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An Inventory Model with Price-, Time- and Greenness-Sensitive Demand and Trade Credit-Based Economic Communications

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Abstract

Background: Price is the most authoritative constituent among the factors shaping consumer demand. Growing consciousness among global communities regarding environmental issues makes greenness one of the key factors controlling demand, along with time, which drives demand in markets. This paper addresses such issues associated with a retail purchase scenario. *Methods:* Consumer's demand for products is hypothesized to be influenced by pricing, time and the green level of the product in the proposed model. Time-dependent inventory carrying cost and green level-induced purchasing cost are considered. The average cost during the decision cycle is the objective function that is analyzed in trade credit phenomena, involving delayed payment by the manufacturer to the supplier. The Convex optimization technique is used to find an optimal solution for the model. *Results:* Once a local optimal solution is found, sensitivity analysis is conducted to determine the optimal value of the objective function and decision variables for other impacting parameters. Results reveal that demand-boosting parameters, for instance, discounts on price and green activity, result in additional average costs. *Conclusions:* Discounts on price and green activity advocate a large supply capacity by boosting demand, creating opportunities for the retailer to earn more revenue.

Keywords: selling price; time- and greenness-dependent demand; time-dependent holding cost; deterioration; trade credit and back-ordering; total average cost optimization

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1. Introduction

Retail phenomena include smart strategies regarding purchasing from the supplier, maintaining the warehouse and supplying products to customers. The economic order quantity (EOQ) model is one of the mathematical representations for economic communications with effective enrollment for ordering, ensuring smooth supply flow of items and minimizing waste. In this regard, demand is the most crucial component connected to an

EOQ model. Demand prediction and acute measures are challenging jobs in the real-world market because demand depends on many impacting parameters. First, pricing is the most influential demand-impacting issue. A reduction in selling price can increase demand. Increased demand can lead to more revenue by meeting customer demand. On the other hand, the retailer may face a loss due to the diminishing selling price. Retail price is a very significant parameter impacting demand, profit or cost goal and associated strategies for maintaining warehouses and retail stores [1]. Second, demand changes with time, either in a continuous or in a discrete manner. In some retail businesses, demand grows as time progresses, while it has the reverse impact in other scenarios. Also, whatever the impact time has on demand, it is not proportionately related to the cost minimization and profit maximization objective. In such cases, time can be included in demand forecasting hypotheses [2,3]. Third, many organizations are now embracing green initiatives as part of their strategies toward reducing their carbon footprints while striving for more ecological methods of operation, and these can create a positive attitude among the customers towards products and retail activities of such organizations [4,5]. Therefore, inventory systems will be among the sectors where sustainability issues are evident. However, it can include additional costs for maintaining the greenness of the product. The complexity of the system requires balancing demand patterns with sustainability goals and financial policies. All of these factors are very regular scenarios impacting retail activities. In the present paper, we accumulate the mentioned intuitions and perspectives for formulating an inventory model, where the influences of pricing, greenness and time on demand are considered. The inventory carrying process includes damage to products in the warehouse as time passes. Therefore, the deterioration of items is listed among the hypotheses for formulating mathematical models representing retail inventory scenarios.

In today's competitive and sustainability-driven markets, the development of effective inventory models is essential to balancing profitability, customer satisfaction and environmental responsibility. Traditional inventory models have primarily focused on cost optimization, demand forecasting and replenishment strategies under deterministic or stochastic conditions [2,3]. However, in recent years, demand has been shown to be significantly influenced by multiple factors, such as selling price, time-dependent market conditions and greenness of products [6,7]. These aspects reflect changing consumer preferences, stricter environmental regulations and the growing importance of corporate social responsibility in supply chain management.

Despite their relevance, existing models often remain limited in scope, for example, focusing on price-sensitive or time-dependent demand or overlooking the role of environmental factors in shaping consumer choices [8]. Moreover, the seller–buyer relationship can be defined by adding trade credit [9]. Hence, there are always gaps between industrial applications and mathematical models for sustainable production. The present study addresses these issues by developing an inventory model with price-, time- and greenness-sensitive demand under trade credit policy. This approach not only extends existing theoretical models but also provides practical insights for decision makers operating in sustainability-oriented markets. By explicitly considering greenness, the model captures the growing role of environmental awareness in consumer purchasing decisions. Furthermore, the integration of trade credit enables a more realistic representation of financial flows in supply chains. The improvements offered by this model lie in its ability to (i) bridge methodological gaps in the current literature, (ii) highlight the impact of greenness on supply chain performance and (iii) provide a more comprehensive decision-making tool for balancing profitability with sustainability. The motivations and objectives of this paper are given in detail in the next section.

The remaining parts of this paper are arranged as follows: Section 2 summarizes an extensive literature survey and motivations of the present work. Section 3 describes fundamental hypotheses and associated notations for formulating the proposed mathematical model. The mathematical representation of the lot-size optimization process is designed, and the convex optimization of the objective functions in different cases is analyzed in Section 4. After taking a few associated parameters as input, the numerical results regarding the convexity of the average cost functions and sensitivity of the optimal solutions in multiple scenarios are given in tables and figures in Section 5. This paper concludes in Section 6 with concluding remarks on the overall investigation and its significant findings.

2. Literature Review

Strategies and decision making in supply chain scenarios include inventory control as one of the authoritative concerns. In the context of a retail organization, inventory lot management comes into play, where the retailer optimizes the order-lot decision cycle to maximize profit (equivalently, minimize cost), thereby providing a smooth supply to meet customer demand. So, demand assessment is a precious component in a retailer's strategy. Numerous variables in various circumstances influence the behavior and purchasing habits of consumers. Among such issues, price is a leading one. Discounts on the selling price may bring more customers towards certain products. Time can also control the demand. Environmental sensitivity may have an influence on demand as well as on induced costs. Moreover, financial facilities and agreements, such as trade credit, shape retail activities. With all these real-time complexities, deterioration of products during inventory carrying represents another challenge. Back-ordering, pricing tactics, time-sensitive demand, environmental factors (greenness) and time-dependent holding costs constitute a few of the aspects that integrated inventory systems aim to optimize in relation to inventory levels. The effects of price, demand trends, holding costs, trade credit, back-ordering and green initiatives in relation to lot-size optimization are the main topics of this literature review, which examines these subjects in the context of deteriorating items.

2.1. Price- and Time-Driven Demand Inventory Model for Deteriorating Items

A pioneering work regarding inventory decision optimization came with the development of fundamental concepts by Harris [10], who introduced foundational concepts. Later, Wilson [11] independently derived the widely recognized economic order quantity (EOQ) formula. These early models assumed a constant demand rate; this, however, is often considered unrealistic in practical scenarios. Following the development of the EOQ model, numerous researchers have explored variable demand rates to better reflect market dynamics. Among these, time-varying demand has emerged as a significant focus of study. Khanra et al. [12] pioneered the concept of a quadratic demand rate in inventory modeling, which was subsequently expanded upon by Ghosh and Chaudhuri [13]. Later, Begum et al. [14] proposed an inventory model featuring partial backlogging, where the rate of consumption was taken as a quadratic function of time. Several studies [15–17] extended the mentioned concept by introducing complex phenomena such as deterioration, financial agreements, pricing and shortages. Saha and Sen [18] proposed a model in which the negatively proportional relation between demand and price was addressed. Moreover, the deterioration of products was also considered as a function of time in their study. Chen et al. [19] also considered a model that emphasizes demand which depends on price, time and stock levels. The work highlighted the dynamic and multifaceted nature of real-world demand, especially for perishable goods.

2.2. Trade Credit in Inventory Systems

Trade credit is a financial agreement between a supplier and a buyer where payment for goods is deferred for a short time. This financing mechanism plays a critical role in managing inventory for deteriorating items by alleviating the buyer's cash flow constraints. Goyal [20] initially developed a financial agreement-based inventory management model. He considered demand to be a constant and allowed late payment. Later, Aggarwal and Jaggi [21] expanded upon [20]'s model by incorporating the aspect of deteriorating items. Jamal et al. [22] further expanded on this concept by introducing allowance for shortages in the model. Numerous studies, including [23,24], examined the impacts of trade credit on inventory decisions. They found that trade credit allowed firms to maintain higher inventory levels without immediate payment obligations. It mitigated stockouts and reduced risks associated with deterioration. However, this benefit must be balanced with the costs of carrying excessive inventory, which may increase over time due to deterioration. Some models integrated trade credit into pricing strategies. For example, in a model proposed by Chang et al. [25], it was confirmed that trade credit could affect demand and inventory turnover rates. The optimal inventory strategy, therefore, needs to consider not only the costs of deterioration but also the financial terms associated with trade credit. Numerous research articles [26–29] addressed the trade credit problem. Trade credit policy in this model is not only a financial tool but also a decision that connects price-, time- and greenness-dependent demand with environmental responsibility.

2.3. Back-Ordering in Inventory Systems

Ensuring seamless supply in response to consumer demand is a difficult endeavor, and as a result, shortages occur in retail and manufacturing operations. The standard inventory management thought process implies that when there is a scarcity of supply, customers will either wait for back orders or migrate to an alternate shop. As a result, researchers and real-world decision makers make different assumptions and strategies for scarcity policies depending on the nature of the items and consumer attitudes. Datta and Pal [30] accounted for shortages with demand as a linear function of time . Sarkar and Pan [31] discussed inflation as a primary focus in a financial context for an inventory model allowing for shortages and finite replenishment as a response. Later, the literature on inventory models in the context of shortages evolved. The influence of time on demand was considered in the study by Bose et al. [32]. Tripathy and Pradhan [33] considered an order-lot optimization model based on the Weibull distribution, incorporating partial backlogging. Yang [34] presented a partial backlogging inventory model representing a manufacturing process. Maihami and Kamalabad [1] considered a model with the hypotheses of time- and price-dependent demand, incorporating partial backlogging. Recent studies [35–37] have developed models where back-ordering is considered alongside deterioration rates, recognizing the need to balance customer satisfaction and inventory costs. Back-ordering in this model helps to reduce costs and acts as a link between trade credit policy and sustainability.

2.4. Greenness Considerations in Inventory Systems

The growing awareness of eco-friendly and sustainable consumerism has resulted in enormous changes in the worldwide market. As a result, inventory management includes green activities that help shape decision making. Green inventory systems seek to reduce the carbon footprint of inventory procedures, optimize energy use and implement measures such as waste reduction, recycling and the use of environmentally friendly packaging. Taleizadeh et al. [38] explored the optimal combination of product selling price and replenishment frequency in an inventory model designed for green product manufacturing. Khatua et al. [39] examined the influence of product greenness on demand and analyzed the relationship between profit and pollution within an imperfect production inventory system. Saxena et al. [4] proposed a green inventory model based on fuzzy logic. Yavari et al. [40] introduced a heuristic approach to address the resilient optimization challenges of green supply chains. Sarkar [5] explored the environmental and economic sustainability of a manufacturing company producing green products. A few studies [27,41] focused on how green practices could be incorporated into inventory systems for deteriorating items. For example, firms might adopt storage techniques that reduce energy consumption or use biodegradable materials to package perishable products, thereby decreasing environmental impact. Green technology is also contemporary in a carbon emission reduction context. Hasan et al. [42] compared inventory models with and without green technology and concluded that green employment is an effective strategy for earning revenue. Jauhari et al. [43] discussed inventory decisions in the context of the coexistence of traditional facilities and green facilities in the manufacturing process for avoiding penalty due to immoderate carbon emissions. Their investigation implied that green technology installation preserved environmental concerns and favored profit goal. Another recent publication by Jauhari et al. [44] explained advantages in terms of profit making by adding trade credit policies in their previous study. The production and warehousing of eco-friendly herbal products also bring the notion of greenness in another context. Bhavani et al. [45] addressed managerial implications of green and preservation consideration for uncertain decision making associated with herbal production–inventory phenomena. Ansu-Mensah [46] discussed impacts of green awareness on consumption. Ogiemwonyi et al. [47] revealed the fact that consciousness about the environment and sustainability might drive consumers' attitudes and consumption patterns. In a recent article by Baca and Reshidi [48], issues regarding green branding and its consequences to control consumers' behavior are addressed. It has been established that green branding provides a competitive edge by leveraging consumers' attitude towards eco-friendly products.

In addition, one can observe that traditional inventory models have primarily focused on cost optimization, time management and price-sensitive demand, but they often overlooked the crucial role of environmental consciousness in shaping purchasing behavior. By incorporating greenness explicitly, the presented model highlights the economic–environmental trade-offs faced by decision makers. Moreover, the theoretical foundation is strengthened by integrating the above literature on the impact of greenness on production, supply chains and consumer demand, positioning this study in the broader context of sustainable operations research.

2.5. Motivations and Objectives

An extensive survey of the existing literature on order and production lot models revealed a few significant points. First, much of the literature discussed the impacts of time and pricing on demand and their consequences on cost or profit goals. Several research articles also described the combined impact of time and pricing on demand from multiple perspectives. The impact of greenness on demand has not been discussed much in this regard. Furthermore, we did not find instances of investigations that amalgamated all three demand-impacting parameters. In this paper, we consider the cumulative influences of selling price, discount on selling price, time and greenness on customers' consumption patterns. We have motivations from real-world retail phenomena for formulating such a model. We consider a newly installed retail store in the market . Now, the demand for the products in that store gradually increases as time passes through communication and smart dealing. Moreover, smart pricing and, more specifically, the retailer can attract customers' attention towards products in their store by offering discounts on the selling price. Furthermore, assurances of the freshness can be another booster of demand for products in a newly opened retail store. Therefore, all these components can exist simultaneously in very regular retail phenomena. In the proposed model, we consider demand to be a decreasing function of selling price and an increasing function of discount rate and green level. We include the influences of time, which increases carrying costs, and that of the green level, which increases purchasing costs, in the hypotheses associated with the model. Second, the newly opened retail organization may have a shortage of purchasing capacity due to small initial capital. In such a case, the retailer will develop a trade credit-based economic transaction with the supplier to ensure initial survival in the retail industry. Therefore, we also include the trade credit phenomena in the mentioned retail activity, where the supplier allows the retailer to delay payment for purchasing for a limited period of time; then, charges are applied as interest. Based on the above-mentioned motivations, we develop an economic order quantity model within the framework of trade credit and analyze the local optimality through a convex optimization technique for minimizing average costs. The gaps in the literature and contributions of this paper are summarized in Table 1.

Table 1. Comparative analysis of existing inventory systems.

Authors	EPQ/EOQ	Demand Rate			Backlogging Type	Deterioration	Trade Credit
		Greenness Dependent	Price Dependent	Time Dependent			
Chen et al. [19]	EOQ	☒	☑	☑	☒	☑	☒
Saha and Sen [18]	EOQ	☒	☑	☑	☑	☑	☒
Maihami and Kamalabadi [1]	EOQ	☒	☑	☑	☑	☑	☒
Bhunia and Shaikh [49]	EOQ	☒	☑	☒	☑	☑	☒
Kumar et al. [50]	EOQ	☒	☑	☒	☒	☑	☑
Shaikh [51]	EOQ	☒	☑	☒	☑	☑	☑
Tiwari et al. [52]	EOQ	☒	☒	☑	☒	☑	☑
Tripathi [53]	EOQ	☒	☒	☑	☑	☑	☑
Shah et al. [54]	EOQ	☒	☑	☑	☒	☑	☑
Shaikh et al. [55]	EOQ	☒	☑	☒	☑	☑	☑
Khanra et al. [56]	EOQ	☒	☒	☑	☑	☒	☑
Shah et al. [57]	EOQ	☒	☑	☒	☒	☑	☑
Rameswari and Uthayakumar [58]	EOQ	☒	☑	☒	☒	☑	☑
Mishra et al. [59]	EOQ	☒	☑	☒	☒	☑	☑
Hakim et al. [60]	EOQ	☑	☑	☒	☒	☑	☒
Katariya and Shukla [9]	EOQ	☑	☑	☒	☒	☑	☑
Wang and Huang [61]	EPQ	☒	☑	☑	☒	☑	☒
Shah and Vaghela [62]	EPQ	☒	☑	☒	☒	☑	☒
Shekhar et al. [63]	EOQ	☒	☑	☑	☑	☒	☑
Akbar et al. [8]	EPQ	☒	☑	☒	☒	☑	☑
This study	EOQ	☑	☑	☑	☑	☑	☑

Based on the literature described above, the role and importance of our work in society can be listed as follows:

- As it includes greenness in demand, the model encourages environmentally friendly practices by promoting sustainable consumption and production.
- The model guides firms to understand the influence of pricing, time and environmental factors on consumer behavior, supporting greener market policies.
- The integration of trade credit policy provides practical insights to improve supplier–buyer relationships and financial sustainability.
- The model reflects the growing awareness and demand of society for eco-friendly products.
- The model provides evidence that can support trade credit policy, pricing strategies and green supply chain management.
- The model enhances the competitiveness of industries to adapt to sustainability, which is essential to long-term survival in global markets.
- With the help of green demand, the model reduces overproduction, waste and environmental degradation.

3. Primary Presumptions and Notations

3.1. Presumptions

(i) A single type of green item is allowed in the model.

(ii) Green initiatives are now playing a central role in the current business environment. Here, for a deteriorating green product, it is assumed that the demand is influenced by price, time and the green level of items (non-linear). Therefore, the demand is mathematically represented as $D = D_1(p, g) + \beta t,\ D_1(p, g) = a - bp(1 - d) + \gamma g^c$, where $a > 0, b > 0$ and $\gamma > 0$.

(iii) The purchasing cost is assumed to rise with the increase in the greenness level of the items, and expressed as $C_p(g) = k_0 + k_1 g^{\xi}$.

(iv) Shortages are permitted in the model and are assumed to be replenished at a fixed backlogging rate, represented by

$$B = \delta$$

(v) The holding cost is assumed to vary linearly with time, $h(t) = h_0 + h_1 t$, where $h_0, h_1 > 0$ are the scale parameters.

(vi) The supplier considers the delayed payment by the retailer for up to $t = M$ to be without any interests. After the permissible time is over, the retailer pays interest for delayed payment, with a rate of interest I_p over $[M, T]$, where $M \leq T$. Over the period $M \geq T$, no interest needs to be paid for stocked items.

(vii) The retailer is able to accumulate revenue and earn interest from the beginning of the inventory cycle until the expiry of the supplier's trade credit period. In other words, revenue can generate interest at rate I_e during the interval $t = 0$ to $t = M$ under trade credit terms.

(viii) Constant deterioration is assumed throughout the paper.

(ix) The replenishment rate is finite.

(x) The lead time is omitted.

(xi) An infinite time horizon is considered in this model.

3.2. Notations

The notations which are used in this paper and their corresponding descriptions with appropriate units are provided in Table 2.

Table 2. The notations and their description with appropriate units.

Notation	Unit	Description
p	USD /unit	Selling price
h_0	USD /unit/unit time	Holding cost
A	USD /cycle	Setup cost
C_p	USD/unit	Cost of unit production
k_0, k_1		Parameters related to purchasing cost
$I(t)$	units	Inventory level at time t
Q	units	Quantity of items ordered in a cycle
S	units	Maximum level of inventory
B	units	Maximum allowable shortage amount
D	units /time unit	Demand rate
M	time unit	Trade credit time of retailer provided by supplier
I_e	USD/USD/time unit	Interest earned
I_p	USD/USD/time unit	Interest paid
g		Greenness
d	%	Price discount on selling price $(0 < d < 1)$
ϕ	constant	Deterioration rate
γ		Greenness-sensitive parameter related to the demand for the product

Table 2. *Cont.*

Notation	Unit	Description
c		Index of greenness in demand
ξ	%	Index of greenness in purchasing cost
δ		Backlogging rate $(0 < \delta < 1)$
Decision variables		
T	time	Total time cycle
t_0	time	Time at which inventory vanishes
Objective function		
TC	USD	Total average cost

4. Formulation of the Model

At the start of inventory, no shortages are present, and after that, shortages occur due to demand and deterioration, as graphically illustrated in Figure 1. At time $t = 0$, the inventory model starts with Q units of items ordered by the retailer. Inventory depletion occurs due to demand exclusively within the interval $[0, t_0)$, while shortages occur with backordering during $[t_0, T)$. At $t = t_0$, the inventory level vanishes and reaches 0. Consequently, the variation in inventory over time is denoted by the differential equations below.

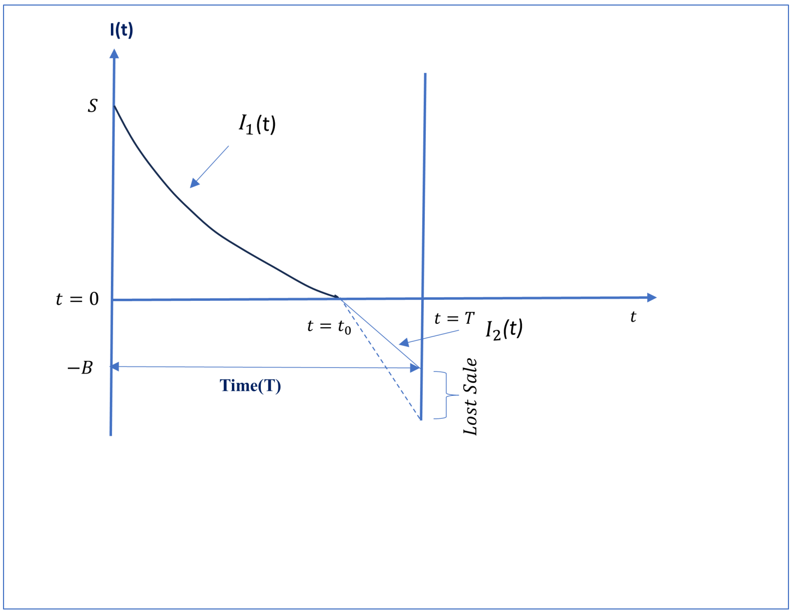

Figure 1. An illustration of the inventory management system.

$$\frac{dI_1(t)}{dt} + \phi I_1(t) = -D_1(p,g) - \beta t; \quad 0 < t \le t_0 \tag{1}$$

$$\frac{dI_2(t)}{dt} = -\delta D_1(p,g); \qquad t_0 \le t \le T, \tag{2}$$

with boundary conditions $I_1(0) = S$ and $I_2(t) = 0$ at $t = T$.

The solutions to Equations (1) and (2) provide the stock function in two distinct phases as follows:

$$I_1(t) = -\frac{D_1}{\phi} - \frac{t\beta}{\phi} + \frac{\beta}{\phi^2} + \left[\frac{D_1}{\phi} + \frac{t_0\beta}{\phi} - \frac{\beta}{\phi^2}\right]e^{\phi(t_0 - t)} \qquad \text{if } 0 < t \le t_0 \tag{3}$$

and

$$I_2(t) = \delta D_1(t_0 - t); \qquad \text{if } \ t_0 \le t \le T \tag{4}$$

The maximum inventory level can be calculated using the following formula:

$$S = I_{max} = I_1(0) = -\frac{D_1}{\phi} + \frac{\beta}{\phi^2} + \left[\frac{D_1}{\phi} + \frac{t_0\beta}{\phi} - \frac{\beta}{\phi^2}\right]e^{\phi t_0}$$

By substituting $t = T$ into Equation (4), the maximum demand backlog per cycle is derived as follows:

$$B = -I_2(T) = -D_1\delta(t_0 - T)$$

Therefore, the order quantity per cycle is

$$Q = S + B = -\frac{D_1}{\phi} + \frac{\beta}{\phi^2} + \left[\frac{D_1}{\phi} + \frac{t_0\beta}{\phi} - \frac{\beta}{\phi^2}\right]e^{\phi t_0} - D_1\delta(t_0 - T) \tag{5}$$

4.1. Itemize the Components of the Different Costs and Revenue for the Proposed Model

The inventory holding cost per cycle is

$$\begin{aligned} HC &= \int_0^{t_0}(h_0 + h_1 t)I_1(t)dt \\ &= \int_0^{t_0}(h_0 + h_1 t)\left[-\frac{D_1}{\phi} - \frac{t\beta}{\phi} + \frac{\beta}{\phi^2} + \left(\frac{D_1}{\phi} + \frac{t_0\beta}{\phi} - \frac{\beta}{\phi^2}\right)e^{\phi(t_0 - t)}\right]dt \\ &= h_0\int_0^{t_0}\left[-\frac{D_1}{\phi} - \frac{t\beta}{\phi} + \frac{\beta}{\phi^2} + Xe^{\phi(t_0-t)}\right]dt + h_1\int_0^{t_0} t\left[-\frac{D_1}{\phi} - \frac{t\beta}{\phi} + \frac{\beta}{\phi^2} + Xe^{\phi(t_0-t)}\right]dt \\ \text{Where } X &= \left(\frac{D_1}{\phi} + \frac{t_0\beta}{\phi} - \frac{\beta}{\phi^2}\right) \\ &= h_0\left[-\frac{D_1 t_0}{\phi} - \frac{\beta t_0^2}{2\phi} + \frac{\beta t_0}{\phi^2} - \frac{X}{\phi}(1 - e^{\phi t_0})\right] + h_1\left[-\frac{D_1 t_0^2}{2\phi} - \frac{\beta t_0^3}{3\phi} + \frac{\beta t_0^2}{2\phi^2}\right] + \\ &\quad h_1 X\left[-\frac{t_0}{\phi} - \frac{1}{\phi^2}(1 - e^{\phi t_0})\right] \end{aligned} \tag{6}$$

The inventory purchasing cost per cycle is

$$\begin{aligned} PC &= C_p(g)Q \\ &= (k_0 + k_1 g^{\xi})\left[-\frac{D_1}{\phi} + \frac{\beta}{\phi^2} + \left(\frac{D_1}{\phi} + \frac{t_0\beta}{\phi} - \frac{\beta}{\phi^2}\right)e^{\phi t_0} - D_1\delta(t_0 - T)\right] \end{aligned} \tag{7}$$

The back-ordering cost/shortage cost per cycle is

$$\begin{aligned} SC &= -C_b\int_{t_0}^{T} I_2(t)dt \\ &= -C_b\int_{t_0}^{T}\delta D_1(t_0 - t)dt \\ &= \frac{1}{2}C_b\delta D_1(T - t_0)^2 \end{aligned} \tag{8}$$

The lost sale cost per cycle is

$$\begin{aligned} LSC &= C_l\int_{t_0}^{T}(1-\delta)D_1(p,g)dt \\ &= C_l(1-\delta)D_1(p,g)(T - t_0) \end{aligned} \tag{9}$$

The ordering cost per cycle is

$$OC = A \tag{10}$$

Based on the values of T and M, two cases are possible for interest earned and interest charged, and these are illustrated graphically in Figure 1.

Case 1: $0 < M \leq t_0$

During this period, the buyer sells the product and earns revenue at an interest rate I_e. Hence, the interest earned is given by

$$\begin{aligned} IE_1 &= pI_e\int_0^M (D_1+\beta t)(M-t)dt \\ &= pI_e\int_0^M (D_1M - D_1t + M\beta t - \beta t^2)dt \\ &= pI_e\left(\frac{M^2}{2}+\frac{M^3\beta}{6}\right) \end{aligned} \tag{11}$$

The buyer sells a total of DM units after the period M and owes c_pDM to the supplier. Interest is charged at a rate I_p by the supplier to the retailer. Therefore, the interest charged is given by

$$\begin{aligned} IP_1 &= c_pI_e\int_M^{t_0} I_1(t)dt \\ &= c_pI_e\int_M^{t_1}\left[-\frac{D_1}{\phi}-\frac{t\beta}{\phi}+\frac{\beta}{\phi^2}+\left(\frac{D_1}{\phi}+\frac{t_0\beta}{\phi}-\frac{\beta}{\phi^2}\right)e^{\phi(t_0-t)}\right]dt \\ &= C_pI_p\left\{\left(-\frac{D_1t_0}{\phi}-\frac{\beta t_0^2}{2\phi}+\frac{\beta t_0}{\phi^2}+\frac{D_1M}{\phi}+\frac{M^2\beta}{2\phi}-\frac{\beta M}{\phi^2}\right)+\left(\frac{D_1}{\phi}+\frac{\beta t_0}{\phi}-\frac{\beta}{\phi^2}\right)\left(e^{\phi(t_0-M)}-1\right)\right\} \end{aligned} \tag{12}$$

Therefore, the total average cost is obtained as

$$\begin{aligned} TC_1 &= \frac{1}{T}\Big[\langle PC\rangle+\langle HC\rangle+\langle LSC\rangle+\langle SC\rangle+\langle OC\rangle+\langle IP_1\rangle-\langle IE_1\rangle\Big] \\ &= \frac{1}{T}\bigg[C_p\left\{-\frac{D_1}{\phi}+\frac{\beta}{\phi^2}+\left(\frac{D_1}{\phi}+\frac{t_0\beta}{\phi}-\frac{\beta}{\phi^2}\right)e^{\phi t_0}-D_1\delta(t_0-T)\right\}+h_0\bigg(-\frac{D_1t_0}{\phi}-\frac{\beta t_0^2}{2\phi}+\frac{\beta t_0}{\phi^2} \\ &\quad -\frac{X}{\phi}(1-e^{\phi t_0})\bigg)+h_1\left(-\frac{D_1t_0^2}{2\phi}-\frac{\beta t_0^3}{3\phi}+\frac{\beta t_0^2}{2\phi^2}\right)+h_1X\left(-\frac{t_0}{\phi}-\frac{1}{\phi^2}(1-e^{\phi t_0})\right) \\ &\quad +C_l(1-\delta)D_1(p,g)(T-t_0)+\frac{1}{2}C_b\delta D_1(T_0-t)^2+A+C_pI_p\bigg\{\bigg(-\frac{D_1t_0}{\phi}-\frac{\beta t_0^2}{2\phi}+\frac{\beta t_0}{\phi^2}+\frac{D_1M}{\phi} \\ &\quad +\frac{M^2\beta}{2\phi}-\frac{\beta M}{\phi^2}\bigg)+\left(\frac{D_1}{\phi}+\frac{\beta t_0}{\phi}-\frac{\beta}{\phi^2}\right)\left(e^{\phi(t_0-M)}-1\right)\bigg\}-pI_e\left(\frac{M^2}{2}+\frac{M^3\beta}{6}\right)\bigg] \end{aligned} \tag{13}$$

For $\phi << 1$, the Taylor series expansion provides the following second-degree approximations:

$$\left.\begin{aligned} e^{\phi t_0} &\approx 1+\phi t_0+\tfrac{t_0^2\phi^2}{2}, \\ e^{\phi(t_0-M)} &\approx 1+\phi(t_0-M)+\tfrac{(t_0-M)^2\phi^2}{2}. \end{aligned}\right\} \tag{14}$$

Using Equation (14), Equation (13) can be written as

$$\begin{aligned} TC_1 &= \frac{1}{T}\bigg[D_1(T-t_0)\left(\delta C_p+\frac{C_b\delta}{2}(T-t_0)+C_l(1-\delta)\right)+(D_1+\beta t_0)\left(h_0+C_pt_0+\frac{C_pI_p}{2}(t_0-M)^2\right) \\ &\quad +\frac{t_0^2}{2}C_p(D_1\phi+\beta t_0\phi-\beta)+h_1\beta\frac{t_0^3}{6\phi}+A-pI_e\left(\frac{D_1M^2}{2}+\frac{M^3\beta}{6}\right)\bigg] \end{aligned} \tag{15}$$

In this model, the global minima and convexity of the average cost function TC_1 are investigated with respect to inventory vanish time t_0 and the time cycle T.

Here, the optimization problem is as follows:

$$\begin{cases} \min TC_1(t_0, T) \\ \text{subject to } 0 < M \leq t_0 \\ \text{where, } TC_1 = \frac{1}{T}\Big[D_1(T-t_0)\Big(\delta C_p + \frac{C_b\delta}{2}(T-t_0) + C_l(1-\delta)\Big) + (D_1+\beta t_0)\Big(h_0 + C_p t_0\Big) \\ +\frac{t_0^2}{2}C_p(D_1\phi + \beta t_0\phi - \beta) + h_1\beta\frac{t_0^3}{6\phi} + A - pI_eMt_0\Big(D_1 + \frac{t_0\beta}{2}\Big) + pI_et_0^2\Big(\frac{D_1}{2} + \frac{t_0\beta}{3}\Big)\Big] \end{cases} \quad (16)$$

4.2. Theoretical Results and Optimal Solutions

The primary goal of the current study is to minimize the total average cost of the model by simultaneously optimizing the inventory vanish time (t_0) and total cycle time (T). Thus, in this section, we wish to demonstrate the convexity of the total cost function by establishing the theorems. In the subsequent section, we verify the convexity of the function by numerical illustrations following the graphs (via Mathematica 13.1 software).

Therefore, to validate the optimality of t_0 and T, it is essential to examine the necessary and sufficient conditions.

$$\frac{\partial TC_1(t_0,T)}{\partial t_0} = 0 \text{ and } \frac{\partial TC_1(t_0,T)}{\partial T} = 0. \quad (17)$$

Since the expression for total cost is

$$\begin{aligned} TC_1 &= \frac{1}{T}\Big[D_1(T-t_0)\Big(\delta C_p + \frac{C_b\delta}{2}(T-t_0) + C_l(1-\delta)\Big) + (D_1+\beta t_0)\Big(h_0 + C_p t_0 + \frac{C_pI_p}{2}(t_0-M)^2\Big) \\ &+\frac{t_0^2}{2}C_p(D_1\phi + \beta t_0\phi - \beta) + h_1\beta\frac{t_0^3}{6\phi} + A - pI_e\Big(\frac{D_1M^2}{2} + \frac{M^3\beta}{6}\Big)\Big] \end{aligned} \quad (18)$$

then,

$$\begin{aligned} \frac{\partial TC_1(t_0,T)}{\partial t_0} &= \frac{1}{2T\phi}\Big[h_1t_0^2\beta + \phi\Big\{2h_0\beta + 2C_lD_1(\delta-1) + 2D_1C_b\delta(t_0-T) - 2C_pD_1\Big(-1 + I_p(M-t_0) \\ &+\delta - t_0\phi\Big) + C_p\beta\Big(I_p(M^2 - 4Mt_0 + 3t_0^2) + t_0(2 + 3t_0\phi\Big)\Big\}\Big] = 0 \end{aligned} \quad (19)$$

and

$$\begin{aligned} \frac{\partial TC_1(t_0,T)}{\partial T} &= -\frac{1}{6T^2\phi}\Big[h_1t_0^3\beta + +\phi\Big\{6A - I_eM^3p\beta + 3t_0\beta\Big(2h_0 + C_pI_p(M-t_0)^2 + C_pt_0(1+t_0\phi)\Big) \\ &+3D_1\Big(2h_0 - I_epM^2 + C_pI_p(M-t_0)^2 - 2C_lt_0 - C_bT^2\delta + 2C_lt_0\delta + C_bt_0^2\delta + C_pt_0 \\ &(2 - 2\delta + t_0\phi)\Big)\Big\}\Big] = 0 \end{aligned} \quad (20)$$

The optimal values of T and t_0, named T^* and t_0^*, can be found by solving Equations (19) and (20), respectively. After substituting those values in Equation (18), the total cost of the system can be found. Additionally, the Hessian matrix with respect to t_0 and T is provided as

$$H = \begin{bmatrix} \frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2} & \frac{\partial^2 TC_1(t_0,T)}{\partial t_0 \partial T} \\ \frac{\partial^2 TC_1(t_0,T)}{\partial t_0 \partial T} & \frac{\partial^2 TC_1(t_0,T)}{\partial T^2} \end{bmatrix}$$

and the conditions on principle minors are as follows:

$$|H_{11}| = \frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2} > 0 \text{ and } |H| = \begin{vmatrix} \frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2} & \frac{\partial^2 TC_1(t_0,T)}{\partial t_0 \partial T} \\ \frac{\partial^2 TC_1(t_0,T)}{\partial t_0 \partial T} & \frac{\partial^2 TC_1(t_0,T)}{\partial T^2} \end{vmatrix} > 0 \quad \text{at } (t_0^*, T^*). \tag{21}$$

Due to the complicated nature of second-order derivatives, establishing the sufficiency criteria mathematically is challenging. Therefore, a graphical method is employed to verify the convexity.

Theorem 1. *The objective function TC_1 in Equation (16) is convex with respect to T and t_0 if $\frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2} > 0$, $\frac{\partial^2 TC_1(t_0,T)}{\partial T^2} > 0$ and $\left(\frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2}\right)\left(\frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2}\right) > \left(\frac{\partial^2 TC_1(t_0,T)}{\partial t_0 \partial T}\right)^2$ at (t_0^*, T^*).*

Proof. Using Equation (18), we have the following:

$$\frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2} = \frac{1}{T\phi}\left[h_1 t_0 \beta + \phi\left\{D_1 C_b \delta + C_p D_1 (I_p + \phi) + C_p \beta (1 - 2 I_p M + 3 I_p t_0 + 3 t_0 \phi)\right\}\right] \tag{22}$$

$$\begin{aligned} \frac{\partial^2 TC_1(t_0,T)}{\partial T^2} = {} & \frac{1}{3T^3\phi}\Big[h_1 t_0^3 \beta + \phi\Big\{6A - I_e M^3 p \beta + 3 t_0 \beta \Big(2h_0 + C_p I_p (M - t_0)^2 + C_p t_0 (1 + t_0 \phi)\Big) \\ & + 3D_1\Big(2h_0 - I_e M^2 p + C_p I_p (M - t_0)^2 - 2C_l t_0 + 2C_l t_0 \delta + C_b t_0^2 \delta + C_p t_0 (2 - 2\delta + t_0 \phi)\Big)\Big] \end{aligned} \tag{23}$$

and

$$\begin{aligned} \frac{\partial^2 TC_1(t_0,T)}{\partial t_0 \partial T} = {} & -\frac{1}{2T^2\phi}\Big[h_1 t_0^2 \beta + \phi\Big\{2C_l D_1(-1+\delta) + 2(h_0 \beta + C_b D_1 t_0 \delta) - 2C_p D_1\Big(-1 + I_p(M - t_0) + \delta - t_0 \phi\Big) \\ & + C_p \beta \Big(I_p (M^2 - 4 M t_0 + 3 t_0^2) + t_0 (2 + 3 t_0 \phi)\Big)\Big\}\Big] \end{aligned} \tag{24}$$

For minimization of the given function, we must have $\frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2} > 0$, $\frac{\partial^2 TC_1(t_0,T)}{\partial T^2} > 0$ and $\left(\frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2}\right)\left(\frac{\partial^2 TC_1(t_0,T)}{\partial T^2}\right) > \left(\frac{\partial^2 TC_1(t_0,T)}{\partial t_0 \partial T}\right)^2$ at (t_0^*, T^*).

Furthermore, since the cost function is highly non-linear, its convexity is demonstrated graphically in the following section.

Therefore, the total average profit $TC_1(t_0, T)$ is convex with respect to t_0, T. Thus, the proof of the theorem is complete. □

Case 2: $t_0 < M \le T$

The amount of interest earned by the retailer is obtained as

$$\begin{aligned} IE_2 &= pI_e \int_0^{t_0} (D_1 + \beta t)(t_0 - t)dt + pI_e (M - t_0) \int_0^{t_0} (D_1 + \beta t)dt \\ &= pI_e \int_0^{t_0} (D_1 t_0 - D_1 t + \beta t t_0 - \beta t^2)dt + pI_e (M - t_0) \int_0^{t_0} (D_1 + \beta t)dt \\ &= pI_e t_0 M \left(D_1 + \frac{\beta t_0}{2}\right) - pI_e t_0^2 \left(\frac{D_1}{2} + \frac{\beta t_0}{3}\right) \end{aligned} \tag{25}$$

The amount of interest paid by the retailer is given by

$$IP_2 = 0 \tag{26}$$

Therefore, the total average cost is

$$\begin{aligned} TC_2 &= \frac{1}{T}\Big[\langle PC\rangle + \langle HC\rangle + \langle LSC\rangle + \langle SC\rangle + \langle OC\rangle + \langle IP_2\rangle - \langle IE_2\rangle\Big] \\ &= \frac{1}{T}\Bigg[C_p(g)\Bigg\{-\frac{D_1}{\phi} + \frac{\beta}{\phi^2} + \left(\frac{D_1}{\phi} + \frac{t_0\beta}{\phi} - \frac{\beta}{\phi^2}\right)e^{\phi t_0} - D_1\delta(t_0 - T)\Bigg\} + h_0\Bigg(-\frac{D_1 t_0}{\phi} - \frac{\beta t_0^2}{2\phi} + \frac{\beta t_0}{\phi^2} \\ &\quad - \frac{X}{\phi}(1 - e^{\phi t_0})\Bigg) + h_1\left(-\frac{D_1 t_0^2}{2\phi} - \frac{\beta t_0^3}{3\phi} + \frac{\beta t_0^2}{2\phi^2}\right) + h_1 X\left(-\frac{t_0}{\phi} - \frac{1}{\phi^2}(1 - e^{\phi t_0})\right) \\ &\quad + C_l(1-\delta)D_1(p,g)(T - t_0) + \frac{1}{2}C_b\delta D_1(T_0 - t)^2 + A - pI_e t_0 M\left(D_1 + \frac{\beta t_0}{2}\right) + pI_e t_0^2\left(\frac{D_1}{2} + \frac{\beta t_0}{3}\right)\Bigg] \end{aligned} \quad (27)$$

Using Equation (14), Equation (27) can be written as

$$\begin{aligned} TC_2 &= \frac{1}{T}\Bigg[D_1(T - t_0)\left(\delta C_p + \frac{C_b\delta}{2}(T - t_0) + C_l(1-\delta)\right) + (D_1 + \beta t_0)\Big(h_0 + C_p t_0\Big) \\ &\quad + \frac{t_0^2}{2}C_p(D_1\phi + \beta t_0\phi - \beta) + h_1\beta\frac{t_0^3}{6\phi} + A - pI_e M t_0\left(D_1 + \frac{t_0\beta}{2}\right) + pI_e t_0^2\left(\frac{D_1}{2} + \frac{t_0\beta}{3}\right)\Bigg] \end{aligned} \quad (28)$$

In this model, the global minima and convexity of the average cost function TC_2 are investigated with respect to inventory vanish time t_0 and the total time cycle T.

Here, the optimization problem is as follows:

$$\begin{cases} \min TC_2(t_0, T) \\ \text{subject to } 0 < t_0 < M \le T \\ \text{where, } TC_2 = \frac{1}{T}\Big[D_1(T - t_0)\Big(\delta C_p + \frac{C_b\delta}{2}(T - t_0) + C_l(1-\delta)\Big) + (D_1 + \beta t_0)\Big(h_0 + C_p t_0\Big) \\ + \frac{t_0^2}{2}C_p(D_1\phi + \beta t_0\phi - \beta) + h_1\beta\frac{t_0^3}{6\phi} + A - pI_e M t_0\Big(D_1 + \frac{t_0\beta}{2}\Big) + pI_e t_0^2\Big(\frac{D_1}{2} + \frac{t_0\beta}{3}\Big)\Big] \end{cases} \quad (29)$$

4.3. Theoretical Results and Optimal Solutions

The primary goal of the current study is to minimize the total average cost of the model by simultaneously optimizing the inventory vanish time (t_0) and total cycle time (T). Thus, in this section, we wish to demonstrate the convexity of the total profit function by establishing the theorems. In the subsequent section, we verify the convexity of the function by numerical illustrations following the graphs (via Mathematica 13.1 software). Therefore, to validate the optimality of t_0 and T, we need to examine the necessary and sufficient conditions.

$$\frac{\partial TC_2(t_0, T)}{\partial t_0} = 0 \text{ and } \frac{\partial TC_2(t_0, T)}{\partial T} = 0, \quad (30)$$

Since the expression for total cost is

$$\begin{aligned} TC_2 &= \frac{1}{T}\Bigg[D_1(T - t_0)\left(\delta C_p + \frac{C_b\delta}{2}(T - t_0) + C_l(1-\delta)\right) + (D_1 + \beta t_0)\Big(h_0 + C_p t_0\Big) \\ &\quad + \frac{t_0^2}{2}C_p(D_1\phi + \beta t_0\phi - \beta) + h_1\beta\frac{t_0^3}{6\phi} + A - pI_e M t_0\left(D_1 + \frac{t_0\beta}{2}\right) + pI_e t_0^2\left(\frac{D_1}{2} + \frac{t_0\beta}{3}\right)\Bigg] \end{aligned} \quad (31)$$

then

$$\begin{aligned} \frac{\partial TC_2(t_0, T)}{\partial t_0} &= \frac{1}{2T\phi}\Bigg[h_1 t_0^2\beta + \phi\Bigg\{2C_l D_1(-1+\delta) + 2\Big(D_1 I_e p(-M + t_0) + h_0\beta + I_e p t_0(-M + t_0)\beta + C_b D_1(-T + t_0)\delta\Big) \\ &\quad + C_p\Big(D_1(2 - 2\delta + 2t_0\phi) + t_0\beta(2 + 3t_0\phi)\Big)\Bigg\}\Bigg] = 0 \end{aligned} \quad (32)$$

and

$$\frac{\partial TC_2(t_0,T)}{\partial T} = -\frac{1}{6T^2\phi}\Big[h_1t_0^3\beta+\phi\Big\{6A+3D_1\Big(2h_0+2C_pt_0-2I_eMpt_0+I_ept_0^2+2C_lt_0(-1+\delta)-C_bT^2\delta-2C_pt_0\delta$$
$$+C_bt_0^2\delta+C_pt_0^2\phi\Big)+t_0\beta\Big(6h_0+I_ept_0(-3M+2t_0)+3C_pt_0(1+t_0\phi)\Big)\Big\}\Big]=0 \quad (33)$$

By solving Equations (32) and (33), the optimal values T^* and t_0^* of T and t_0 can be determined. After substituting those optimum values in Equation (31), the total cost of the system can be calculated. Additionally, the Hessian matrix, which is required to verify the sufficient condition of optimality with respect to t_0 and T, is provided below.

$$H=\begin{bmatrix}\frac{\partial^2TC_2(t_0,T)}{\partial t_0^2} & \frac{\partial^2TC_2(t_0,T)}{\partial t_0\partial T}\\ \frac{\partial^2TC_2(t_0,T)}{\partial t_0\partial T} & \frac{\partial^2TC_2(t_0,T)}{\partial T^2}\end{bmatrix}$$

And the conditions on all principle minors are

$$|H_{11}|=\frac{\partial^2TC_2(t_0,T)}{\partial t_0^2}>0 \text{ and } |H|=\begin{vmatrix}\frac{\partial^2TC_2(t_0,T)}{\partial t_0^2} & \frac{\partial^2TC_2(t_0,T)}{\partial t_0\partial T}\\ \frac{\partial^2TC_2(t_0,T)}{\partial t_0\partial T} & \frac{\partial^2TC_2(t_0,T)}{\partial T^2}\end{vmatrix}>0 \text{ at } (t_0^*,T^*). \quad (34)$$

Due to the complicated nature of second-order derivatives, establishing the sufficiency criteria mathematically is challenging. Therefore, a graphical method is employed to verify the convexity of the objective function.

Theorem 2. *The objective function TC_1 in Equation (16) is convex with respect to cycle length T and the shortage time t_0 if $\frac{\partial^2TC_2(t_0,T)}{\partial t_0^2}>0$, $\frac{\partial^2TC_2(t_0,T)}{\partial T^2}>0$ and $\left(\frac{\partial^2TC_2(t_0,T)}{\partial t_0^2}\right)\left(\frac{\partial^2TC_2(t_0,T)}{\partial t_0^2}\right)>\left(\frac{\partial^2TC_2(t_0,T)}{\partial t_0\partial T}\right)^2$ at (t_0^*,T^*).*

Proof. Using Equation (31), we obtain

$$\frac{\partial^2TC_2(t_0,T)}{\partial t_0^2} = \frac{h_1t_0\beta+D_1\phi(I_ep+C_b\delta+C_p\phi)+\beta\phi(C_p-I_ep(M-2t_0)+3C_pt_0\phi)}{T\phi}, \quad (35)$$

$$\frac{\partial^2TC_2(t_0,T)}{\partial T^2} = \frac{1}{3T^3\phi}\Big[h_1t_0^3\beta+\phi\Big\{6A+t_0\beta\Big(6h_0+I_ept_0(-3M+2t_0)+3C_pt_0(1+t_0\phi)\Big)+3D_1\Big(2h_0$$
$$+t_0[-2I_eMp+I_ept_0+2C_l(-1+\delta)+C_bt_0\delta+C_p(2-2\delta+t_0\phi)]\Big)\Big\}\Big], \quad (36)$$

and

$$\frac{\partial^2TC_2(t_0,T)}{\partial t_0\partial T} = -\frac{1}{2T^2\phi}\Big[h_1t_0^2\beta+\phi\Big\{2C_lD_1(-1+\delta)+2\Big(D_1I_ep(-M+t_0)+h_0\beta+I_ept_0(-M+t_0)\beta+C_bD_1t_0\delta\Big)$$
$$+C_p\Big(D_1(2-2\delta+2t_0\phi)+t_0\beta(2+3t_0\phi)\Big)\Big\}\Big]\phi)\Big)\Big\}\Big] \quad (37)$$

For the minimization of the given function, we must have $\frac{\partial^2TC_2(t_0,T)}{\partial t_0^2}>0$, $\frac{\partial^2TC_2(t_0,T)}{\partial T^2}>0$ and $\left(\frac{\partial^2TC_2(t_0,T)}{\partial t_0^2}\right)\left(\frac{\partial^2TC_2(t_0,T)}{\partial T^2}\right)>\left(\frac{\partial^2TC_2(t_0,T)}{\partial t_0\partial T}\right)^2$ at (t_0^*,T^*).

Furthermore, since the cost function is highly non-linear, its convexity is demonstrated graphically in the following section.

Therefore, the total average profit $TC_2(t_0, T)$ is convex with respect to t_0, T. The proof of the theorem is complete. □

5. Numerical Experiment

Here, a numerical simulation is performed on the proposed model, and the variability of the optimal result with respect to the decision-influencing parameters is discussed in tabular and graphical forms.

Example 1. *The following values of parameters with appropriate units are considered for Case 1:*

$a = 250$, $b = 0.5$, $d = 0.15$, $\delta = 0.7$, $\gamma = 0.8$, $c = 1.5$, $\beta = 50$, $K_0 = 15$, $K_1 = 0.9$, $\xi = 1.2$, $A = 120$, $M = 0.3$ *year*, $I_e = 0.05/year$, $I_p = 0.12/year$, $h_0 = 2$, $h_1 = 0.3$, $\phi = 0.05$, $C_b = 15$, $C_l = 5$, $g = 0.92$, $p = 45$.

Solving the problem with the help of Algorithm 1 in Mathematica 13.1 yields $t_0 = 0.458186$ *year,* $T = 1.04177$ *year,* $Q = 203.381$ *and* $TC_1 = USD4330.01$.

Algorithm 1: Algorithm 1 for solution of the model: The ensuing scheme will be used to acquire the minimum overall cost of our model:

Step 1. Find t_0 and T such that $\frac{\partial TC_1(t_0,T)}{\partial t_0} = 0$ and $\frac{\partial TC_1(t_0,T)}{\partial T} = 0$.
Step 2. If $M < t_0$, t_0 is feasible; then go to step 3.
Step 3. If $M > t_0$, t_0 is not feasible. Set $t_0 = M$, and evaluate T from (20); then go to step 4.
Step 4. Check if $\frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2} > 0$ for the value of t_0^*.
Step 5. Check if $\left(\frac{\partial^2 TC_1(t_0,T)}{\partial t_0^2}\right) * \left(\frac{\partial^2 TC_1(t_0,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_1(t_0,T)}{\partial t_0 \partial T}\right)^2 > 0$ for the values of t_0, T.
Step 6. Use the value of t_0, T to compute the total average cost function $TC_1(t_0, T)$ as a minimization one.

Example 2. *The following values of parameters with appropriate units are considered for Case 2.*

In this example, we assume the same parameters used in Example 1, except $M = 0.8$.

By solving the problem with the help of Algorithm 2 in Mathematica 13.1, the optimal solution is obtained as follows: $t_0 = 0.542121$ *year,* $T = 1.07291$ *year,* $Q = 215.391$ *and* $TC_2 = USD\ 4201.64$.

Additionally, the convexity of the objective function is illustrated graphically. All three-dimensional plots were generated using Mathematica software and are displayed in Figures 2–7.

Algorithm 2: Algorithm 2 for solution of the model: The ensuing scheme will be used to acquire the minimum overall cost of our proposed model:

Step 1. Find t_0 and T such that $\frac{\partial TC_2(t_0,T)}{\partial t_0} = 0$, $\frac{\partial TC_2(t_0,T)}{\partial T} = 0$.
Step 2. If $M > t_0$, t_0 is feasible; then go to step 3.
Step 3. If $M < t_0$, t_0 is not feasible. Set $T = M$, and evaluate the corresponding values of t_0 from Equation (32); then go to step 4.
Step 4. Check if $\frac{\partial^2 TC_2(t_0,T)}{\partial t_0^2} > 0$ for the value of t_0^*.
Step 5. Check if $\left(\frac{\partial^2 TC_2(t_0,T)}{\partial t_0^2}\right) * \left(\frac{\partial^2 TC_2(t_0,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_2(t_0,T)}{\partial t_0 \partial T}\right)^2 > 0$ for the values of t_0, T.
Step 6. Use the value of t_0, T to compute the total average cost function $TC_2(t_0, T)$ as a minimization one.

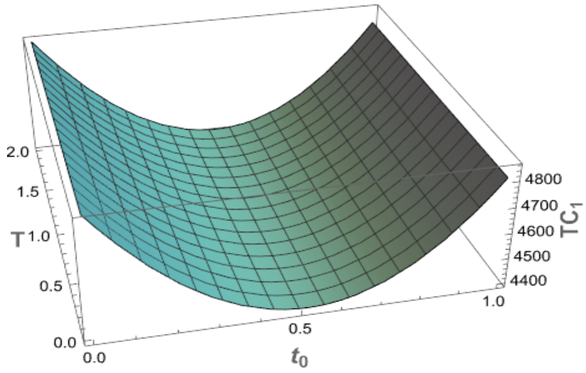

Figure 2. Graphical view of TC_1 vs. t_0 vs. T.

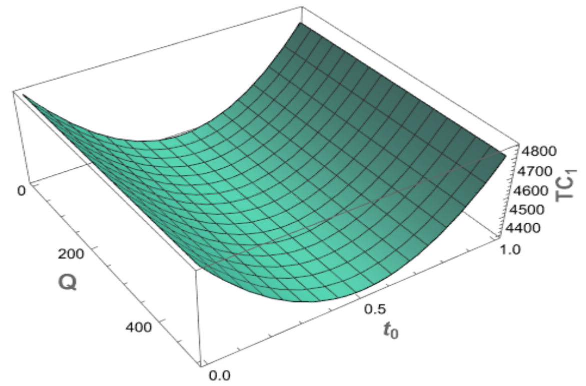

Figure 3. Graphical view of TC_1 vs. t_0 vs. Q.

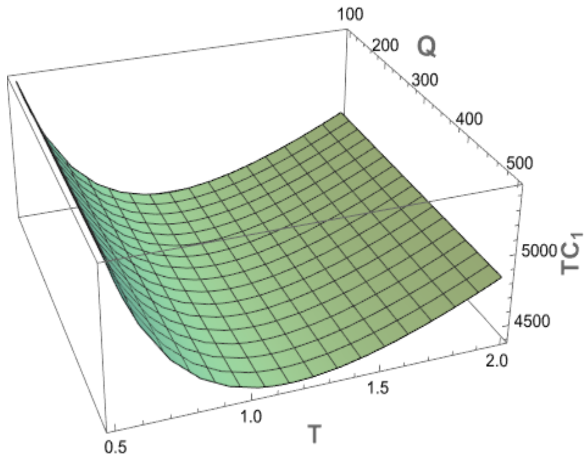

Figure 4. Graphical view of TC_1 vs. T vs. Q.

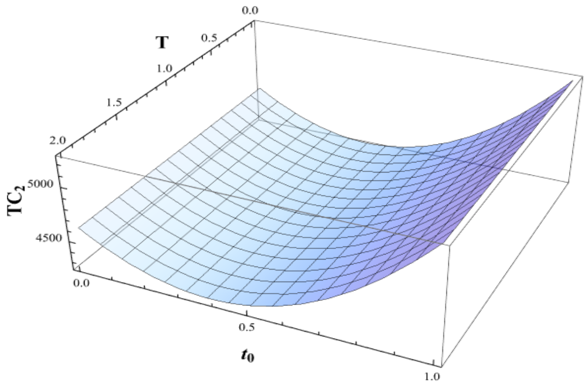

Figure 5. Graphical view of TC_2 vs. t_0 vs. T.

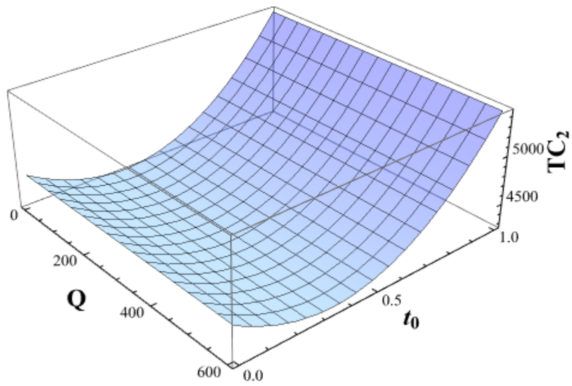

Figure 6. Graphical view of TC_2 vs. t_0 vs. Q.

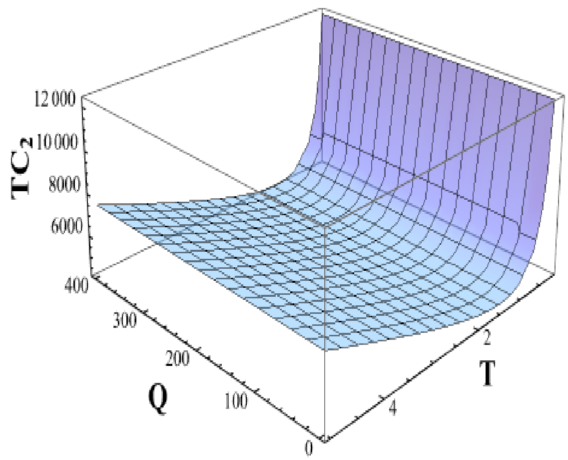

Figure 7. Graphical view of TC_2 vs. T vs. Q.

We now examine the sensitivity of the optimal solution with respect to variations in different parameters within the inventory system in Examples 1 and 2. To simplify this analysis, the parameter values are selected at random. The optimal solutions for varying parameters are summarized in Table 3, manifesting the influence of each parameter on the inventory model's performance.

Table 3. Sensitivity of the optimal results to the impacting coefficients and parameters.

Parameters	% Changes	Optimal Results for Case 1				Optimal Results for Case 2			
		t_0	T	Q	TC_1	t_0	T	Q	TC_2
a	−40	0.36759	1.02158	109.618	2557.51	0.44342	1.05246	115.954	2495.42
	−20	0.41907	1.02295	155.464	3446.32	0.49998	1.06073	164.644	3351.59
	+20	0.48929	1.05452	252.849	5210.60	0.57512	1.08536	267.648	5048.03
	+40	0.51482	1.06662	303.516	6089.11	0.60186	1.09705	321.054	5892.01
δ	−40	0.001	0.89088	87.00	3509.53	0.11028	0.99755	111.987	3504.29
	−20	0.25219	0.99182	155.10	3999.16	0.374794	1.07229	179.009	3917.19
	+20	0.593831	1.04024	228.966	4564.17	0.65394	1.04470	233.105	4401.78
	+40	0.68622	1.0149	239.769	4731.08	0.72907	1.00184	237.867	4540.76
b	−40	0.46338	1.04373	210.859	4464.91	0.54767	1.07484	223.299	4331.34
	−20	0.46081	1.04275	207.116	4397.47	0.54492	1.07387	219.34	4266.50
	+20	0.45551	1.04079	199.655	4262.54	0.53926	1.07195	211.45	4136.76
	+40	0.45379	1.03981	195.939	4195.04	0.53636	1.07098	207.518	4071.86

Table 3. *Cont.*

Parameters	% Changes	Optimal Results for Case 1				Optimal Results for Case 2			
		t_0	T	Q	TC_1	t_0	T	Q	TC_2
h_0	−40	0.35274	0.84869	163.636	4116.93	0.44003	0.88890	177.72	3992.72
	−20	0.40902	0.95078	184.654	4228.33	0.49607	0.98570	197.409	4101.57
	+20	0.50209	1.12449	220.406	4424.39	0.583629	1.15281	231.866	4294.99
	+40	0.54192	1.12078	236.103	4513.05	0.62154	1.22691	247.145	4382.97
p	−40	0.46892	1.05302	212.883	4474.34	0.54718	1.09816	227.161	4391.13
	−20	0.46356	1.04739	208.109	4402.13	0.54456	1.08547	221.220	4296.05
	+20	0.45279	1.03616	198.699	4258.00	0.53982	1.06044	209.662	4107.91
	+40	0.44737	1.03058	194.065	4186.09	0.53764	1.04803	204.024	4014.86
k_0	−40	0.74307	1.20891	255.017	3071.05	0.76005	1.16206	248.840	2915.84
	−20	0.58701	1.11864	226.617	3717.34	0.64804	1.11982	232.097	3571.84
	+20	0.34856	0.97275	183.419	4915.09	0.44177	1.02171	198.793	4807.45
	+40	0.25310	0.90831	165.610	5476.80	0.34640	0.96627	182.20	5390.90
d	−40	0.45724	1.04142	202.065	4306.00	0.54122	1.07257	213.999	4178.75
	−20	0.45771	1.04159	202.723	4318.10	0.54162	1.07274	214.695	4190.19
	+20	0.45865	1.04194	204.039	4341.92	0.54261	1.07308	216.087	4213.09
	+40	0.45911	1.04221	204.618	4353.82	0.54311	1.07325	216.784	4224.54
β	−40	0.57146	1.13008	226.956	4269.30	0.65671	1.15692	238.588	4127.27
	−20	0.50871	1.08078	213.803	4302.01	0.59365	1.11019	225.712	4166.99
	+20	0.41624	1.00983	194.84	4354.34	0.49883	1.04221	206.854	4232.25
	+40	0.38062	0.98300	187.666	4375.73	0.46171	1.01633	199.644	4254.57
g	−40	0.47277	1.05071	205.69	4248.87	0.55469	1.07889	217.068	4118.37
	−20	0.465621	1.04634	204.552	4288.31	0.54855	1.07598	216.244	4158.85
	+20	0.450532	1.03704	202.184	4373.61	0.53545	1.0697	214.514	4246.39
	+40	0.442707	1.03219	209.969	4418.86	0.52859	1.06636	213.618	4292.00
k_1	−40	0.471133	1.04966	205.72	4264.92	0.55334	1.07822	217.198	4134.46
	−20	0.46463	1.0457	204.546	4297.50	0.5400	1.07557	216.294	4168.08
	+20	0.451797	1.03786	202.225	4362.44	0.53653	1.07024	214.488	4235.13
	+40	0.445463	1.03397	201.08	4394.78	0.53096	1.06755	213.585	4268.54
M	−40	0.438131	1.03259	200.187	4356.47	0.48	1.0958	205.823	4286.77
	−20	0.448351	1.03797	201.882	4343.97	0.514052	1.06272	211.384	4245.22
	+20	0.467926	1.04548	204.755	4316.08	0.569368	1.08141	219.075	4156.05
	+40	0.476669	1.04859	206.001	4301.66	0.6142	1.12	229.178	4108.92
ϕ	−40	0.465613	1.0471	203.748	4324.93	0.548348	1.07655	215.948	4195.35
	−20	0.463269	1.04565	203.879	4327.09	0.547016	1.07625	215.571	4197.87
	+20	0.451854	1.03677	202.579	4333.26	0.535596	1.06811	214.801	4205.97
	+40	0.444945	1.03126	201.603	4333.65	0.528149	1.0626	213.985	4210.55

5.1. Sensitivity Analysis and Managerial Implications

In this section, we study the impacts of different system parameters $(a, b, p, \delta, h_0, k_0, k_1, d, g$ and $\beta)$ on the optimum values of average cost $TC_i(t_0, T)$, $i = 1, 2$, active retail cycle t_0, complete decision cycle T and lot size Q by changing the values of the impacting parameters from −40% to +40%. The detailed results are summarized in Table 3. From Table 3, we can note the following points:

- The fixed part of the demand, which is called demand potential, exhibits a sharp enhancement in the cost accumulation TC_i and enlargement in the economic lot size Q. We have taken the cost function to be optimized, and the demand-impacting component a seems to harm the cost minimization goal. However, a robust growth in demand must induce the earned revenue to suppress the increasing nature of the average cost. In comparison, the impact of demand on the active retail cycle t_0 seems to be less sensitive.
- In the demand function, b represents the coefficient for the demand-impacting pricing parameter. The low selling price boosts demand, resulting in negative impressions on the cost goal. However, if we consider the profit maximization objective, the scenario must be reversed due to the sharper impact of lowering the selling price on the average profit through demand and earned revenue.
- The presence of greenness in products is a demand-enhancing parameter in the lot-size phenomenon. So, the impact of the greenness level g on the total average cost TC_i seems to be identical to that of the demand potential. Moreover, the greenness measure includes additional costs for purchasing, which has a negative impact on the cost goal. The numerical results reveal less sensitivity of t_0, T and Q to the green level g.
- The average cost TC_i increases as the rate of backlogging δ of shortage increases. It is also perceived that the shortage time t_0 and order quantity Q are highly sensitive to changes in the backlogging parameter δ. On the other side, the total time cycle T shows less sensitivity to the changes in δ.
- The sensitivity computation demonstrates that the total average cost TC_i, shortage time t_0 and total cycle time T exhibit moderate sensitivity to changes in the holding cost parameter h_0. Alternatively, the optimal order quantity Q demonstrates high sensitivity. An increase in the holding cost parameter h_0 results in a rise in both the optimal order quantity Q and the inventory cost TC_i.
- The sensitivity analysis shows that both the total average cost TC_i and order quantity Q are moderately sensitive to changes in the selling price p, whereas the shortage time t_0 and total cycle time T exhibit lower sensitivity to these changes. If we increase the selling price p, then the associated inventory cost TC_i decreases, obviously.
- The demand for products in a newly installed retail system gradually increases as time progresses. Therefore, the time-varying coefficient β increases the average cost through demand. The reverse impression may be identified by taking profit as the objective function. There are interesting observations about the sensitivity of the shortage time t_0, total cycle time T and order quantity Q to the demand-controlling time coefficient β. The lot size, decision and retail cycles are reduced moderately with the growing dependence of demand on time.
- In the proposed model, d represents the rate of discount on the selling price, which impacts demand positively. Since discount is one of the demand growth-impacting factors, it exhibits a cost-increasing effect like the other parameters mentioned above. However, the sensitivity of the variance is moderate in this case. Also, the decision cycle, retail cycle and optimal lot size increase as discounts are offered to the consumers.
- The purchasing cost per unit item is controlled by the parameters k_0 and k_1. Between them, k_1 stands for the additional component in purchasing cost due to greenness maintenance. It is an expected phenomenon that k_0 and k_1 contribute to the cost enhancement and are reflected in the numerical outcome accordingly. However, increasing purchasing results in a downward trend for shortage time t_0, total time T and order quantity Q.

- When g is negligible, the demand function is only influenced by price and time. In such a case, the model turns into a traditional price- and time-dependent inventory model with trade credit considerations. If $g = 0$, the demand function reflects purely economic demand. In this situation, the model remains mathematically consistent and provides optimal solutions but loses its environmental application.
- The deterioration of products has a minor impact on the inventory decision in the proposed model. The average cost increases with deterioration. To avoid more deterioration during inventory management, the retailer tries to lessen lot size and cycle, which is reflected in the obtained results.
- It is perceived from the sensitivity analysis that a retailer can lessen the average cost when the supplier allows the retailer an extension of the credit cycle, irrespective of the cases of trade credit phenomena. The second case of a trade credit phenomenon provides superior results compared with the first one because the span of the trade credit facility for the retailer provided by the supplier is bigger in the second case. The retailer can enjoy the opportunity to make the lot size and cycle larger.

5.2. Discussion

Unlike classic EOQ, our model jointly optimizes price, replenishment timing, greenness effort, and trade credit terms under a triply influenced demand function. If one compares our model with [6,7], the total cost acquired in our model is less than that in [6,7] under a modified environment. This model provides better results than [28], with a better modification in parameters. In one word, our contribution is unique and provides a clear pathway for extension to a stochastic model with explicit carbon regulation.

The proposed model has broad applications across industries where demand is jointly influenced by price, time and environmental consciousness. For example, the model is highly suitable for fast-moving consumer goods (FMCG) and retail sectors. With growing demand, the model supports electronics and durable goods sectors. Organic products have significant market potential for the food and agricultural sectors. By following sustainable fashion trends, the model has a direct connection with the textiles and fashion sectors.

The model may face challenges in markets where greenness is not valued and cost is the only factor. Differences in environmental regulations across countries may complicate the uniform application of the model. The incorporation of green technology often requires a higher investment, which is not always possible. Trade credit periods may differ widely across industries, which can be complex in practice.

Improvements over previous models:

The proposed model improves upon earlier inventory models in several important ways:

- Unlike traditional models that generally consider demand to be a function of either price or time, this model simultaneously incorporates price, time and greenness sensitivity, thereby offering a more realistic representation of modern consumer behavior where sustainability plays a growing role in purchasing decisions.
- While many prior studies optimize inventory without considering financial interactions, our model embeds trade credit-based economic communications, which capture how supplier–retailer credit terms influence inventory decisions, cash flows and profitability.
- By explicitly modeling greenness-sensitive demand, this study bridges the gap between operational efficiency and environmental responsibility, which have often been treated separately in earlier works. This addition links sustainable production choices directly with consumer demand and profitability.

- The framework provides joint optimization of pricing, order quantity and credit period decisions, improving the managerial applicability of the model compared with the fragmented approaches in the existing literature.

Impact on production systems and sustainability under real-world conditions:

The impact of this study on production systems and sustainability under real-world conditions is described below:

- By considering demand elasticity with respect to both economic (price and credit) and environmental (greenness) factors, firms can align their production schedules more closely with consumer expectations, reducing overproduction and stockouts.
- The integration of trade credit enables retailers to manage cash flows better, invest in green production initiatives and reduce the financial risks typically associated with adopting sustainable practices.
- Since greener products influence demand positively in the model, firms are incentivized to adopt environmentally friendly production technologies, creating a competitive advantage while contributing to reduced carbon footprints.
- By linking profitability with sustainability, the model demonstrates that environmentally conscious production is not just a regulatory or ethical requirement but also a financially viable strategy in competitive markets.
- At a macro-level, the model supports supply chain resilience and sustainability by reducing waste, optimizing resource use and encouraging industry-wide adoption of green practices under realistic financial settings.

6. Conclusions

In this paper, we have designed and analyzed an economic lot management model with several intuitive and practical perspectives. Boosting demand for products is an important issue, shaped by several impacting factors. Pricing is an important factor associated with demand, as a lower selling price can make customers enthusiastic for products. We have included discount phenomena regarding pricing, which impact demand positively. Greenness is a matter of concern for the sustainable sourcing of the economic lot. Therefore, it generates additional demand, making an impact on the retail environment , and this intuition is reflected in the hypothesis of the model formulation. However, retailers have to pay additional costs in addition to regular purchasing costs to maintain greenness. Also, demand for the product must be considered an increasing function of time in a newly installed retail phenomenon. The inventory cost per unit can be taken as a time-variant increasing function. Based on the mentioned assumptions regarding demand and associated costs, the economic order quantity model is analyzed in the trade credit phenomenon, where the supplier allows the retailer to delay payment. After the permissible period of delayed credit, the retailer must pay interest to the supplier; meanwhile, the retailer can earn interest through the opportunity of delayed credit. We have discussed two cases for the permissible delayed period. We have proved the average cost as a convex objective function of active retail and decision cycles, irrespective of the mentioned two cases, and have provided some feasible values of the other parameters involved. With suitable values of input parameters, the numerical and graphical analysis portions follow the analytical findings. It has been established from the numerical outcomes that all demand-boosting parameters result in a negative impact on the cost reduction goal. The negative impact on the retailer's goal can be decoded in a way that more supply of products is required to meet the additional demand; therefore, the average cost increases due to purchase, maintenance and supply activities. However, if we consider the average profit as the objective function, the demand-boosting parameters will show positive impacts on the profit goal, suppressing the negative impact on the cost reduction objective.

In this paper, we have accumulated many real-world impacting issues related to the retail process in a single mathematical model and established the local optimality of the proposed objective function in a complicated environment. This paper is the first to design and investigate the impact of pricing, greenness and time on cost goal through demand in a trade credit scenario. On the contrary, it is perceived after completing the numerical simulation that a more precise conclusion could be made if profit replaced the cost function as the objective function on the same set of hypotheses. However, we have noticed that considering average profit as the objective function makes the analysis more complicated and thus can be regarded as a future challenge in research building on the present study. The model may be extended to capture interactions among multiple suppliers, manufacturers and retailers, especially where greenness propagates through multiple tiers. The extension of this model for addressing reverse logistics, recycling and re-manufacturing in industries like electronics and fashion would provide practical and adaptable applications and might guide industry managers in balancing profitability with sustainability objectives.

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References

1. Maihami, R.; Kamalabadi, I.N. Joint pricing and inventory control for non-instantaneous deteriorating items with partial backlogging and time and price dependent demand. *Int. J. Prod. Econ.* **2012**, *136*, 116–122. [CrossRef]
2. Pervin, M.; Roy, S.K.; Weber, G.W. Analysis of inventory control model with shortage under time-dependent demand and time-varying holding cost including stochastic deterioration. *Ann. Oper. Res.* **2018**, *260*, 437–460. [CrossRef]
3. Roy, S.K.; Pervin, M.; Weber, G.W. A two-warehouse probabilistic model with price discount on backorders under two levels of trade-credit policy. *J. Ind. Manag. Optim.* **2020**, *16*, 553. [CrossRef]
4. Saxena, S.; Gupta, R.K.; Singh, V.; Singh, P.; Mishra, N.K. Environmental Sustainability with eco-friendly green inventory model under Fuzzy logics considering carbon emission. *J. Emerg. Technol. Innov. Res.* **2018**, *5*, 1–12.
5. Sarkar, B.; Ullah, M.; Sarkar, M. Environmental and economic sustainability through innovative green products by remanufacturing. *J. Clean. Prod.* **2022**, *332*, 129813. [CrossRef]
6. Pervin, M. A sustainable deteriorating inventory model with backorder and controllable carbon emission by using green technology. *Environ. Dev. Sustain.* **2024**, 1–37. [CrossRef]
7. Haque, R.; Pervin, M.; Mondal, S.P. A sustainable manufacturing–remanufacturing inventory model with price and green sensitive demand for defective and usable items. *RAIRO-Oper. Res.* **2024**, *58*, 3439–3467. [CrossRef]
8. Akbar, S.A.; Cárdenas-Barrón, L.E.; Kumar, M.A.; Céapedes-Mota, A. An economic production quantity (EPQ) model for a deteriorating item with partial trade credit policy for price dependent demand under inflation and reliability. *Yugosl. J. Oper. Res.* **2021**, *31*, 139–151. [CrossRef]
9. Katariya, D.K.; Shukla, K.T. An EOQ model for deteriorating products with green technology investment and trade credit financing. *Int. J. Procure. Manag.* **2023**, *18*, 300–320. [CrossRef]
10. Harris, F. *Operations and Cost. Factory Management Series*; A.W. Shaw Company: Winnetka, IL, USA, 1915.
11. Wilson, R. *A Scientific Routine for Stock Control*; Harvard University: Cambridge, MA, USA, 1934.

12. Khanra, S.; Chaudhuri, K. A note on an order-level inventory model for a deteriorating item with time-dependent quadratic demand. *Comput. Oper. Res.* **2003**, *30*, 1901–1916. [CrossRef]
13. Ghosh, S.K.; Chaudhuri, K. An EOQ model with a quadratic demand, time-proportional deterioration and shortages in all cycles. *Int. J. Syst. Sci.* **2006**, *37*, 663–672. [CrossRef]
14. Begum, R.; Sahu, S.; Sahoo, R. An inventory model for deteriorating items with quadratic demand and partial backlogging. *Br. J. Appl. Sci. Technol.* **2012**, *2*, 112–131. [CrossRef]
15. Lakshmidevi, P.; Maragatham, M. An inventory model with three rates of production and time dependent deterioration rate for quadratic demand rate. *Int. J. Fuzzy Math. Arch.* **2015**, *6*, 99–103.
16. Sharmila, D.; Uthayakumar, R. Inventory model for deteriorating items with quadratic demand, partial backlogging and partial trade credit. *Oper. Res. Appl. Int. J. (ORAJ)* **2015**, *2*, 51–70.
17. Sharma, P.; Sharma, A.; Jain, S. Inventory model for deteriorating items with price and time-dependent seasonal demand. *Int. J. Procure. Manag.* **2019**, *12*, 363–375. [CrossRef]
18. Saha, S.; Sen, N. An inventory model for deteriorating items with time and price dependent demand and shortages under the effect of inflation. *Int. J. Math. Oper. Res.* **2019**, *14*, 377–388. [CrossRef]
19. Chen, L.; Chen, X.; Keblis, M.F.; Li, G. Optimal pricing and replenishment policy for deteriorating inventory under stock-level-dependent, time-varying and price-dependent demand. *Comput. Ind. Eng.* **2019**, *135*, 1294–1299. [CrossRef]
20. Goyal, S.K. Economic order quantity under conditions of permissible delay in payments. *J. Oper. Res. Soc.* **1985**, *36*, 335–338. [CrossRef]
21. Aggarwal, S.; Jaggi, C. Ordering policies of deteriorating items under permissible delay in payments. *J. Oper. Res. Soc.* **1995**, *46*, 658–662. [CrossRef]
22. Jamal, A.; Sarker, B.; Wang, S. An ordering policy for deteriorating items with allowable shortage and permissible delay in payment. *J. Oper. Res. Soc.* **1997**, *48*, 826–833. [CrossRef]
23. Min, J.; Zhou, Y.W.; Zhao, J. An inventory model for deteriorating items under stock-dependent demand and two-level trade credit. *Appl. Math. Model.* **2010**, *34*, 3273–3285. [CrossRef]
24. Sarkar, B.; Saren, S.; Cárdenas-Barrón, L.E. An inventory model with trade-credit policy and variable deterioration for fixed lifetime products. *Ann. Oper. Res.* **2015**, *229*, 677–702. [CrossRef]
25. Chang, H.C.; Ho, C.H.; Ouyang, L.Y.; Su, C.H. The optimal pricing and ordering policy for an integrated inventory model when trade credit linked to order quantity. *Appl. Math. Model.* **2009**, *33*, 2978–2991. [CrossRef]
26. Mahato, C.; Mahata, G.C. Optimal ordering policy under order-size dependent trade credit and complete backlogging derived algebraically. *Opsearch* **2023**, *60*, 420–444. [CrossRef]
27. Paul, A.; Pervin, M.; Roy, S.K.; Maculan, N.; Weber, G.W. A green inventory model with the effect of carbon taxation. *Ann. Oper. Res.* **2022**, *309*, 233–248. [CrossRef]
28. Paul, A.; Pervin, M.; Pinto, R.V.; Roy, S.K.; Maculan, N.; Weber, G.W. Effects of multiple prepayments and green investment on an EPQ model. *J. Ind. Manag. Optim.* **2023**, *19*, 6688–6704. [CrossRef]
29. Tiwari, S.; Cárdenas-Barrón, L.E.; Shaikh, A.A.; Goh, M. Retailer's optimal ordering policy for deteriorating items under order-size dependent trade credit and complete backlogging. *Comput. Ind. Eng.* **2020**, *139*, 105559. [CrossRef]
30. Datta, T.; Pal, A. Effects of inflation and time-value of money on an inventory model with linear time-dependent demand rate and shortages. *Eur. J. Oper. Res.* **1991**, *52*, 326–333. [CrossRef]
31. Sarker, B.R.; Pan, H. Effects of inflation and the time value of money on order quantity and allowable shortage. *Int. J. Prod. Econ.* **1994**, *34*, 65–72. [CrossRef]
32. Bose, S.; Goswami, A.; Chaudhuri, K. An EOQ model for deteriorating items with linear time-dependent demand rate and shortages under inflation and time discounting. *J. Oper. Res. Soc.* **1995**, *46*, 771–782. [CrossRef]
33. Tripathy, C.; Pradhan, L. An EOQ model for Weibull deteriorating items with power demand and partial backlogging. *Int. J. Contemp. Math. Sci.* **2010**, *5*, 1895–1904.
34. Yang, H.L. A partial backlogging production-inventory lot-size model for deteriorating items with time-varying production and demand rate over a finite time horizon. *Int. J. Syst. Sci.* **2011**, *42*, 1397–1407. [CrossRef]
35. Mondal, R.; Das, S.; Akhtar, M.; Shaikh, A.A.; Bhunia, A.K. A two-warehouse inventory model for deteriorating items with partially backlogged demand rate under trade credit policies. *Int. J. Syst. Assur. Eng. Manag.* **2024**, *15*, 3350–3367. [CrossRef]
36. Singh, R.; Mishra, V.K. Inventory model using Machine Learning for demand forecast with imperfect deteriorating products and partial backlogging under carbon emissions. *Ann. Oper. Res.* **2024**, *335*, 361–383. [CrossRef]
37. De, P.K.; Devi, S.P.; Narang, P. Inventory model for deteriorating goods with stock and price-dependent demand under inflation and partial backlogging to address post Covid-19 supply chain challenges. *Results Control Optim.* **2024**, *14*, 100369. [CrossRef]
38. Taleizadeh, A.A.; Noori-Daryan, M.; Cárdenas-Barrón, L.E. Joint optimization of price, replenishment frequency, replenishment cycle and production rate in vendor managed inventory system with deteriorating items. *Int. J. Prod. Econ.* **2015**, *159*, 285–295. [CrossRef]
39. Khatua, D.; Maity, K. Research on relationship between economical profit and environmental pollution of imperfect production inventory control problem. *J. Niger. Math. Soc.* **2016**, *35*, 560–579.

40. Yavari, M.; Geraeli, M. Heuristic method for robust optimization model for green closed-loop supply chain network design of perishable goods. *J. Clean. Prod.* **2019**, *226*, 282–305. [CrossRef]
41. Kundu, A.; Guchhait, P.; Maiti, M.; Castillo, O. Inventory of a deteriorating green product with preservation technology cost using a hybrid algorithm. *Soft Comput.* **2021**, *25*, 11621–11636. [CrossRef]
42. Hasan, S.M.; Mashud, A.H.M.; Miah, S.; Daryanto, Y.; Lim, M.K.; Tseng, M.L. A green inventory model considering environmental emissions under carbon tax, cap-and-offset, and cap-and-trade regulations. *J. Ind. Prod. Eng.* **2023**, *40*, 538–553. [CrossRef]
43. Jauhari, W.A.; Ramadhany, S.C.N.; Rosyidi, C.N.; Mishra, U.; Hishamuddin, H. Pricing and green inventory decisions for a supply chain system with green investment and carbon tax regulation. *J. Clean. Prod.* **2023**, *425*, 138897. [CrossRef]
44. Jauhari, W.A.; Cahyaningtyas, S.; Rosyidi, C.N. Investment and pricing decisions for a manufacturer-retailer system with hybrid production and carbon cap-and-trade policy. *Int. J. Manag. Sci. Eng. Manag.* **2025**, *20*, 440–457. [CrossRef]
45. Bhavani, G.D.; Meidute-Kavaliauskiene, I.; Mahapatra, G.S.; Činčikaitė, R. A sustainable green inventory system with novel eco-friendly demand incorporating partial backlogging under fuzziness. *Sustainability* **2022**, *14*, 9155. [CrossRef]
46. Ansu-Mensah, P. Green product awareness effect on green purchase intentions of university students': An emerging market's perspective. *Future Bus. J.* **2021**, *7*, 48. [CrossRef]
47. Ogiemwonyi, O.; Alam, M.N.; Alshareef, R.; Alsolamy, M.; Azizan, N.A.; Mat, N. Environmental factors affecting green purchase behaviors of the consumers: Mediating role of environmental attitude. *Clean. Environ. Syst.* **2023**, *10*, 100130. [CrossRef]
48. Baca, G.; Reshidi, N. Green branding and consumer behavior, unveiling the impact of environmental marketing strategies on purchase decisions. *Bus. Strategy Environ.* **2025**, *34*, 3701–3713. [CrossRef]
49. Bhunia, A.; Shaikh, A. A deterministic inventory model for deteriorating items with selling price dependent demand and three-parameter Weibull distributed deterioration. *Int. J. Ind. Eng. Comput.* **2014**, *5*, 497–510. [CrossRef]
50. Kumar, M.; Chauhan, A.; Kumar, R. A deterministic inventory model for deteriorating items with price dependent demand and time varying holding cost under trade credit. *Int. J. Soft Comput. Eng.* **2012**, *2*, 99–105.
51. Shaikh, A.A. An inventory model for deteriorating item with frequency of advertisement and selling price dependent demand under mixed type trade credit policy. *Int. J. Logist. Syst. Manag.* **2017**, *28*, 375–395. [CrossRef]
52. Tiwari, S.; Wee, H.M.; Sarkar, S. Lot-sizing policies for defective and deteriorating items with time-dependent demand and trade credit. *Eur. J. Ind. Eng.* **2017**, *11*, 683–703. [CrossRef]
53. Tripathi, R. EOQ model for time dependent demand with deterioration, inflation, shortages and trade credits. *Int. J. Inf. Decis. Sci.* **2024**, *16*, 73–89.
54. Shah, N.H.; Chaudhari, U.; Jani, M.Y. Inventory model with expiration date of items and deterioration under two-level trade credit and preservation technology investment for time and price sensitive demand: DCF approach. *Int. J. Logist. Syst. Manag.* **2017**, *27*, 420–437. [CrossRef]
55. Shaikh, A.A.; Cárdenas-Barrón, L.E.; Bhunia, A.K.; Tiwari, S. An inventory model of a three parameter Weibull distributed deteriorating item with variable demand dependent on price and frequency of advertisement under trade credit. *RAIRO-Oper. Res.* **2019**, *53*, 903–916. [CrossRef]
56. Khanra, S.; Mandal, B.; Sarkar, B. An inventory model with time dependent demand and shortages under trade credit policy. *Econ. Model.* **2013**, *35*, 349–355. [CrossRef]
57. Shah, N.H.; Keswani, M.; Khedlekar, U.K.; Prajapati, N.M. Non-instantaneous controlled deteriorating inventory model for stock-price-advertisement dependent probabilistic demand under trade credit financing. *Opsearch* **2024**, *61*, 421–459. [CrossRef]
58. Rameswari, M.; Uthayakumar, R. An integrated inventory model for deteriorating items with price-dependent demand under two-level trade credit policy. *Int. J. Syst. Sci. Oper. Logist.* **2018**, *5*, 253–267. [CrossRef]
59. Mishra, U.; Wu, J.Z.; Tseng, M.L. Effects of a hybrid-price-stock dependent demand on the optimal solutions of a deteriorating inventory system and trade credit policy on re-manufactured product. *J. Clean. Prod.* **2019**, *241*, 118282. [CrossRef]
60. Abdul Hakim, M.; Hezam, I.M.; Alrasheedi, A.F.; Gwak, J. Pricing policy in an inventory model with green level dependent demand for a deteriorating item. *Sustainability* **2022**, *14*, 4646. [CrossRef]
61. Wang, C.; Huang, R. Pricing for seasonal deteriorating products with price-and ramp-type time-dependent demand. *Comput. Ind. Eng.* **2014**, *77*, 29–34. [CrossRef]
62. Shah, N.H.; Vaghela, C.R. An EPQ model for deteriorating items with price dependent demand and two level trade credit financing. *Investigación Oper.* **2018**, *39*, 170–180.
63. Shekhar, C.; Yadav, V.; Saurav, A. Optimal analysis of time-price discount-advertisement dependent demand with credit policy for inventory models. *Ann. Oper. Res.* **2024**, 1–48. [CrossRef]