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Fuzzy Analytical Solution of Horizontal Diffusion Equation into the Vadose Zone

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Abstract: The process of how soil moisture profiles evolve into the soil and reach the root zone could be estimated by solving the appropriate strong nonlinear Richards' equation. The nonlinearity of the equation occurs because diffusivity D is generally an exponential function of water content. In this work, the boundary conditions of the physical problem are considered fuzzy for various reasons (e.g., machine impression, human errors, etc.), and the overall problem is encountered with a new approximate fuzzy analytical solution, leading to a system of crisp boundary value problems. According to the results, the proposed fuzzy analytical solution is in close agreement with Philip's semi-analytical method, which is used as a reference solution, after testing 12 different types of soils. Additionally, possibility theory is applied, enabling the decision-makers to take meaningful actions and gain knowledge of various soil and hydraulic properties (e.g., sorptivity, infiltration, etc.) for rational and productive engineering studies (e.g., irrigation systems).

Keywords: fuzzy solution; vadose zone; diffusion equation; infiltration; possibility

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1. Introduction

Infiltration is a common physical phenomenon of water movement in porous media that is of great interest in many earth and plant sciences. Soil water flow plays an important role in understanding the phenomena of runoff, groundwater recharge, and the transport of contaminants. Especially in the vadose zone, soil moisture strongly influences plants' growing processes, and for irrigation problems, it is important to know how the soil moisture profiles evolve into the soil and reach the root zone. Historically, Buckingham (1907) [1], Gardner and Widsoe (1921) [2], and Richards (1931) [3] are the pioneers in the development of soil water movement, introducing the ideas of capillary potential, capillary conductivity, and diffusion phenomena in the concept of soil water movement, which were completed later by Childs (1936a, 1936b) [4,5]. Klute (1952) [6] presented the equation of flow arising from Darcy's (1856) law [7], while the law of conservation of mass is as follows:

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (K \nabla \Phi)$$

where θ = the moisture content (cm^3/cm^3), K = the unsaturated hydraulic conductivity (cm/s), Φ = the total potential (cm) $\Phi = \Psi - z$, Ψ = the pressure potential or capillary potential (cm), and z = the gravitational component (cm). In the case of one-dimensional horizontal flow or in cases where gravity may be neglected (fine-textured soils), in which the moisture gradient's influence is much more important than gravity, the phenomenon is described by the following equation:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left(D \frac{\partial \theta}{\partial x} \right)$$

where D = diffusivity (L^2/T). According to Philip (1957, 1969) [8,9], this equation represents the water movement in a horizontal column and is called the absorption equation because it describes the wetting up of the column under tension.

Many methods exist with their corresponding advantages and disadvantages, such as semi-analytic methods [9], finite difference [10–15], finite element methods [16,17], finite control volume methods [18,19], flux concentration methods [20], and approximate analytical methods [21–25]. Henceforth, our research is focused on approximate analytical methods.

Tolikas et al. (1984) [26] presented a simple analytical solution to the problem of horizontal absorption under the assumption that diffusivity is an exponential function of soil water content. Brutsaert (1982) [27] presented a group of 14 solutions with different diffusivities for the problem of desorption, considering, as pointed out by Crank (1975) [28], that sorption and desorption are related by the correspondence principle. Specifically, if the solution for sorption with a given $D(\theta)$ is $\theta = f(x,t)$ in normalized form, then the function $f_1 = 1 - f(x,t)$ is the solution for the desorption problem with the diffusivity $D(1 - \theta)$. Lisle and Parlange (1993) [29] used a large class of transformations in conjunction with symmetry properties to obtain an analytical reduction of the transformed equation for certain diffusivities, emphasizing that the symmetry leaves the equation invariant. Hang and Zhiqiang (1997) [30] presented an approximated analytical solution with the Boltzmann transformation method for the case of an exponential diffusivity form and a profile of soil water content of a finite extent under the assumption that the wetting front is known. Prevedello et al. (2008) [31] derived a new analytical solution of the Boltzmann transformation φ as a function of matrix potential for horizontal water infiltration into a sand soil sample without invoking the concept or use of $D(\theta)$. The derivation assumes that a similarity exists between the soil water retention function and the Boltzmann transformation φ . Tzimopoulos et al. (2015) [32] presented a new approximate analytical solution of the nonlinear diffusion equation, and the solution is given for the transformed equation through the Boltzmann transformation. It is considering the case of an exponential function of the diffusivity under the following assumptions: (a) the profiles of soil water content have a finite extent; (b) the concentration at the boundaries is constant; and (c) the reduced flux of Philip (1973) [20] is of a special form [33]. More recent works have implemented the diffusion equation in applications for volatile organic compound transport [34], mass transfer [35], and Turing pattern formation [36].

The calculation of water flow in the unsaturated zone requires exact knowledge of the initial and boundary conditions and the various soil parameters, and this assumption is based principally on in situ measurements. In general, this assumption is subject to different kinds of uncertainty due to human and machine imprecision. In many cases, the uncertainties were considered in statistical terms as random variables with given mean values, variances, and correlations. However, these methods require exact knowledge of mean values, variances, and correlations and often suffer from an insufficient amount of accurate in situ measurement data. Zadeh (1965) [37] introduced the theory of fuzzy sets and fuzzy logic and covered all these kinds of uncertainty. A significant number of research studies presenting uncertainties were carried out in that field, especially regarding the fuzzy differentiation of functions. Puri and Ralescu (1983) [38] generalized and extended Hukuhara's fundamental study [39] (H-derivative) of a set of values appearing in fuzzy sets. Vorobiev and Seikkala (2002) [40], O'Regan et al. (2003) [41], and Nieto et al. (2006) [42] have worked in the theoretical and applied research fields on fuzzy differential equations with an H-derivative, but their method has presented certain drawbacks since it has led to solutions with increasing support along with increasing time. To overcome this drawback, the generalized derivative gH (gH -derivative) was

introduced by [43–46]. The gH-derivative will be used henceforth for a more extensive degree of fuzzy functions than the Hukuhara derivative. In addition, a recent review work by Mazandarani and Xiu (2021) [47] presents a chronological survey on fuzzy differential equations of integer and fractional orders, with the corresponding future perspectives and challenges to be discussed.

In the present article, the problem of fuzzy, unsteady water movement in the vadose zone in the horizontal direction is examined and described by a nonlinear partial diffusion equation with fuzzy conditions. In this context, we developed and propose a new fuzzy analytical solution that has never been presented before in the literature, coupled with the innovative possibility theory (see Section 2.2.1). Specifically, the nonlinear equation was first fuzzified and subsequently translated to a system of second-order crisp boundary value problems called the corresponding system for the fuzzy problem. Subsequently, using the Boltzmann transformation, the crisp problem was transformed to a system of two classical ordinary differential equations [32]. With the help of α -cuts, profiles of moisture content θ versus Boltzmann transformation φ were estimated, as were the storativity (S) and cumulative moisture content (I). In addition, for evaluation purposes, the present analytical solution and the approximated solution of Hang and Zhiqiang (1997) [30] were tested using the semi-analytical solution of Philip as a reference solution.

2. Mathematical Model

2.1. Classical (Crisp) Case

The one-dimensional horizontal nonlinear flow of water in unsaturated soil with initial and boundary conditions for absorption is described by the following equation [6]:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left(D \frac{\partial \theta}{\partial x} \right) \quad (1)$$

under constant head condition, with initial and boundary conditions as follows:

- Initial condition

$$x \geq 0, t = 0, \theta = \theta_r, \quad (2)$$

- Boundary condition

$$x = 0, t > 0, \theta = \theta_s, x \rightarrow \infty, t > 0, \theta = \theta_r. \quad (3)$$

where θ_s = the saturated soil moisture content at $x = 0$ and θ_r = the residual soil moisture content. The diffusivity $D(\theta)$ can be closely expressed by an exponential function [48–50] as follows:

$$D(\theta) = D(\theta_r) e^{\beta(\theta - \theta_r)} \quad (4)$$

The non-dimensional variables are now introduced in Equation (1),

$$\theta = \frac{\theta - \theta_r}{\theta_s - \theta_r}, X = \frac{x}{L}, \tau = \frac{D_r t}{L^2}, D^* = \frac{D}{D_r} = e^{\lambda_1 \theta}, \quad (5)$$

where L = a characteristic length usually equal to the wetting front, $D_r = D(\theta_r)$ = the diffusivity at the residual soil moisture content, τ = the non-dimensional time, D^* = the non-dimensional diffusivity, and $\lambda_1 = \beta(\theta_s - \theta_r)$. The advantage of introducing non-dimensional variables is that they allow the generalization of the theory, speeding up the computational processes.

Then, Equation (1) with initial and boundary conditions (2) and (3) becomes the following:

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial X} \left(D^*(\theta) \frac{\partial \theta}{\partial X} \right) \quad (6)$$

Subjacent to the following:

- Initial condition

$$X \geq 0, \tau = 0, \theta = 0, \quad (7)$$

- Boundary condition

$$X = 0, \tau > 0, \theta = 1, X \rightarrow \infty, \tau > 0, \theta = 0. \quad (8)$$

The Boltzmann transformation $\varphi^* = X\tau^{-1/2}$ is now introduced in Equation (6), and due to conditions (7) and (8), Equation (6) is reduced to the following ordinary differential equation:

$$\frac{d}{d\varphi^*} (D^*(\theta) \frac{d\theta}{d\varphi^*}) + \frac{\varphi^*}{2} \frac{d\theta}{d\varphi^*} = 0, \quad (9)$$

The conditions (2) and (3) are transformed into the following:

$$\varphi^* = 0, \theta = 1 \quad \varphi^* \rightarrow \varphi_f^*, \theta \rightarrow 0, \quad (10)$$

where φ_f^* is the wetting front.

Philip (1957) [8] introduced the term sorptivity in order to define the capacity of the medium to absorb or desorb water by capillarity. He expresses the term as follows:

$$S^* = \int_0^1 \varphi^* d\theta. \quad (11)$$

The non-dimensional Boltzmann transformation φ^* and the non-dimensional sorptivity S^* are related to the dimensional Boltzmann transformation φ and the dimensional sorptivity S by the following expressions:

$$\varphi^* = \frac{\varphi}{\sqrt{D_r}}, \quad S^* = \frac{S}{(\theta_s - \theta_r)\sqrt{D_r}} \quad (12)$$

Tzimopoulos et al. (2015) [32] proposed the following: (a) an explicit analytical solution, which has the Boltzmann transformation as the dependent variable and the soil water moisture as the independent variable; (b) a simple form for the sorptivity. Both solutions are presented in normalized form. The solutions were tested with 12 soils and showed a close agreement with Philip's semi-analytical method [9], which has been considered a reference method. The solution concerning sorptivity is given as follows:

$$S^* = \sqrt{e^{\lambda_1} \left(\frac{2}{\lambda_1} - \frac{1}{\lambda_1^2} \right)} \quad (13)$$

or in dimensional form:

$$S = (\theta_s - \theta_r) \sqrt{D_r e^{\lambda_1} \left(\frac{2}{\lambda_1} - \frac{1}{\lambda_1^2} \right)} \quad (14)$$

The solution concerning Boltzmann transformation is given as follows:

$$\varphi^* = \frac{1}{S^*} \left\{ \frac{1}{\lambda_1} (e^{\lambda_1} - e^{\lambda_1 \theta}) + E_i(\lambda_1) - E_i(\lambda_1 \theta) \right\} \quad (15)$$

or in dimensional form:

$$\varphi = \frac{(\theta_s - \theta_r) D_r}{S} \left\{ \frac{1}{\lambda_1} (e^{\lambda_1} - e^{\lambda_1 \theta}) + E_i(\lambda_1) - E_i(\lambda_1 \theta) \right\} \quad (16)$$

where $E_i(\lambda_1), E_i(\lambda_1 \theta)$ are the exponential integrals [51,52]. Equation (16) satisfies the boundary conditions (10). Indeed, for the following:

$$\theta = 1 \rightarrow \varphi = \frac{(\theta_s - \theta_r) D_r}{S} \left\{ \frac{1}{\lambda_1} (e^{\lambda_1} - e^{\lambda_1}) + E_i(\lambda_1) - E_i(\lambda_1) \right\} = 0. \quad (17)$$

$$\begin{aligned}
\theta &= 1 \\
\rightarrow \lim_{\Theta \rightarrow 0}(\phi) &= \lim_{\Theta \rightarrow 0} \left[\frac{(\theta_s - \theta_r) D_r}{S} \left\{ \frac{1}{\lambda_1} (e^{\lambda_1} - e^{\lambda_1 \Theta}) + E_i(\lambda_1) - E_i(\lambda_1 \Theta) \right\} \right] = \\
&= \frac{(\theta_s - \theta_r) D_r}{S} \left\{ \frac{1}{\lambda_1} (e^{\lambda_1} - 1) + E_i(\lambda_1) - \lim_{\Theta \rightarrow 0} E_i(\lambda_1 \Theta) \right\}
\end{aligned} \quad (18)$$

According to Ramanujan et al. (2000) [53], the function $E_i(x) - \gamma - \ln x$, $\gamma \approx 0.5772156649 \dots$ for small values of x is included in the following interval:

$$\left[1 - \frac{3x}{4}, 1 - \frac{3x}{4} + \frac{11x^2}{36} \right], x = \lambda_1 \theta \quad (19)$$

If we choose $x = 5 \times 10^{-5}$ as the minimum value, the above relation gives a value of $E_i(\lambda_1 \theta) | \lambda_1 \theta = 5 \times 10^{-5} = -8.32631$. With this value of integral, the value of ϕ is equal to the moisture content front (ϕ_{front}), which is as follows:

$$\phi_{front} = \frac{(\theta_s - \theta_r) D_r}{S} \left\{ \frac{1}{\lambda_1} (e^{\lambda_1} - 1) + E_i(\lambda_1) + 8.32631 \right\} \quad (20)$$

Now the boundary condition $\phi^* \rightarrow \infty, \theta \rightarrow 0$, becomes the following:

$$\text{For } \phi \geq \phi_{front}, \theta \approx 0. \quad (21)$$

From the above expressions (14) and (16), it is clear that by knowing the soil moisture contents θ_s, θ_r , and the diffusivity as an exponential function of θ , it is possible to find the following:

- The sorptivity S ;
- The moisture content front ϕ_{front} ;
- The water profiles in the form of $\phi = \phi(\theta) = \phi(\theta)$.

Table 1 shows the parameter values for the 12 different soils under consideration and a comparison of the wetting fronts and sorptivities between the values obtained by the analytical values in this paper and Philip's semi-analytical solution, which is in close agreement with the experimental values and was taken as a reference profile [32].

Table 1. Estimated values of moisture content front and sorptivity for 12 soil types.

a/a Soil Type	θ_r (cm ³ /cm ³)	θ_s (cm ³ /cm ³)	D_r cm ² /min	λ_1	$\phi^{\text{exp}_{front}}$ (cm/min ^{0.5})	$\phi^{\text{anal}_{front}}$ (cm/min ^{0.5})	S^{exp} (cm/min ^{0.5})	S^{anal} (cm/min ^{0.5})	Ref.
1 Hagenet sand	0.0540	0.370	9.443×10^{-2}	6.756	5.857	5.870	1.490	1.490	[54]
2 Hayden sandy loam	0.0146	0.510	6.824×10^{-3}	7.256	1.906	1.908	0.780	0.780	[55]
3 Manawatu fine sand. loam	0.0800	0.360	1.168×10^{-3}	11.337	4.490	4.490	1.138	1.138	[56]
4 Adelanto loam	0.0380	0.387	6.678×10^{-8}	18.690	1.010	1.010	0.333	0.333	[57]
5 Edina silt loam	0.0560	0.500	4.214×10^{-3}	7.995	2.013	2.014	0.760	0.760	[54]
6 Nicollet sandy loam	0.0380	0.364	2.151×10^{-5}	13.094	1.346	1.348	0.404	0.404	[58]
7 Fayette silty clay loam	0.0480	0.448	3.418×10^{-4}	9.223	0.960	0.960	0.337	0.337	[58]
8 Panoche clay loam	0.0480	0.443	2.554×10^{-3}	8.731	2.129	2.129	0.730	0.730	[49]
9 Pine silty clay	0.0500	0.463	8.349×10^{-9}	20.262	0.750	0.750	0.294	0.294	[57]
10 Yolo clay	0.0400	0.490	3.265×10^{-4}	6.869	0.360	0.360	0.131	0.131	[59]
11 sample 1	0.0175	0.295	1.13×10^{-3}	9.170	1.710	1.709	0.416	0.416	[60]
12 sample 2	0.0080	0.500	2.992×10^{-1}	6.618	9.930	9.910	3.890	3.890	[60]

The relation

$$S^* = \int_0^1 \varphi^* d\theta, \quad (22)$$

easily follows the dimensional form of I

$$S = \{\int (\theta - \theta_r) dx\} t^{-1/2} = I t^{-1/2} \rightarrow I = S t^{1/2} \quad (23)$$

where I = cumulative absorption, L (L³/L³). From this one, it follows the flow velocity at x = 0:

$$v|_{x=0} = 0 = \frac{dl}{dt} = S t^{-1/2} \quad (24)$$

2.2. Fuzzy Case

In order to facilitate an understanding for readers unfamiliar with the fuzzy and possibility theories, we describe definitions concerning some preliminaries in the fuzzy and possibility theories and definitions about the differentiability of fuzzy numbers.

2.2.1. Fuzzy Theory

Definition 1. A fuzzy set $\tilde{U}$ on a universe set X is a mapping $\tilde{U}: X \rightarrow [0,1]$ or $\mu_{\tilde{U}}: X \rightarrow [0,1]$, which is called membership function. The value $\mu_{\tilde{U}}(x), x \in X$ is the membership value of $x \in X$ and expresses the membership degree of $x \in X$, i.e., the truth of fuzzy logical proposition: $T(xis\tilde{U}|X = x) = \mu_{\tilde{U}}(x)$.

The membership function $\mu_{\tilde{U}}(x)$, defined also as $\tilde{U}(x)$, has the following properties: (i) $\mu_{\tilde{U}}$ is upper semicontinuous; (ii) $\mu_{\tilde{U}}(x) = 0$ outside of some interval $[c, d]$; (iii) there are real numbers $c \leq a \leq b \leq d$, such that $\mu_{\tilde{U}}$ is increasing on $[c, a]$, decreasing on $[b, d]$, and $\mu_{\tilde{U}}(x) = 1$ for each $x \in [a, b]$; (iv) $\tilde{U}$ is a convex fuzzy set (i.e., $\mu_{\tilde{U}}(\lambda x + (1 - \lambda)x) \geq \min\{\mu_{\tilde{U}}(\lambda x), \mu_{\tilde{U}}((1 - \lambda)x)\}$).

Definition 2. Let X be a Banach space and $\tilde{U}$ be a fuzzy set on X . We define the α -cuts of $\tilde{U}$ as $[\tilde{U}(x)]_{\alpha} = \{x \in R | \tilde{U}(x) \geq \alpha\}, \alpha \in (0,1]$, and for $\alpha = 0$, we define the following closure:

$$[\tilde{U}(x)]_0 = \overline{\{x \in R | \tilde{U}(x) > 0\}}.$$

Definition 3. Let $K(X)$ be the family of all nonempty compact convex subsets of a Banach space. A fuzzy set $\tilde{U}$ on X is called compact if $[\tilde{U}]_{\alpha} \in K(X), \forall \alpha \in [0,1]$. The space of all compact and convex fuzzy sets on X is denoted as $F(X)$.

Definition 4. Let $[\tilde{U}] \in F(R)$. The α -cuts of $\tilde{U}$ are as follows: $[\tilde{U}]_{\alpha} = [U_{\alpha}^-, U_{\alpha}^+]$. According to the representation theorem of [61,62], the membership function and the α -cut form of a fuzzy number $\tilde{U}$ are equivalent, and in particular, the α -cuts $[\tilde{U}]_{\alpha} = [U_{\alpha}^-, U_{\alpha}^+]$ uniquely represent $\tilde{U}$, provided that the two functions are monotonic (U_{α}^- increasing, U_{α}^+ decreasing) and $U_1^- \leq U_1^+$ for $\alpha = 1$. For every $\tilde{U}, \tilde{V} \in F(R)$, the metric structure is given by the Hausdorff distance as follows:

$$D: F(R) \times F(R) \rightarrow R_+ \cup \{0\}, \text{ by } D(U, V) = \sup_{\alpha \in [0,1]} \max\{|U_{\alpha}^- - V_{\alpha}^-|, |U_{\alpha}^+ - V_{\alpha}^+|\}$$

Definition 5. gH-Differentiability [63]. Let $\tilde{U}: [a, b] \rightarrow R_F$ be such that $[\tilde{U}]_{\alpha} = [U_{\alpha}^-, U_{\alpha}^+]$. Suppose that the functions U_{α}^- and U_{α}^+ are real-valued functions, differentiable w.r.t. (with respect to) x , uniformly w.r.t. $\alpha \in [0,1]$. Then the function $\tilde{U}(x)$ is gH-differentiable at a fixed $x \in [a, b]$ only if one of the following two cases holds:

- (a) $(U_{\alpha}^-)'(x)$ is increasing, $(U_{\alpha}^+)'(x)$ is decreasing as functions of α , and $(U_1^-)'(x) \leq (U_1^+)'(x)$, or

- (b) $(U_{\alpha}^{+})'(x)$ is decreasing, $(U_{\alpha}^{-})'(x)$ is increasing as functions of α , and $(U_1^{+})'(x) \leq (U_1^{-})'(x)$.

Notation: $(U_{\alpha}^{-})'(x) = \frac{\partial U_{\alpha}^{-}(x)}{\partial x}$, $(U_{\alpha}^{+})'(x) = \frac{\partial U_{\alpha}^{+}(x)}{\partial x}$. In both of the above cases, the $\tilde{U}'_{\alpha}(x)$ derivative is a fuzzy number.

Definition 6. *gH-Differentiable at x_0 . Let $\tilde{U}: [a, b] \rightarrow F(R)$ and $x_0 \in [a, b]$ with $U_{\alpha}^{-}(x)$ and $U_{\alpha}^{+}(x)$ both being differentiable at x_0 . We say that [63]:*

- $\tilde{U}$ is (i)-gH-differentiable at x_0 if
- (i) $[U'_{gH}(x_0)]_{\alpha} = [(U_{\alpha}^{-})'(x_0), (U_{\alpha}^{+})'(x_0)], \forall \alpha \in [0, 1]$
- $\tilde{U}$ is (ii)-gH-differentiable at x_0 if
- (ii) $[U'_{gH}(x_0)]_{\alpha} = [(U_{\alpha}^{+})'(x_0), (U_{\alpha}^{-})'(x_0)], \forall \alpha \in [0, 1]$

Definition 7. *g-Differentiability Let $\tilde{U}: [a, b] \rightarrow F(R)$ be such that $[\tilde{U}]_{\alpha} = [U_{\alpha}^{-}, U_{\alpha}^{+}]$. If $U_{\alpha}^{-}(x)$ and $U_{\alpha}^{+}(x)$ are differentiable real-valued functions with respect to x , uniformly for $\alpha \in [0, 1]$, then $f(x)$ is g-differentiable and we have the following [63]:*

$$[U'_g(x)]_{\alpha} = \left[\inf_{\beta \geq \alpha} \min\{ (U_{\alpha}^{-})'(x), (U_{\alpha}^{+})'(x) \}, \sup_{\beta \geq \alpha} \max\{ (U_{\alpha}^{-})'(x), (U_{\alpha}^{+})'(x) \} \right],$$

Definition 8. *The gH-Differentiability implies g-differentiability, but the inverse is not true.*

Definition 9. *[gH-p]-Differentiability. A fuzzy-valued function $\tilde{U}$ of two variables is a rule that assigns to each ordered pair of real numbers (x, t) in a set D a unique fuzzy number denoted by $\tilde{U}(x, t)$. Let $\tilde{U}(x_0, t_0): D \rightarrow F(R)$, $(x_0, t_0) \in D$ and $U_{\alpha}^{-}(x_0, t_0)$, $U_{\alpha}^{+}(x_0, t_0)$ are real-valued functions and partially differentiable w.r.t. x . We say that [46,64]:*

- $\tilde{U}(x, t)$ is [(i)-p]-differentiable w.r.t. x at (x_0, t_0) if:

$$\frac{\partial \tilde{U}_{\alpha}(x_0, t_0)}{\partial x_{i.gH}} = \left[\frac{\partial U_{\alpha}^{-}(x_0, t_0)}{\partial x}, \frac{\partial U_{\alpha}^{+}(x_0, t_0)}{\partial x} \right]$$

- $\tilde{U}(x, t)$ is [(ii)-p]-differentiable w.r.t. x at (x_0, t_0) if:

$$\frac{\partial \tilde{U}_{\alpha}(x_0, t_0)}{\partial x_{ii.gH}} = \left[\frac{\partial U_{\alpha}^{+}(x_0, t_0)}{\partial x}, \frac{\partial U_{\alpha}^{-}(x_0, t_0)}{\partial x} \right]$$

Notation: The same is valid for $\frac{\partial \tilde{U}_{\alpha}(x_0, t_0)}{\partial t_{p.gH}}$.

Definition 10. *Let $\tilde{U}(x, t): D \rightarrow F(R)$, and $\frac{\partial \tilde{U}_{\alpha}(x_0, t_0)}{\partial x_{p.gH}}$ be [gH-p]-differentiable at $(x_0, t_0) \in D$ with respect to x . We say that [46,64] $\frac{\partial \tilde{U}_{\alpha}(x_0, t_0)}{\partial x_{p.gH}}$ is*

- [(i)-p]-differentiable w.r.t. x , if the type of [gH-p]-differentiability of both $\tilde{U}_{\alpha}(x_0, t_0)$ and $\frac{\partial \tilde{U}_{\alpha}(x_0, t_0)}{\partial x_{p.gH}}$ is the same, then $\frac{\partial^2 \tilde{U}_{\alpha}(x_0, t_0)}{\partial x^2_{i.gH}} = \left[\frac{\partial^2 U_{\alpha}^{-}(x_0, t_0)}{\partial x^2}, \frac{\partial^2 U_{\alpha}^{+}(x_0, t_0)}{\partial x^2} \right]$
- [(ii)-p]-differentiable w.r.t. x , if the type of [gH-p]-differentiability of both $\tilde{U}_{\alpha}(x_0, t_0)$ and $\frac{\partial \tilde{U}_{\alpha}(x_0, t_0)}{\partial x_{p.gH}}$ is different, then $\frac{\partial^2 \tilde{U}_{\alpha}(x_0, t_0)}{\partial x^2_{ii.gH}} = \left[\frac{\partial^2 U_{\alpha}^{+}(x_0, t_0)}{\partial x^2}, \frac{\partial^2 U_{\alpha}^{-}(x_0, t_0)}{\partial x^2} \right]$

2.2.2. Possibility Theory

Definition 11. *A possibility measure Π on a set X (e.g., a set of reels) is characterized by a possibility distribution $\pi: X \rightarrow [0, 1]$ and is defined by the following:*

$$\forall A \subseteq X, \Pi(A) = \sup\{\pi(x), x \in A\}.$$

For finite sets, this definition reduces to the following:

$$\forall A \subseteq X, \Pi(A) = \max\{\pi(x), x \in A\}.$$

Definition 12. A degree of necessity $Ness_X$ on a set X (e.g., a set of reals) is characterized by the non-possibility (one minus possibility) of A complement (A^c).

$$\forall A \subseteq X, Ness_X(A) = 1 - \Pi(A^c).$$

Definition 13. A probability distribution p and a possibility distribution π are said to be consistent only if $\pi(u) \geq p(u), \forall u$ [65–68].

Definition 14. Two possibility distributions π_X, π'_X are consistent with the probability distribution p_X . The π_X distribution is more specific than π'_X if it is $\pi_X < \pi'_X$.

A possibility distribution π_X^* consistent with the probability distribution p_X is called *maximal specificity* if it is more specific than each other possibility distribution $\pi_X: \pi_X^*(x) < \pi_X(x), \forall x$.

Definition 15. For a number Y with a known and continuous probability distribution function p , the fuzzy number $\tilde{Y}$, which has a possibility measure $\Pi(\tilde{Y}) = \mu_{\tilde{Y}}$ is the fuzzy estimator of Y and has an α -cut of $\Pi_{\tilde{Y}}[\alpha] = \tilde{Y}[\alpha]$. This fuzzy number satisfies the consistency principle and verifies $P(\tilde{Y}[\alpha]) = Ness_{\tilde{Y}}[\alpha] = 1 - \alpha$, so that the probability of the possibility α -cut is equal to $1 - \alpha$. The α -cuts $\tilde{Y}[\alpha]$ are the confidence intervals of P , and the confidence level is α .

Definition 16. Conjecture [68]. For a function $Y = Y(X_1, X_2, \dots, X_n)$ with unknown probability distribution function, a fuzzy number may be constructed $\tilde{Y}^* = \tilde{Y}(\tilde{X}_1^*, \tilde{X}_2^*, \dots, \tilde{X}_n^*)$, and the α -cut is equal to the following:

$$\tilde{Y}^*[a] = \tilde{Y}(\tilde{X}_1^*[a], \tilde{X}_2^*[a], \dots, \tilde{X}_n^*[a]).$$

In this case, the fuzzy number $\tilde{Y}^*$ is the fuzzy estimator of Y and verifies the following:

$$P(\tilde{Y}^*[a]) \geq Ness_{\tilde{Y}^*}[\alpha] = 1 - \alpha,$$

so that the probability of the possibility α -cut is greater than $1 - \alpha$.

2.2.3. Fuzzy Model

We write Equation (1) in its fuzzy form according to [69]:

$$\frac{\partial \tilde{\theta}}{\partial \tau} = \frac{\partial}{\partial X} \left(\tilde{D} \frac{\partial \tilde{\theta}}{\partial X} \right) \quad (25)$$

with the following:

- Initial condition

$$X \geq 0, \tau = 0, \tilde{\theta} = (\tilde{0}) \quad (26)$$

- Boundary conditions

$$X = 0, \tau > 0, \tilde{\theta} = \tilde{1}, X \rightarrow \infty, \tau > 0, \tilde{\theta} = \tilde{0}. \quad (27)$$

The fuzzy parameters were considered to be the following:

- The soil moisture contents θ_s (saturated soil moisture content);
- θ_r (the residual soil moisture content);
- The parameter λ_1 (in the exponent of the diffusivity D).

These parameters are considered symmetrical triangular fuzzy numbers and are illustrated in Figure 1.

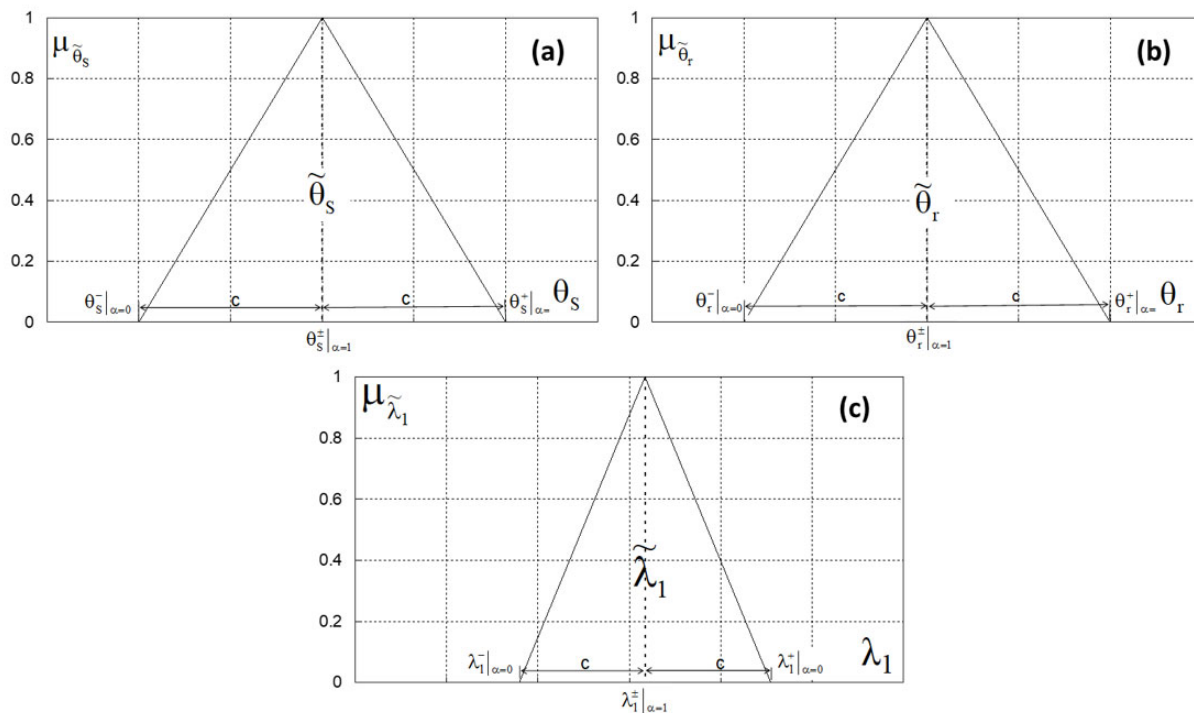


Figure 1. Membership function of (a) moisture content θ_s , (b) moisture content θ_r , and (c) parameter λ_1 .

We can find solutions to the fuzzy problem of Equation (25) and the initial and boundary conditions (26, 27) by utilizing the theory of [22,46,63,64,70] and translating the above fuzzy problem to a system of second-order crisp boundary value problems, hereafter called the corresponding system for the fuzzy problem. Therefore, many crisp BVP systems are possible for the fuzzy problem, but in the current work, the solution is restricted to the following systems (Cases A and B), which describe exactly the physical problem.

The following fuzzy expression is valid and is referring to α -cuts:

$$L(\tilde{\theta}_\alpha) = [L(\theta_\alpha^-), L(\theta_\alpha^+)]$$

In this case, for more convenience, the above expression could be written as follows, related to fuzzy partial differential equations:

$$\begin{array}{cc} \text{Case A} & \text{Case B} \\ \frac{\partial \tilde{\theta}}{\partial \tau} = \frac{\partial}{\partial X} \left(\tilde{D} \frac{\partial \tilde{\theta}}{\partial X} \right) \Leftrightarrow \frac{\partial \theta_\alpha^-}{\partial \tau} = \frac{\partial}{\partial X} \left(D_\alpha^- \frac{\partial \theta_\alpha^-}{\partial X} \right), & \frac{\partial \theta_\alpha^+}{\partial \tau} = \frac{\partial}{\partial X} \left(D_\alpha^+ \frac{\partial \theta_\alpha^+}{\partial X} \right) \end{array}$$

The main difference between the above two cases is that the operator (-) describes the left side of the α -cut while the (+) describes the right side of the α -cut, constructing in this way a possible range in which the desired value will be.

Case A

For Case A, the following expression arises:

$$\frac{\partial \theta_\alpha^-}{\partial \tau} = \frac{\partial}{\partial X} \left(D_\alpha^- \frac{\partial \theta_\alpha^-}{\partial X} \right) \quad (28)$$

Following the same procedure described above for the classical case, the non-dimensional and dimensional forms are presented below:

- Non-dimensional form:

$$\frac{d}{d\varphi^*}(D^*(\theta^-))\frac{d\theta^-}{d\varphi^*} + \frac{\varphi^*}{2}\frac{d\theta^-}{d\varphi^*} = 0, \quad (29)$$

$$(S^*)_{\alpha}^- = \left\{ e^{\lambda_1^-} \left(\frac{2}{\lambda_1^-} - \frac{1}{(\lambda_1^-)^2} \right) \right\}_{\alpha}^{1/2} \quad (30)$$

$$(\varphi^*)_{\alpha}^- = \frac{1}{(S^*)_{\alpha}^-} \left\{ \frac{1}{\lambda_1^-} (e^{\lambda_1^-} - e^{\lambda_1^- \theta}) + E_i(\lambda_1^-) - E_i(\lambda_1^- \theta^-) \right\}_{\alpha} \quad (31)$$

- Dimensional form:

$$S_{\alpha}^- = [(\theta_s)_{\alpha}^- - (\theta_r)_{\alpha}^-] \sqrt{D_r^- e^{(\lambda_1)_{\alpha}^-} \left(\frac{2}{(\lambda_1)_{\alpha}^-} - \frac{1}{((\lambda_1)_{\alpha}^-)^2} \right)} \quad (32)$$

$$\varphi_{\alpha}^- = \frac{[(\theta_s)_{\alpha}^- - (\theta_r)_{\alpha}^-](D_r)_{\alpha}^-}{S_{\alpha}^-} \left\{ \frac{1}{(\lambda_1)_{\alpha}^-} (e^{(\lambda_1)_{\alpha}^-} - e^{(\lambda_1)_{\alpha}^- \theta_{\alpha}^-}) + E_i(\lambda_1)_{\alpha}^- - E_i((\lambda_1)_{\alpha}^- \theta_{\alpha}^-) \right\} \quad (33)$$

Case B

For Case B, the following expression arises:

$$\frac{\partial \theta_{\alpha}^+}{\partial \tau} = \frac{\partial}{\partial X} \left(D_{\alpha}^+ \frac{\partial \theta_{\alpha}^+}{\partial X} \right) \quad (34)$$

Following the same procedure described above for the classical case, the non-dimensional and dimensional forms are presented below:

- Non-dimensional form:

$$\frac{d}{d\varphi^*}(D^*(\theta^+))\frac{d\theta^+}{d\varphi^*} + \frac{\varphi^*}{2}\frac{d\theta^+}{d\varphi^*} = 0, \quad (35)$$

$$(S^*)_{\alpha}^+ = \left\{ e^{\lambda_1^+} \left(\frac{2}{\lambda_1^+} - \frac{1}{(\lambda_1^+)^2} \right) \right\}_{\alpha}^{1/2} \quad (36)$$

$$(\varphi^*)_{\alpha}^+ = \frac{1}{(S^*)_{\alpha}^+} \left\{ \frac{1}{\lambda_1^+} (e^{\lambda_1^+} - e^{\lambda_1^+ \theta}) + E_i(\lambda_1^+) - E_i(\lambda_1^+ - \theta^+) \right\}_{\alpha} \quad (37)$$

- Dimensional form:

$$S_{\alpha}^+ = [(\theta_s)_{\alpha}^+ - (\theta_r)_{\alpha}^+] \sqrt{D_r^+ e^{(\lambda_1)_{\alpha}^+} \left(\frac{2}{(\lambda_1)_{\alpha}^+} - \frac{1}{((\lambda_1)_{\alpha}^+)^2} \right)} \quad (38)$$

$$\varphi_{\alpha}^+ = \frac{[(\theta_s)_{\alpha}^+ - (\theta_r)_{\alpha}^+](D_r)_{\alpha}^+}{S_{\alpha}^+} \left\{ \frac{1}{(\lambda_1)_{\alpha}^+} (e^{(\lambda_1)_{\alpha}^+} - e^{(\lambda_1)_{\alpha}^+ \theta_{\alpha}^+}) + E_i(\lambda_1)_{\alpha}^+ - E_i((\lambda_1)_{\alpha}^+ \theta_{\alpha}^+) \right\} \quad (39)$$

3. Results and Discussion

3.1. Construction of the Hang and Zhiqiang Solution

For comparison purposes, the Hang and Zhiqiang (1997) [30] solution was selected mainly due to its simplicity in applications. However, this solution is valid under the following two certain conditions:

1. Diffusivity is an exponential function of Θ ;
2. The value of the moisture wetting front is a priori known.

The Hang and Zhiqiang analytical solution initially had the following form:

$$\theta(x, t) = \theta(\varphi) = \frac{1}{\beta} \ln [e^{\beta \theta_0} - \left(\frac{A(e^{\beta \theta_0} - e^{\beta \theta_r}) - 0.5 \varphi_{front}^2}{A \varphi_{front}} \right) \varphi] - \frac{1}{2A} \varphi^2$$

In order to make a comparison with Equation (6) and with Philip's reference solution, the above equation is transformed into the following expression:

$$\varphi = \varphi(\theta) = -\alpha + \sqrt{\alpha^2 + 2A(e^{\beta \theta_0} - e^{\beta \theta})},$$

with

$$\alpha = \frac{A(e^{\beta\theta_0} - e^{\beta\theta_r}) - 0.5\varphi_{front}^2}{\varphi_{front}}, \quad A = D_r e^{-\theta_r}$$

The numerical values for φ_{front} were taken from Equation (20).

3.2. Results

Figures 2 and 3 illustrate the profiles of this study versus (vs.) the profiles of the Hang and Zhiqiang solution and Philip's semi-analytical method (the reference method). It is found that the mean relative errors ε_{mean} between the present analytical method and the Hang and Zhiqiang solution are as follows:

(a) Soil sample 1 $\varepsilon_{mean} = 2.669 \times 10^{-2}$;

(b) Soil sample 2 $\varepsilon_{mean} = 8.04 \times 10^{-2}$.

In both cases, these values are the mean of the values φ , $\theta_{\alpha=1}$, $\theta_{\alpha=0}^-$, and $\theta_{\alpha=0}^+$.

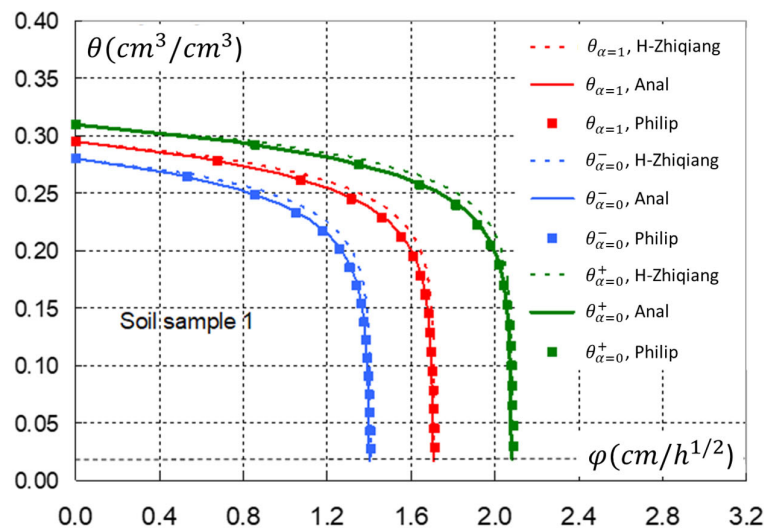


Figure 2. Profiles of moisture content θ versus Boltzmann transformation φ in the case of soil sample 1. The red color corresponds to $\theta_{\alpha=1}$, the blue color to $\theta_{\alpha=0}^-$, and the green color to $\theta_{\alpha=0}^+$.

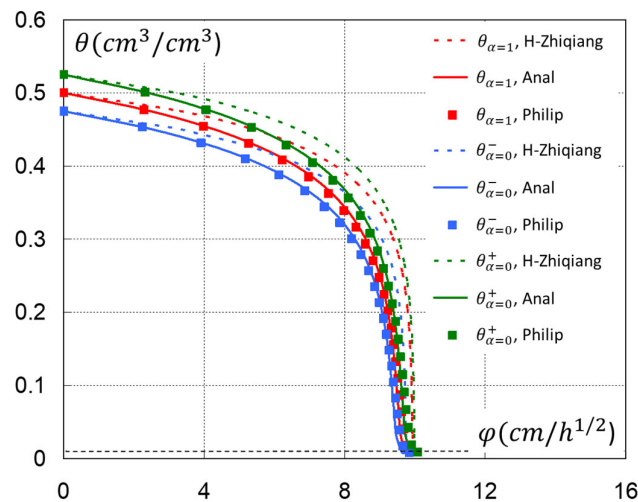


Figure 3. Profiles of moisture content θ versus Boltzmann transformation φ in the case of soil sample 2. The red color corresponds to $\theta_{\alpha=1}$, the blue color to $\theta_{\alpha=0}^-$, and the green color to $\theta_{\alpha=0}^+$.

Figure 4 is considered very informative for drawing conclusions since it illustrates the membership functions of the profiles (Figure 4a,b) and the sorptivity S (Figure 4c,d) for both samples (1, 2), also giving space for the application of the possibility theory and the meaningful interpretation of the fuzzy results. In that regard and by carefully observing the different cases in Figure 4, we can conclude that, as an example, at α -cut = 0.05, we have valuable information regarding the θ and S intervals of confidence for all cases, which, according to the possibility theory, could provide us with confidence for a probability greater than 95%. For example, in Figure 4b, the interval $[0.394, 0.438]$ denotes that the desired value will be in this range with a probability of > 95%.

Furthermore, the cumulative infiltration I (cm) vs. time t (h) for the proposed solution, Hang and Zhiqiang solution, and Philip's semi-analytical method is illustrated in Figure 5. Even from the visualization of the results, it is easily observable that the proposed solution is in close agreement with Philip's semi-analytical reference solution. This agreement could allow us to state that the proposed solution is accurate and reliable and can be used for decision-making and planning needs. In contrast, the solution by Hang and Zhiqiang presents some deviations (Table 2).

In addition, Figure 6 illustrates the membership functions for the aforementioned methods, while for the case of the proposed solution, the possibility theory was implemented, leading to the conclusion that for α -cut = 0.05, we have a probability of > 95% for our value to be in the interval $[1.460, 2.370]$.

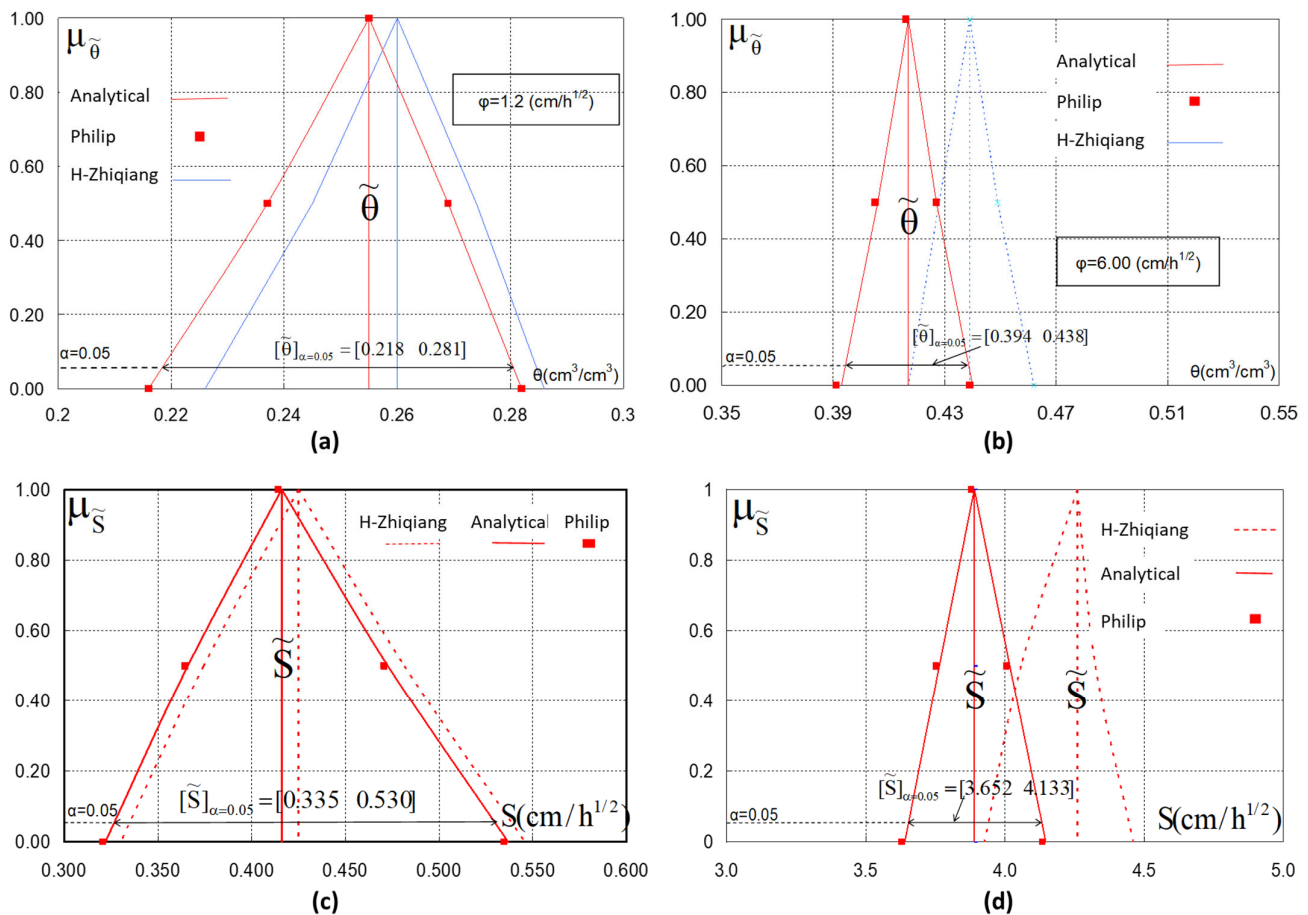


Figure 4. Membership function of (a) moisture content θ at $\phi = 1.2 \text{ cm/h}^{1/2}$ in the case of soil sample 1; (b) moisture content θ at $\phi = 6 \text{ cm/h}^{1/2}$ in the case of soil sample 2; (c) sorptivity in the case of soil sample 1; and (d) sorptivity in the case of soil sample 2.

To compare the results produced by the three different methods and for both samples 1 and 2, the mean relative error was chosen as a metric.

Table 2 presents the differences in an analytical way to validate the strong agreement of our proposed solution with Philip's reference solution. Specifically, regarding sample 1, the difference is $\varepsilon_{\text{mean}} = 3.7 \times 10^{-3}$, while for sample 2, the difference is $\varepsilon_{\text{mean}} = 3.7 \times 10^{-3}$. As it emerged from the visualized results, the deviation between the Hang and Zhiqiang with Philip's reference solution is confirmed here as well, where for samples 1 and 2, we have $\varepsilon_{\text{mean}} = 2.5 \times 10^{-2}$ and $\varepsilon_{\text{mean}} = 7.8 \times 10^{-2}$, respectively.

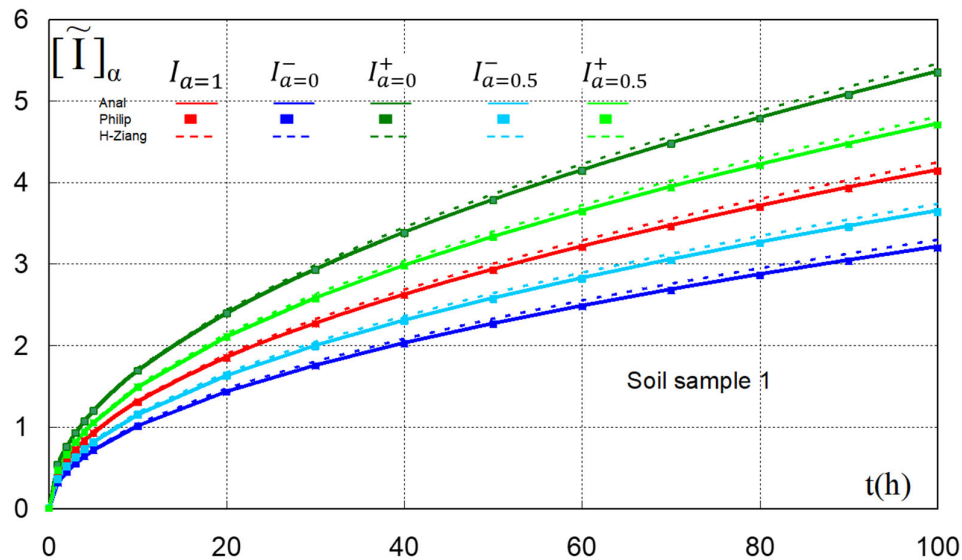


Figure 5. Cumulative absorption I for soil sample 1.

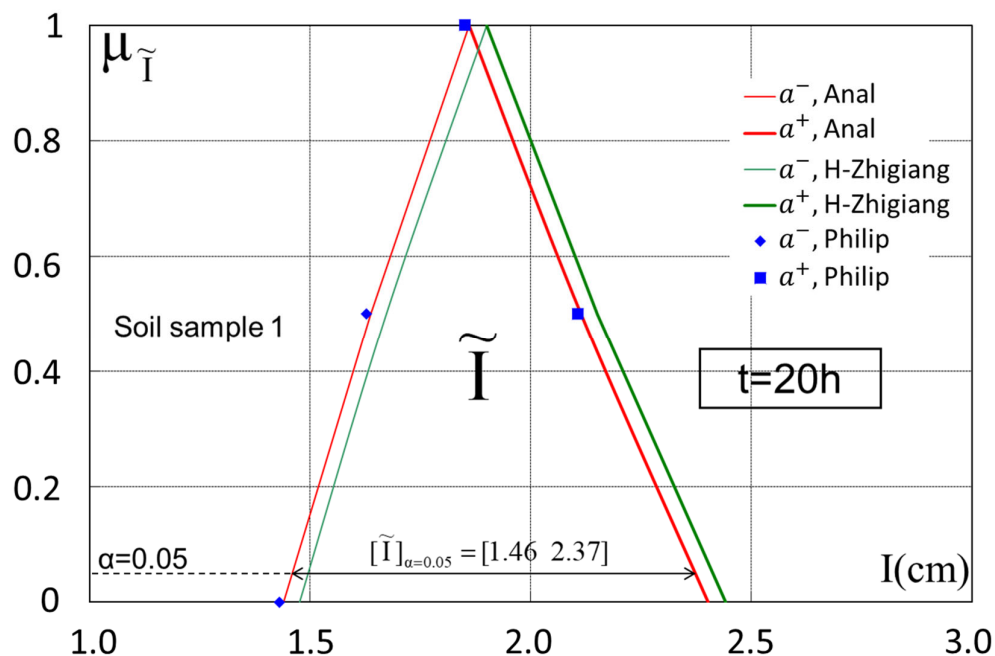


Figure 6. Membership function of cumulative absorption I.

Table 2. Relative error of S between the different solutions related to soil samples 1 and 2.

		Method			Difference	
S	α -Cut	Hang and Zhiqiang (1)	Analytical (2)	Philip (3)	(2) vs. (3)	(1) vs. (3)
Soil sample 1						
S	1	0.425	0.416	0.416	0	0.0254
S-	0.5	0.374	0.366	0.364	0.005	0.0268
S+	0.5	0.481	0.473	0.471	0.0044	0.0225
S-	0	0.33	0.322	0.32	0.0046	0.0293
S+	0	0.546	0.537	0.535	0.0043	0.0206
				mean =	0.0037	0.0249
Soil sample 2						
S	1	4.26	3.89	3.878	0.003	0.09
S-	0.5	4.057	3.764	3.753	0.003	0.075
S+	0.5	4.325	4.017	4.005	0.003	0.074
S-	0	3.926	3.64	3.629	0.003	0.076
S+	0	4.461	4.146	4.134	0.003	0.073
				mean =	0.003	0.078

4. Conclusions

The proposed fuzzy solution of Richard's equation is based on the Hukuhara theory [39] with the generalized Hukuhara (gH-derivative), as well as its extension to partial differential equations. Today, practical problems useful for engineers can be solved, considering the fuzziness of various sizes.

In the current work, real measurements of 12 different types of soils were used in order to draw reliable conclusions. The application of our proposed solution to real cases of soils has proved that the present analytical solution is in close agreement with Philip's semi-analytical method, which is used as a reference solution that proves the accuracy and reliability of the new fuzzy analytical method. In addition, it was observed to be extremely easy and simple to calculate in comparison to other methods, without affecting the accuracy of the results.

Furthermore, it should be highlighted that according to the possibility theory, for practical cases, such as irrigation, drainage, and water resources projects, the engineers and designers could now have a better understanding of the real physical conditions (including the uncertainties), knowing the confidence intervals of the crisp value of sorptivity and cumulative infiltration with a certain strong probability.

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References

1. Buckingham, E. *Studies on the Movement of Soil Moisture*; Government Publishing Office: Washington, DC, USA, 1907.
2. Gardner, W.; Widtsoe, J.A. The Movement of Soil Moisture. *Soil Sci.* **1921**, *11*, 215–232. <https://doi.org/10.1097/00010694>.

3. Richards, L.A. Capillary conduction of liquids through porous mediums. *Physics* **1931**, *1*, 318–333. <https://doi.org/10.1063/1.1745010>.
4. Childs, E.C. The Transport of Water through Heavy Clay Soils. I. *J. Agric. Sci.* **1936**, *26*, 114–127. <https://doi.org/10.1017/S0021859600021833>.
5. Childs, E.C. The Transport of Water through Heavy Clay Soils. III. *J. Agric. Sci.* **1936**, *26*, 527–545. <https://doi.org/10.1017/S0021859600023078>.
6. Klute, A. A numerical method for solving the flow equation for water in unsaturated materials. *Soil Sci.* **1952**, *73*, 105–116. <https://doi.org/10.1097/00010694-195202000-00003>.
7. Darcy, H. *Les Fontaines Publiques de La Ville de Dijon*; Victor Dalmont: Paris, France, 1956.
8. Philip, J.R. The theory of infiltration. *Soil Sci.* **1957**, *83*, 345–358. <https://doi.org/10.1097/00010694-195705000-00002>.
9. Philip, J.R. Theory of Infiltration. In *Advances in Hydrosience*; Elsevier: Amsterdam, The Netherlands, 1969; pp. 215–296.
10. Shahraiyini, H.T.; Ashtiani, B.A. Comparison of Finite Difference Schemes For Water Flow in Unsaturated Soils. *Int. J. Mech. Ind. Aerosp. Eng.* **2009**, *3*, 10–14.
11. Chávez-Negrete, C.; Domínguez-Mota, F.J.; Santana-Quinteros, D. Numerical Solution of Richards' Equation of Water Flow by Generalized Finite Differences. *Comput. Geotech.* **2018**, *101*, 168–175. <https://doi.org/10.1016/j.compgeo.2018.05.003>.
12. Samarinas, N.; Tzimopoulos, C.; Evangelides, C. Fuzzy Numerical Solution to Horizontal Infiltration. *Int. J. Circuits Syst. Signal Process.* **2018**, *12*, 326–332.
13. Savović, S.; Drljača, B.; Djordjević, A. A Comparative Study of Two Different Finite Difference Methods for Solving Advection–Diffusion Reaction Equation for Modeling Exponential Traveling Wave in Heat and Mass Transfer Processes. *Ric. Mat.* **2022**, *71*, 245–252. <https://doi.org/10.1007/s11587-021-00665-2>.
14. Samarinas, N.; Tzimopoulos, C.; Evangelides, C. Fuzzy Numerical Solution to the Unconfined Aquifer Problem under the Boussinesq Equation. *Water Supply* **2021**, *21*, 3210–3224. <https://doi.org/10.2166/ws.2021.115>.
15. Samarinas, N.; Tzimopoulos, C.; Evangelides, C. An Efficient Method to Solve the Fuzzy Crank–Nicolson Scheme with Application to the Groundwater Flow Problem. *J. Hydroinformatics* **2022**, *24*, 590–609. <https://doi.org/10.2166/hydro.2022.150>.
16. Crevoisier, D.; Chanzy, A.; Voltz, M. Evaluation of the Ross Fast Solution of Richards' Equation in Unfavourable Conditions for Standard Finite Element Methods. *Adv. Water Resour.* **2009**, *32*, 936–947. <https://doi.org/10.1016/j.advwatres.2009.03.008>.
17. Fahs, M.; Younes, A.; Lehmann, F. An Easy and Efficient Combination of the Mixed Finite Element Method and the Method of Lines for the Resolution of Richards' Equation. *Environ. Model. Softw.* **2009**, *24*, 1122–1126. <https://doi.org/10.1016/j.envsoft.2009.02.010>.
18. Arampatzis, G.K.; Tzimopoulos, C.D.; Evangelides, C.H. Numerical Solution of Richards' Equation With Control Volume Method. *J. Mech. Behav. Mater.* **2004**, *15*, 291–308. <https://doi.org/10.1515/JMBM.2004.15.4-5.291>.
19. Pour, M.; Shoshtari, M.; Adid, A. Numerical Solution of Richards Equation by Using of Finite Volume Method. *World Appl. Sci. J.* **2011**, *14*, 1832–1842.
20. Philip, J.R. On Solving the Unsaturated Flow Equation 1: The Flux Concentration Relation. *Soil Sci.* **1973**, *116*, 328–335.
21. Tzimopoulos, C.; Papadopoulos, K.; Evangelides, C.; Papadopoulos, B. Fuzzy Solution to the Unconfined Aquifer Problem. *Water* **2018**, *11*, 54. <https://doi.org/10.3390/w11010054>.
22. Tzimopoulos, C.; Papadopoulos, K.; Evangelides, C. Fuzzy Analytical Solution to Horizontal Infiltration. *J. Comput. Methods Sci. Eng.* **2019**, *19*, 913–927. <https://doi.org/10.3233/JCM-190023>.
23. Tzimopoulos, C.; Papaevangelou, G.; Papadopoulos, K.; Evangelides, C.; Arampatzis, G. Fuzzy Analytical Solution to Vertical Infiltration. *J. Softw. Eng. Appl.* **2020**, *13*, 41–46.
24. Tzimopoulos, C.; Papadopoulos, K.; Evangelides, C.; Papadopoulos, B. Analytical Solutions of Advection-Dispersion Equation Using Fuzzy Theory. *Desalin. WATER Treat.* **2020**, *193*, 302–312. <https://doi.org/10.5004/dwt.2020.25809>.
25. Tzimopoulos, C.; Samarinas, N.; Papadopoulos, K.; Evangelides, C. Fuzzy Unsteady-State Drainage Solution for Land Reclamation. *Hydrology* **2023**, *10*, 34. <https://doi.org/10.3390/hydrology10020034>.
26. Tolikas, P.K.; Sidiropoulos, E.G.; Tzimopoulos, C.D. A Simple Analytical Solution for the Boussinesq One-Dimensional Groundwater Flow Equation. *Water Resour. Res.* **1984**, *20*, 24–28. <https://doi.org/10.1029/WR020i001p00024>.
27. Brutsaert, W. Some Exact Solutions for Nonlinear Desorptive Diffusion. *Z. Angew. Math. Phys.* **1982**, *33*, 540–546. <https://doi.org/10.1007/BF00955702>.
28. Crank, J. *The Mathematics of Diffusion*, 1st ed.; Oxford University Press: London, UK, 1975.
29. Lisle, I.G.; Parlange, J.-Y. Analytical Reduction for a Concentration Dependent Diffusion Problem. *ZAMP Z. Angew. Math. Phys.* **1993**, *44*, 85–102. <https://doi.org/10.1007/BF00914355>.
30. Hang, L.; Zhiqiang, L. Almost Analytic Solutions and Their Tests of the Horizontal Diffusion Equation for the Movement of Water in Unsaturated Soil. *Appl. Math. Mech.* **1997**, *18*, 647–655. <https://doi.org/10.1007/BF00127012>.
31. Prevedello, C.L.; Loyola, J.M.T.; Reichardt, K.; Nielsen, D.R. New Analytic Solution of Boltzmann Transform for Horizontal Water Infiltration into Sand. *Vadose Zo. J.* **2008**, *7*, 1170–1177. <https://doi.org/10.2136/vzj2007.0181>.
32. Tzimopoulos, C.; Evangelides, C.; Arampatzis, G. Explicit Approximate Analytical Solution of the Horizontal Diffusion Equation. *Soil Sci.* **2015**, *180*, 47–53. <https://doi.org/10.1097/SS.0000000000000113>.
33. Vauclin, M.; Haverkamp, R. Solutions Quasi Analytiques de l'équation d'absorption de l'eau Par Les Sols Non Saturés. *Agronomie* **1985**, *5*, 597–606.

34. Huang, J. Analytical Model for Volatile Organic Compound Transport in the Coupled Vadose Zone–Groundwater System. *J. Hydrol. Eng.* **2021**, *26*, 04020058. [https://doi.org/10.1061/\(ASCE\)HE.1943-5584.0002029](https://doi.org/10.1061/(ASCE)HE.1943-5584.0002029).
35. Ali, I.; Saleem, M.T. Applications of Orthogonal Polynomials in Simulations of Mass Transfer Diffusion Equation Arising in Food Engineering. *Symmetry* **2023**, *15*, 527. <https://doi.org/10.3390/sym15020527>.
36. Ali, I.; Saleem, M.T. Spatiotemporal Dynamics of Reaction–Diffusion System and Its Application to Turing Pattern Formation in a Gray–Scott Model. *Mathematics* **2023**, *11*, 1459. <https://doi.org/10.3390/math11061459>.
37. Zadeh, L.A. Fuzzy logic. *Computer* **1988**, *21*, 83–93. <https://doi.org/10.1109/2.53>.
38. Puri, M.L.; Ralescu, D.A. Differentials of Fuzzy Functions. *J. Math. Anal. Appl.* **1983**, *91*, 552–558. [https://doi.org/10.1016/0022-247X\(83\)90169-5](https://doi.org/10.1016/0022-247X(83)90169-5).
39. Hukuhara, M. Integration Des Applications Mesurables Dont La Valeur Est Un Compact Convexe. *Funkc. Ekvacioj* **1967**, *10*, 205–233.
40. Vorobiev, D.; Seikkala, S. Towards the Theory of Fuzzy Differential Equations. *Fuzzy Sets Syst.* **2002**, *125*, 231–237.
41. O'Regan, D.; Lakshmikantham, V.; Nieto, J. Initial and Boundary Value Problem for Fuzzy Differential Equations. *Nonlinear Anal.* **2003**, *54*, 405–415.
42. Nieto, J.J.; Rodríguez-López, R. Bounded Solutions for Fuzzy Differential and Integral Equations. *Chaos Solitons Fractals* **2006**, *27*, 1376–1386. <https://doi.org/10.1016/j.chaos.2005.05.012>.
43. Bede, B.; Gal, S.G. Generalizations of the Differentiability of Fuzzy-Number-Valued Functions with Applications to Fuzzy Differential Equations. *Fuzzy Sets Syst.* **2005**, *151*, 581–599. <https://doi.org/10.1016/j.fss.2004.08.001>.
44. Bede, B. A Note on “Two-Point Boundary Value Problems Associated with Non-Linear Fuzzy Differential Equations”. *Fuzzy Sets Syst.* **2006**, *157*, 986–989. <https://doi.org/10.1016/j.fss.2005.09.006>.
45. Stefanini, L. A Generalization of Hukuhara Difference and Division for Interval and Fuzzy Arithmetic. *Fuzzy Sets Syst.* **2010**, *161*, 1564–1584. <https://doi.org/10.1016/j.fss.2009.06.009>.
46. Allahviranloo, T.; Gouyandeh, Z.; Armand, A.; Hasanoglu, A. On Fuzzy Solutions for Heat Equation Based on Generalized Hukuhara Differentiability. *Fuzzy Sets Syst.* **2015**, *265*, 1–23. <https://doi.org/10.1016/j.fss.2014.11.009>.
47. Mazandarani, M.; Xiu, L. A Review on Fuzzy Differential Equations. *IEEE Access* **2021**, *9*, 62195–62211. <https://doi.org/10.1109/ACCESS.2021.3074245>.
48. Gardner, W.R.; Mayhugh, M.S. Solutions and Tests of the Diffusion Equation for the Movement of Water in Soil. *Soil Sci. Soc. Am. J.* **1958**, *22*, 197–201. <https://doi.org/10.2136/sssaj1958.03615995002200030003x>.
49. Reichardt, K.; Nielsen, D.R.; Biggar, J.W. Scaling of Horizontal Infiltration into Homogeneous Soils. *Soil Sci. Soc. Am. J.* **1972**, *36*, 241–245. <https://doi.org/10.2136/sssaj1972.03615995003600020014x>.
50. Miller, R.D.; Bresler, E. A Quick Method for Estimating Soil Water Diffusivity Functions. *Soil Sci. Soc. Am. J.* **1977**, *41*, 1020–1022. <https://doi.org/10.2136/sssaj1977.03615995004100050044x>.
51. Meyer zur Capellen, W. *Integralktafel—Sammlung Unbestimmter Integrale Elementarer Functionen*; Springer: Berlin/Heidelberg, Germany, 1950.
52. Oldham, K.; Myland, J.; Spanier, J. *An Atlas of Functions*, 2nd ed.; Springer: New York, NY, USA, 2009.
53. Ramanujan, S.; Hardy, G.H.; Seshu Aiyar, P.V.; Wilson, B.M.; Berndt, B.C. *Collected Papers of Srinivasa Ramanujan*; AMS Chelsea Pub.: Providence, RI, USA, 2000.
54. Selim, H.M.; Kirkham, D.; Amemiya, M. A Comparison of Two Methods for Determining Soil Water Diffusivity. *Soil Sci. Soc. Am. J.* **1970**, *34*, 14–18. <https://doi.org/10.2136/sssaj1970.03615995003400010010x>.
55. Whisler, F.D.; Klute, A.; Peters, D.B. Soil Water Diffusivity from Horizontal Infiltration. *Soil Sci. Soc. Am. J.* **1968**, *32*, 6–11. <https://doi.org/10.2136/sssaj1968.03615995003200010003x>.
56. Clothier, B.E.; Scotter, D.R.; Green, A.E. Diffusivity and One-Dimensional Absorption Experiments. *Soil Sci. Soc. Am. J.* **1983**, *47*, 641–644. <https://doi.org/10.2136/sssaj1983.03615995004700040006x>.
57. Jackson, R.D. Porosity and Soil-Water Diffusivity Relations. *Soil Sci. Soc. Am. J.* **1963**, *27*, 123–126. <https://doi.org/10.2136/sssaj1963.03615995002700020011x>.
58. McBride, J.F.; Horton, R. An Empirical Function to Describe Measured Water Distributions From Horizontal Infiltration Experiments. *Water Resour. Res.* **1985**, *21*, 1539–1544. <https://doi.org/10.1029/WR021i010p01539>.
59. Nofziger, D.L. Errors in Gamma-Ray Measurements of Water Content and Bulk Density in Nonuniform Soils. *Soil Sci. Soc. Am. J.* **1978**, *42*, 845–850. <https://doi.org/10.2136/sssaj1978.03615995004200060001x>.
60. Evangelides, C.; Arampatzis, G.; Tzimopoulos, C. Estimation of Soil Moisture Profile and Diffusivity Using Simple Laboratory Procedures. *Soil Sci.* **2010**, *175*, 118–127. <https://doi.org/10.1097/SS.0b013e3181d53bb6>.
61. Negoitǎ, C.V. and Ralescu, D.A. (1975) Representation Theorems for Fuzzy Concepts. *Kybernetes* **1975**, *4*, 169–174. <https://doi.org/10.1108/eb005392>.
62. Goetshel, R.; Voxman, W. Elementary Fuzzy Calculus. *Fuzzy Sets Syst.* **1986**, *18*, 31–43. [https://doi.org/10.1016/0165-0114\(86\)90026-6](https://doi.org/10.1016/0165-0114(86)90026-6).
63. Bede, B.; Stefanini, L. Generalized Differentiability of Fuzzy-Valued Functions. *Fuzzy Sets Syst.* **2013**, *230*, 119–141. <https://doi.org/10.1016/j.fss.2012.10.003>.
64. Khastan, A.; Nieto, J.J. A Boundary Value Problem for Second Order Fuzzy Differential Equations. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 3583–3593. <https://doi.org/10.1016/j.na.2009.12.038>.

65. Dubois, D.; Prade, H. When Upper Probabilities Are Possibility Measures. *Fuzzy Sets Syst.* **1992**, *49*, 65–74. [https://doi.org/10.1016/0165-0114\(92\)90110-P](https://doi.org/10.1016/0165-0114(92)90110-P).
66. Dubois, D.; Foulloy, L.; Mauris, G.; Prade, H. Probability-Possibility Transformations, Triangular Fuzzy Sets, and Probabilistic Inequalities. *Reliab. Comput.* **2004**, *10*, 273–297. <https://doi.org/10.1023/B:REOM.0000032115.22510.b5>.
67. Dubois, D.; Prade, H. Possibility Theory and Possibilistic Logic: Tools for Reasoning under and about Incomplete Information. In *Intelligence Science III*; Springer: Cham, Switzerland, 2021; pp. 79–89.
68. Mylonas, N. Applications in Fuzzy Statistic and Approximate Reasoning. Ph.D. Thesis, University of Thrace, Komotini, Greece, 2022.
69. Nguyen, H.T. A Note on the Extension Principle for Fuzzy Sets. *J. Math. Anal. Appl.* **1978**, *64*, 369–380. [https://doi.org/10.1016/0022-247X\(78\)90045-8](https://doi.org/10.1016/0022-247X(78)90045-8).
70. Tzimopoulos, C.; Papadopoulos, K.; Papadopoulos, B.; Samarinas, N.; Evangelides, C. Fuzzy Solution of Nonlinear Boussinesq Equation. *J. Hydroinformatics* **2022**, *24*, 1127–1147. <https://doi.org/10.2166/hydro.2022.026>.

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