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The Dirichlet Problem for the Nonstationary Stokes System in a Domain with Angular or Conical Points

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Abstract

The paper deals with the Dirichlet problem for the nonstationary Stokes system in a bounded two- or three-dimensional domain with angular or conical points on the boundary. The author proves the existence and uniqueness of solutions in weighted Sobolev spaces. The main result can also be used to obtain existence and uniqueness results in non-weighted spaces.

Keywords: stokes system; conical points; existence and uniqueness results; weighted Sobolev spaces

MSC: 35B45; 35B65; 35K51; 35Q35

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N = 2$ or $N = 3$. We assume that the boundary $\partial\Omega$ contains a set $\mathcal{P} = \{P_1, \dots, P_n\}$ of angular or conical points. This means that for each point P_j there exist a neighborhood $\mathcal{U}_j$ and a C^2-mapping $y = \kappa(x)$ such that $\kappa(P_j)$ is the origin O and $\kappa(\Omega \cap \mathcal{U}_j) = K_j \cap \mathcal{U}_j'$, where K_j is an angle or cone (see Section 2) and $\mathcal{U}_j'$ is a neighborhood of the origin. Without loss of generality, we may assume that the Jacobian matrix κ' coincides with the identity matrix at P_j. The set $\Gamma = \partial\Omega \backslash \mathcal{P}$ is assumed to be smooth (of class C^2). In the case $N = 2$, we denote the opening of the angle K_j by α_j.

The present paper deals with the initial-boundary value problem

$$\frac{\partial \mathfrak{u}}{\partial t} - \Delta \mathfrak{u} - \nabla\nabla \cdot \mathfrak{u} + \nabla \mathfrak{p} = \mathfrak{f}, \quad -\nabla \cdot \mathfrak{u} = \mathfrak{g} \text{ in } Q_T = \Omega \times (0, T), \tag{1}$$

$$\mathfrak{u}(x, t) = 0 \text{ for } x \in \Gamma,\ 0 < t < T, \tag{2}$$

$$\mathfrak{u}(x, 0) = 0 \text{ for } x \in \Omega. \tag{3}$$

The stationary Stokes system in domains with nonsmooth boundaries is well studied (see, e.g., the papers [1–4] for Lipschitz domains and the papers [5–14] for domains with corners and edges). The nonstationary problem (1)–(3) in an infinite cone was studied in [15,16], the same problem in an infinite two-dimensional angle was handled in [17]. It should also be mentioned that V. A. Kozlov [18,19] studied a class of general parabolic problems in a cone. However, the results of these papers cannot be applied to the Stokes system. Moreover, there is a significant difference between the results for the Stokes system and the results for the class of parabolic problems in [18,19]. It is a feature of the nonstationary Stokes system that the bounds for the weight parameter in the existence and solvability results depend on the eigenvalues of two operator pencils (the pencils $\mathcal{L}_K$ and $\mathcal{N}_K$ introduced in Section 2). For the class of parabolic problems in [18,19], one needs only information on the eigenvalues of one operator pencil.

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As in [15–17], the major part of the present paper deals with the parameter-depending problem

$$s\,u - \Delta u - \nabla\nabla\cdot u + \nabla p = f, \quad -\nabla\cdot u = g \text{ in } \Omega, \tag{4}$$

$$u = 0 \text{ on } \Gamma. \tag{5}$$

We are interested in solutions of this problem in the class of the weighted Sobolev spaces $V^l_\beta(\Omega)$ which are defined for nonnegative integer l and real n-tuples $\beta = (\beta_1, \dots, \beta_n)$ as the spaces (closure of the set $C^\infty_0(\overline{\Omega}\backslash\mathcal{P})$) with the norm

$$\|u\|_{V^l_\beta(\Omega)} = \Big(\int_\Omega \sum_{|\alpha|\le l} \prod_{j=1}^n r_j^{2(\beta_j - l + |\alpha|)} \, |\partial^\alpha_x u(x)|^2 \, dx \Big)^{1/2}, \tag{6}$$

where $r_j(x)$ denotes the distance of the point x from the corner point P_j. We suppose that $f \in V^0_\beta(\Omega)$ and $g \in V^1_\beta(\Omega)$. In the case $N = 2$, we assume that the components β_j of β satisfy the inequalities

$$\max\big(1 - \operatorname{Re}\Lambda(\alpha_j), -\frac{\pi}{\alpha_j}\big) < \beta_j < \min\big(1, \frac{\pi}{\alpha_j}\big), \quad \beta_j \neq 0. \tag{7}$$

Here, $\Lambda(\alpha)$ is the solution of the equation

$$\sin(\lambda\alpha) + \lambda \sin\alpha = 0 \tag{8}$$

with smallest positive real part. A similar condition is imposed in the case $N = 3$. It is shown in Section 3 (see Theorem 3) that the problem (4), (5) has a unique solution (u, p) satisfying the condition $\int_\Omega p(x)\,dx = 0$ and the estimate

$$\|u\|_{V^2_\beta(\Omega)} + |s|\,\|u\|_{V^0_\beta(\Omega)} + \|p\|_{W^1_\beta(\Omega)} \le c\,\Big(\|f\|_{V^0_\beta(\Omega)} + \|g\|_{V^1_\beta(\Omega)} + |s|\,\|g\|_{(W^1_{-\beta}(\Omega))^*} \Big)$$

with a constant c independent of f, g and s provided that $\operatorname{Re} s \ge 0$ and $|s| > \gamma_0$, where γ_0 is a sufficiently large positive number. Here, $W^1_\beta(\Omega) = \{p \in L_2(\Omega) : \nabla p \in V^0_\beta(\Omega)\}$.

Applying the inverse Laplace transform, we prove first the existence of solutions of the problem (1)–(3) for $T = \infty$ in weighted spaces with the additional weight function $e^{-\gamma t}$, $\gamma > \gamma_0$ (see Theorem 4). After this it is not difficult to prove the solvability of this problem on a finite t-interval. In the case $N = 2$, the main result of the paper (Theorem 5) is the following:

Suppose that $\mathfrak{f} \in L_2(0, T; V^0_\beta(\Omega))$, $\mathfrak{g} \in L_2(0, T; V^1_\beta(\Omega))$ and $\partial_t\mathfrak{g} \in L_2(0, T; (W^1_{-\beta}(\Omega))^*)$, where the inequalities *(7)* are satisfied for all j. Furthermore, we assume that $\mathfrak{g}(x, 0) = 0$ for $x \in \Omega$ and

$$\int_\Omega \mathfrak{g}(x, t)\,dx = 0 \text{ for almost all } t. \tag{9}$$

Then there exists a uniquely determined solution $(\mathfrak{u}, \mathfrak{p})$ of the problem (1)–(3) such that $\mathfrak{u} \in L_2(0, T; V^2_\beta(\Omega))$, $\mathfrak{u}_t \in L_2(0, T; V^0_\beta(\Omega))$, $\mathfrak{p} \in L_2(0, T; W^1_\beta(\Omega))$ and

$$\int_\Omega \mathfrak{p}(x, t)\,dx = 0 \textit{ for almost all } t. \tag{10}$$

This result can be extended to the problem with nonzero initial condition $\mathfrak{u}(x, 0) = u_0(x)$ for $x \in \Omega$ (see Corollary 1).

The condition $\beta_j > 1 - \operatorname{Re}\Lambda(\alpha_j)$ for all j is not surprising. This condition is also necessary for the existence und uniqueness of solutions $(u, p) \in V^2_\beta(\Omega) \times W^1_\beta(\Omega)$ of the Dirichlet problem for the stationary Stokes system. We show at the end of the paper that

the condition $\beta_j > -\frac{\pi}{\alpha_j}$ is necessary, too. For small α_j, this condition is stronger than the condition $\beta_j > 1 - \operatorname{Re}\Lambda(\alpha_j)$.

Note that $\operatorname{Re}\Lambda(\alpha_j) > 1$ if $\alpha_j < \pi$. Then the condition (7) allows β_j to be negative. If $\beta_j \le 0$ for all j, then $V^l_\beta(\Omega)$ and $W^l_\beta(\Omega)$ are subspaces of the Sobolev space $W^l(\Omega)$, and we obtain an existence result in nonweighted Sobolev spaces.

In the 3-dimensional case, the assertion of Theorem 5 is valid for $\beta_j = 0$ if there exists an open half-space H_j such that $P_j \in \overline{H}_j$ and $\overline{K}_j \backslash \{P_j\} \subset H_j$. Suppose that this is true for all j. Then Theorem 5 implies the following result. If $\mathfrak{f} \in L_2(\Omega \times (0,T))$, $\mathfrak{g} \in L_2(0,T;W^1(\Omega))$, $\mathfrak{g}_t \in L_2(0,T;(W^1(\Omega))^*)$, $\mathfrak{g}$ satisfies the conditions $\mathfrak{g}(x,0) = 0$ for $x \in \Omega$ and (9), then the problem (1)–(3) has a unique solution $(\mathfrak{u},\mathfrak{p})$, where $\mathfrak{u} \in L_2(0,T;W^2(\Omega))$, $\mathfrak{u}_t \in L_2(\Omega \times (0,T))$, $\mathfrak{p} \in L_2(0,T;W^1(\Omega))$ and $\mathfrak{p}$ satisfies (10).

2. The Parameter-Dependent Problem in an Angle/Cone

Let K be an infinite angle or cone with vertex at the origin. More precisely,

$$K = \{x = (x_1, x_2) : \ 0 < r < \infty,\ 0 < \varphi < \alpha\} \ \text{ or } \ K = \{x \in \mathbb{R}^3 : \ \omega = x/|x| \in Q\}.$$

Here, r, φ denote the polar coordinates of the point $x = (x_1, x_2)$ and Q is a subdomain of the unit sphere S^2 with smooth (of class C^2) boundary ∂Q. We consider the problem

$$s\,u - \Delta u - \nabla\nabla\cdot u + \nabla p = f, \quad -\nabla\cdot u = g \text{ in } K, \quad u = 0 \text{ on } \Gamma, \tag{11}$$

where $\Gamma = \partial K \backslash \{0\}$. In this section, we present two theorems which were proved in [17] (Theorem 4.12 and Corollary 4.13) for the 2-dimensional case and in [15] (Theorem 2.4) for the 3-dimensional case.

2.1. The Operator Pencils $\mathcal{L}_K$ and $\mathcal{N}_K$

We introduce the following two operator pencils $\mathcal{L}_K$ and $\mathcal{N}_K$ for the cases $N = 2$ (the case of an angle) and $N = 3$ (the case of a cone).

(1) We start with the case $N = 3$. Then the operator $\mathcal{L}_K(\lambda)$ is defined for every complex λ as the mapping

$$\begin{aligned} \overset{\circ}{W}{}^1(Q) \times L_2(Q) \ni \begin{pmatrix} U \\ P \end{pmatrix} \\ \to \begin{pmatrix} r^{2-\lambda}\big(-\Delta r^\lambda U(\omega) + \nabla r^{\lambda-1} P(\omega)\big) \\ -r^{1-\lambda}\nabla\cdot\big(r^\lambda U(\omega)\big) \end{pmatrix} \in W^{-1}(Q) \times L_2(Q), \end{aligned}$$

where $r = |x|$ and $\omega = x/|x|$. The properties of this pencil are studied in [20] (Chapter 5). In particular, it is known that the eigenvalues in the strip $-2 \le \operatorname{Re}\lambda \le 1$ are real [20] (Theorem 5.3.1, Remark 5.3.2)), the strip $-1 \le \operatorname{Re}\lambda \le 0$ does not contain eigenvalues ([20] (Theorem 5.5.6)) and $\lambda = 1$ is always an eigenvalue with the corresponding eigenfunction $(U,P) = (0,1)$. If $\overline{K}\backslash\{0\}$ is a subset of the half-space $x_3 > 0$, then the strip $-2 \le \operatorname{Re}\lambda \le 1$ contains only the eigenvalues $\lambda = -2$ and $\lambda = 1$, and these eigenvalues are simple (see [20] (Theorem 5.5.5)). We denote by $\lambda_1(K)$ the smallest positive eigenvalue of the pencil $\mathcal{L}_K$. Furthermore, let $\lambda_1^*(K)$ be the eigenvalue with smallest real part greater than 1.

The operator $\mathcal{N}_K(\lambda)$ is defined on the space $W^2(Q)$ as

$$\mathcal{N}_K(\lambda)\,U = \Big(-\delta U - \lambda(\lambda+N-2)U\,,\ \frac{\partial U}{\partial \vec{n}}\big|_{\partial Q}\Big),$$

where δ denotes the Beltrami operator. As is known (see, e.g., [20] (Section 2.3)), the eigenvalues of this pencil are real, and generalized eigenfunctions do not exist. The spectrum

contains, in particular, the simple eigenvalue $\mu_0 = 0$ with constant eigenfunctions. The interval $(-1, 0)$ is free of eigenvalues. Let $\mu_1(K)$ be the smallest positive eigenvalue of the pencil $\mathcal{N}_K$.

(2) Analogously, the operators $\mathcal{L}_K$ and $\mathcal{N}_K$ are defined in the case $N = 2$. It is known (see, e.g., [20] (Section 5.1)) that the eigenvalues of $\mathcal{L}_K$ are the solutions of the equation

$$\sin^2(\lambda\alpha) - \lambda^2 \sin^2\alpha = 0.$$

The eigenvalues in the strip $-1 \le \operatorname{Re}\lambda \le 1$ are real. Obviously $\lambda = 1$ is an eigenvalue for every α with the constant eigenvector $(U, P) = (0, 1)$. It is a solution of the equation $\sin(\lambda\alpha) - \lambda \sin\alpha = 0$, and it is simple for $\alpha < \pi$. As in the case $N = 3$, we denote the smallest positive eigenvalue of the pencil $\mathcal{L}_K$ by $\lambda_1(K)$. Note that $\lambda_1(K) > 1/2$ for $\alpha < 2\pi$. If $\alpha < \pi$, then $\lambda_1(K) = 1$. In this case, we consider the eigenvalue $\lambda_1^*(K)$ with smallest positive real part greater than 1. By [20] (Theorem 5.1.1), both $\lambda_1(K)$ if $\alpha > \pi$ and $\lambda_1^*(K)$ if $\alpha < \pi$ are solutions of Equation (8). There exists an angle $\alpha_0 \approx 0.355\pi$ such that $\operatorname{Re}\lambda_1^*(K) > 1 + \frac{\pi}{\alpha}$ for $\alpha < \alpha_0$ and $\frac{\pi}{\alpha} < \operatorname{Re}\lambda_1^*(K) < 1 + \frac{\pi}{\alpha}$ for $\alpha_0 < \alpha < \pi$ (cf. [17] (Lemma 4.7)).

The eigenvalues of the pencil $\mathcal{N}_K$ are the numbers $\mu_k(K) = \frac{k\pi}{\alpha}$, $k = 0, \pm1, \pm2, \ldots$, if $N = 2$.

2.2. The Problem in an Angle

Let K be the angle introduced above. We introduce the following weighted Sobolev spaces. For nonnegative integer l and real β, we define the weighted Sobolev spaces $V_\beta^l(K)$ as the set of all functions (or vector functions) with finite norm

$$\|u\|_{V_\beta^l(K)} = \Big(\int_K \sum_{|\alpha|\le l} r^{2(\beta-l+|\alpha|)} \left|\partial_x^\alpha u(x)\right|^2 dx \Big)^{1/2}. \tag{12}$$

The intersection $V_\beta^2(K) \cap V_\beta^0(K)$ is denoted by $E_\beta^2(K)$. Furthermore, let $W_\beta^1(K)$ be the space with the norm

$$\|g\|_{W_\beta^1(K)} = \Big(\int\limits_{\substack{K \\ 1<|x|<2}} |g|^2\, dx + \int_K r^{2\beta} |\nabla g|^2\, dx \Big)^{1/2}. \tag{13}$$

Theorem 1. *Suppose that $f \in V_\beta^0(K)$, $g \in V_\beta^1(K) \cap (W_{-\beta}^1(K))^*$, $\operatorname{Re} s \ge 0$ and $s \ne 0$. In the case $0 < \beta < 2$, we assume that the integral of g over K is zero.*

(1) If $1 - \lambda_1(K) < \beta < \min(2, \frac{\pi}{\alpha})$, then the problem (11) has a unique solution $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$ satisfying the estimate

$$\|u\|_{V_\beta^2(K)} + |s|\, \|u\|_{V_\beta^0(K)} + \|p\|_{V_\beta^1(K)} \le c \left(\|f\|_{V_\beta^0(K)} + \|g\|_{V_\beta^1(K)} + |s|\, \|g\|_{(W_{-\beta}^1(K))^*} \right) \tag{14}$$

with a constant c independent of f, g and s.

(2) If $\alpha < \pi$, then the above assertion is true for all β satisfying the inequalities

$$\max(-\frac{\pi}{\alpha}, 1 - \operatorname{Re}\lambda_1^*(K)) < \beta < \min(2, \frac{\pi}{\alpha}),\ \beta \ne 0.$$

2.3. The Problem in a Cone

Let K be the cone introduced above. The weighted Sobolev spaces $V_\beta^l(K)$ and $W_\beta^1(K)$ are equipped with the same norms (12) and (13) as in the case of an angle.

Theorem 2. *(1) Suppose that $\frac{1}{2} - \lambda_1(K) < \beta < \min(\mu_1(K) + \frac{1}{2}, \lambda_1(K) + \frac{3}{2})$, $\beta \ne \frac{1}{2}$, $\operatorname{Re} s \ge 0$ and $s \ne 0$. Furthermore, let $f \in V_\beta^0(K)$ and $g \in V_\beta^1(K) \cap (W_{-\beta}^1(K))^*$. In the case $\beta > \frac{1}{2}$,*

we assume that the integral of g over K is zero. Then the problem (11) has a unique solution $(u,p) \in E^2_\beta(K) \times V^1_\beta(K)$ satisfying the estimate (14) with a constant c independent of f, g and s.

(2) If $\lambda_1(K) = 1$ and this eigenvalue is simple, then the above assertion is true for all β satisfying the inequalities

$$\max\left(\frac{1}{2} - \operatorname{Re}\lambda_1^*(K), -\mu_1(K) - \frac{1}{2}\right) < \beta < \min\left(\mu_1(K) + \frac{1}{2}, \lambda_1(K) + \frac{3}{2}\right), \quad \beta \neq \pm\frac{1}{2}.$$

Proof. For the first assertion see [15] (Theorem 2.4).

Suppose that $\lambda_1(K) = 1$, the eigenvalue $\lambda = 1$ is simple and β satisfies the inequalities $\max(\frac{1}{2} - \operatorname{Re}\lambda_1^*(K), -\mu_1(K) - \frac{1}{2}) < \beta < -\frac{1}{2}$. Then the operator

$$E^2_\beta(K) \times V^1_\beta(K) \ni (u,p) \to (f,g) \in V^0_\beta(K) \times \left(V^1_\beta(K) \cap V^1_{-\beta}(K))^*\right)$$

of the problem (11) is injective ([15] (Lemma 2.10)) and has closed range ([15] (Theorem 2.1)). We show that the problem (11) is solvable in $E^2_\beta(K) \times V^1_\beta(K)$ for arbitrary $f \in V^0_\beta(K)$ and $g \in V^1_\beta(K) \cap W^1_{-\beta}(K))^* = V^1_\beta(K) \cap V^1_{-\beta}(K))^*$. Let $f \in C^\infty_0(\overline{K}\backslash\{0\})$ and $g \in C^\infty_0(\overline{K}\backslash\{0\})$ be given. By the first part of the theorem, there exists a solution $(u,p) \in E^2_\gamma(K) \times V^1_\gamma(K)$ of the problem, where $-\frac{1}{2} < \gamma < 0$. Since the strip $\frac{1}{2} - \gamma \leq \operatorname{Re}\lambda \leq \frac{1}{2} - \beta$ contains only the eigenvalue $\lambda = 1$ of the pencil $\mathcal{L}_K$, we can conclude analogously to [17] (Lemma 4.5) that there exists a constant c such that $u \in E^2_\beta(K)$ and $p - c \in V^1_\beta(K)$. Thus, the pair $(u, p-c)$ is a solution of the problem (1) in the space $E^2_\beta(K) \times V^1_\beta(K)$. Since the set $C^\infty_0(\overline{K}\backslash\{0\})$ is dense in $V^0_\beta(K)$ and $V^1_\beta(K) \cap V^1_{-\beta}(K))^*$, it follows that the problem (1) is solvable in $E^2_\beta(K) \times V^1_\beta(K)$ for arbitrary $f \in V^0_\beta(K)$ and $g \in V^1_\beta(K) \cap V^1_{-\beta}(K))^*$. □

As was mentioned above, the condition in item 2 of Theorem 1 is satisfied, e.g., if $\overline{K}\backslash\{0\}$ is a subset of the half-space $x_3 > 0$.

Remark 1. *In [17] (Theorem 4.12) and [15], (Theorem 2.4), the assertion of Theorems 1 and 2 were obtained under the assumption that $f \in V^0_\beta(K)$ and $g \in V^1_\beta(K) \cap (V^1_{-\beta}(K))^*$. But this assumption is the same as in Theorems 1 and 2 since the integral of g is assumed to be zero for $\beta > \frac{N}{2} - 1$. Indeed, if $\beta < \frac{N}{2} - 1$, then it follows from Hardy's inequality that the spaces $W^1_{-\beta}(K)$ and $V^1_{-\beta}(K)$ coincide. Suppose that $\beta > \frac{N}{2} - 1$, $g \in V^1_\beta(K) \cap (V^1_{-\beta}(K))^*$ and $\int_K g\,dx = 0$. Then any $q \in W^1_{-\beta}(K)$ is continuous at the corner point $x = 0$ and*

$$\begin{aligned} \left|\int_K g\,q\,dx\right| &= \left|\int_K g\,(q - q(0))\,dx\right| \leq \|g\|_{(V^1_{-\beta}(K))^*}\,\|q - q(0)\|_{V^1_{-\beta}(K)} \\ &\leq c_1\,\|g\|_{(V^1_{-\beta}(K))^*}\,\|q\|_{W^1_{-\beta}(K)} \end{aligned}$$

(see [21] (Lemma 7.1.3)). This means that $g \in (W^1_{-\beta}(K))^*$,

$$\|g\|_{(W^1_{-\beta}(K))^*} \leq c_1\,\|g\|_{(V^1_{-\beta}(K))^*} \leq c_2\,\|g\|_{(W^1_{-\beta}(K))^*},$$

and the norm of g in $(W^1_{-\beta}(K))^*$ in the estimate (14) can be replaced by the norm in the space $(V^1_{-\beta}(K))^*$.

Remark 2. *The proof of Theorems 1 and 2 in [15,17] contains a small mistake. In [15], (Lemma 2.3), the norm of $\zeta_\nu g$ in $W^{-1}(K_\nu)$ must be replaced by the norm in $(W^1(K_\nu))^*$, and in [15] (Lemma 2.4), one needs the additional assumption that $x \cdot u \in (V^1_{2-\beta}(K))^*$. This mistake was corrected in [16] (Subsection 2.2) (cf. [16] (Lemma 2.9)).*

3. The Parameter-Dependent Problem in a Bounded Domain

We consider the problem (4), (5) in the bounded domain Ω which was described in the introduction. In particular, we assume that for each point P_j there exist a neighborhood $\mathcal{U}_j$ and a C^2-mapping $y = \kappa(x)$ such that $\kappa(P_j)$ is the origin O, the Jacobian matrix $\kappa'(x)$ is equal to the identity matrix for $x = P_j$ and $\kappa(\Omega \cap \mathcal{U}_j) = K_j \cap \mathcal{U}_j'$, where $\mathcal{U}_j'$ is a neighborhood of the origin and K_j is an angle with opening α_j or a cone. In the case $N = 3$ we denote the intersection of K_j with the unit sphere S^2 by Q_j. For every index j, we define the numbers $\lambda_1(K_j)$, $\lambda_1^*(K_j)$ and $\mu_1(K_j)$ as in Section 2.1.

3.1. Existence of Solutions

Let $W^l(\Omega)$ be the Sobolev space with the norm

$$\|u\|_{W^l(\Omega)} = \Big(\int_\Omega \sum_{|\alpha| \le l} |\partial_x^\alpha u(x)|^2 \, dx \Big)^{1/2}.$$

The closure of the set $C_0^\infty(\Omega)$ in this space is denoted by $\overset{\circ}{W}{}^l(\Omega)$, and $W^{-l}(\Omega)$ is defined as the dual space of $\overset{\circ}{W}{}^l(\Omega)$. By $(\cdot,\cdot)_\Omega$ we denote the $L_2(\Omega)$-scalar product and its extension to the product $W^{-l}(\Omega) \times \overset{\circ}{W}{}^l(\Omega)$. Note that $\overset{\circ}{W}{}^1(\Omega) \subset V_0^1(\Omega)$. For $N = 3$ this follows directly from Hardy's inequality, in the case $N = 2$ this can be easily deduced from Friedrich's inequality.

We consider the bilinear form

$$b_s(u,v) = \int_\Omega \Big(su \cdot v + 2 \sum_{i,j=1}^N \varepsilon_{i,j}(u)\, \varepsilon_{i,j}(v) \Big) \, dx,$$

where $\varepsilon(u)$ denotes the strain tensor with the elements

$$\varepsilon_{i,j}(\mathfrak{u}) = \frac{1}{2} \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \Big), \quad i,j = 1, \dots, N,$$

Obviously, this bilinear form is continuous on $W^1(\Omega) \times W^1(\Omega)$. Furthermore, it follows from Korn's inequality that

$$\big| b_s(u, \overline{u}) \big| \ge c \, \|u\|^2_{W^1(\Omega)} \tag{15}$$

for all $u \in W^1(\Omega)$ and for all s, $\operatorname{Re} s \ge 0$, $s \ne 0$, where c depends on $|s|$ but not on u.

Let $\overset{\circ}{L}{}_2(\Omega)$ be the set of all $g \in L_2(\Omega)$ satisfying the condition

$$\int_\Omega g(x) \, dx = 0. \tag{16}$$

The following lemma can be found, e.g., in [22] (Chapter 1, Corollary 2.4) or [11] (Lemma 11.1.1).

Lemma 1. *Let $g \in \overset{\circ}{L}{}_2(\Omega)$. Then there exists a vector function $u \in \overset{\circ}{W}{}^1(\Omega)$ such that $\nabla \cdot u = g$ in Ω and*

$$\|u\|_{W^1(\Omega)} \le c \, \|g\|_{L_2(\Omega)}$$

with a constant c independent of g.

Applying the inequality (15) and Lemma 1, one can prove the following lemma analogously to [11] (Theorem 11.1.2).

Lemma 2. *Suppose that* $f \in W^{-1}(\Omega)$, $g \in \overset{\circ}{L}_2(\Omega)$, $\operatorname{Re} s \geq 0$ *and* $s \neq 0$. *Then there exists a unique solution* $(u,p) \in \overset{\circ}{W}{}^1(\Omega) \times \overset{\circ}{L}_2(\Omega)$ *of the problem*

$$b_s(u,\bar{v}) - \int_\Omega p\,\nabla\cdot\bar{v}\,dx = (f,v)_\Omega \text{ for all } v \in \overset{\circ}{W}{}^1(\Omega), \quad -\nabla\cdot u = g \text{ in } \Omega. \tag{17}$$

satisfying the estimate

$$\|u\|_{W^1(\Omega)} + \|p\|_{L_2(\Omega)} \leq c\left(\|f\|_{W^{-1}(\Omega)} + \|g\|_{L_2(\Omega)}\right), \tag{18}$$

where c *depends on* $|s|$ *but not on* f *and* g.

The solution of the problem (17) is called a weak solution of the problem (4), (5). Let l be a positive integer, $\beta = (\beta_1,\dots,\beta_n)$ a real n-tuple, and let $V^l_\beta(\Omega)$ be the weighted Sobolev space with the norm (6). Then we define $W^l_\beta(\Omega)$ as the weighted Sobolev space with the norm

$$\|u\|_{W^l_\beta(\Omega)} = \left(\|\nabla u\|^2_{V^{l-1}_\beta(\Omega)} + \|u\|^2_{L_2(\Omega)}\right)^{1/2}.$$

Furthermore, we introduce the following notation. If $\beta = (\beta_1,\dots,\beta_n)$ and k is a real number, then $V^l_{\beta+k}(\Omega) = V^l_{\beta'}(\Omega)$ and $W^l_{\beta+k}(\Omega) = W^l_{\beta'}(\Omega)$, where $\beta' = (\beta_1+k,\dots,\beta_n+k)$.

It follows from Hardy's inequality that $W^l_\beta(\Omega) = V^l_\beta(\Omega)$ if $\beta_j > l - \frac{N}{2}$ for all j.

Using the last lemma together with well-known regularity results for elliptic problems (and, in particular, for the stationary Stokes system), we are able to prove the following lemma.

Lemma 3. *Suppose that* $\operatorname{Re} s \geq 0$, $s \neq 0$, $f \in V^0_\beta(\Omega)$ *and* $g \in V^1_\beta(\Omega)$, *where the components* β_j *of* β *satisfy the inequalities*

$$2 - \frac{N}{2} - \lambda_1(K_j) < \beta_j < 1.$$

Furthermore, we assume that g *satisfies the condition* (16). *Then there exists a unique solution* $(u,p) \in V^2_\beta(\Omega) \times (W^1_\beta(\Omega) \cap \overset{\circ}{L}_2(\Omega))$ *of the problem* (4), (5). *If* $\lambda_1(K_j) = 1$ *and this is a simple eigenvalue of the pencil* $\mathcal{L}_{K_j}$, *then the condition on* β_j *can be replaced by the weaker condition*

$$2 - \frac{N}{2} - \operatorname{Re}\lambda^*_1(K_j) < \beta_j < 1, \quad \beta_j \neq 1 - \frac{N}{2}.$$

Proof. Since $\beta_j < 1$ for all j, it follows from Hölder's and Hardy's inequalities that

$$\left|\int_\Omega f\cdot v\,dx\right| \leq \|f\|_{V^0_\beta(\Omega)}\,\|v\|_{V^0_{-\beta}(\Omega)} \leq c\,\|f\|_{V^0_\beta(\Omega)}\,\|v\|_{W^1(\Omega)}.$$

Hence, the functional

$$\overset{\circ}{W}{}^1(\Omega) \ni v \to (f,\bar{v})_\Omega = \int_\Omega f\cdot v\,dx$$

is continuous. Furthermore, $g \in V^1_\beta(\Omega) \subset L_2(\Omega)$. By Lemma 2, there exists a unique solution $(u,p) \in \overset{\circ}{W}{}^1(\Omega) \times \overset{\circ}{L}_2(\Omega) \subset V^1_0(\Omega) \times L_2(\Omega)$ of the problem (17) satisfying the estimate (18). Let ζ_j be two times continuously differentiable functions with support in a sufficiently small neighborhood of P_j which are equal to one near P_j. The vector function $\zeta_j(u,p)$ satisfies the equations

$$-\Delta(\zeta_j u) + \nabla(\zeta_j p) = F_j = \zeta_j(f - \nabla g - su) - u\,\Delta\zeta_j - 2\sum_{i=1}^{2}\partial_{x_i}\zeta_j\,\partial_{x_i}u + p\,\nabla\zeta_j$$

and $-\nabla \cdot (\zeta_j u) = G_j = \zeta_j g - u \cdot \nabla \zeta_j$. From Hardy's inequality it follows that $\zeta_j u \in V^0_\beta(\Omega)$ for $u \in \overset{\circ}{W}{}^1(\Omega)$ and $\beta_j > -1$. Consequently, $F_j \in V^0_\beta(\Omega)$ and $G_j \in V^1_\beta(\Omega)$ if $\beta_j > -1$.

Suppose that $2 - \frac{N}{2} - \lambda_1(K_j) < \beta_j < 1$. Then, in particular, $\beta_j > -1$ (since $\lambda_1(K_j) \le 1$). Furthermore, the strip $1 - \frac{N}{2} \le \operatorname{Re}\lambda \le 2 - \beta_j - \frac{N}{2}$ does not contain eigenvalues of the pencil $\mathcal{L}_{K_j}$. Using well-known regularity results for the stationary Stokes system, we conclude that $\zeta_j(u,p) \in V^2_\beta(\Omega) \times V^1_\beta(\Omega)$.

Suppose now that $\lambda_1(K_j) = 1$ and that this is a simple eigenvalue of the pencil $\mathcal{L}_{K_j}$. Furthermore, we assume that β_j satisfies the inequalities $2 - \frac{N}{2} - \operatorname{Re}\lambda_1^*(K_j) < \beta_j < 1 - \frac{N}{2}$. Then $\lambda = 1$ is the only eigenvalue in the strip $1 - \frac{N}{2} \le \operatorname{Re}\lambda \le 2 - \beta_j - \frac{N}{2}$. If moreover $\beta_j > -1$, then $F_j \in V^0_\beta(\Omega)$, $G_j \in V^1_\beta(\Omega)$ and the vector function $\zeta_j(u,p)$ admits the asymptotics

$$\zeta_j(u,p) = \zeta_j(0,c_j) + (v,q), \text{ where } (v,q) \in V^2_\beta(\Omega) \times V^1_\beta(\Omega).$$

This implies $\zeta_j u \in V^2_\beta(\Omega)$ and $\zeta_j p \in W^1_\beta(\Omega)$. Since then $s\zeta_j u \in V^0_{\beta-2}(\Omega)$, one obtains the same result if $\beta_j \le -1$. This proves the existence of a solution (u,p) in the space $V^2_\beta(\Omega) \times (W^1_\beta(\Omega) \cap \overset{\circ}{L}{}_2(\Omega))$. The uniqueness of this solution follows from the imbedding $V^2_\beta(\Omega) \subset W^1(\Omega)$ (since $\beta_j < 1$ for all j) and Lemma 2. □

3.2. An Estimate for p

Our goal is to obtain an estimate for the solution (u,p) in Lemma 3 which is analogous to (14). For this, we need an estimate for solutions of the Neumann problem for the Poisson equation in the weighted space $W^2_\beta(\Omega)$ introduced in the foregoing subsection.

Lemma 4. *Suppose that $\phi \in V^0_{1-\gamma}(\Omega)$, where $0 < \gamma_j < \frac{N}{2} - 1 + \mu_1(K_j)$ for all j, and that the integral of ϕ over Ω is zero. Then there exists a unique solution $q \in W^2_{1-\gamma}(\Omega) \cap \overset{\circ}{L}{}_2(\Omega)$ of the problem*

$$-\Delta q = \phi \ \text{ in } \Omega, \quad \frac{\partial q}{\partial n} = 0 \text{ on } \Gamma \tag{19}$$

satisfying the estimate

$$\|q\|_{W^2_{1-\gamma}(\Omega)} \le c \, \|\phi\|_{V^0_{1-\gamma}(\Omega)} \tag{20}$$

with a constant c independent of ϕ.

Proof. Since $\gamma_j > 0$ for all j, Hölder's inequality implies the imbedding $V^0_{1-\gamma}(\Omega) \subset L_1(\Omega)$. Furthermore, it follows from Hardy's inequality that

$$\int_\Omega \prod r_j^{2\gamma_j - 2} |v|^2 \, dx \le c \int_\Omega \left(\prod r_j^{2\gamma_j} \, |\nabla v|^2 + |v|^2 \right) dx \le c' \, \|v\|^2_{W^1(\Omega)}$$

for all $v \in W^1(\Omega)$, i.e., $W^1(\Omega) \subset V^0_{\gamma-1}(\Omega)$ and, consequently, $V^0_{1-\gamma}(\Omega) \subset (W^1(\Omega))^*$. As is known, the W^1-norm is equivalent to the norm

$$\|q\| = \|\nabla q\|_{L_2(\Omega)}$$

on the subspace $W^1(\Omega) \cap \overset{\circ}{L}{}_2(\Omega)$. Hence, there exists a unique variational solution $q \in W^1(\Omega) \cap \overset{\circ}{L}{}_2(\Omega)$ of the problem (19), i.e.,

$$\int_\Omega \nabla q \cdot \nabla v \, dx = \int_\Omega \phi \, v \, dx \ \text{ for all } v \in W^1(\Omega).$$

This solution satisfies the estimate

$$\|q\|_{W^1(\Omega)} \le c' \|\phi\|_{(W^1(\Omega))^*} \le c \|\phi\|_{V^0_{1-\gamma}(\Omega)}.$$

We show that $q \in W^2_{1-\gamma}(\Omega)$. For every j, let ζ_j be a smooth (of class C^2) cut-off function with support in the neighborhood $\mathcal{U}_j$ of P_j which is equal to one near P_j and satisfies the condition $\partial\zeta_j/\partial n = 0$ on Γ. Obviously, $\zeta_j q \in V^1_{\vec{\varepsilon}}(\Omega)$, where $\vec{\varepsilon} = (\varepsilon, \dots, \varepsilon)$ with arbitrary positive ε,

$$-\Delta(\zeta_j q) = F_j \ \text{ in } \Omega \quad \text{and} \quad \frac{\partial(\zeta_j q)}{\partial n} = 0 \ \text{ on } \Gamma,$$

where $F_j = \zeta_j \phi - 2\nabla\zeta_j \cdot \nabla q - q\,\Delta\zeta_j \in V^0_{1-\gamma}(\Omega)$. The interval $1 - \frac{N}{2} - \varepsilon \le \lambda \le 1 - \frac{N}{2} + \gamma_j$ contains only the eigenvalue $\mu_0 = 0$ of the pencil $\mathcal{N}_{K_j}$ if $\varepsilon < \frac{1}{2}$ and $\gamma_j < \frac{N}{2} - 1 + \mu_1(K_j)$. This eigenvalue is simple with constant eigenfunctions if $N = 3$ and has multiplicity 2 (constant eigenfunction and constant generalized eigenfunction) if $N = 2$. Hence, $\zeta_j q$ admits the representation (see, e.g., [21] (Theorem 6.4.1))

$$\zeta_j q = c_j + d_j \log r_j + w_j \ \text{ if } N = 2, \quad \zeta_j q = c_j + w_j \ \text{ if } N = 3,$$

where $w_j \in V^2_\beta(\Omega)$, β has the components $\beta_k = 1 + \varepsilon$ for $k \ne j$ and $\beta_j = \max(\varepsilon, 1 - \gamma_j)$. Since $\zeta_j(q - w_j) \in W^1(\Omega)$, it follows that $d_j = 0$. Thus, w_j is a solution of the Neumann problem $-\Delta w_j = F_j$ in Ω, $\partial w_j/\partial n = 0$ on Γ. Using well-known regularity assertions for solutions of elliptic problems in domains with angular or conical points (see, e.g., [21] (Corollary 6.3.2)), we conclude that $w_j \in V^2_{\beta'}(\Omega)$, where $\beta'_j = 1 - \gamma_j$ and $\beta'_k = 1 + \varepsilon$ for $k \ne j$. Consequently, $\nabla(\zeta_j q) = \nabla w_j \in V^1_{\beta'}(\Omega)$. Since this is true for every j, we have $\nabla q \in V^1_{1-\gamma}(\Omega)$, i.e., $q \in W^2_{1-\gamma}(\Omega)$. This proves the existence of solutions in $W^2_{1-\gamma}(\Omega) \cap \overset{\circ}{L}_2(\Omega)$. The uniqueness follows from the imbedding $W^2_{1-\gamma}(\Omega) \subset W^1(\Omega)$. □

In the following lemma, we consider solutions $(u, p) \in V^2_\gamma(\Omega) \times V^1_\gamma(\Omega)$ of the problem (4), (5), where $0 < \gamma_j < \frac{N}{2} + 1$ for all j. In this case, the integral of p over Ω exists. Furthermore, the constant function $p = c$ is an element of the space $V^1_\gamma(\Omega)$.

Lemma 5. *Suppose that* $\operatorname{Re} s \ge 0$, $s \ne 0$, *and that* $(u, p) \in V^2_\gamma(\Omega) \times V^1_\gamma(\Omega)$ *is a solution of the problem* (4), (5). *We assume that*

$$0 < \gamma_j < \min\left(\frac{N}{2} + 1, \frac{N}{2} - 1 + \mu_1(K_j)\right)$$

for all j and that the integral of p over Ω is zero. Then

$$\|p\|_{V^0_{\gamma-1}(\Omega)} \le c\left(\|f\|_{V^0_\gamma(\Omega)} + \|g\|_{V^1_\gamma(\Omega)} + |s|\,\|g\|_{(W^1_{-\gamma}(\Omega))^*} + \|Du\|_{V^0_{\gamma-1/2}(\Gamma)}\right), \tag{21}$$

where Du is the matrix with the elements $\partial_{x_i} u_j$ and

$$\|u\|_{V^0_{\gamma-1/2}(\Gamma)} = \left\| u \prod_j r_j^{\gamma_j - 1/2} \right\|_{L_2(\Gamma)}.$$

The constant c in (21) *is independent of u, p and s.*

Proof. First, note that $V^1_\gamma(\Omega) \subset V^0_{\gamma-1}(\Omega) \subset L_1(\Omega)$ if $\gamma_j < \frac{N}{2} + 1$ for all j. By (4), (5), we have

$$\int_\Omega \nabla p \cdot \nabla \bar{q}\, dx = (\Phi, q)_\Omega \text{ for all } q \in W^2_{1-\gamma}(\Omega),$$

where

$$(\Phi, q)_\Omega = \int_\Omega (f + \Delta u + \nabla\nabla\cdot u)\cdot\nabla\bar{q}\,dx - s\,(g,q)_\Omega\,.$$

Integration by parts yields

$$-\int_\Omega p\,\Delta\bar{q}\,dx + \int_\Gamma p\,\frac{\partial\bar{q}}{\partial n}\,d\sigma = (\Phi,q)_\Omega. \tag{22}$$

Let $\phi \in V^0_{1-\gamma}(\Omega) \subset L_1(\Omega)$. We define $c_0 = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \phi\,dx$. By Lemma 4, there exists a solution $q \in W^2_{1-\gamma}(\Omega)$ of the problem

$$-\Delta q = \phi - c_0 \ \text{ in } \Omega, \quad \frac{\partial q}{\partial n} = 0 \ \text{ on } \Gamma$$

satisfying the estimate

$$\|\nabla q\|_{V^0_{1-\gamma}(\Omega)} \le \|q\|_{W^2_{1-\gamma}(\Omega)} \le c\,\|\phi - c_0\|_{V^0_{1-\gamma}(\Omega)} \le c'\,\|\phi\|_{V^0_{1-\gamma}(\Omega)}\,. \tag{23}$$

Using (22), we obtain

$$\int_\Omega p\,\bar{\phi}\,dx = \int_\Omega p\,\overline{\phi - c_0}\,dx = -\int_\Omega p\,\Delta\bar{q}\,dx = (\Phi,q)_\Omega\,. \tag{24}$$

We set $\phi = r^{2\gamma-2}p$ and obtain

$$\|p\|^2_{V^0_{\gamma-1}(\Omega)} = (\Phi,q)_\Omega\,.$$

There is the decomposition $\Phi = \Phi_1 + \Phi_2$, where

$$(\Phi_1,q)_\Omega = \int_\Omega \Big((f - 2\nabla g)\cdot\nabla\bar{q} - sg\,\bar{q}\Big)\,dx, \quad (\Phi_2,q)_\Omega = \int_\Omega (\Delta u - \nabla\nabla\cdot u)\,\nabla\bar{q}\,dx$$

Obviously,

$$\Big|\int_\Omega (f - 2\nabla g)\,\nabla\bar{q}\,dx\Big| \le c\,\Big(\|f\|_{V^0_\gamma(\Omega)} + \|g\|_{V^1_\gamma(\Omega)}\Big)\,\|\nabla q\|_{V^0_{-\gamma}(\Omega)}\,.$$

and

$$\Big|\int_\Omega g\,\bar{q}\,dx\Big| \le \|g\|_{(W^1_{-\gamma}(\Omega))^*}\,\|q\|_{W^1_{-\gamma}(\Omega)} \le \|g\|_{(W^1_{-\gamma}(\Omega))^*}\,\|q\|_{W^2_{1-\gamma}(\Omega)}\,.$$

Thus,

$$|(\Phi_1,q)_\Omega| \le c\,\Big(\|f\|_{V^0_\gamma(\Omega)} + \|g\|_{V^1_\gamma(\Omega)} + |s|\,\|g\|_{(W^1_{-\gamma}(\Omega))^*}\Big)\,\|p\|_{V^0_{\gamma-1}(\Omega)}\,.$$

Furthermore,

$$\begin{aligned}|(\Phi_2,q)_\Omega| &= \Big|\int_\Gamma \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}\Big(n_j\,\frac{\partial\bar{q}}{\partial x_i} - n_i\,\frac{\partial\bar{q}}{\partial x_j}\Big)\,d\sigma\Big| \\ &\le c\,\|Du\|_{V^0_{\gamma-1/2}(\Gamma)}\,\|\nabla q\|_{V^0_{-\gamma+1/2}(\Gamma)} \le c'\,\|Du\|_{V^0_{\gamma-1/2}(\Gamma)}\,\|\nabla q\|_{V^1_{1-\gamma}(\Omega)}\,.\end{aligned}$$

Here, we used the estimate

$$\|\nabla q\|_{V^0_{-\gamma+1/2}(\Gamma)} \le c\,\|\nabla q\|_{V^{1/2}_{1-\gamma}(\Gamma)} \le c\,\|\nabla q\|_{V^1_{1-\gamma}(\Omega)}$$

($V^{1/2}_{1-\gamma}(\Gamma)$ denotes the trace space for $V^1_{1-\gamma}(\Omega)$ on Γ, cf. [11] (Lemma 2.1.10)). Consequently, (24) together with (20) yields (21). □

3.3. An Estimate for the Solution

First, we prove an estimate for solutions having support in a sufficiently small neighborhood of a corner point P_j.

Lemma 6. *Let $\mathcal{U}_\varepsilon$ be a sufficiently small neighborhood of the corner point P_j and let β_j satisfy the estimate*

$$2-\frac{N}{2}-\lambda_1(K_j)<\beta_j<\min\Big(1,\mu_1(K_j)+\frac{N}{2}-1\Big),\ \ \beta_j\neq\frac{N}{2}-1.$$

Furthermore, let $\operatorname{Re} s\geq 0$ and $s\neq 0$. Then there exists a constant c independent of u, p and s such that every solution $(u,p)\in V^2_\beta(\Omega)\times(W^1_\beta(\Omega)\cap\overset{\circ}{L}_2(\Omega))$ of the problem (4), (5) with support in $\mathcal{U}_\varepsilon$ satisfies the estimate

$$\|u\|^2_{V^2_\beta(\Omega)}+|s|^2\,\|u\|^2_{V^0_\beta(\Omega)}+\|p\|^2_{W^1_\beta(\Omega)}\leq c\left(\|f\|^2_{V^0_\beta(\Omega)}+\|g\|^2_{V^1_\beta(\Omega)}+\ |s|^2\,\|g\|^2_{(W^1_{-\beta}(\Omega))^*}\right).\tag{25}$$

If $\lambda_1(K_j)=1$ and this is a simple eigenvalue of the pencil $\mathcal{L}_{K_j}$, then the condition on β_j can be replaced by the weaker conditions $\beta_j\neq\pm\frac{N-2}{2}$ and

$$\max\Big(2-\frac{N}{2}-\operatorname{Re}\lambda_1^*(K_j)\,,\,1-\frac{N}{2}-\mu_1(K_j)\Big)<\beta_j<\min\Big(1,\mu_1(K_j)+\frac{N}{2}-1\Big).$$

Proof. By our assumptions, there exist a neighborhood $\mathcal{U}$ of P_j and a C^2-mapping $y=\kappa(x)$ such that $\kappa(P_j)=O$, $\kappa'(P_j)=I$ and $\kappa(\Omega\cap\mathcal{U})=K_j\cap\mathcal{U}'$, where K_j is an angle with opening α_j or a cone and $\mathcal{U}'$ is a neighborhood of the origin. Let A be the matrix with the elements $a_{i,j}=\frac{\partial y_j}{\partial x_i}$. In the coordinates $y=(y_1,\dots,y_N)$, the system (4) takes the form

$$su-L_2u+L_1p=f,\quad -L_1'u=g\ \text{ in } K_j\cap\mathcal{U}_j'\,.$$

Here,

$$L_1p=A\,\nabla_y,\quad L_1'u=\sum_{i,j=1}^N a_{i,j}\,\frac{\partial u_i}{\partial x_j}$$

and L_2 has the form

$$L_2u=\nabla_y u+\nabla_y\nabla_y\cdot u+\sum_{1\leq|\alpha|\leq 2}A_\alpha\partial_y^\alpha u,$$

where $A_\alpha\in C^{|\alpha|-1}(\overline{K}_j\cap\overline{\mathcal{U}'})$ for $1\leq|\alpha|\leq 2$ and $A_\alpha(0)=0$ for $|\alpha|=2$. Let M be the maximum of $|\partial_{y_k}a_{i,j}|$ and $|\partial_y^\gamma A_\alpha|$, $|\gamma|\leq|\alpha|-1$, in $\overline{K}_j\cap\overline{\mathcal{U}'}$. Furthermore, let $\mathcal{U}'_\varepsilon=\{y\in\mathcal{U}':\ |y|<\varepsilon\}$. Since $A_\alpha(0)=0$ for $|\alpha|=2$ we have $|A_\alpha|\leq cM\varepsilon$ in $K_j\cap\mathcal{U}'_\varepsilon$ for $|\alpha|=2$. This implies

$$\|L_2u-\Delta u-\nabla\nabla\cdot u\|_{V^0_{\beta_j}(K_j)}\leq cM\varepsilon\,\|u\|_{V^2_{\beta_j}(K_j)}$$

for $u\in V^2_{\beta_j}(K_j)$, supp $u\subset\mathcal{U}'_\varepsilon$. Since $|a_{i,j}(0)-\delta_{i,j}|\leq 2M\varepsilon$ in $\mathcal{U}'_\varepsilon$, we obtain

$$\|L_1p-\nabla_y p\|_{V^0_{\beta_j}(K_j)}\leq cM\varepsilon\,\|p\|_{W^1_{\beta_j}(K_j)}$$

for $p\in V^1_{\beta_j}(K_j)$, supp $p\subset\mathcal{U}'_\varepsilon$, and analogously,

$$\|L_1'u-\nabla_y\cdot u\|_{V^1_{\beta_j}(K_j)}\leq cM\varepsilon\,\|u\|_{V^2_{\beta_j}(K_j)}$$

for $u\in V^2_{\beta_j}(K_j)$, supp $u\subset\mathcal{U}'_\varepsilon$. We estimate the norm of $L_1'u-\nabla_y\cdot u$ in $(W^1_{-\beta_j}(K_j))^*$. Let $u\in V^2_{\beta_j}(K_j)$, supp $u\subset\mathcal{U}'_\varepsilon$, $u=0$ on the boundary of K_j, and $q\in W^1_{-\beta_j}(K_j)$. If $\beta_j>\frac{N}{2}-1$,

then $q(0)$ exists and $q - q(0) \in V^1_{-\beta_j}(K_j)$ (see [21, Lemma 7.1.3]). Since the integral of $\nabla_y \cdot u$ over K_j is zero , we obtain

$$\begin{aligned}
&\int_{K_j} (L_1' u - \nabla_y \cdot u)\, q\, dy \\
&= \int_{K_j} (q - q(0)) \sum_{i,j} (a_{i,j} - \delta_{i,j}) \partial_{y_j} u_i \, dy + q(0) \int_{K_j} \sum_{i,j} a_{i,j}\, \partial_{y_j} u_i \, dy \\
&= -\int_{K_j} \sum_{i,j} u_i\, \partial_{y_j} \big((a_{i,j} - \delta_{i,j})\, (q - q(0)) \big)\, dy - q(0) \int_{K_j} \sum_{i,j} u_i\, \partial_{y_j} a_{i,j}\, dy.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\Big| \int_{K_j} (L_1' u - \nabla_y \cdot u)\, q\, dy \Big| \\
&\le cM\varepsilon \, \|u\|_{V^0_{\beta_j}(K_j)} \, \|q - q(0)\|_{V^1_{-\beta_j}(K_j)} + cM\, \varepsilon^{1-\beta_j} \, |q(0)| \, \|u\|_{V^0_{\beta_j}(K_j)} \\
&\le cM\varepsilon^{1-\beta_j} \, \|u\|_{V^0_{\beta_j}(K_j)} \, \|q\|_{W^1_{-\beta_j}(K_j)},
\end{aligned}$$

i.e.,

$$\|L_1' u - \nabla_y \cdot u\|_{(W^1_{-\beta_j}(K_j))^*} \le cM\varepsilon^{1-\beta_j} \, \|u\|_{V^0_{\beta_j}(K_j)}.$$

If $\beta_j < \frac{N}{2} - 1$, then $W^1_{-\beta_j}(K_j) = V^1_{-\beta_j}(K_j)$, and we obtain the estimate

$$\|L_1' u - \nabla_y \cdot u\|_{(W^1_{-\beta_j}(K_j))^*} \le cM\varepsilon \, \|u\|_{V^0_{\beta_j}(K_j)}.$$

By Theorems 1 and 2, any pair $(u, p) \in E^2_{\beta_j}(K_j) \times V^1_{\beta_j}(K_j)$, $u = 0$ on ∂K_j, satisfies the estimate

$$\|u\|_{V^2_{\beta_j}(K_j)} + |s| \, \|u\|_{V^0_{\beta_j}(K_j)} + \|p\|_{V^1_{\beta_j}(K_j)} \le c \left(\|f\|_{V^0_{\beta_j}(K_j)} + \|g\|_{V^1_{\beta_j}(K_j)} + |s| \, \|g\|_{(W^1_{-\beta_j}(K_j))^*} \right),$$

where $f = su - \Delta u - \nabla \nabla \cdot u + \nabla p$ and $g = -\nabla u$. Here, c is independent of u, p and s. If $\operatorname{supp}(u, p) \subset \mathcal{U}_\varepsilon'$ and ε is sufficiently small, then we can replace f and g by $su - L_2 u + L_1 p$ and $-L_1' u$, respectively. This proves the estimate (25) for solutions of the problem (4), (5) with support in $\mathcal{U}_\varepsilon = \kappa^{-1}(\mathcal{U}_\varepsilon')$. □

Now we prove the main result of this section.

Theorem 3. *Suppose that* $\operatorname{Re} s \ge 0$, $s \ne 0$, $f \in V^0_\beta(\Omega)$ *and* $g \in V^1_\beta(\Omega)$, *where the components* β_j *of* β *satisfy the conditions of Lemma 6. Furthermore, we assume that* g *satisfies the condition* (16). *Then there exists a unique solution* $(u, p) \in V^2_\beta(\Omega) \times (W^1_\beta(\Omega) \cap \overset{\circ}{L}_2(\Omega))$ *of the problem* (4), (5). *Moreover, there exists a positive number* γ_0 *such that the solution* (u, p) *satisfies the estimate* (25) *for* $\operatorname{Re} s \ge 0$, $|s| > \gamma_0$. *Here, the constant* c *is independent of* f, g *and* s.

Proof. The existence and uniqueness of a solution $(u, p) \in V^2_\beta(\Omega) \times (W^1_\beta(\Omega) \cap \overset{\circ}{L}_2(\Omega))$ follows from Lemma 3. We prove the estimate (25).

Let ζ_j, $j = 1, \dots, n$, be smooth ($\in C^2$) cut-off functions with supports in sufficiently small neighborhoods of the corner points P_j which are equal to one near P_j. Furthermore, let ζ_{n+1} be such that $\zeta_1 + \cdots + \zeta_{n+1} = 1$ in Ω. Then the vector function $\zeta_j(u, p)$ satisfies the equations

$$(s - \Delta)(\zeta_j u) - \nabla \nabla \cdot (\zeta_j u) + \nabla(\zeta_j p) = F_j, \quad -\nabla \cdot (\zeta_j u) = G_j \ \text{ in } \Omega,$$

where $F_j = \zeta_j(f + \Delta u + \nabla\nabla\cdot u) - \Delta(\zeta_j u) - \nabla\nabla\cdot(\zeta_j u) + p\nabla\zeta_j$ and $G_j = \zeta_j g - u\cdot\nabla\zeta_j$. Furthermore, $\zeta_j u = 0$ on Γ. By Lemma 6, there is the estimate

$$\begin{aligned}&\|\zeta_j u\|_{V^2_\beta(\Omega)} + |s|\,\|\zeta_j u\|_{V^0_\beta(\Omega)} + \|\nabla(\zeta_j p)\|_{V^0_\beta(\Omega)}\\ &\le c\left(\|F_j\|_{V^0_\beta(\Omega)} + \|G_j\|_{V^1_\beta(\Omega)} + |s|\,\|G_j\|_{(W^1_{-\beta}(\Omega))^*}\right)\end{aligned} \tag{26}$$

for $j = 1,\dots,n$. The function ζ_{n+1} is zero in a neighborhood of any corner point $P_1,\dots,P_n$. Using existence and uniqueness results for the Dirichlet problem for the nonstationary Stokes system in domains with smooth boundaries (see [23] (Theorem 3.1)), one can prove the estimate (26) with the additional term $\|\zeta_{n+1}u\|_{L_2(\Omega)}$ on the right hand side for $j = n+1$. Here, we refer to [15] (Lemma 2.2) and [16] (Lemma 2.6) (in [15], the norm of g in W^{-1} must be replaced by the norm in $(W^1)^*$). Obviously, one can omit the norm of $\zeta_{n+1}u$ in $L_2(\Omega)$ on the right hand side if $|s|$ is large enough. Thus, the estimate (26) is valid for $j = 1,\dots,n+1$. One can easily verify the estimate

$$\begin{aligned}&\|F_j\|_{V^0_\beta(\Omega)} + \|G_j\|_{V^1_\beta(\Omega)} + |s|\,\|G_j\|_{(W^1_{-\beta}(\Omega))^*} \le \|\zeta_j f\|_{V^0_\beta(\Omega)} + \|\zeta_j g\|_{V^1_\beta(\Omega)}\\ &\quad + |s|\,\|\zeta_j g\|_{(W^1_{-\beta}(\Omega))^*} + c\left(\|u\|_{W^1(\Omega_j)} + \|p\|_{L_2(\Omega_j)} + |s|\,\|u\cdot\nabla\zeta_j\|_{(W^1_{-\beta}(\Omega))^*}\right),\end{aligned} \tag{27}$$

where $\Omega_j = \Omega \cap \operatorname{supp}\nabla\zeta_j$ has a positive distance to the set $\mathcal{P}$ of the corner points. By Ehrling's lemma, there is the inequality

$$\|u\|_{W^1(\Omega_j)} \le \varepsilon\,\|u\|_{W^2(\Omega_j)} + c(\varepsilon)\,\|u\|_{L_2(\Omega_j)} \le c_1\,\varepsilon\,\|u\|_{V^2_\beta(\Omega)} + c_1\,c(\varepsilon)\,\|u\|_{V^0_\beta(\Omega)}$$

with an arbitrarily small positive ε. We estimate the norm of $u\cdot\nabla\zeta_j$ in $(W^1_{-\beta}(\Omega))^*$. Let $q \in W^1_{-\beta}(\Omega)$. Then

$$\begin{aligned}\int_\Omega squ\cdot\nabla\zeta_j\,dx &= \int_\Omega (f - \nabla g + \Delta u - \nabla p)\cdot(q\nabla\zeta_j)\,dx = \int_\Omega (f-\nabla g)\cdot(q\nabla\zeta_j)\,dx\\ &\quad - \int_\Omega\Big(\sum_{i=1}^N \nabla u_i\cdot\nabla(q\partial_{x_i}\zeta_j) + p\,\nabla\cdot(q\nabla\zeta_j)\Big)\,dx + \int_\Gamma\Big(\frac{\partial u}{\partial n}\cdot(q\nabla\zeta_j) - pq\nabla\zeta_j\cdot n\Big)\,d\sigma\end{aligned}$$

and, consequently,

$$\begin{aligned}&\|su\cdot\nabla\zeta_j\|_{(W^1_{-\beta}(\Omega))^*}\\ &\le c\left(\|f-\nabla g\|_{V^0_\beta(\Omega)} + \|u\|_{W^1(\Omega_j)} + \|p\|_{L_2(\Omega_j)} + \Big\|\frac{\partial u}{\partial n}\cdot\nabla\zeta_j\Big\|_{L_2(\Gamma)} + \Big\|p\,\frac{\partial\zeta_j}{\partial n}\Big\|_{L_2(\Gamma)}\right).\end{aligned}$$

Here,

$$\Big\|\frac{\partial u}{\partial n}\cdot\nabla\zeta_j\Big\|_{L_2(\Gamma)} + \Big\|p\,\frac{\partial\zeta_j}{\partial n}\Big\|_{L_2(\Gamma)} \le c\left(\|u\|_{W^{1+\sigma}(\Omega_j)} + \|p\|_{W^\sigma(\Omega_j)}\right)$$

with an arbitrary $\sigma \in (\frac{1}{2}, 1)$ (the constant c depends on σ but not on u and p). Using Ehrling's lemma, we obtain

$$\begin{aligned}&\Big\|\frac{\partial u}{\partial n}\cdot\nabla\zeta_j\Big\|_{L_2(\Gamma)} + \Big\|p\,\frac{\partial\zeta_j}{\partial n}\Big\|_{L_2(\Gamma)}\\ &\le \varepsilon\left(\|u\|_{V^2_\beta(\Omega)} + \|p\|_{W^1_\beta(\Omega)}\right) + c(\varepsilon)\left(\|u\|_{V^0_\beta(\Omega)} + \|p\|_{L_2(\Omega)}\right).\end{aligned}$$

It remains to estimate the norm of p in $L_2(\Omega)$. Let γ_j be real numbers, $\max(0, \beta_j) < \gamma_j < \min\left(1, \frac{N}{2} - 1 + \mu_1(K_j)\right)$ for $j = 1, \ldots, n$. By Lemma 5, p satisfies the estimate

$$\begin{aligned} \|p\|_{L_2(\Omega)} &\leq c \, \|p\|_{V^0_{\gamma-1}(\Omega)} \\ &\leq c' \left(\|f\|_{V^0_\gamma(\Omega)} + \|g\|_{V^1_\gamma(\Omega)} + |s| \, \|g\|_{(W^1_{-\gamma}(\Omega))^*} + \|Du\|_{V^0_{\gamma-1/2}(\Gamma)} \right). \end{aligned}$$

Let ε' be an arbitrarily small positive number. Since $V^{1/2}_\beta(\Gamma) \subset V^0_{\beta-1/2}(\Gamma)$ and $\beta_j < \gamma_j$ for all j, one can choose a subset Γ' of Γ with positive distance to $\mathcal{P}$ such that

$$\begin{aligned} \|Du\|_{V^0_{\gamma-1/2}(\Gamma)} &\leq \varepsilon' \, \|Du\|_{V^{1/2}_\beta(\Gamma)} + c(\varepsilon') \, \|Du\|_{L_2(\Gamma')} \\ &\leq \varepsilon' \, \|Du\|_{V^1_\beta(\Omega)} + c(\varepsilon') \left(\varepsilon'' \, \|u\|_{V^2_\beta(\Omega)} + c_1(\varepsilon'') \, \|u\|_{V^0_\beta(\Omega)} \right), \end{aligned}$$

where ε'' can be chosen arbitrarily small. Hence,

$$\|p\|_{L_2(\Omega)} \leq c \left(\|f\|_{V^0_\gamma(\Omega)} + \|g\|_{V^1_\gamma(\Omega)} + |s| \, \|g\|_{(W^1_{-\gamma}(\Omega))^*} \right) + \varepsilon' \, \|u\|_{V^2_\beta(\Omega)} + C(\varepsilon') \, \|u\|_{V^0_\beta(\Omega)},$$

where ε' can be chosen arbitrarily small. Thus, the estimates (26), (27) together with the above estimates for the W^1-norm of u, the L_2-norm of p on Ω_j and the norm of $u \cdot \nabla \zeta_j$ lead to the estimate

$$\begin{aligned} &\|u\|_{V^2_\beta(\Omega)} + |s| \, \|u\|_{V^0_\beta(\Omega)} + \|p\|_{W^1_\beta(\Omega)} \leq c \left(\|f\|_{V^0_\beta(\Omega)} + \|g\|_{V^1_\beta(\Omega)} + |s| \, \|g\|_{(W^1_{-\beta}(\Omega))^*} \right) \\ &+ \frac{1}{2} \left(\|u\|_{V^2_\beta(\Omega)} + \|p\|_{W^1_\beta(\Omega)} \right) + C \, \|u\|_{V^0_\beta(\Omega)} \end{aligned}$$

with constants c and C independent of u, p and s. For $|s| > 2C$, the inequality (25) holds. □

4. The Time-Dependent Problem on Ω

Now, we consider the problem (1), (2). We employ Theorem 3 in order to obtain an existence and uniqueness result for solutions in weighted Sobolev spaces.

4.1. Weighted Sobolev Spaces in $\Omega \times (0, T)$

Let $0 < T \leq \infty$ and let l be a nonnegative integer. Then $L_2(0, T; V^l_\beta(\Omega))$ is defined as the space of all functions (vector functions) on $\Omega \times (0, T)$ with finite norm

$$\|\mathfrak{u}\|_{L_2(0,T;V^l_\beta(\Omega))} = \left(\int_0^T \|\mathfrak{u}(\cdot, t)\|^2_{V^l_\beta(\Omega)} \, dt \right)^{1/2}.$$

Furthermore, let $W^{2,1}_\beta(\Omega \times (0, T))$ be the space of all functions (vector functions) $\mathfrak{u}(x, t)$ on $\Omega \times (0, T)$ with finite norm

$$\|\mathfrak{u}\|_{W^{2,1}_\beta(\Omega\times(0,T))} = \left(\int_0^T \left(\|\mathfrak{u}(\cdot, t)\|^2_{V^2_\beta(\Omega)} + \|\partial_t \mathfrak{u}(\cdot, t)\|^2_{V^0_\beta(\Omega)} \right) dt \right)^{1/2}.$$

Analogously, $W^{1,1}_\beta(\Omega \times (0, T))$ is defined as the space of all functions (vector functions) on $\Omega \times (0, T)$ with finite norm

$$\|\mathfrak{u}\|_{W^{1,1}_\beta(\Omega\times(0,T))} = \left(\int_0^T \left(\|\mathfrak{u}(\cdot, t)\|^2_{V^1_\beta(\Omega)} + \|\partial_t \mathfrak{u}(\cdot, t)\|^2_{(W^1_{-\beta}(\Omega))^*} \right) dt \right)^{1/2}.$$

The subspaces $\overset{\circ}{W}{}^{2,1}_\beta(\Omega \times (0, T))$ and $\overset{\circ}{W}{}^{1,1}_\beta(\Omega \times (0, T))$ are defined by the additional condition $\mathfrak{u}(x, 0) = 0$ for $x \in \Omega$.

For $T = \infty$, it makes sense to introduce function spaces with the additional weight function $e^{-\gamma t}$. We define the spaces $\overset{\circ}{W}{}^{l,1}_{\beta}(\Omega \times \mathbb{R}_+, e^{-\gamma t})$ for $l = 1, 2$ and $L_{2,-\gamma}(0, \infty; V^l_\beta(\Omega))$ as the sets of functions $\mathfrak{u} = \mathfrak{u}(x,t)$ such that $e^{-\gamma t}\mathfrak{u} \in \overset{\circ}{W}{}^{l,1}_{\beta}(\Omega \times (0,\infty))$ and $e^{-\gamma t}\mathfrak{u} \in L_2(0,\infty; V^l_\beta(\Omega))$, respectively. The space $\overset{\circ}{W}{}^{l,1}_{\beta}(\Omega \times \mathbb{R}_+, e^{-\gamma t})$ is provided with the norm

$$\|\mathfrak{u}\|_{W^{l,1}_\beta(\Omega\times\mathbb{R}_+, e^{-\gamma t})} = \|e^{-\gamma t}\mathfrak{u}\|_{W^{l,1}_\beta(\Omega\times(0,\infty))}.$$

Analogously, the norm in $L_{2,-\gamma}(0,\infty; V^l_\beta(\Omega))$ is defined.

We consider the Laplace transforms. Let $H_\beta(\Omega,\gamma)$ be the space of holomorphic functions $u(x,s)$ for $\operatorname{Re} s > \gamma$ with values in $V^0_\beta(\Omega)$ for which the norm

$$\|u\|_{H_\beta(\Omega,\gamma)} = \sup_{\sigma>\gamma}\Big(\frac{1}{i}\int_{\operatorname{Re} s=\sigma}\|u(\cdot,s)\|^2_{V^0_\beta(\Omega)}\,ds\Big)^{1/2}$$

is finite. The spaces $H^l_\beta(\Omega,\gamma)$, $l = 1, 2$, are the sets of holomorphic functions $u(x,s)$ for $\operatorname{Re} s > \gamma$ with values in $V^l_\beta(\Omega)$ for which the norms

$$\|u\|_{H^1_\beta(\Omega,\gamma)} = \sup_{\sigma>\gamma}\Big(\frac{1}{i}\int_{\operatorname{Re} s=\sigma}\big(\|u(\cdot,s)\|^2_{V^1_\beta(\Omega)} + |s|^2\|u(\cdot,s)\|^2_{(W^1_{-\beta}(\Omega))^*}\big)\,ds\Big)^{1/2}$$

and

$$\|u\|_{H^2_\beta(\Omega,\gamma)} = \sup_{\sigma>\gamma}\Big(\frac{1}{i}\int_{\operatorname{Re} s=\sigma}\big(\|u(\cdot,s)\|^2_{V^2_\beta(\Omega)} + |s|^2\|u(\cdot,s)\|^2_{V^0_\beta(\Omega)}\big)\,ds\Big)^{1/2}$$

are finite. The proof of the following lemma is essentially the same as for nonweighted spaces in [24] (Theorem 8.1). It is based on Plancherel's theorem for the Laplace transform (see, e.g., [25] (Formula (1.5.5))).

Lemma 7. *Let γ be a positive number. Then the Laplace transform realizes isomorphisms between the spaces $L_{2,-\gamma}(0,\infty; V^0_\beta(\Omega))$ and $\overset{\circ}{W}{}^{l,1}_{\beta}(\Omega\times\mathbb{R}_+, e^{-\gamma t})$ on one side and $H_\beta(\Omega,\gamma)$ and $H^l_\beta(\Omega,\gamma)$, $l = 1, 2$, on the other side.*

4.2. Solvability in $\overset{\circ}{W}{}^{2,1}_{\beta}(\Omega\times\mathbb{R}_+, e^{-\gamma t}) \times L_{2,-\gamma}(0,\infty; W^1_\beta(\Omega))$

Using Theorem 3, we can easily prove the following theorem.

Theorem 4. *Suppose that $\mathfrak{f} \in L_{2,-\gamma}(0,\infty; V^0_\beta(\Omega))$ and $\mathfrak{g} \in \overset{\circ}{W}{}^{1,1}_{\beta}(\Omega\times\mathbb{R}_+, e^{-\gamma t})$, where the components β_j of β satisfy the conditions of Lemma 6, and $\gamma \geq \gamma_0$ with a sufficiently large positive number γ_0. Furthermore, we assume that $\mathfrak{g}$ satisfies the condition*

$$\int_\Omega \mathfrak{g}(x,t)\,dx = 0 \ \text{ for almost all } t. \tag{28}$$

Then there exists a uniquely determined solution $(\mathfrak{u},\mathfrak{p})$ of the problem (1)–(3) satisfying the estimate

$$\|\mathfrak{u}\|_{W^{2,1}_\beta(\Omega\times\mathbb{R}_+, e^{-\gamma t})} + \|\mathfrak{p}\|_{L_{2,-\gamma}(\mathbb{R}_+; W^1_\beta(\Omega))} \leq c\left(\|\mathfrak{f}\|_{L_{2,-\gamma}(\mathbb{R}_+; V^0_\beta(\Omega))} + \|\mathfrak{g}\|_{W^{1,1}_\beta(\Omega\times\mathbb{R}_+, e^{-\gamma t})}\right) \tag{29}$$

and the condition (10). The constant c is independent of γ for $\gamma \geq \gamma_0$.

Proof. Let $f \in H_\beta(\Omega,\gamma)$ and $g \in H^1_\beta(\Omega,\gamma)$ be the Laplace transforms of $\mathfrak{f}$ and $\mathfrak{g}$. If $|s| \geq \operatorname{Re} s > \gamma_0$, then there exists a unique solution $(u,p) \in V^2_\beta(\Omega) \times (W^1_\beta(\Omega) \cap \overset{\circ}{L}_2(\Omega))$ of the problem (4), (5) satisfying the estimate (25) (see Theorem 3). Integrating over the line

$\operatorname{Re} s = \sigma$ and taking the supremum with respect to $\sigma > \gamma$, we obtain the estimate (29) for the inverse Laplace transforms $\mathfrak{u}, \mathfrak{p}$ of u, p. Obviously, the pair $(\mathfrak{u}, \mathfrak{p})$ is a solution of the problem (1)–(3). The uniqueness of the solution follows directly from Theorem 3. □

4.3. Solvability in a Finite t-Interval

Theorem 5. *Suppose that* $\mathfrak{f} \in L_2(0,T;V^0_\beta(\Omega))$ *and* $\mathfrak{g} \in \overset{\circ}{W}{}^{1,1}_\beta(\Omega \times (0,T))$, *where the components* β_j *of* β *satisfy the conditions of Lemma 6. Furthermore, we assume that* $\mathfrak{g}$ *satisfies the condition* (28). *Then there exists a uniquely determined solution* $(\mathfrak{u}, \mathfrak{p})$ *of the problem* (1)–(3) *satisfying the estimate*

$$\|\mathfrak{u}\|_{W^{2,1}_\beta(\Omega\times(0,T))} + \|\mathfrak{p}\|_{L_2(0,T;W^1_\beta(\Omega))} \le c\left(\|\mathfrak{f}\|_{L_2(0,T;V^0_\beta(\Omega))} + \|\mathfrak{g}\|_{W^{1,1}_\beta(\Omega\times(0,T))}\right) \tag{30}$$

and the condition (10).

Proof. Let $\mathfrak{F} \in L_2(0,\infty;V^0_\beta(\Omega))$ and $\mathfrak{G} \in \overset{\circ}{W}{}^{1,1}_\beta(\Omega \times \mathbb{R}_+)$ be extensions of $\mathfrak{f}$ and $\mathfrak{g}$ to $\Omega \times \mathbb{R}_+$ such that the integral of $\mathfrak{G}(\cdot,t)$ over Ω is zero for all t and the estimates

$$\|\mathfrak{F}\|_{L_2(0,\infty;V^0_\beta(\Omega))} \le c\,\|\mathfrak{f}\|_{L_2(0,T;V^0_\beta(\Omega))}, \quad \|\mathfrak{G}\|_{W^{1,1}_\beta(\Omega\times\mathbb{R}_+)} \le c\,\|\mathfrak{g}\|_{W^{1,1}_\beta(\Omega\times(0,T))}$$

are satisfied. Obviously, $\mathfrak{F} \in L_{2,-\gamma}(0,\infty;V^0_\beta(\Omega))$ and $\mathfrak{G} \in \overset{\circ}{W}{}^{1,1}_\beta(\Omega \times \mathbb{R}_+, e^{-\gamma t})$ for arbitrary $\gamma \ge 0$. By Theorem 4, there exist a solution $(\mathfrak{u}, \mathfrak{p})$ of the problem (1)–(3) satisfying the estimate (29) with $\mathfrak{F}, \mathfrak{G}$ instead of $\mathfrak{f}, \mathfrak{g}$ and arbitrary $\gamma \ge \gamma_0$. Then

$$\begin{aligned}
\|\mathfrak{u}\|_{W^{2,1}_\beta(\Omega\times(0,T))} + \|\mathfrak{p}\|_{L_2(0,T;W^1_\beta(\Omega))} &\le e^{\gamma T}\left(\|\mathfrak{u}\|_{W^{2,1}_\beta(\Omega\times\mathbb{R}_+,e^{-\gamma t})} + \|\mathfrak{p}\|_{L_{2,-\gamma}(0,\infty;W^1_\beta(\Omega))}\right)\\
&\le c' e^{\gamma T}\left(\|\mathfrak{F}\|_{L_{2,-\gamma}(\mathbb{R}_+;V^0_\beta(\Omega))} + \|\mathfrak{G}\|_{W^{1,1}_\beta(\Omega\times\mathbb{R}_+,e^{-\gamma t})}\right)\\
&\le c\, e^{\gamma T}\left(\|\mathfrak{f}\|_{L_2(0,T;V^0_\beta(\Omega))} + \|\mathfrak{g}\|_{W^{1,1}_\beta(\Omega\times(0,T))}\right).
\end{aligned}$$

By Theorem 4, the constant c is independent of γ. We show that the solution $(\mathfrak{u}, \mathfrak{p})$ depends only on $\mathfrak{f}$ and $\mathfrak{g}$ on $\Omega \times (0,T)$ but not on the extensions $\mathfrak{F}$ and $\mathfrak{G}$. Let $(\mathfrak{F}', \mathfrak{G}')$ be another extension of $(\mathfrak{f}, \mathfrak{g})$, and let $(\mathfrak{u}', \mathfrak{p}')$ be the corresponding solution. Then Theorem 4 yields

$$\begin{aligned}
&\|\mathfrak{u} - \mathfrak{u}'\|_{W^{2,1}_\beta(\Omega\times\mathbb{R}_+,e^{-\gamma t})} + \|\mathfrak{p} - \mathfrak{p}'\|_{L_{2,-\gamma}(0,\infty;W^1_\beta(\Omega))}\Big)\\
&\le c\left(\|\mathfrak{F} - \mathfrak{F}'\|_{L_{2,-\gamma}(\mathbb{R}_+;V^0_\beta(\Omega))} + \|\mathfrak{G} - \mathfrak{G}'\|_{W^{1,1}_\beta(\Omega\times\mathbb{R}_+,e^{-\gamma t})}\right)\\
&\le c\, e^{-\gamma T}\left(\|\mathfrak{F} - \mathfrak{F}'\|_{L_2(0,\infty;V^0_\beta(\Omega))} + \|\mathfrak{G} - \mathfrak{G}'\|_{W^{1,1}_\beta(\Omega\times\mathbb{R}_+)}\right)
\end{aligned}$$

for $\gamma > \gamma_0$, where c is independent of γ. Since $e^{\gamma(T-\varepsilon-t)} \ge 1$ for $t \le T - \varepsilon$, we have

$$\|\mathfrak{u} - \mathfrak{u}'\|_{W^{2,1}_\beta(\Omega\times(0,T-\varepsilon))} \le e^{\gamma(T-\varepsilon)}\,\|\mathfrak{u} - \mathfrak{u}'\|_{W^{2,1}_\beta(\Omega\times\mathbb{R}_+,e^{-\gamma t})}$$

and

$$\|\mathfrak{p} - \mathfrak{p}'\|_{L_2(0,T-\varepsilon;W^1_\beta(\Omega))} \le e^{\gamma(T-\varepsilon)}\,\|\mathfrak{p} - \mathfrak{p}'\|_{L_{2,-\gamma}(0,\infty;W^1_\beta(\Omega))}$$

Consequently,

$$\begin{aligned}
&\|\mathfrak{u} - \mathfrak{u}'\|_{W^{2,1}_\beta(\Omega\times(0,T-\varepsilon))} + \|\mathfrak{p} - \mathfrak{p}'\|_{L_2(0,T-\varepsilon;W^1_\beta(\Omega))}\\
&\le c\, e^{-\gamma\varepsilon}\left(\|\mathfrak{F} - \mathfrak{F}'\|_{L_2(0,\infty;V^0_\beta(\Omega))} + \|\mathfrak{G} - \mathfrak{G}'\|_{W^{1,1}_\beta(\Omega\times\mathbb{R}_+)}\right),
\end{aligned}$$

where c is independent of γ for $\gamma > \gamma_0$. If we let γ tend to infinity, we conclude that $\mathfrak{u} = \mathfrak{u}'$ and $\mathfrak{p} = \mathfrak{p}'$ in $\Omega \times (0, T-\varepsilon)$ for arbitrary $\varepsilon > 0$. This proves the theorem. □

4.4. Nonzero Initial Conditions

Let $u_0 \in V^1_\beta(\Omega)$, $u_0 = 0$ on Γ. Then there exists a function $\mathfrak{v} \in W^{2,1}_\beta(\Omega \times (0,T))$ satisfying the conditions $\mathfrak{v}(x,0) = u_0(x)$ for $x \in \Omega$, $\mathfrak{v}(x,t) = 0$ for $x \in \Gamma$, $0 < t < T$, and the estimate

$$\|\mathfrak{v}\|_{W^{2,1}_\beta(\Omega\times(0,T))} \le c\,\|u_0\|_{V^1_\beta(\Omega)} \tag{31}$$

with a constant c independent of u_0 (cf. [18] (Proposition 3.1)). Thus, the following result can be deduced from Theorem 5.

Corollary 1. *Suppose that* $\mathfrak{f} \in L_2(0,T;V^0_\beta(\Omega))$, $\mathfrak{g} \in W^{1,1}_\beta(\Omega\times(0,T))$ *and* $u_0 \in V^1_\beta(\Omega)$, $u_0 = 0$ *on* Γ. *We assume that the components* β_j *of* β *satisfy the inequalities* (7), $\mathfrak{g}$ *satisfies the condition* (28) *and* $-\nabla\cdot u_0(x) = \mathfrak{g}(x,0)$ *for* $x \in \Omega$. *Then there exists a uniquely determined solution* $(\mathfrak{u},\mathfrak{p})$ *of the problem* (1), (2) *with the initial condition* $\mathfrak{u}(x,0) = u_0(x)$ *satisfying the estimate*

$$\|\mathfrak{u}\|_{W^{2,1}_\beta(\Omega\times(0,T))} + \|\mathfrak{p}\|_{L_2(0,T;W^1_\beta(\Omega))} \le c\left(\|\mathfrak{f}\|_{L_2(0,T;V^0_\beta(\Omega))} + \|\mathfrak{g}\|_{W^{1,1}_\beta(\Omega\times(0,T))} + \|u_0\|_{V^1_\beta(\Omega)}\right) \tag{32}$$

and the condition (10).

Proof. Let $\mathfrak{v} \in W^{2,1}_\beta(\Omega\times(0,T)$ be a vector function satisfying the conditions $\mathfrak{v}(x,0) = u_0(x)$ for $x \in \Omega$, $\mathfrak{v}(x,t) = 0$ for $x \in \Gamma$, $0 < t < T$, and the estimate (31). Then

$$\int_\Omega \big(\mathfrak{g}(x,t) + \nabla\cdot\mathfrak{v}(x,t)\big)\,dx = 0 \ \text{ for almost all } t$$

$\mathfrak{g}(x,0) + \nabla\cdot\mathfrak{v}(x,0) = \mathfrak{g}(x,0) + \nabla\cdot u_0(x) = 0$ for $x \in \Omega$, i.e., $\mathfrak{g} + \nabla\cdot\mathfrak{v} \in \overset{\circ}{W}{}^{1,1}_\beta(\Omega\times(0,T))$. By Theorem 5, there exists a unique solution $(\mathfrak{w},\mathfrak{p}) \in \overset{\circ}{W}{}^{2,1}_\beta(\Omega\times(0,T)) \times L_2(0,T;W^1_\beta(\Omega))$ of the problem

$$\begin{aligned}
&\frac{\partial\mathfrak{w}}{\partial t} - \Delta\mathfrak{w} - \nabla\nabla\cdot\mathfrak{w} + \nabla\mathfrak{p} = \mathfrak{f} - \mathfrak{v}_t + \Delta\mathfrak{v} + \nabla\nabla\cdot\mathfrak{v},\\
&-\nabla\cdot\mathfrak{w} = \mathfrak{g} + \nabla\cdot\mathfrak{v} \ \text{ in } Q_T = \Omega\times(0,T),\\
&\mathfrak{w}(x,t) = 0 \ \text{ for } x \in \Gamma,\ 0 < t < T, \quad \mathfrak{w}(x,0) = 0 \ \text{ for } x \in \Omega
\end{aligned}$$

satisfying the condition (10). Then the pair $(\mathfrak{u},\mathfrak{p}) = (\mathfrak{v}+\mathfrak{w},\mathfrak{p})$ is a solution of the problem (1), (2) with the initial condition $\mathfrak{u}(x,0) = u_0$ and satisfies (32). □

4.5. Necessity of the Condition on β

The conditions $\beta_j > 2 - \frac{N}{2} - \operatorname{Re}\lambda_1(K_j)$ and $\beta_j > 2 - \frac{N}{2} - \operatorname{Re}\lambda_1^*(K_j)$, respectively, for all j in Theorems 4 and 5 are also necessary for the existence und uniqueness of solutions $(u,p) \in V^2_\beta(\Omega) \times W^1_\beta(\Omega)$ of the Dirichlet problem for the stationary Stokes system. In contrast to the stationary Stokes system, the existence and uniqueness theorem for the instationary Stokes system requires the additional condition $\beta_j > 1 - \frac{N}{2} - \mu_1(K_j)$. We show this in the following lemma for the case $N = 2$.

Lemma 8. *Suppose that* $N = 2$ *and* $1 - \operatorname{Re}\Lambda(\alpha_j) < \beta_j < -\frac{\pi}{\alpha_j}$ *for at least one* j. *Then the assertion of Theorem 5 is not true.*

Proof. Suppose that $1 - \operatorname{Re}\Lambda(\alpha_1) < \beta_1 < -\frac{\pi}{\alpha_1}$. This condition implies in particular that $\operatorname{Re}\Lambda(\alpha_1) > 1 + \frac{\pi}{\alpha_1}$, i. e., $\alpha_1 < \alpha_0 \approx 0.355\,\pi$.

For simplicity, we assume that P_1 is the origin and that Ω coincides with the angle K in the neighborhood $B_\rho = \{x : |x| < \rho\}$ of P_1. Let $v \in V^2_\beta(\Omega)$, $v|_\Gamma = 0$, $q \in V^1_\beta(\Omega)$, $\int_\Omega q(x)\,dx = 0$, and $\operatorname{supp}(v,q) \subset \overline{K} \cap B_\varepsilon$, where $\varepsilon < \rho$. Furthermore, let $\delta > 0$ and $\operatorname{Re} s \geq 0$. Obviously, the functions $\mathfrak{u}(x,t) = e^{\delta^2 st} v(x)$ and $\mathfrak{p}(x,t) = e^{\delta^2 st} q(x)$ are in $W^{2,1}_\beta(\Omega \times (0,1))$ and $L_2(0,1; V^1_\beta(\Omega))$, respectively. Furthermore,

$$\frac{\partial \mathfrak{u}}{\partial t} - \Delta \mathfrak{u} - \nabla\nabla\cdot\mathfrak{u} + \nabla\mathfrak{p} = e^{\delta^2 st}\phi(x), \quad -\nabla\cdot\mathfrak{u} = e^{\delta^2 st}\psi(x) \ \text{ in } Q = \Omega\times(0,1),$$

where $\phi = \delta^2 s\, v - \Delta v - \nabla\nabla\cdot v + \nabla q$ and $\psi = -\nabla\cdot v$. Moreover $\mathfrak{u}$ satisfies the initial condition $\mathfrak{u}(x,0) = v(x)$. We assume that the assertion of Theorem 4 is true. Then $(\mathfrak{u},\mathfrak{p})$ satisfies the estimate (32). Since $\int_0^1 e^{2\delta^2 t \operatorname{Re} s}\,dt \geq 1$ for $\operatorname{Re} s \geq 0$, it follows that

$$\begin{aligned} &\|v\|_{V^2_{\beta_1}(K)} + \delta^2|s|\,\|v\|_{V^0_{\beta_1}(K)} + \|q\|_{V^1_{\beta_1}(K)} \\ &\leq c\left(\|\phi\|_{V^0_{\beta_1}(K)} + \|\psi\|_{V^1_{\beta_1}(K)} + \delta^2|s|\,\|\psi\|_{(V^1_{-\beta_1}(K))^*} + \|v\|_{V^1_{\beta_1}(K)}\right) \end{aligned} \tag{33}$$

with a constant c independent of v, q, δ and s. Here, we used the estimate

$$\|q\|_{V^1_{\beta_1}(K)} \leq c\,\|\nabla q\|_{V^0_{\beta_1}(K)} \ \text{ for } q \in V^1_{\beta_1}(K),\ \beta_1 \neq 0,$$

and Remark 1. If $\operatorname{supp} v \in \overline{K} \cap B_\varepsilon$ and ε is small enough, then

$$\|v\|_{V^1_{\beta_1}(K)} \leq \varepsilon\,\|v\|_{V^2_{\beta_1}(K)}$$

and we can omit the norm of v in $V^1_{\beta_1}(K)$ on the right-hand side of (33).

Now, let $u \in V^2_{\beta_1}(K)$, $u|_{\partial K} = 0$, $p \in V^1_{\beta_1}(K)$, $\int_K p(x)\,dx = 0$ and $\operatorname{supp}(u,p) \subset \overline{K} \cap B_R$. Furthermore, let $f = su - \Delta u - \nabla\nabla\cdot u + \nabla p$ and $g = -\nabla\cdot u$. We define $v(x) = u(\delta x)$, $q(x) = \delta\, p(\delta x)$, $\phi(x) = \delta^2 f(\delta x)$ and $\psi(x) = \delta\, g(\delta x)$, where $\delta = \varepsilon^{-1}R$ and ε is sufficiently small. Then $\operatorname{supp}(v,q) \subset \overline{K} \cap B_\varepsilon$ and (v,q) satisfies the equations

$$\delta^2 s\, v - \Delta v - \nabla\nabla\cdot v + \nabla q = \phi, \quad -\nabla\cdot v = \psi \ \text{ in } K.$$

Thus (v,q) satisfies (33). Using the equalities

$$\|v\|_{V^2_{\beta_1}(K)} = \delta^{1-\beta_1}\,\|u\|_{V^2_{\beta_1}(K)}, \quad \|v\|_{V^0_{\beta_1}(K)} = \delta^{-1-\beta_1}\,\|u\|_{V^0_{\beta_1}(K)}$$

and analogous equalities for the norms of ϕ and ψ, we obtain the estimate

$$\|u\|_{V^2_{\beta_1}(K)} + |s|\,\|u\|_{V^0_{\beta_1}(K)} + \|p\|_{V^1_{\beta_1}(K)} \leq c\left(\|f\|_{V^0_{\beta_1}(K)} + \|g\|_{V^1_{\beta_1}(K)} + |s|\,\|g\|_{(V^1_{-\beta_1}(K))^*}\right). \tag{34}$$

Here, c is independent of R and s.

It remains to show the estimate (34) for all $u \in V^2_{\beta_1}(K)$, $u|_{\partial K} = 0$, and $p \in V^1_{\beta_1}(K)$ without the condition $C = \int_K p(x)\,dx = 0$. Assume first that $u(x) = 0$ and $p(x) = 0$ for $|x| > 1$. Then

$$|C| = \left|\int_K p(x)\,dx\right| \leq c\,\|p\|_{V^1_{\beta_1}(K)}.$$

Let $\zeta \in C^\infty_0(\overline{K})$, $\operatorname{supp}\zeta \subset B_2 \backslash B_1$, and $\int_K \zeta(x)\,dx = 1$. We define $\zeta_n(x) = n^{-2}\,\zeta(n^{-1}x)$. Then

$$\|\zeta_n\|_{V^1_{\beta_1}(K)} = n^{\beta_1 - 2}\,\|\zeta\|_{V^1_{\beta_1}(K)} \quad \text{and} \quad \int_K \zeta_n(x)\,dx = 1.$$

Obviously, the function $p_0(x) = p(x) - C\,\zeta_n(x)$ satisfies the condition $\int_K p_0(x)\,dx = 0$. Furthermore,

$$\|p - p_0\|_{V^1_{\beta_1}(K)} = \|C\zeta_n\|_{V^1_{\beta_1}(K)} \le c\,n^{\beta-2}\,\|p\|_{V^1_{\beta_1}(K)}. \tag{35}$$

Hence, the pair (u, p_0) satisfies the estimate (cf. (34))

$$\begin{aligned}&\|u\|_{V^2_{\beta_1}(K)} + |s|\,\|u\|_{V^0_{\beta_1}(K)} + \|p_0\|_{V^1_{\beta_1}(K)}\\ &\le c\left(\|f - \nabla(p - p_0)\|_{V^0_{\beta_1}(K)} + \|g\|_{V^1_{\beta_1}(K)} + |s|\,\|g\|_{(V^1_{-\beta_1}(K))^*}\right),\end{aligned}$$

where $f = su - \Delta u - \nabla\nabla\cdot u + \nabla p$ and $g = -\nabla\cdot u$. This together with (35) implies (34) if n is sufficiently large. Arguing as above, one obtains the same estimate (with the same constant c) for functions $u \in V^2_{\beta_1}(K)$, $u|_{\partial K} = 0$, and $p \in V^1_{\beta_1}(K)$ with support in $\overline{K} \cap B_R$. Hence, this estimate is valid for arbitrary $(u, p) \in V^2_{\beta_1}(K) \times V^1_{\beta_1}(K)$, $u|_{\partial K} = 0$. But this contradicts [17] (Lemma 4.15). □

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