.* **1976**, *16*, 181–191. [\[CrossRef\]](#)
24. Pouquet, A.; Patterson, G.S. Numerical simulation of helical magnetohydrodynamic turbulence. *J. Fluid Mech.* **1978**, *85*, 305–323. [\[CrossRef\]](#)
25. Shebalin, J.V. Anisotropy in MHD Turbulence Due to a Mean Magnetic Field. Ph.D. Thesis, College of William and Mary, Williamsburg, VA, USA, 1982.

26. Shebalin, J.V. Broken ergodicity and coherent structure in homogeneous turbulence. *Phys. D* **1989**, *37*, 173–191. [[CrossRef](#)]
27. Palmer, R.G. Broken ergodicity. *Adv. Phys.* **1982**, *31*, 669–735. [[CrossRef](#)]
28. Shebalin, J.V. Broken symmetry in ideal magnetohydrodynamic turbulence. *Phys. Plasmas* **1994**, *1*, 541–547. [[CrossRef](#)]
29. Shebalin, J.V. Broken ergodicity in magnetohydrodynamic turbulence. *Geophys. Astrophys. Fluid Dyn.* **2013**, *107*, 411–466. [[CrossRef](#)]
30. Shebalin, J.V. The Statistical Mechanics of Ideal Magnetohydrodynamic Turbulence and a the Dynamo Problem. *Fluids* **2024**, *9*, 46. [[CrossRef](#)]
31. Khinchin, A.I. *Mathematical Foundations of Statistical Mechanics*; Dover: New York, NY, USA, 1949; pp. 137–145.
32. Lee, T.D. On some statistical properties of Hydrodynamical and magneto-hydrodynamical fields. *Q. Appl. Math.* **1952**, *10*, 69–74. [[CrossRef](#)]
33. Pathria, R.K. *Statistical Mechanics*, 2nd ed.; Elsevier: Oxford, UK, 1972.
34. Landau, L.D.; Lishitz, E.M. *Statistical Physics, Part 1*, 3rd ed.; Pergamon: Oxford, UK, 1980.
35. Shebalin, J.V. Plasma relaxation and the turbulent dynamo. *Phys. Plasmas* **2009**, *16*, 072301. [[CrossRef](#)]
36. Shebalin, J.V. Dynamo action in dissipative, forced, rotating MHD turbulence. *Phys. Plasmas* **2016**, *23*, 062318. [[CrossRef](#)]
37. Shebalin, J.V. Magnetohydrodynamic turbulence and the geodynamo. *Phys. Earth Planet. Inter.* **2018**, *285*, 59–75. [[CrossRef](#)]
38. Shebalin, J.V. Magnetic Helicity and the Geodynamo. *Fluids* **2021**, *6*, 99. [[CrossRef](#)]
39. Dehant, V.; Campuzano, S.A.; De Santis, A.; van Westrenen, W. Structure, Materials and Processes in the Earth's Core and Mantle. *Surv. Geophys.* **2022**, *43*, 263–302. [[CrossRef](#)]
40. Garnero, G.; Richardson, C. The mysterious, massive structures in Earth's deep mantle. *Phys. Today* **2024**, *77*, 37–43. [[CrossRef](#)]
41. Vanyo, J.P.; Dunn, J.R. Core precession: Flow structures and energy. *Geophys. J. Int.* **2000**, *142*, 409–425. [[CrossRef](#)]
42. Gottlieb, D.; Orszag, S.A. *Numerical Analysis of Spectral Methods: Theory and Applications*; SIAM: Philadelphia, PA, USA, 1977.
43. Canuto, C.; Hussaini, M.Y.; Quarteroni, A.; Zang, T.A. *Spectral Methods in Fluid Dynamics*; Springer: New York, NY, USA, 1987.
44. Fornberg, B. *A Practical Guide to Pseudospectral Methods*; Cambridge U.P.: Cambridge, UK, 1996; pp. 7–8.
45. Batchelor, G.K. *The Theory of Homogeneous Turbulence*; Cambridge U.P.: Cambridge, UK, 1953; pp. 28–33.
46. Mininni, P.D.; Montgomery, D. Magnetohydrodynamic activity inside a sphere. *Phys. Fluids* **2006**, *18*, 116602. [[CrossRef](#)]
47. Mininni, P.D.; Montgomery, D.; Turner, L. Magnetohydrodynamic activity inside a sphere. *New J. Phys.* **2007**, *9*, 303–330. [[CrossRef](#)]
48. Shebalin, J.V. Broken ergodicity, magnetic helicity, and the MHD dynamo. *Geophys. Astrophys. Fluid Dyn.* **2013**, *107*, 353–375. [[CrossRef](#)]
49. Chandrasekhar, S. On Force-Free Magnetic Fields. *Proc. Nat. Acad. Sci. USA* **1956**, *42*, 1–5. [[CrossRef](#)]
50. Chandrasekhar, S.; Kendall, P.C. On Force-Free Magnetic Fields. *Astrophys. J.* **1957**, *12*, 457–460. [[CrossRef](#)]
51. Jackson, J.D. *Classical Electrodynamics*, 3rd ed.; Wiley: New York, NY, USA, 1999; pp. 18, 37–38.
52. Kuang, W.; Bloxham, J. An Earth-like numerical dynamo model. *Nature* **1997**, *389*, 371–374. [[CrossRef](#)]
53. Montgomery, D.; Turner, L.; Vahala, G. Three-dimensional magnetohydrodynamic turbulence in a cylindrical geometry. *Phys. Fluids* **1978**, *21*, 757–764. [[CrossRef](#)]
54. Pozzo, M.; Davies, C.; Gubbins, D.; Alfè, D. Thermal and electrical conductivity of solid iron and iron–silicon mixtures at Earth's core conditions. *Earth Planet. Sci. Lett.* **2014**, *393*, 159–164. [[CrossRef](#)]
55. Pozzo, M.; Alfè, D. Saturation of electrical resistivity of solid iron at Earth's core conditions. *SpringerPlus* **2016**, *5*, 256. [[CrossRef](#)] [[PubMed](#)]
56. Civet, F.; Thébaud, E.; Verhoeven, O.; Langlais, B.; Saturnino, D. Electrical conductivity of the Earth's mantle from the first Swarm magnetic field measurements. *Geophys. Res. Lett.* **2015**, *42*, 3338–3346. [[CrossRef](#)]
57. Grayver, A.V.; Munch, F.D.; Kuvshinov, A.V.; Khan, A.; Sabaka, T.J.; Tøffner-Clausen, L. Joint inversion of satellite-detected tidal and magnetospheric signals constrains electrical conductivity and water content of the upper mantle and transition zone. *Geophys. Res. Lett.* **2017**, *44*, 6074–6081. [[CrossRef](#)]
58. Moreau, R. *Magnetohydrodynamics*; Kluwer Academic: Dordrecht, The Netherlands, 1990; p. 47.
59. Shebalin, J.V. Mantle Electrical Conductivity and the Magnetic Field at the Core-Mantle Boundary. *Fluids* **2021**, *6*, 403. [[CrossRef](#)]
60. Olson, P. Overview. In *Core Dynamics*; Schubert, G., Olson, P., Eds.; Elsevier: Amsterdam, The Netherlands, 2009; pp. 1–30.
61. Roberts, P.H. *An Introduction to Magnetohydrodynamics*; Longmans: London, UK, 1971; Chapter 1.
62. Polovin, R.V.; Demutskii, V.P. *Fundamentals of Magnetohydrodynamics*; Consultants Bureau: New York, NY, USA, 1990.
63. Biskamp, D. *Magnetohydrodynamic Turbulence*; Cambridge U.P.: Cambridge, UK, 2003; Chapter 2.
64. Goedbloed, J.P.; Keppens, R.; Poedts, S. *Magnetohydrodynamics of Laboratory and Astrophysical Plasmas*; Cambridge U.P.: New York, NY, USA, 2019.
65. Davidson, P.A. *An Introduction to Magnetohydrodynamics*; Cambridge U.P.: Cambridge, UK, 2001; pp. 30–31.
66. Woltjer, L. A theorem on force-free magnetic fields. *Proc. Nat. Acad. Sci. USA* **1958**, *44*, 489–491. [[CrossRef](#)] [[PubMed](#)]

67. Shebalin, J.V. Ideal homogeneous magnetohydrodynamic turbulence in the presence of rotation and a mean magnetic field. *J. Plasma Phys.* **2006**, *72*, 507–524. [\[CrossRef\]](#)
68. Cattaneo, F.; Emonet, T.; Weiss, N. On the Interaction Between Convection and Magnetic Fields. *Astrophys. J.* **2003**, *588*, 1183–1198. [\[CrossRef\]](#)
69. Kraichnan, R.H. Irreversible Statistical Mechanics of Incompressible Hydromagnetic Turbulence. *Phys. Rev.* **1958**, *109*, 1407–1422. [\[CrossRef\]](#)
70. Kraichnan, R.H. Helical turbulence and absolute equilibrium. *J. Fluid Mech.* **1973**, *59*, 745–752. [\[CrossRef\]](#)
71. Morse, E.C. Eigenfunctions of the curl in cylindrical geometry. *J. Math. Phys.* **2005**, *46*, 113511. [\[CrossRef\]](#)
72. Morse, E.C. Eigenfunctions of the curl in annular cylindrical and rectangular geometry. *J. Math. Phys.* **2007**, *48*, 083504. [\[CrossRef\]](#)
73. Glatzmaier, G.A. *Introduction to Modeling Convection in Planets and Stars*; Princeton U.P.: Princeton, NJ, USA, 2014; p. 267.
74. Ballani, L.; Greiner-Mai, H.; Stromeyer, D. Determining the magnetic field in the core–mantle boundary zone by non-harmonic downward continuation. *Geophys. J. Int.* **2002**, *149*, 374–389. [\[CrossRef\]](#)
75. Shebalin, J.V. Transition to Equilibrium and Coherent Structure in Ideal MHD Turbulence. *Fluids* **2023**, *8*, 107. [\[CrossRef\]](#)
76. Shebalin, J.V. Transition to Equilibrium and Coherent Structure in Ideal MHD Turbulence, Part 2. *Fluids* **2023**, *8*, 181. [\[CrossRef\]](#)
77. Holme, R. Large-Scale Flow in the Core. In *Treatise on Geophysics 8: Core Dynamics*; Olson, P., Ed.; Elsevier: Amsterdam, The Netherlands, 2009; pp. 107–130.
78. Arfken, G.B.; Weber, H.J. *Mathematical Methods for Physicists*, 6th ed.; Academic Press: New York, NY, USA, 2005; pp. 557–560. 725–732.
79. Barrera, R.G. Estévez, G.A.; Giraldo, J. Vector spherical harmonics and their application to magnetostatics. *Eur. J. Phys.* **1995**, *6*, 287–294. [\[CrossRef\]](#)
80. Liu, H.; Zou, J. Zeros of the Bessel and spherical Bessel functions and their applications for uniqueness in inverse acoustic obstacle scattering. *IMA J. Appl. Math.* **2007**, *72*, 817–831. [\[CrossRef\]](#)
81. Gray, A.; Mathews, G.B. *A Treatise on Bessel Functions and Their Application to Physics*; MacMillan: London, UK, 1895; p. 53.
82. Arfken, G.B.; Weber, H.J. *Mathematical Methods for Physicists*, 5th ed.; Academic Press: New York, NY, USA, 1995; pp. 1085–1088.
83. Alken, P.; Thébaud, E.; Beggan, C.D.; Aubert, J.; Baerenzung, J.; Brown, W.J.; Califf, S.; Chulliat, A.; Cox, G.A.; Finlay, C.C.; et al. International Geomagnetic Reference Field: The 13th generation. *Earth Planets Space* **2021**, *73*, 49. [\[CrossRef\]](#)
84. Ziener, C.H.; Kurz, F.T.; Buschle, L.R.; Kampf, T. Orthogonality, Lommel integrals and cross product zeros of linear combinations of Bessel functions. *SpringerPlus* **2015**, *4*, 390. [\[CrossRef\]](#)
85. Winch, D.E.; Ivers, D.J.; Turner, J.P.R.; Stening, R.J. Geomagnetism and Schmidt quasi-normalization. *Geophys. J. Int.* **2005**, *160*, 487–504.
86. Kolmogorov, A.N. The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds Numbers. *Dokl. Akad. Nauk SSSR* **1941**, *30*, 299–303; reprinted in *Proc. Roy. Soc. Lond. A* **1991**, *434*, 9–13.
87. Verma, M.K. Statistical theory of magnetohydrodynamic turbulence: Recent results. *Phys. Rep.* **2004**, *401*, 229–380. [\[CrossRef\]](#)
88. Alexandrova, O.; Saur, J.; Lacombe, C.; Mangeney, A.; Mitchell, J.; Schwartz, S.J.; Robert, P. Universality of Solar-Wind Turbulent Spectrum from MHD to Electron Scales. *Phys. Rev. Lett.* **2009**, *103*, 487–504. [\[CrossRef\]](#)

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