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A New Odd Beta Prime-Burr X Distribution with Applications to Petroleum Rock Sample Data and COVID-19 Mortality Rate

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Abstract: In this article, we pioneer a new Burr X distribution using the odd beta prime generalized (OBP-G) family of distributions called the OBP-Burr X (OBPBX) distribution. The density function of this model is symmetric, left-skewed, right-skewed, and reversed-J, while the hazard function is monotonically increasing, decreasing, bathtub, and N-shaped, making it suitable for modeling skewed data and failure rates. Various statistical properties of the new model are obtained, such as moments, moment-generating function, entropies, quantile function, and limit behavior. The maximum-likelihood-estimation procedure is utilized to determine the parameters of the model. A Monte Carlo simulation study is implemented to ascertain the efficiency of maximum-likelihood estimators. The findings demonstrate the empirical application and flexibility of the OBPBX distribution, as showcased through its analysis of petroleum rock samples and COVID-19 mortality data, along with its superior performance compared to well-known extended versions of the Burr X distribution. We anticipate that the new distribution will attract a wider readership and provide a vital tool for modeling various phenomena in different domains.

Keywords: T-X family; odd beta prime-G family; Burr X distribution; quantile function; maximum likelihood estimation; Monte Carlo simulation; infectious disease; COVID-19; public health

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1. Introduction

Statistical distributions play a crucial role in statistics, probability theory, and various domains of science and engineering in describing and predicting the characteristics of natural phenomena. However, the classical probability distributions do not always adequately capture many naturally existing asymmetric data sets [1]. To overcome this limitation, there is a need to enhance the flexibility of existing distributions in modeling data, particularly in reliability analysis, where the hazard rate can have various shapes. Consequently, in recent years, there have been several attempts to introduce new distributions that can generalize the established models, thereby improving their flexibility for modeling a variety of real-world problems, including problems in engineering, finance, health, biology, agriculture, demography, economics, the environment, and many other fields [2,3]. Many studies have also focused on introducing new families of probability distributions by generalizing and extending existing families of distributions that are more flexible and useful in explaining diverse natural phenomena [4,5].

Here, we explore a variety of classes of distributions that have been established and defined in recent statistical literature. Reference [6] developed and defined a general class of distribution that was used to study the beta-G class by incorporating two shape parameters into the logit of a beta random variable. Reference [7] proposed the Kumaraswamy-G through the addition of two parameters to the Kumaraswamy distribution, which is based on the unit interval (0, 1). Reference [8] suggested the modified Kies family and extensively discussed its special case. Furthermore, reference [9] presented the Marshall–Olkin–Weibull-H class, which competes with the established classes. Reference [10] constructed the odd extended exponential-G family generated from the extended exponential distribution. Reference [11] defined the exponentiated odd exponential half logistic-G power series class through the compounding of an established class. Another notable contribution is the odd JCA-G family studied by reference [12], whose main mathematical properties were explored. Reference [13] developed the new Kumaraswamy-G family as an alternative to a class of distributions. Reference [14] pioneered the new Kumaraswamy-G class through a one-parameter exponential-logarithmic transformation. Reference [15] produced the Topp-Leone Gompertz-G class via the extended form of the distribution. Reference [16] introduced the new-G class, illustrating its accuracy in modeling COVID-19 mortality rates. Reference [17] developed the type II exponentiated half-logistic Gompertz-G class, studying some of its main features. Moreover, reference [18] initiated the reciprocal Burr X-G class derived from the T-X approach, describing its capability to fit virtually all types of failure rates. Reference [19] offered the odd Muth-G class using the T-X method and derived various of its properties. Reference [20] presented the exponentiated-exponentiated Weibull-X class, which has the ability to model any complex data. Reference [21] pioneered the gamma-G class. Reference [22] established and examined the Weibull-G class by adding two additional parameters to the odds ratio. These generalized distributions have the well-known feature that they contain more parameters and are used to model complicated data structures. According to reference [23], four-parameter distributions have proven to be most suitable for modeling the majority of practical applications. Hence, in this paper, we aim to explore the efficacy of such four-parameter distributions within the context of petroleum rock samples and COVID-19 mortality data.

In the statistical literature, reference [24] established twelve distinct versions of distributions for modeling data. Among these, the Burr X and Burr XII distributions have gained significant popularity [25]. Interestingly, the Burr X model has recently attracted considerable attention and holds a special place in the statistical literature. This model has been extensively used in engineering studies to model the lifetime of natural processes and has proven to provide reasonable fits for data modeling in medicine, biology, agriculture, engineering, survival analysis, finance, and other fields; see References [26–32]. The classical Burr X model has a nonmonotone hazard rate, contrasting with the gamma, Weibull, and generalized exponential distributions [33]. This peculiar feature makes it highly flexible in fitting different hazard functions, making it suitable for modeling infectious diseases such as COVID-19. The Burr X distribution provides practitioners with a desirable tool for capturing the complexity of real-world data, delivering enhanced predictions in areas of application.

Consider X has the Burr X distribution with a shape parameter $(\alpha > 0)$ and scale parameter $(\beta > 0)$, if its cumulative distribution function (CDF) $T(x;\alpha,\beta)$ is given by

$$T(x;\alpha,\beta) = \left(1 - \exp -(\beta x)^2\right)^{\alpha}; \quad x > 0, \tag{1}$$

The probability density function (PDF) $t(x;\alpha,\beta)$ corresponding to $T(x;\alpha,\beta)$ is given by

$$t(x;\alpha,\beta) = 2\alpha\beta^2 x\left(\exp -(x\beta)^2\right)\left[1 - \exp -(\beta x)^2\right]^{\alpha-1}; \quad x > 0. \tag{2}$$

Furthermore, other families of distributions contributed to the recent literature on the extension of the Burr X model to improve its fitting power for modeling various natural processes. For references, see some well-known generalized forms of the Burr X distributions, such as the gamma-Burr X [34], Weibull-Burr X [35], Marshall–Olkin Burr X [36], type I half-logistic Burr X [37], power Burr X [38], transmuted Burr X [39], exponentiated Burr X [40], exponentiated Weibull Burr X [41], sine-exponentiated Weibull Burr X [42], Kumaraswamy Burr X [43], Maxwell Burr X [44], and (recently) the Kavya–Manoharan Burr X distributions [25]. Table 1offers a comprehensive overview of the various extensions of the Burr X distribution and their extensive utilization across diverse fields of application.

Recently, reference [45] introduced the odd beta prime generalized (OBP-G) class based on the approach developed in [46], in which they introduced a general class to develop a noble class of distributions called the transformed–transformer (T-X) class. The general T-X methodology is derived by employing any continuous random variable as a generator. Reference [47] introduced and studied the OBP-logistic distribution. For more recent generalized versions of distributions using the T-X class, we refer to the Maxwell–Weibull distribution [48], the Maxwell exponential distribution [49], the alpha power exponentiated distribution [50], the Weibull exponential distribution [51], the generalized odd inverse exponential Weibull distribution [52], the odd Perks–Weibull distribution [53], the odd Chen Burr III distribution [54], the odd F-Weibull distribution [55], the odd Weibull inverse Topp–Leone distribution [56], the generalized OBP-Weibull distribution [57], the new generalized logarithmic Weibull distribution [58], the odd Lomax–Dagum distribution [59], and the OBP-Fréchet distribution [60]. The interested reader can explore more about the T-X class in [61].

Based on the literature highlighted above regarding the generalization of the Burr X model, we notice that the development of noble generalized versions of this model is a hot topic in contemporary statistics, particularly within parametric statistical modeling. This article presents another generalized version of the Burr X model. The new Burr X distribution is developed by implementing a novel OBP-G class of distributions defined in [47]. The OBP-G class is proven to generate more flexible compound distributions. For instance, reference [45] provided a flexible OBP-exponential distribution through this class. In particular, the CDF $Q(x;\theta,\lambda,\delta)$ of the OBP-G family for the random variable X is written as

$$Q(x;\theta,\lambda,\delta)=\frac{B_{\frac{T(x,\delta)}{1-T(x,\delta)}}(\theta,\lambda)}{B(\theta,\lambda)}\quad;\quad x>0,\ \theta,\lambda>0, \tag{3}$$

where δ is a vector parameter. The odds ratio $\frac{T(x,\delta)}{1-T(x,\delta)}$ implies that for every baseline distribution $T(x,\delta)$, we have a different distribution $Q(x;\theta,\lambda,\delta)$. In addition, the generator $\frac{T(x,\delta)}{1-T(x,\delta)}$ meets the following prerequisites:

1. $\frac{T(x,\delta)}{1-T(x,\delta)}\in[c,d]$ for $0<c<d<\infty$.
2. $\frac{T(x,\delta)}{1-T(x,\delta)}$ is differentiable and monotonically non-decreasing.
3. $\frac{T(x,\delta)}{1-T(x,\delta)}\Rightarrow c$ as $x\Rightarrow 0$ but $\frac{T(x,\delta)}{1-T(x,\delta)}\Rightarrow d$ as $x\Rightarrow\infty$.

The corresponding PDF $q(x;\theta,\lambda,\delta)$ is given by:

$$q(x;\theta,\lambda,\delta)=\frac{t(x,\delta)}{B(\theta,\lambda)\{1-T(x,\delta)\}^2}\frac{\left\{\frac{T(x,\delta)}{1-T(x,\delta)}\right\}^{\theta-1}}{\left\{1+\left(\frac{T(x,\delta)}{1-T(x,\delta)}\right)\right\}^{\theta+\lambda}};\quad x>0,\theta,\lambda>0, \tag{4}$$

where $t(x,\delta)$ is the baseline PDF.

The present paper aims to introduce a fresh generalization of the Burr X model using the OBP-G class. This novel generalized distribution proposed in this paper is referred to as the odd beta prime-Burr X (OBPBX) distribution. The proposed model offers greater flexibility compared to other versions of the Burr X models. The proposed model enables the modelling of various kinds of data, such as COVID-19 mortality data. The overall structure of this article is outlined in Figure 1.

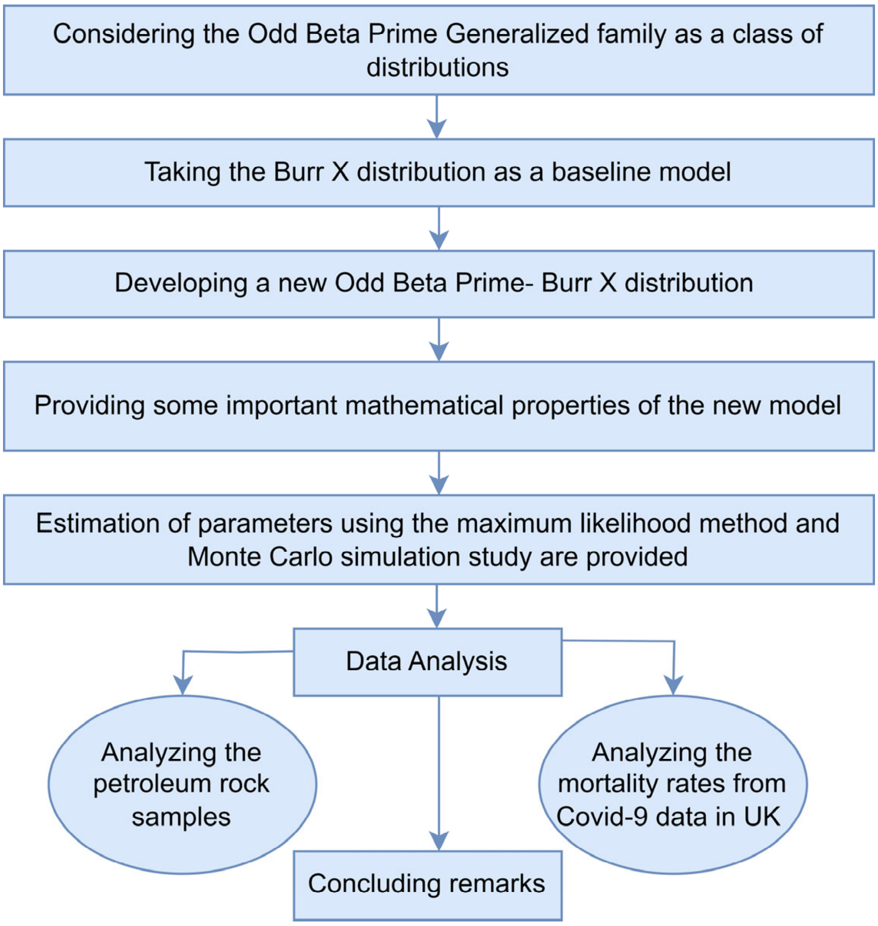

Figure 1. A complete visual summary of the paper.

The motivation and justification for introducing the OBPBX distribution are as follows:

(i) to improve the general performance of the classical Burr X distribution, which can handle skewed and heavy-tailed data sets when compared to other competitive models;
(ii) to improve a model kurtosis that is more flexible in contrast to the referenced models;
(iii) to develop a model with different shapes, such as left-skewed, right-skewed, reversed-J, and symmetric;
(iv) to introduce a new model with various hazard functions that can capture increasing, decreasing, bathtub, and concave-convex shapes; and
(v) to consistently offer superior fit in comparison to well-established, generated distributions for the same baseline distribution.

For these reasons, we proposed the odd beta prime Burr X (OBPBX) distribution, made up of the combination of the odd beta prime family of distributions proposed by [47] and the Burr X distribution.

The subsequent sections of this paper are as follows. Section 2 presents the relevant literature. Section 3 expresses the CDF and PDF of the OBPBX model. Section 4 derives various desirable features from the OBPBX distribution. Section 5 estimates the parameters of the model through the maximum-likelihood-estimation technique. Section 6 provides the competence of the estimates through Monte Carlo simulation. Section 7 investigates the usefulness of the new model for petroleum rock sample data and COVID-19 mortality rates. Finally, Section 8 provides concluding remarks.

2. Overview of the Previous Studies

In this section, some of the studies that have been done in the last decade are reviewed. Probabilistic models serve as valuable tools for describing and predicting real-world phenomena. Over the past few decades, a range of generalized distributions have found widespread application in data modeling across various fields. Many researchers have explored different techniques to enhance the flexibility and adaptability of the classical Burr X distribution. These extended models have been useful in modeling diverse real-world datasets from a variety of disciplines, including engineering, physics, finance, medicine, failure time analysis, reliability assessment, survival analysis, and hydrology, and even in the context of COVID-19.

Table 1 provides a comprehensive overview of the significant literature concerning the numerous extensions of the Burr X distribution, showcasing their practical applications in modeling real-world data sets. It is noteworthy that in the works of the aforementioned authors who delved into extending the Burr X distribution, none ventured into utilizing these distributions for the modeling of geological data. However, the main aim of this article is to develop a novel version of the Burr X distribution with the hope of providing an adequate fit to the petroleum rock samples and COVID-19 data sets. COVID-19 is a new viral disease caused by the severe acute respiratory syndrome coronavirus-2 (SARS-CoV-2) that generated a global epidemic. Several mathematical and statistical models have been proposed to explain the path of the pandemic [62]. It is important to point out that the characteristics of the pandemic data can fluctuate, making it unable to fit classical probability distributions in all cases. Similarly, fluctuations in rock sample data are expected due to the inherent heterogeneity of rock formations. Careful data analysis, appropriate model selection, and consideration of geological context are essential when modeling probability distributions for rock properties. As a result, we have developed the OBPBX distribution to model the geological data and the mortality rate of the infectious COVID-19 disease in the United Kingdom.

The new Burr X model was developed specifically with the aim of accurately capturing certain characteristics of both petroleum rock samples and COVID-19 data. Therefore, this research introduces a novel Burr X distribution, providing a competitive alternative to the established variants of Burr X distributions present in the statistical literature. In summary, the main contributions and new findings of this study include the following:

1. pioneering the use of the extended Burr X distribution for modelling geological data, a discipline that had not previously been investigated within this framework;
2. introducing a new Burr X distribution version developed to offer a good fit for petroleum rock samples and COVID-19 data sets; and
3. applying the new distribution to different kinds of data, because the three shape parameters can control the tail of data.

Table 1. Relevant literature review.

Year	Model	Application	Authors
2023	Odd beta prime Burr X distribution	Geological and COVID-19 data	New
	Unit–power Burr X distribution	COVID-19 data	[63]
	Exponentiated beta Burr X distribution	Failure time data	[64]
	Maxwell Burr X distribution	COVID-19 data	[44]
	Kavya–Manoharan Burr X distribution	Survival, waiting time, and financial data	[25]
	Exponentiated Kavya–Manoharan Burr X model	Medical and survival data	[65]
2022	Exponentiated Weibull Burr X distribution	Survival data	[41]
	Gamma odd Burr X Weibull distribution	Taxes revenue and repair time data	[66]
	Type I half-logistic Burr X Weibull distribution	COVID-19 data	[67]
	Burr X logistic exponential distribution	Engineering and physics data	[68]
	Kumaraswamy Burr X distribution	Physics, engineering, and medical data	[43]
	Generalized Burr X Lomax distribution	Failure time	[69]
	Sine-exponentiated Weibull Burr X distribution	Food chain, wholesale, and physics data	[42]

Table 1. *Cont.*

Year	Model	Application	Authors
2021	Exponentiated Burr X distribution	Physics data	[40]
	Transmuted Burr X exponential distribution	Physics and failure time data	[70]
	Truncated Burr X exponential distribution	Actuarial and financial data	[71]
	Odd log-logistic Burr-X normal distribution	Agricultural and medical data	[72]
	Type I half-logistic Burr X Lomax distribution	COVID-19 data	[73]
	Type I half-logistic Burr X exponential distribution	COVID-19 data	[73]
	Type I half-logistic Burr X Rayleigh distribution	COVID-19 data	[73]
2020	Transmuted Burr X distribution	Reliability data	[39]
	Power Burr X distribution	Physics and hydrological data	[38]
	Poisson Burr X inverse Rayleigh distribution	Physics and engineering data	[74]
	Odd Burr–Burr X distribution	Failure time, medical, survival, and physics data	[75]
2019	Odd log-logistic Burr X distribution	Reliability data	[76]
	Type I half-logistic Burr X distribution	Physics data	[37]
	Burr X Fréchet distribution	Survival data	[77]
	Zero truncated Poisson Burr X Weibull distribution	Reliability and medical data	[78]
	Poisson Burr X Weibull distribution	Failure time and survival data	[79]
	Burr X exponentiated exponential distribution	Physics data	[80]
	Burr X exponentiated Weibull distribution	Failure time and survival data	[81]
	Burr X Nadarajah Haghighi distribution	Hydrological data	[82]
	Marshall–Olkin exponentiated Burr X distribution	Physics data	[83]
2018	Exponentiated generalized Burr X distribution	Physics data	[84]
	Burr X exponentiated exponential distribution	Failure time and survival data	[85]
	Burr X Lomax distribution	Survival data	[86]
	Beta Kumaraswamy Burr X distribution	Physics and medical data	[87]
2017	Weibull Burr X distribution	Reliability data	[88]
	Burr X Lomax distribution	Medical data	[32]
	Burr X Pareto distribution	Financial time series data	[89]
	Burr X exponentiated Fréchet distribution	Survival and hydrological data	[90]
	Marshall–Olkin extended Burr X distribution	Physics data	[91]
	Marshall–Olkin Burr X Lomax distribution	Physics, hydrological, and survival data	[36]
	Weibull Burr X distribution	Hydrological and failure time data	[35]
2016	Gamma Burr X distribution	Failure data	[34]
	Beta Burr X distribution	Physics data	[92]

3. Development of Odd Beta Prime-Burr X Distribution

Here, the novel four-parameter odd beta prime-Burr X (OBPBX) model is presented.

Let $T(x)$ represent the CDF of the Burr X distribution. By substituting (1) in (3), we get the CDF of the OBPBX distribution, expressed in (5) as

$$Q(x;\theta,\lambda,\alpha,\beta)=\frac{B_{\frac{\left(1-e^{-(x\beta)^2}\right)^{\alpha}}{\left(1-\left(1-e^{-(x\beta)^2}\right)^{\alpha}\right)}}(\theta,\lambda)}{B(\theta,\lambda)};\quad x>0,\theta>0,\lambda>0,\alpha>0,\beta>0, \tag{5}$$

where $\theta,\lambda,\alpha>0$ are shape parameters and $\beta>0$ is the scale parameter.

The associated PDF of (5) is presented as follows:

$$q(x;\theta,\lambda,\alpha,\beta)=\frac{2\alpha\beta^2 x e^{-(x\beta)^2}\left\{1-e^{-(x\beta)^2}\right\}^{\theta\alpha-1}}{B(\theta,\lambda)\left\{1-\left(1-e^{-(x\beta)^2}\right)^{\alpha}\right\}^{\theta+1}}\times\left[1+\left(\frac{\left(1-e^{-(x\beta)^2}\right)^{\alpha}}{1-\left(1-e^{-(x\beta)^2}\right)^{\alpha}}\right)\right]^{-(\theta+\lambda)};\quad x>0. \tag{6}$$

We denote the random variable defined in (6) as $X\sim OBPBX(\theta,\lambda,\alpha,\beta)$.

Figure 2a–d shows several different PDF shapes of the new OBPBX model for distinct parameter combinations. The different-shaped behaviors of the OBPBX PDF include (a) symmetric, (b) left-skewed, (c) right-skewed, and (d) reversed-J shaped. It is evident that the OBPBX distribution provides crucial characteristics that could make it more suitable for modeling COVID-19 as well as petroleum rock sample data sets. The long tails, skewness, and unimodality, which are frequently seen in such data sets, can be efficiently modeled by this distribution. In addition, the distribution can capture the unpredictable nature of virus dissemination. These are the justifications for why the OBPBX distribution could be regarded as a more accurate model for analyses of petroleum rock sample and COVID-19 mortality rate data sets.

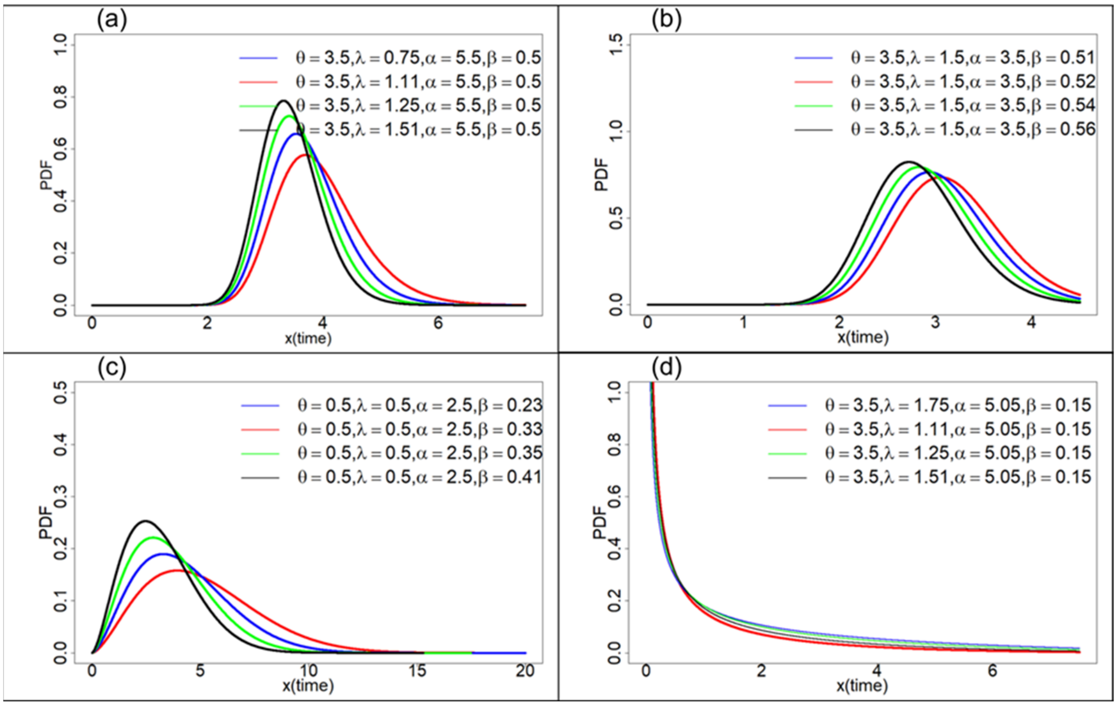

Figure 2. Plots showing the PDF of the OBPBX model for distinct parameter combinations. Subfigures (**a**–**d**) illustrate that the PDF of the OBPBX model has symmetric, left-skewed, right-skewed, and reversed-J shaped density functions.

The survival function $S(x;\theta,\lambda,\alpha,\beta)$ of X is given by

$$S(x;\theta,\lambda,\alpha,\beta) = 1 - \frac{B_{\frac{(1-\exp-(\beta x)^2)^\alpha}{(1-(1-\exp-(\beta x)^2)^\alpha)}}(\theta,\lambda)}{B(\theta,\lambda)}. \tag{7}$$

The hazard function $h(x;\theta,\lambda,\alpha,\beta)$ of X is expressed by

$$h(x;\theta,\lambda,\alpha,\beta) = \frac{\frac{2\alpha\beta^2 x e^{-(x\beta)^2}\left\{1-e^{-(x\beta)^2}\right\}^{\theta\alpha-1}}{B(\theta,\lambda)\left\{1-\left(1-e^{-(x\beta)^2}\right)^\alpha\right\}^{\theta+1}} \times \left[1+\left(\frac{\left(1-e^{-(x\beta)^2}\right)^\alpha}{1-\left(1-e^{-(x\beta)^2}\right)^\alpha}\right)\right]^{-(\theta+\lambda)}}{1-\frac{B_{\frac{(1-\exp-(\beta x)^2)^\alpha}{(1-(1-\exp-(\beta x)^2)^\alpha)}}(\theta,\lambda)}{B(\theta,\lambda)}}; \quad x > 0. \tag{8}$$

Figure 3a–d displays different hazard function shapes of the OBPBX model for distinct parameter combinations. The figure shows different hazard function shapes, such as (a) decreasing, (b) increasing, (c) bathtub-shaped, and (d) N-shaped behavior. These structures of the OBPBX hazard function allow it to model any kind of data that is characterized by various hazard shapes, including COVID-19 mortality rates and petroleum rock samples.

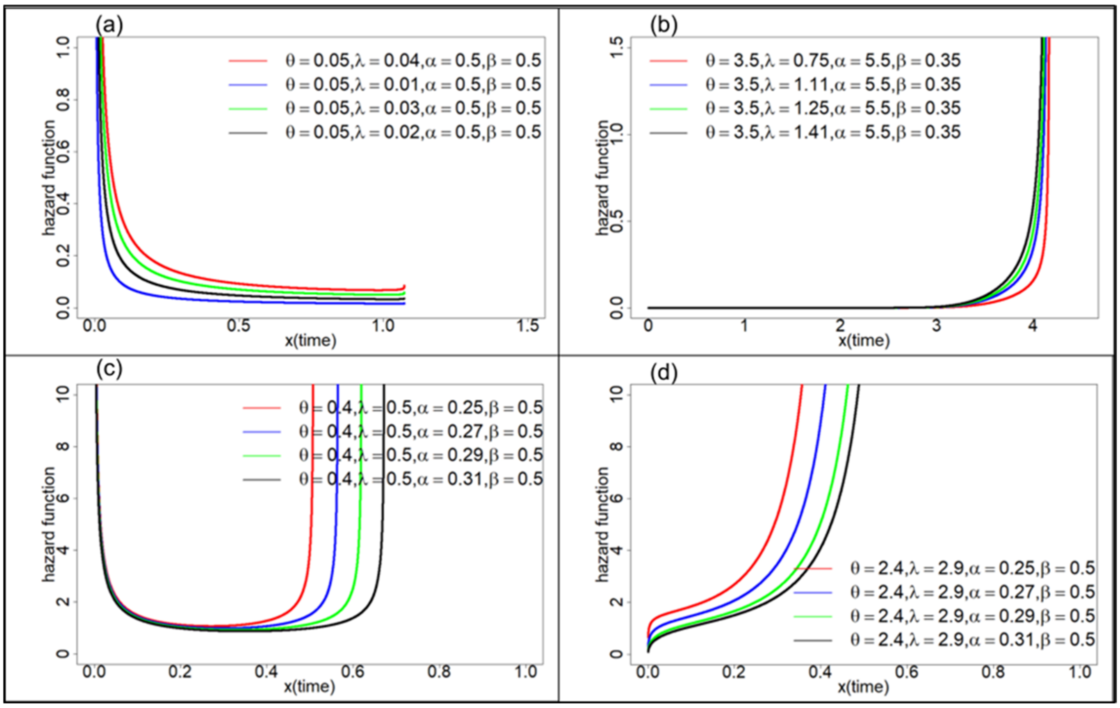

Figure 3. Plots illustrating the hazard function of the OBPBX model for various parameter combinations. Subfigures (**a**–**d**) demonstrate that the hazard functions of the OBPBX model are decreasing, increasing, bathtub, and N-shaped hazard functions.

Linear Representations

Here, we provide the expression of the PDF of the OBPBX model in terms of the linear representation. This expression is utilized to derive some desirable features of the OBPBX model. The expression of the PDF is derived by considering the binomial expansion given by

$$(1+w)^{-n} = \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} w^i. \tag{9}$$

Inserting (9) into (6), we obtain

$$q(x;\theta,\lambda,\alpha,\beta) = \frac{2\alpha\beta^2 x e^{-(x\beta)^2}\left\{1-e^{-(\beta x)^2}\right\}^{\theta\alpha-1}}{B(\theta,\lambda)\left\{1-\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right\}^{\theta+1}} \sum_{i=1}^{\infty} (-1)^i \binom{\theta+\lambda+i-1}{i} \frac{\left\{\left(1-e^{-(\beta x)^2}\right)\right\}^{\theta\alpha-1+\alpha i}}{\left\{1-\left(1-e^{-(\beta x)^2}\right)^{\alpha}\right\}^{\theta+1+i}}. \tag{10}$$

By repeating the binomial expansion in (9) into (10), (10) becomes

$$q(x;\theta,\lambda,\alpha,\beta) = 2\alpha\beta^2 x e^{-(x\beta)^2} \sum_{i,j=1}^{\infty} (-1)^i \binom{\theta+\lambda+i-1}{i} \binom{\theta+i+j}{j} \left\{1-e^{-(\beta x)^2}\right\}^{\alpha(\theta+i+j)-1}. \tag{11}$$

According to reference [93], the generalized binomial for any real number $y > 0$ is expressed as

$$(1-m)^{y-1} = \sum_{i=1}^{\infty} (-1)^i \binom{y}{i} m^i, \tag{12}$$

where $|m| < 1$.

Applying (12) to (11), we obtain the linear representation for the OBPBX distribution as

$$q(x;\theta,\lambda,\alpha,\beta) = 2\alpha\beta^2 x \sum_{i,j,k=1}^{\infty} \Omega_{i,j,k} e^{-(1+k)(\beta x)^2}, \tag{13}$$

where $\Omega_{i,j,k} = (-1)^{i+k} \begin{pmatrix} \theta+\lambda+i-1 \\ i \end{pmatrix} \begin{pmatrix} \theta+i+j \\ j \end{pmatrix} \begin{pmatrix} \alpha(\theta+i+j)-1 \\ k \end{pmatrix}$.

Equation (13) can be applied directly to derive several features of the OBPBX distribution.

4. Statistical Features

In this section, we explore several key statistical features associated with the OBPBX distribution. The derived features include the moments, moment-generating function, Rényi entropy, quantile function, and limit behavior.

4.1. Moments

In this subsection, we focus on deriving the r^{th} moment of the OBPBX model. Moments play a significance role in parametric statistics, especially in applied analysis, as they have been utilized to study several important features, including the mean, median, skewness, and kurtosis. For OBPBX distribution, the expression for the r^{th} moment is given as

$$E(X^r) = \int_0^\infty x^r q(x;\theta,\lambda,\alpha,\beta)dx. \tag{14}$$

By considering the PDF of the OBPBX model defined in (13), (14) becomes

$$E(X^r) = 2\alpha\beta^2 \sum_{i,j,k=1}^{\infty} \Omega_{i,j,k} \int_0^\infty x^{r+1} e^{-(1+k)(\beta x)^2} dx, \tag{15}$$

where $\Omega_{i,j,k} = (-1)^{i+k} \begin{pmatrix} \theta+\lambda+i-1 \\ i \end{pmatrix} \begin{pmatrix} \theta+i+j \\ j \end{pmatrix} \begin{pmatrix} \alpha(\theta+i+j)-1 \\ k \end{pmatrix}$.

Let $z = (1+k)(\beta x)^2$ such that

$$dx = \frac{dz}{2\beta^2 x(1+k)}. \tag{16}$$

Substituting (16) into (15), we have

$$E(X^r) = 2\alpha\beta^2 \sum_{i,j,k=1}^{\infty} \Omega_{i,j,k} \int_0^\infty x^{r+1} e^{-z} \frac{dz}{2\beta^2 x(1+k)}.$$

After some algebra, the r^{th} moment of the OBPBX model is given as

$$E(X^r) = \frac{\alpha}{\beta^r (1+k)^{1+\frac{r}{2}}} \Gamma\left(\frac{r}{2}+1\right) \sum_{i,j,k=1}^{\infty} \Omega_{i,j,k}. \tag{17}$$

Equation (17) is employed to obtain the moment-generating function of the OBPBX model.

4.2. Moment-Generating Function

The moment-generating function (MGF) for the OBPBX model is expressed as

$$E\left(e^{tX}\right) = M_X(X) = \int_0^\infty e^{tx} q(x;\theta,\lambda,\alpha,\beta)dx. \tag{18}$$

By considering the exponential series expansion,

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^l x^l}{l!}.$$

Then, (18) can be written by

$$M_X(X) = \sum_{i=0}^{\infty} \frac{t^l}{l!} \int_0^{\infty} x^l q(x;\theta,\lambda,\alpha,\beta)dx.$$

Adopting [94] and using (17), the MGF for the OBPBX model is obtained as

$$M_X(X) = \sum_{l=0}^{\infty} \frac{t^l}{l!} \frac{\alpha}{\beta^r (1+k)^{1+\frac{r}{2}}} \Gamma\left(\frac{r}{2}+1\right) \sum_{i,j,k=1}^{\infty} \Omega_{i,j,k}, \tag{19}$$

where $\Omega_{i,j,k} = (-1)^{i+k} \binom{\theta+\lambda+i-1}{i} \binom{\theta+i+j}{j} \binom{\alpha(\theta+i+j)-1}{k}$.

4.3. Rényi and q Entropies

Entropy serves as a tool for quantifying the variance or uncertainty of a phenomena and finds applications in several scientific domains, including probability, communication theory, and engineering. Here, we present the Rényi and q-entropies of the OBPBX distribution.

(i) Rényi entropy

The Rényi entropy is given as

$$R_\eta(X) = \frac{1}{1-\eta} \log\left[\int_0^{\infty} q^\eta(x;\theta,\lambda,\alpha,\beta)dx\right];\ \eta > 0,\ \eta \neq 1, \tag{20}$$

where $q(x;\theta,\lambda,\alpha,\beta)$ is the PDF of the OBPBX model expressed in (6). The integrand in (20) is given as

$$q^\eta(x;\theta,\lambda,\alpha,\beta) = \left\{\frac{2\alpha\beta^2}{B(\theta,\lambda)}\right\}^\eta \frac{x^\eta e^{-\eta(x\beta)^2}\left\{1-e^{-(\beta x)^2}\right\}^{\eta(\theta\alpha-1)}}{\left\{1-\left(1-e^{-(\beta x)^2}\right)^\alpha\right\}^{\eta(\theta+1)}} \left[1+\left(\frac{\left(1-e^{-(\beta x)^2}\right)^\alpha}{1-\left(1-e^{-(\beta x)^2}\right)^\alpha}\right)\right]^{-\eta(\theta+\lambda)}. \tag{21}$$

Using binomial expansion, (21) yields

$$q^\eta(x;\theta,\lambda,\alpha,\beta) = \left\{\frac{2\alpha\beta^2}{B(\theta,\lambda)}\right\}^\eta x^\eta e^{-\eta(x\beta)^2} \sum_{m=0}^{\infty} (-1)^m \binom{\eta(\theta+\lambda)+m-1}{m} \frac{\left\{1-e^{-(\beta x)^2}\right\}^{\eta(\theta\alpha-1)+\alpha m}}{\left\{1-\left(1-e^{-(\beta x)^2}\right)^\alpha\right\}^{\eta(\theta+1)+m}}. \tag{22}$$

For simplification, it is expressed as

$$q^\eta(x;\theta,\lambda,\alpha,\beta) = \Bbbk\, x^\eta e^{-\eta(x\beta)^2} \sum_{m,p=0}^{\infty} (-1)^m \binom{\eta(\theta+\lambda)+m-1}{m} \binom{\eta(\theta+\lambda)+m+p-1}{p} \left\{1-e^{-(\beta x)^2}\right\}^{\alpha(\theta\eta+m+p)-\eta}, \tag{23}$$

where $\Bbbk = \left\{\frac{2\alpha\beta^2}{B(\theta,\lambda)}\right\}^\eta$.

By considering the exponential series expansion, (23) becomes

$$q^\eta(x;\theta,\lambda,\alpha,\beta) = \Bbbk x^\eta \sum_{m,p,q=0}^{\infty} \mathbb{F}_{m,p,q} e^{-(\eta+q)(x\beta)^2}, \tag{24}$$

where

$\mathbb{F}_{m,p,q} = (-1)^{m+q} \binom{\eta(\theta+\lambda)+m-1}{m} \binom{\eta(\theta+\lambda)+m+p-1}{p} \binom{\alpha(\theta\eta+m+p)-\eta}{q}$.

Thus, we have

$$R_\eta(X) = \frac{1}{1-\eta}\log\left[\Bbbk \sum_{m,p,q=0}^{\infty} \mathbb{F}_{m,p,q} \int_0^\infty x^\eta e^{-(\eta+q)(x\beta)^2} dx\right].$$

After some algebra, the Rényi entropy for the OBPBX model is found to be

$$R_\eta(X) = \frac{1}{1-\eta}\log\left[\frac{\Bbbk \sum\limits_{m,p,q=0}^{\infty} \mathbb{F}_{m,p,q}}{2\beta^{1+\eta}(\eta+q)^{\frac{1+\eta}{2}}}\Gamma\left(\tfrac{1+\eta}{2}\right)\right]. \tag{25}$$

The primary result of this subsection is (25), and it can be used to derive some kinds of entropies such as q-entropy.

(ii) q-entropy

The q-entropy is expressed as

$$Q_\eta(X) = \frac{1}{\eta-1}\log\left[1 - \int_0^\infty q^\eta(x;\theta,\lambda,\alpha,\beta)dx\right];\ \ \eta > 0,\ \eta \neq 1, \tag{26}$$

By using (25) and (26), we have

$$Q_\eta(X) = \frac{1}{\eta-1}\log\left[1 - \frac{\Bbbk \sum\limits_{m,p,q=0}^{\infty} \mathbb{F}_{m,p,q}}{2\beta^{1+\eta}(\eta+q)^{\frac{1+\eta}{2}}}\Gamma\left(\tfrac{1+\eta}{2}\right)\right].$$

After some simplifications, the q-entropy for the OBPBX distribution is formulated in the following form:

$$Q_\eta(X) = \frac{1}{\eta-1}\log\left[1 - \frac{2^{\eta-1}\alpha^\eta \sum\limits_{m,p,q=0}^{\infty} \mathbb{F}_{m,p,q}}{B(\theta,\lambda)\beta^{1-\eta}(\eta+q)^{\frac{1+\eta}{2}}}\Gamma\left(\tfrac{1+\eta}{2}\right)\right]. \tag{27}$$

4.4. Quantile Function

In probability theory, the quantile function (QF) is utilized to evaluate various characteristics, such as the coefficient of quartile, the coefficient of variation, Bowley's skewness, and Moor's kurtosis.

The QF of the OBPBX model is derived by inverting the CDF of the OBPBX model in (5), as given by

$$\frac{\left(1-e^{-(\beta x)^2}\right)^\alpha}{\left(1-\left(1-e^{-(\beta x)^2}\right)^\alpha\right)} = I^{-1}(u,\theta,\lambda), \tag{28}$$

so that

$$\left(1-e^{-(\beta x)^2}\right)^\alpha = \frac{I^{-1}(u,\theta,\lambda)}{1+I^{-1}(u\,,\,\theta,\lambda)},$$

this gives $e^{-(\beta x)^2} = 1 - \mathbb{S}^{\frac{1}{\alpha}}$, where $\mathbb{S} = \frac{I^{-1}(u\,,\theta,\lambda)}{1+I^{-1}(u\,,\theta,\lambda)}$.

So that

$$x = \frac{\left\{-\log\left(1-\mathbb{S}^{\frac{1}{\alpha}}\right)\right\}^{\frac{1}{2}}}{\beta}. \tag{29}$$

Assuming that U is a uniform random variable on the interval (0, 1), the simulation of the OBPBX random variable is simple. The random variable can be expressed through the inverse transformation approach, which is obtained as

$$X = Q(U) = \frac{\left\{-\log\left(1-\mathbb{S}^{\frac{1}{\alpha}}\right)\right\}^{\frac{1}{2}}}{\beta}. \tag{30}$$

Equation (30) can be applied to simulate experimental samples from the OBPBX model when the parameters θ, λ, α and β are defined.

4.5. Quantile Based on Bowley's Skewness and Moor's Kurtosis

Here, we present the quantile measures based on Bowley's skewness and Moor's kurtosis. By considering the QF defined in (30), Bowley's skewness (B) based on quantile [95] is expressed as

$$B = \frac{Q_{0.75} - 2Q_{0.50} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}.$$

Moor's kurtosis (M) based on octiles is expressed by

$$M = \frac{(Q_{0.875} - Q_{0.625}) + (Q_{0.375} - Q_{0.125})}{Q_{0.75} - Q_{0.25}}.$$

In particular, $Q(0.25)$, $Q(0.50)$, and $Q(0.75)$ are the first, second, and third quartiles of the OBPBX distribution.

4.6. Limit Behavior

Here, we explore the limit behavior of the PDF of the OBPBX model. We are interested to examining how the PDF behaves as $x \to 0$ and $x \to \infty$.

The $\lim\limits_{x\to 0} q(x;\theta,\lambda,\alpha,\beta) = 0$, because $\lim\limits_{x\to 0}\left(1-e^{-(x\beta)^2}\right) = 0$, and the $\lim\limits_{x\to\infty} q(x;\theta,\lambda,\alpha,\beta) = 0$, because $\lim\limits_{x\to\infty}\left(e^{-(x\beta)^2}\right) = 0$.

5. Parameter Estimation

Here, the maximum-likelihood-estimation (MLE) technique is utilized to estimate the unknown parameters of the OBPBX distribution.

Maximum-Likelihood Function

Suppose $x_1, x_2, ..., x_n$ represent experimental samples of size n from the OBPBX model with parameters θ, λ, α and β. The expression of the likelihood function for a vector parameter $\Psi = (\theta, \lambda, \alpha, \beta)^T$ is found as follows:

$$L(\Psi) = \left\{\frac{2\alpha\beta^2}{B(\theta,\lambda)}\right\}^n \prod_{i=1}^{n} \frac{x_i e^{-(x_i\beta)^2}\left\{1-e^{-(\beta x_i)^2}\right\}^{\theta\alpha-1}}{\left\{1-\left(1-e^{-(\beta x_i)^2}\right)^\alpha\right\}^{\theta+1}} \times \prod_{i=1}^{n}\left[1+\left(\frac{\left(1-e^{-(\beta x_i)^2}\right)^\alpha}{1-\left(1-e^{-(\beta x_i)^2}\right)^\alpha}\right)\right]^{-(\theta+\lambda)}. \tag{31}$$

The log-likelihood function of (31) is written by

$$\begin{aligned}\ell(\Psi) = {} & n\log(2) + n\log(\alpha) + 2n\log(\beta) - n\log[B(\theta,\lambda)] + \sum_{i=1}^{n}\log(x_i) \\ & -\beta^2\sum_{i=1}^{n}\left(x_i^2\right) + (\theta\alpha-1)\sum_{i=1}^{n}\log\left\{1-e^{-(\beta x_i)^2}\right\} - (\theta+1)\sum_{i=1}^{n}\log\left\{1-\left(1-e^{-(\beta x_i)^2}\right)^\alpha\right\} \\ & -(\theta+\lambda)\sum_{i=1}^{n}\log\left[1+\left(\frac{\left(1-e^{-(\beta x_i)^2}\right)^\alpha}{1-\left(1-e^{-(\beta x_i)^2}\right)^\alpha}\right)\right].\end{aligned} \tag{32}$$

The MLE of Ψ, say $\hat{\Psi}$, is derived by taking the partial derivative of (32) with respect to parameters θ, λ, α and β as follows:

$$\frac{\partial \ell(\Psi)}{\partial \alpha} = \frac{n}{\alpha} + \theta \sum_{i=1}^{n} \log\left\{1 - e^{-(\beta x_i)^2}\right\} + (1-\lambda) \sum_{i=1}^{n} \left(\frac{\left(1 - e^{-(\beta x_i)^2}\right)^{\alpha} \log\left\{1 - e^{-(\beta x_i)^2}\right\}}{1 - \left(1 - e^{-(\beta x_i)^2}\right)^{\alpha}} \right). \tag{33}$$

$$\begin{aligned} \frac{\partial \ell(\Psi)}{\partial \beta} = {} & \frac{2n}{\beta} + 2\beta \sum_{i=1}^{n} \left(x_i^2\right) + 2\beta(\theta\alpha - 1) \sum_{i=1}^{n} \left(\frac{x_i^2 e^{-(\beta x_i)^2}}{\left\{1 - e^{-(\beta x_i)^2}\right\}} \right) \\ & + (1 - 2\alpha\lambda\beta) \sum_{i=1}^{n} \left(\frac{x_i^2 e^{-(\beta x_i)^2} \left\{1 - e^{-(\beta x_i)^2}\right\}^{\alpha-1}}{1 - \left(1 - e^{-(\beta x_i)^2}\right)^{\alpha}} \right). \end{aligned} \tag{34}$$

$$\frac{\partial \ell(\Psi)}{\partial \theta} = -n\psi(\theta) + n\psi(\theta+\lambda) + \alpha \sum_{i=1}^{n} \log\left\{1 - e^{-(\beta x_i)^2}\right\}. \tag{35}$$

$$\frac{\partial \ell(\Psi)}{\partial \lambda} = -n\psi(\lambda) + n\psi(\theta+\lambda) + \sum_{i=1}^{n} \log\left(1 - \left(1 - e^{-(\beta x_i)^2}\right)^{\alpha}\right), \tag{36}$$

where ψ is the digamma function.

The estimators for the parameters θ, λ, α and β can be obtained by equating (33)–(36) to zero. Although it is tedious to find the system's solutions analytically, computational software, such as the R programming language, MATLAB, Maple, and Python, provides the numerical solutions of the system, using iterative methods.

6. Monte Carlo Simulation Study

Here, we investigate the accuracy of the MLEs for the parameters of the OBPBX distribution through Monte Carlo simulations. The accuracy of the MLEs is assessed for different sample sizes and for selected parameter combinations. We conducted the simulations, which consisted of 1000 replications. The study involved evaluating the performance of MLEs by computing various statistics, such as the mean, biases, and mean squared errors (MSEs), based on two distinct parameter cases: case 1 and case 2. The procedure for the Monte Carlo simulation algorithm (Algorithm 1) is stated in Table 2.

Algorithm 1. Algorithm of Monte Carlo simulation for various sample sizes and selected parameter values.

(1) The simulation is repeated for $N = 1000$ replications based on the QF defined in (30).
(2) Various sample sizes n are considered, such as 15, 25, 50, 75, 100, 150, and 200.
(3) Distinct cases of actual values are employed as follows:

Case I: $\theta = 0.52, \lambda = 1.07, \alpha = 1.25$ and $\beta = 0.03$.
Case II: $\theta = 1.5, \lambda = 0.2, \alpha = 0.75$ and $\beta = 1.5$.

(4) Generate $u_i \sim U(0,1)$, $i = 1, 2, ..., n$.
(5) Determine the random samples from the OBPBX distribution.
(6) Use MLE method to analyze the parameters of the OBPBX model.
(7) Repeat steps 4 to 6 for 1000 replications to obtain the estimates.
(8) Compute different measures, such as the bias and mean square error (MSE) to investigate the precision of the MLEs as follows:

Bias $= \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\Psi}_i - \Psi\right)$ and MSE $= \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\Psi}_i - \Psi\right)^2$, where $\Psi = (\theta, \lambda, \alpha, \beta)^T$.

(9) The results of the analysis were obtained using the R-studio software version 4.2.1. Table 2 presents the summary of the experiment.

Table 2. The simulated MLE, bias, and MSE for the OBPBX distribution for case 1 and case 2.

		Case 1: θ = 0.52, λ = 1.07, α = 1.25, β = 0.03			**Case 2: θ = 1.5, λ = 0.2, α = 0.75, β = 1.5**		
Parameter	n	**Mean**	**Bias**	**MSE**	**Mean**	**Bias**	**MSE**
$\hat{\theta}$	15	0.541404	0.025403	0.002630	1.545356	0.005950	0.008764
	25	0.540392	0.025215	0.002429	1.524563	0.005755	0.008384
	50	0.537420	0.021420	0.002134	1.515567	0.005452	0.007845
	75	0.532419	0.021053	0.002035	1.509634	0.005358	0.005647
	100	0.531406	0.011402	0.000130	1.506465	0.004251	0.002637
	150	0.528139	0.011213	0.000126	1.502745	0.003351	0.000864
	200	0.523918	0.011191	0.000122	1.501351	0.002935	0.000176
$\hat{\lambda}$	15	1.067935	−0.00108	0.004533	0.042320	−0.15761	0.024862
	25	1.068648	−0.00117	0.002436	0.042314	−0.15768	0.022435
	50	1.068823	−0.00119	0.001624	0.042260	−0.15779	0.021534
	75	1.068931	−0.00122	0.000764	0.075674	−0.15946	0.020455
	100	1.069034	−0.00126	0.000663	0.093452	−0.16374	0.019674
	150	1.069532	−0.00132	0.000534	0.142654	−0.16747	0.014868
	200	1.069720	−0.00139	0.000243	0.195432	−0.20125	0.011367
$\hat{\alpha}$	15	1.255833	0.005832	0.093542	0.755199	0.005199	0.009354
	25	1.254828	0.004828	0.054232	0.755200	0.005201	0.008464
	50	1.254245	0.004739	0.023547	0.755197	0.005197	0.005631
	75	1.253837	0.004236	0.008452	0.755201	0.005205	0.003569
	100	1.253132	0.004132	0.005432	0.755201	0.005201	0.002345
	150	1.252833	0.003830	0.003745	0.755204	0.005197	0.000935
	200	1.250829	0.002828	0.009543	0.755205	0.005141	0.000438
$\hat{\beta}$	15	0.225331	0.687564	0.885534	1.348640	0.019592	0.019354
	25	0.225196	0.683885	0.821651	1.404465	0.015593	0.017457
	50	0.201484	0.655484	0.765274	1.426411	0.013588	0.013219
	75	0.182574	0.605392	0.652271	1.436756	0.008592	0.006351
	100	0.129271	0.555271	0.615527	1.446405	0.006594	0.003238
	150	0.102522	0.465486	0.565141	1.468564	0.005597	0.000948
	200	0.092255	0.495570	0.486535	1.494783	0.003599	0.000374

The bias and MSEs of the estimates $\hat{\theta}, \hat{\lambda}, \hat{\alpha}$ and $\hat{\beta}$ for the OBPBX model are given in Table 2. The findings exhibit that both the bias and MSEs decrease as the sample size n increases. Thus, the MLEs effectively estimate the parameters of the OBPBX model, indicating accurate performance.

7. Applications to Petroleum Rock Samples and COVID-19 Mortality Rates

In this section, we examine the significance of the OBPBX model for modeling petroleum rock samples and COVID-19 data sets. To verify its performance, we compare the fit of the OBPBX distribution with other competitive distributions, such as (i) the Weibull-Burr X (WBX) distribution [35], (ii) the exponential generalized-Burr X (EGBX) distribution [94], and (iii) the beta-Burr X (BBX) distribution [96]. To assess the suitability of the competing models, we consider five goodness-of-fit statistics, such as (a) the maximized log-likelihood $\left(\hat{\ell}\right)$, (b) the Akaike information criterion (AIC), (c) the corrected AIC (CAIC), (d) the Bayesian information criterion (BIC), and (e) the Hannan–Quinn information criterion (HQIC). For interested readers regarding these criteria, see [97,98]. We compute the MLEs and the standard errors for the candidate distributions.

7.1. First Data Set: Petroleum Rock Sample Data

The first data set consisted of 48 rock samples extracted from a petroleum reservoir. This data set was analyzed in a study conducted by [99]. These data are presented below:

0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

A descriptive summary of the petroleum rock samples is provided in Table 3. The table indicates that the data set is right-skewed due to a positive skewness coefficient, and the kurtosis value less than three reveals that the data set has a platykurtic distribution. The histogram, kernel density, box, and violin plots for petroleum rock samples are displayed in Figure 4. The histogram and kernel density plots show that the petroleum rock sample data are right-skewed, while the box and violin plots suggest evidence of extreme values. Therefore, the OBPBX model is well-suited for modeling this data set. In addition, the petroleum rock sample data exhibit an increasing failure rate, evident from the shape of the total time on test (TTT) plot depicted in Figure 5. This validates that the shape of the hazard function of the proposed OBPBX model provided in Figure 3 is appropriate for modeling this type of data.

Table 4 highlights the MLEs and their standard errors for the OBPBX model and other competitive models alongside the statistical measures, including the $\left(\hat{\ell}\right)$, AIC, CAIC, BIC, and HQIC for the petroleum rock samples. From Table 4, it is quite clear that the proposed OBPBX model is the most suitable model, as it offered the smallest values for all statistical measures compared to the other competing models.

Figure 6 displays the plots for the petroleum rock sample data, such as (a) empirical and fitted densities and (b) empirical and fitted CDFs. From this figure, it can be inferred that the new model is closely fitted to petroleum rock sample data. Hence, the OBPBX model is the best-fit distribution to model the petroleum rock samples.

Table 3. Descriptive summary of petroleum rock sample data.

Data	Min	Q1	Q3	Median	Mean	Max	Variance	Skewness	Kurtosis
Petroleum	0.090	0.162	0.263	0.199	0.218	0.464	0.007	1.133	0.940

Table 4. MLEs, standard errors (in parentheses), and statistical measures for petroleum rock sample data.

Model	Estimates				Fitted Measures				
	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\ell}$	AIC	CAIC	BIC	HQIC
OBPBX	0.3744 (0.0536)	1.0933 (0.3242)	1.4102 (0.5343)	1.9231 (0.6301)	25.659	−43.318	−42.388	−35.833	−40.490
WBX	0.2535 (0.0342)	0.9017 (0.1425)	0.8647 (0.4326)	0.9985 (0.5362)	16.072	−24.143	−23.213	−16.658	−21.315
EGBX	0.6911 (0.3242)	1.0313 (0.0746)	0.9975 (0.8625)	0.4525 (0.0240)	5.063	−2.126	−1.196	5.359	0.702
BBX	0.5662 (0.2614)	1.0680 (0.5342)	0.6806 (0.4231)	0.9948 (0.2015)	−5.191	−2.383	−1.452	5.102	0.446

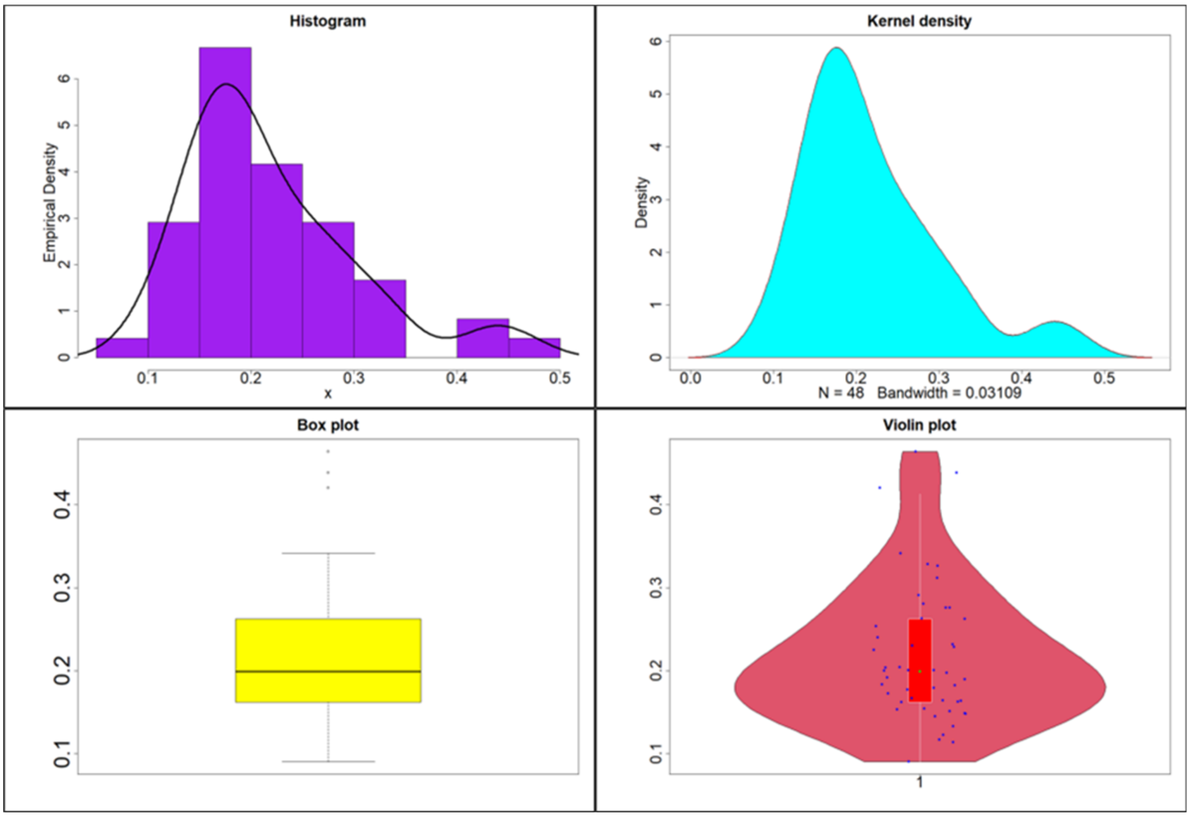

Figure 4. Histogram, kernel density, box plot, and violin plot for the petroleum rock sample data.

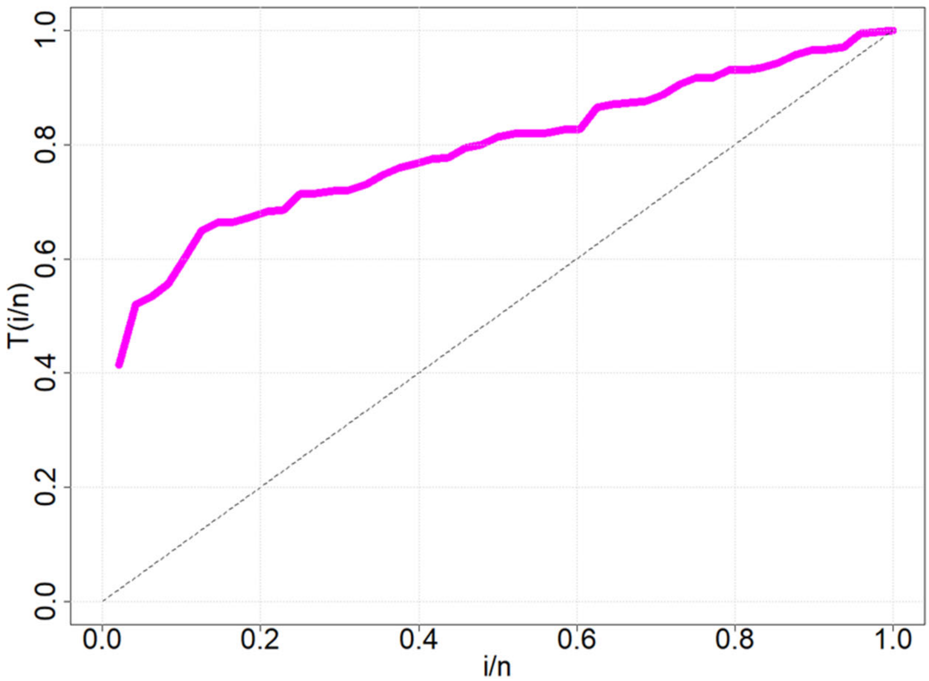

Figure 5. TTT plot for the petroleum rock sample data.

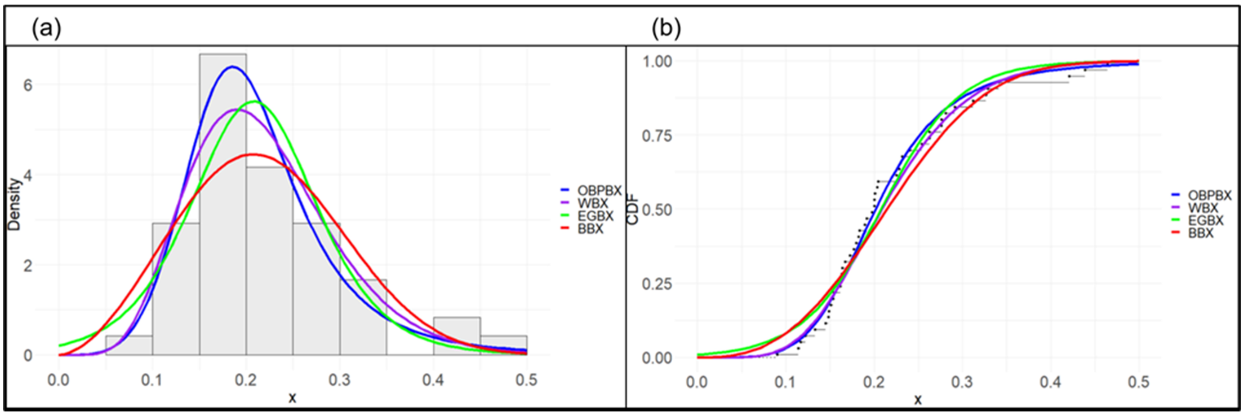

Figure 6. (**a**) Plot of empirical and fitted PDFs; (**b**) plot of empirical and fitted CDFs of the contesting models for petroleum sample data.

7.2. Second Data Set: United Kingdom COVID-19 Mortality Rate

The second data set provides information on COVID-19 mortality in the United Kingdom for the 24-day interval from October 15 to November 7, 2020. These data were utilized by [16] and are listed as follows:

0.2240, 0.2189, 0.2105, 0.2266, 0.0987, 0.1147, 0.3353, 0.2563, 0.2466, 0.2847, 0.2150, 0.1821, 0.1200, 0.4206, 0.3456, 0.3045, 0.2903, 0.3377, 0.1639, 0.1350, 0.3866, 0.4678, 0.3515, 0.3232.

A descriptive summary of COVID-19 mortality data is illustrated in Table 5. This table illustrates that the data set has a positive skewness and platykurtic distribution. Figure 7 shows the histogram, kernel density, box, and violin plots for the COVID-19 mortality rates. The figure shows that the mortality rates from COVID-19 are positively skewed and contain evidence of platykurtic behavior. As a result, the OBPBX model is suitable to model these kinds of data. Moreover, the TTT plot in Figure 8 portrays the shape of the failure rate for the data, indicating that the COVID-19 mortality rate data follow an increasing hazard function.

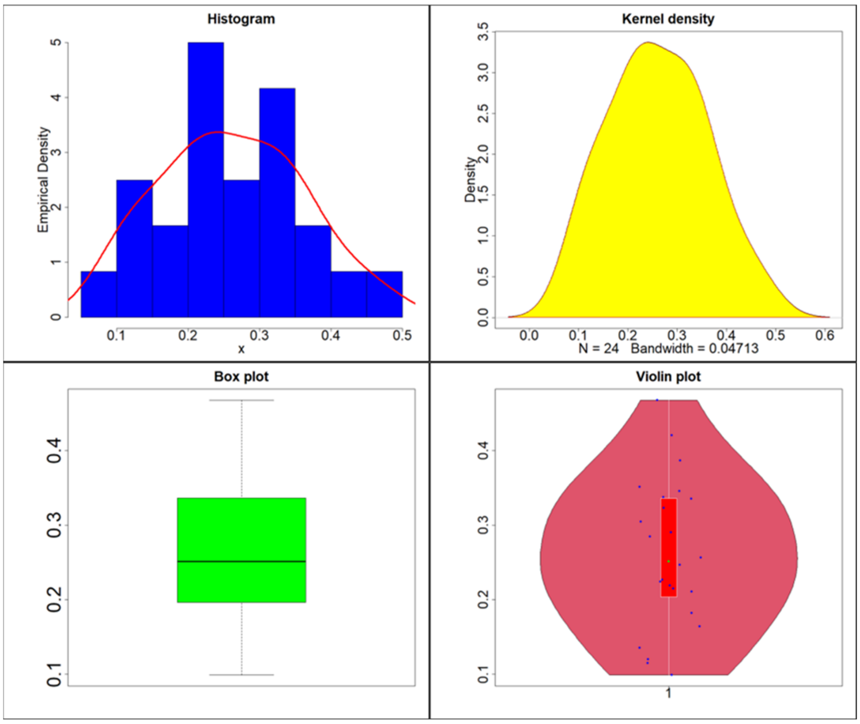

Figure 7. Histogram, kernel density, box plot, and violin plot for the COVID-19 mortality data.

Table 5. Descriptive summary of the COVID-19 mortality rates.

Data	Min	Q1	Q3	Median	Mean	Max	Variance	Skewness	Kurtosis
COVID-19	0.099	0.203	0.336	0.252	0.261	0.4678	0.010	0.152	−0.902

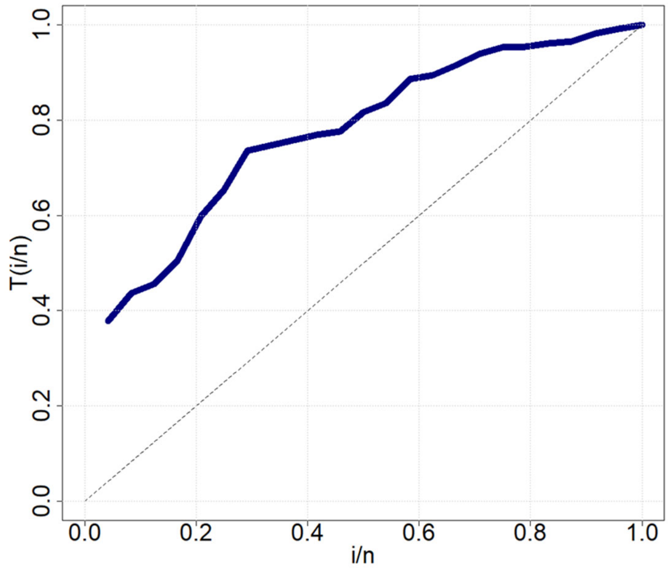

Figure 8. TTT plot for the COVID-19 mortality data.

Table 6 shows the MLEs and their standard errors for the OBPBX model and other competitive models, together with the statistical measures, such as the $\left(\hat{\ell}\right)$, AIC, CAIC, BIC, and HQIC, for the COVID-19 mortality rates. As shown in Table 6, the proposed OBPBX distribution is the best-fitting distribution, as it provides the smallest values for all statistical measures compared to the other competing models.

Figure 9 displays the plots for the data from the COVID-19 mortality data, such as (a) empirical and fitted densities and (b) empirical and fitted CDFs. This figure shows how well the suggested model fits the COVID-19 mortality data. Therefore, the OBPBX model is more suitable for modeling these kinds of data.

Table 6. MLEs, standard errors (in parentheses), and statistical measures for the COVID-19 mortality data.

Model	Estimates				Fitted Measures				
	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\ell}$	AIC	CAIC	BIC	HQIC
OBPBX	0.5662 (0.2853)	1.0680 (0.0546)	1.3613 (0.7034)	1.9897 (1.0745)	11.549	−15.097	−12.992	−10.385	−13.847
WBX	0.1550 (0.0234)	0.9267 (0.5362)	0.7583 (0.1901)	0.9495 (0.5462)	2.114	3.772	5.878	8.485	5.022
EGBX	0.7293 (0.2324)	1.0833 (0.3425)	0.9026 (0.4319)	0.4920 (0.0183)	0.670	6.661	8.766	11.373	7.911
BBX	0.5662 (0.2340)	1.0680 (0.5362)	0.6806 (0.3425)	0.9948 (0.3211)	1.257	5.486	7.592	10.199	6.736

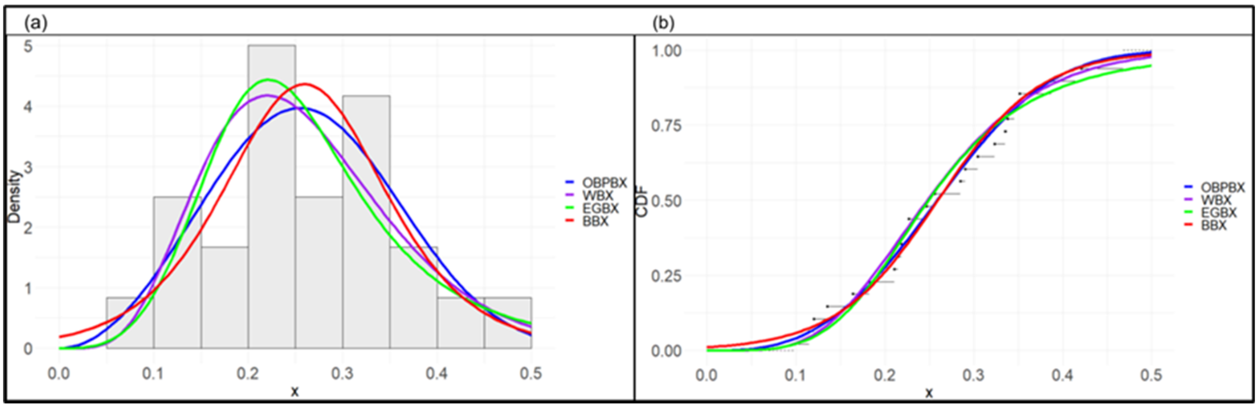

Figure 9. (**a**) Plot of empirical and fitted PDFs; (**b**) plot of empirical and fitted CDFs of the contesting models for COVID-19 mortality data.

8. Concluding Remarks

This paper introduced a novel Burr X distribution called the odd beta prime Burr X (OBPBX). This model is a generalization of the Burr X model by utilizing the odd beta prime generalized class. The distribution, density, survival, and hazard functions were presented. The OBPBX distribution has different shapes, such as symmetric, right-skewed, left-skewed, increasing, decreasing, and bathtub. Some of its basic statistical features were explored, such as the moments, the moment-generating function, entropies, the quantile function, and limit behavior. The model parameters were estimated based on the maximum-likelihood-estimation approach. The simulation study was conducted based on randomly generated samples of various sizes. Finally, we performed the numerical analysis on the petroleum rock samples and COVID-19 mortality rates in order to assess the usefulness of the suggested model over its competitors. The findings exhibit that the OBPBX distribution offers superior fits, for both data sets, than those of other competing models, such as Weibull-Burr X, exponential generalized-Burr X, and beta-Burr X distributions. We expect the proposed distribution will obtain a wider readership and provide a vital tool for modeling various phenomena in different domains.

In future studies, the proposed model will be improved by exploring more estimation procedures, such as the Bayesian, maximum product of spacing, least squares, and weighted least squares procedures. The usefulness of the OBPBX distribution will be evaluated by utilizing diverse data sets from other domains that are not covered in this article. Furthermore, the next direction will focus on the bivariate and multivariate versions of the OBPBX distribution.

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Nomenclature

X	Random variable
$T(x;\alpha,\beta)$	Cumulative distribution function of the Burr X distribution
$t(x;\alpha,\beta)$	Probability density function of Burr X distribution
$Q(x;\theta,\lambda,\delta)$	Cumulative distribution function of the odd beta prime generalized class
$q(x;\theta,\lambda,\delta)$	Probability density function of the odd beta prime generalized family
$T(x,\delta)$	Cumulative distribution function of the baseline distribution
$t(x,\delta)$	Probability density function of the baseline distribution
$\frac{T(x,\delta)}{1-T(x,\delta)}$	Odd ratio
$Q(x;\theta,\lambda,\alpha,\beta)$	Cumulative distribution function of the odd beta prime-Burr X distribution
$q(x;\theta,\lambda,\alpha,\beta)$	Probability density function of the odd beta prime-Burr X distribution
θ	Shape parameter
λ	Shape parameter
α	Shape parameter
β	Scale parameter
δ	Vector parameter
$S(x;\theta,\lambda,\alpha,\beta)$	Survival function
$h(x;\theta,\lambda,\alpha,\beta)$	Hazard function
$E(X^r)$	The rth moment
$E(e^{tX}), M_X(X)$	Moment generating function
$R_\eta(X)$	The Rényi entropy
$Q_\eta(X)$	The q-entropy
$Q(U)$	Quantile function
U	Continuous uniform variable
$Q\left(\frac{1}{2}\right)$	Median
n	sample size
Ψ	Vector parameter
L	The likelihood function
ℓ	The logarithm of likelihood function
N	The number of samples
ψ	The digamma function
Abbreviations	
OBP-G	Odd Beta Prime Generalized
OBPBX	Odd Beta Prime Burr X
CDF	Cumulative distribution function
PDF	Probability density function
MGF	Moment generating function
QF	Quantile function
B	Bowley's skewness
M	Moor's kurtosis
MLE	Maximum likelihood estimation
MSE	Mean squared error
WBX	Weibull-Burr X
EGBX	Exponential generalized-Burr X
BBX	Beta-Burr X
AIC	Akaike information criterion
CAIC	Corrected Akaike information criterion
BIC	Bayesian information criterion
HQIC	Hannan–Quinn information criterion
TTT	Total time on test

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