Quantum Chaos in Time Series of Single Photons as a Superposition of Wave and Particle States

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Abstract: We build a time series of single photons with quantum chaos statistics, using a version of the Grangier anti-correlation experiment. The criteria utilized to determine the presence of quantum chaos is the frame of the Fano factor and the power spectrum. We also show that photons with chaotic statistics are in a balanced superposition of photons with both wave-like and particle like behaviors. To support the presence of quantum chaos, we study both Shannon's entropy, and the complexity of single photons time series.

Keywords: quantum chaos; wave-particle duality; Fano factor

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1. Introduction

The dichotomy between the wave and corpuscular nature of light has gone a long way in the history of physics, but we do not really yet understand the meaning of waves and particles at the quantum level. We only have a classical representation of these concepts in mind and some intuitive definitions about them: a quantum wave can produce interference, while a quantum particle can track a path. J. Wheeler proposed the now-famous delayed-choice gedankenexperiment [1] to show that the nature of light, wave, or particle depends on how it is measured. This experiment was carried out in 2006 by Jacques et al. [2,3], which confirmed Bohr's complementary principle. Other similar experiments have been carried out with some variants, and the same results [4–6] showing that the states of light can be considered a superposition of wave and particle where the measuring device collapses the state in one of the two behaviors. Radu et al. investigated what happens in the thought experiment if the delayed choice is made through quantum-controlled experiments [7]. This proposal was developed by Jian-Shun et al. [8], who derive a quantum superposition of single-photon wave and particle properties by selecting the quantum detecting device in a superposition state rather than the eigenstates of the delayed-choice experiment. This superposition can be measured indirectly through interference visibility. Then, it is possible to measure simultaneously wave and particle behaviors in single photons with potential applications in coding quantum information [8,9]. In this article, we show that the superposition of wave and particle behavior of quantum systems is linked with quantum chaos statistics.

Among the various definitions that exist about quantum chaos, one, in particular, is well known: A quantum system behaves chaotically if there is a classical analog system that exhibits chaos [10].

However, this is not the only definition because quantum chaos appears to have an elusive behavior in comparison with classical chaos. In order to explain the essence of quantum chaos from the superposition of wave and particle behaviors, we considered the following: There is a relationship between the interference visibility of quantum particles and its transition from regular to chaotic behavior [11,12]. On the other hand, there is a change in the interference visibility depending on its degree of superposition between wave and particle behaviors, as mentioned above [8]. This reasoning is closed if quantum chaos is related to the superposition of both wave and particle behaviors in some systems. We will explore the latter relationship experimentally.

The methodology that we use to identify quantum chaos relies on the framework of the Fano factor [13–16] and the power spectrum [16–18]. This article is organized as follows. In Section 2, we explain the construction of wave and particle states on single photons. In Section 3, we discuss the relationship between the second-order correlation function and the Fano factor in order to show that this experiment always preserves the single-photon properties. Next, in Section 4, we present the experimental details. Our results are presented in Section 5, where we analyze the different statistical limits obtained. In Section 6, we describe the results in terms of regularity-complexity parameters, and Shannon's entropy. Finally, in the last section, we present our conclusions.

2. Wave and Particle Superposition with the Statistical Criterion

Gedanken-experiments help us understand the dual nature of quantum particles. Moreover, thanks to this, we are able to understand the role of varying the degree of superposition of these behaviors. The state of a photon can be defined as a quantum superposition of a wave ($|w\rangle$) and a particle ($|p\rangle$) [7–9,19]:

$$|\psi\rangle = C_w|w\rangle + C_p|p\rangle \quad (1)$$

where C_w and C_p are probability amplitudes, with $P_w = |C_w|^2$, $P_p = |C_p|^2$ the probabilities for each photon to be detected in one or the other behavior.

In order to analyze the particle–wave superposition, we reproduce a version of the anti-correlation experiment of Grangier [20] where single photons cross a polarizing beam splitter (PBS). Selecting the beam splitter proportions (P_T, P_R), transmission, and reflecting probabilities, depending on the polarizing angle of the incoming single photons, we will have predictable trajectories for (1,0) and (0,1). For any other case, we will have some degree of unpredictability, having the maximal for (0.5,0.5). We use 'wave behavior' to refer to this maximal unpredictability and 'particle behavior' to refer to the predictable trajectories of both cases (1,0) and (0,1). A general quantum superposition of both limits represents a certain degree of unpredictability that can be measured by photon-counting fluctuations under the shot-noise limit. Because the Grangier experiment lacks the second beam splitter in comparison with the delayed-choice experiment, apparently, the behavior of each photon will always be detected as 'particle,' but we can show that there is an equivalence. Rotating the linear polarization of the incoming single photons (using a $\lambda/2$ plate, the probabilities in the two output ports of the PBS are

$$P_T = \cos^2(\phi/2) \quad (2)$$

$$P_R = \sin^2(\phi/2) \quad (3)$$

where $\phi/2$ is the angle of the $\lambda/2$ wave plate. These probabilities are equivalent to the output probabilities in the Mach–Zender interferometer if the usual phase $e^{i\theta}$ in one of the two arms is $\theta = \phi/2$. The trigonometric identity for the double angle indicates that $P_T = \frac{1}{2}(1 + \cos(\phi))$ and $P_R = \frac{1}{2}(1 - \cos(\phi))$, the interference pattern of single photons [21]. Then, the wave or particle behaviors of single photons must also be codified in the photon statistics, particularly in the noise associated with the counting of single photons.

3. Sub Shot-Noise, Second-Order Correlation Function, and Fano Factor

The shot noise of electrons emitted in vacuum tubes has a standard deviation equal to the average of electrons emitted in a fixed time Δt . Therefore, the Fano factor in this case is $F = 1$. This is the Poisson limit. The photons emitted by a laser fulfill this condition. To obtain noise with statistics below the shot noise, a control over the emission of photons is necessary; in other words, a single photon source is necessary. The second-order correlation function for photons crossing a beam splitter is defined as [21]:

$$g^2(\tau) = \frac{\langle \hat{a}^\dagger(0)\hat{a}^\dagger(\tau)\hat{a}(\tau)\hat{a}(0) \rangle}{\langle \hat{a}^\dagger(0)\hat{a}(0) \rangle^2}, \quad (4)$$

$$= 1 + \frac{\langle (\Delta \hat{n})^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle^2}. \quad (5)$$

where $\langle \rangle$ implies a temporal average in the interval ΔT , $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators respectively, and τ is the delay produced by the difference in the optical path between the two signal detected. The Fano factor is then defined as [16,22,23]:

$$F = \frac{\langle (\Delta \hat{n})^2 \rangle}{\langle \hat{n} \rangle}. \quad (6)$$

From Equations (4) and (5) we find that there is a relation between the second-order correlation function and the Fano factor,

$$g^2(\tau) = 1 + \frac{F - 1}{\langle \hat{n} \rangle}. \quad (7)$$

When $\langle \hat{n} \rangle = 1$, the second-order correlation function has the same value as the Fano factor. In this case, the Grangier's anti-correlation experiment [20] allow us to control the noise in the interval $[0 \leq g^{(2)}(\tau) \leq 1]$. One way to control the transition from quantum to classic statistics is by increasing the coincidence window size τ ; or by increasing the average number of photons $\bar{n}$ in the interval ΔT . In both cases $g^2(0) = 0$ must be satisfied. To know the dependence between the transmitted Fano factor and the angle of the half-wave plate (HWP), we consider that $g^{(2)}(\tau) = 0$ for single photons, and that $\langle \hat{n} \rangle_{T,R} = P_{T,R}$ in Equation (7). Then, using Equations (2) and (3) we find that $F_T = \sin^2(\phi/2)$, and $F_R = \cos^2(\phi/2)$. Moreover, from Equation (6) the variance can be expressed as $\langle (\Delta \hat{n})^2 \rangle = \sin^2(\phi/2)\cos^2(\phi/2)$.

Theoretically, if the single photons are horizontally polarized, the probability of being transmitted in a PBS is $P_T = 1$. In this case, we say that the trajectory of the photons is well-defined, and it presents 'particle' behavior. The same argument is valid when vertically polarizing photons are reflected in the PBS with $P_R = 1$. If the polarization of photons can be controlled by one HWP, the photons' particle-like states outcoming from the PBS can be written as:

$$|\psi\rangle_p^{\phi=0} = |H, 0\rangle^{\phi=0} \quad (8)$$

$$|\psi\rangle_p^{\phi=\pi/2} = i|0, V\rangle^{\phi=\pi/2} \quad (9)$$

where we labeled each state with the polarization angle ϕ , and the labels inside the ket, correspond to transmitted and reflected states $|H, V\rangle$, respectively. When $\phi = 0$ the photon that goes through the HWP is horizontally polarized and it keeps its original polarization. If $\phi = \pi/2$, the original state $|H, 0\rangle$ is converted to $|0, V\rangle$. On the other hand, when the horizontal polarization is rotated $\pi/4$, the final state is:

$$|\psi\rangle_w^{\phi=\pi/4} = \frac{1}{\sqrt{2}}(|H, 0\rangle^{\phi=\pi/4} + i|0, V\rangle^{\phi=\pi/4}) \quad (10)$$

In this case, each photon has the same probability of being transmitted or reflected, so it has the maximum delocalization and presents wave-like behavior. Hence, the superposition of wave and particle states can be rewritten as:

$$|\psi\rangle_{wp} = C_p|\psi\rangle_p^{\phi=0,\pi/2} + C_w|\psi\rangle_w^{\phi=\pi/4} \quad (11)$$

where C_p and C_w are the probability amplitudes in which the photon is registered as a particle or as a wave; however, we need to be careful here because a wave-like state $|\psi\rangle_w^{\phi=\pi/4}$ corresponds to a single photon that can be detected with equal probability in one of two places, which are positioned at the detector as transmitted or reflected (D_T, D_R). In the case, we use only one detector, D_T or D_R , we can characterize both wave or particle behaviors through its statistical noise, as we will see.

The fundamental step in the realization of this experiment consists of the separate analysis of photon counting of the time series of photons transmitted or reflected separately. The reason is simple, as we have anti-correlation at each moment, which implies $g^{(2)} = 0$, and thus $F = 0$, i.e., the noise associated to the photon counting when adding both outputs (T and R) is $\langle(\Delta\hat{n})^2\rangle = 0$. Analyzing without correlations (separately), this noise suppression is no longer valid. We then, analyze one (or the other) of the two output time series of photons. The superposition of particle and wave photon-behavior expressed in Equation (11) can be written as a function of the particle trajectory (transmitted or reflected):

$$|\psi\rangle_{wp}^{\phi=0} = C_p|H, 0\rangle^{\phi=0} + \frac{C_w}{\sqrt{2}}|H, 0\rangle^{\phi=\pi/4} + \frac{iC_w}{\sqrt{2}}|0, V\rangle^{\phi=\pi/4} \quad (12)$$

$$|\psi\rangle_{wp}^{\phi=\pi/2} = iC_p|0, V\rangle^{\phi=\pi/2} + \frac{C_w}{\sqrt{2}}|H, 0\rangle^{\phi=\pi/4} + \frac{iC_w}{\sqrt{2}}|0, V\rangle^{\phi=\pi/4} \quad (13)$$

We place particular focus on the transmitted states in Equation (12), where it must be clear that $|H, 0\rangle^{\phi=\pi/4}$ and $|H, 0\rangle^{\phi=0}$ are not statistically equivalent and, as a consequence, cannot be factored as $(C_p + \frac{C_w}{\sqrt{2}})|H, 0\rangle$. The same argument is valid for the reflected state.

4. Experiment

The proposed experiment is a version of the Grangier experiment [20,24]. We send individual photons with linear polarization to a polarizing beam splitter. The state of the photons that cross the HWP whose fast axis is at an angle $\phi/2$ are prepared in the state $|\psi\rangle = \cos(\phi)|H\rangle + \sin(\phi)|V\rangle$. The probabilities of detecting photons that exit the transmitted and reflected output ports are $P_T = \cos^2(\phi)$ and $P_R = \sin^2(\phi)$, respectively.

Figure 1 shows the experimental setup. We used a photon pair source based on the SPDC – I spontaneous parametric down-conversion process. A violet laser, (Crystalaser, $\lambda = 405$ nm), excites a non-linear crystal type I (Newlight photonics), with a thickness of 2 mm. The infrared photons come out, in a 3 degree angle with respect to the axis of the experiment. The signal photons are directed towards a HWP and a PBS. At the output ports of the PBS, we placed two polarizers to reinforce the polarization selection of the PBS. At the output ports of the beam splitter and after the two polarizers, two avalanche photodiodes (APDs, Excelitas) with a quantum efficiency of 60% register the photons (D_T and D_R , respectively). The idler photons are directed directly to the idler APD (D_I).

The three APDs are connected to homemade electronics that count individual events and coincidences. Our electronics are prepared to count only coincidences between the idler photon and the signal photon at the beam splitter output ports T and R , which verify the preservation of the probability of the presence of the photon. The coincidences are detected in a time window of $\tau = 10$ ns. We start with $g^2(\tau) = 0.025 \pm 0.015$, which implies anti-correlation in the photon detection over the T and R outputs. Our electronics detect all the photons that present the anti-correlation condition. For this to happen, the presence of the idler photon is necessary. Therefore, two binary sequences are stored for a given ϕ , for example, $(1, 0, 0, 1, 1, 0, \dots)_T$ and $(0, 1, 1, 0, 0, 1, \dots)_R$, where 1 indicates a detected photon, and 0—no photon detection. In the end, we have collected the time series of transmitted and reflected photons.

Figure 1. Experimental setup, where a pair of photons signal and idler (S, I) are produced when a non-linear crystal (NLC) *BBO-I* is pumping by a violet laser (VL), through spontaneous parametric down conversion process. Idler photons are sent to the idler single photons detector (D_I). Signal photons are sent to a $\lambda/2$ wave plate followed by a PBS. In order to compensate for the polarizing beam splitter imperfections, we placed two polarizers with their horizontal and vertical axes on the transmitted and reflected outputs, respectively. Finally, interference filters (F) for 810 nm were placed before the avalanche photodiodes (D_R , D_T).

5. Results

Figure 2 shows the counting tests of 1's and 0's registered by D_T traduced to the probabilities P_1 and P_0 to detect or not photons. For each phase $\phi/2$ of the HWP, we take 102,400 bits. Theoretically, for $\phi = 0$, all the bits detected by D_T should be one, while for $\phi/2 = \pi/4$, all the bits detected by D_T should be zero. We draw special attention to the cases in which the 1's and 0's curves intersect because at these points, we expect a wave-like behavior and equal probability of transmission and reflection $P = 1/2$.

Figure 2. Probability P_1 Equation (1) to detect or not (P_0) single photons in the detector P_T . P_0 (green x symbols) and P_1 (red + symbols) as a function of the HWP angle $\phi/2$ can be interpreted as the normalized proportions of 1's and 0's stored in the time series to make the statistical analysis of the photon counting.

Figure 3 shows the Fano factor F for the output ports T and R , as function of the probability $P_T = \cos^2(\phi/2)$ and $P_R = \sin^2(\phi/2)$. The statistic curve goes from $F = 0$ to $F = 1$, showing a transition from particle to wave behavior in the sector $0 \leq F \leq 1/2$. The maximal transmitted probability, $P_T = 1$, gives us $F = 0$, described by the state: $|\psi\rangle_p^{\phi=0} = |H, 0\rangle^{\phi=0}$. In this case, the photons are localized in D_T and have a particle-like behavior. The Fano factor $F = 1/2$ is found when the probability that the photon is transmitted (or reflected) is $P_T = 1/2$, i.e., when photons behave like waves (Equation (10)). We note that the prediction for quantum chaos in the sub-shot noise sector has a Fano factor of $F = 1/4$ [13–16]. Because the probability P_T is related with the average of the number of detecting photons in D_T , experimentally, the chaotic Fano factor for the transmitted time series of photons is located at $P_T = 3/4$. In such case, in order to accomplish with the probabilities of the quantum chaos statistics, the HWP must rotate the polarization angle of photons to $\phi = \pi/6$. In this way, the photons crossing the PBS can be described by:

$$|\psi\rangle^{\phi=\pi/6} = \sqrt{\frac{3}{4}}|H, 0\rangle^{\pi/6} + i\sqrt{\frac{1}{4}}|0, V\rangle^{\pi/6} \quad (14)$$

However, it is interesting to look closely at the value $F = 3/4$, obtained for $P_T = 1/4$, because it implies quantum chaos for the reflected output.

$$|\psi\rangle^{\phi=\pi/3} = \sqrt{\frac{1}{4}}|H, 0\rangle^{\pi/3} + i\sqrt{\frac{3}{4}}|0, V\rangle^{\pi/3} \quad (15)$$

Here, we have noise symmetry because $P_T = 3/4$ produces quantum chaos statistics ($F = 1/4$) in the transmitted output, but $F = 3/4$ in the reflected one, and vice versa for $P_T = 1/4$. We will see below that complementary probabilities have the same variance for photon counting measurements. This is why we argue that $F = 3/4$ is also a criterion

of quantum chaos. The experimental values that we obtained for the Fano factor for chaotic probability of transmission are the following (see Figure 3): $F = 0.25 \pm 0.01$ for $P_T = 0.75 \pm 0.01$; $F = 0.74 \pm 0.01$ for $P_T = 0.26 \pm 0.01$. On the other hand, the Fano factor obtained for the wave behavior of photons was $F = 0.53 \pm 0.01$ for $P_T = 0.47 \pm 0.01$, while for the particle behavior, $F = 0.005 \pm 0.001$ for $P_T = 0.995 \pm 0.001$ was obtained.

Figure 3. Fano factor as function of the transmitted and reflected probabilities P_T (red circles) and P_R (green circles). The time series of photons obtained in the transmitted and reflected detectors displays quantum chaos at $F = 0.25$ (0.75) for $P = 0.75$ (0.25).

We also applied the power spectrum criterion to verify that there is indeed a quantum chaos behavior [17,18] beyond the criterion $F = 1/4, 3/4$. We apply a simplified version of the power spectrum for binary time series. We proceeded as follows: We separated the time series in partitions of 2^n combinations of n bits, bearing in mind that each bit is detected in the coincidence time τ , then each combination implies the time $n\tau$. Then, the time series of photons have a frequency f related to the number of photons counted during time $t = n\tau$. For $n = 2$ we can do the partition of the time series with 102,400 bits into 2-bit combinations, of which there are 4: $(0,0), (1,0), (0,1)$ and $(1,1)$. Each combination has a frequency based on the number of photons registered. For example, the element $(1,1)$ has two photons, and therefore it corresponds to $f = 2$ with the amplitude A_f related to number of times that appears $f = 2$. The power spectrum is defined by $PS(f) = |A_f|^2$. Since $(1,0)$ and $(0,1)$ have the same number of photons gives $f = 1$, they represent the same point in the power spectrum plot with degeneracy equal to 2. The state $(0,0)$ corresponds to $f = 0$. If the power spectrum follows a power law, the $\ln(PS(f))$ as a function of the frequency f outline a straight line whose slope β is the power of the frequency in $PS(f) \propto f^\beta$. Next, we obtain β as a function of P_T .

In Figure 4, β is displayed as a function of the transmission probability. When the time series is composed of only 0s or 1s (probabilities 0 and 1, respectively, in the transmitted port) the slopes will be $\beta_0, \beta_1 = \infty$, with Fano factors $F = 1$ and $F = 0$, respectively. Experimentally, we obtain $\beta_{0,1} \approx \pm 6$ (our experimental approximation to ∞). We can also verify that the slopes for $P = 1/4$ and $P = 3/4$ have approximately the theoretically expected powers $\beta_{1/4,3/4} = -1, +1$, for $F = 3/4, 1/4$, respectively, indicating the presence of quantum chaos in both series. Experimentally, we get the adjusted values $\beta_{1/4} = -1.05 \pm 0.03$ and $\beta_{3/4} = +1.08 \pm 0.01$, respectively. It is also clear that the series that has $F = 1/2$, corresponds to $\beta_{1/2} = 0$, since the adjusted value we get is $\beta_{1/2} = 0.058 \pm 0.003$.

Figure 4. Slopes behavior obtained from the Power Spectrum analysis (β) as function of the transmission probability for different basis (n), $\beta \approx -1, +1$ corresponds to $P \approx 0.25, 0.75$, respectively, for quantum chaos behavior. Clearly, $\beta \approx 0$ corresponds to $P \approx 1/2$, the quantum random, or wave-like behavior.

The chaotic $1/f$ or pink noise appears for $P_T = 1/4$, while its complementary signal f appears for $P_T = 3/4$. This complementarity can be showed also through the counting photon variance using the definition (4). For $P_T = 1/4, 3/4$, and its respective averages $\langle \hat{n} \rangle = 1/4, 3/4$, the variance for the quantum chaos signals have the same value $\langle (\Delta \hat{n})^2 \rangle = 3/16$. Experimentally, for $P_T = 0.75 \pm 0.01$, and $P_T = 0.26 \pm 0.01$, we obtain $\langle (\Delta \hat{n})^2 \rangle = 0.18 \pm 0.01, 0.19 \pm 0.01$, respectively (see Figure 5), in agreement with the two chaotic times series having the same level of information as we will see in Section 6.

Figure 5. Variance of the experimental photon counting versus the probability of transmission. The horizontal line at $\langle(\Delta n)^2\rangle = 3/16$, indicating the theoretical values for the noise associated to the photon counting noise for quantum chaos $F = 1/4, 3/4$, corresponding to the transmission probabilities $P_T = 3/4, 1/4$, respectively.

6. Complexity, Shannon's Information, and Quantum Chaos

We have shown that there is quantum chaos behavior in the time series of single photons for the probabilities $P_T = 1/4, 3/4$. Now, comparing Equations (12) and (13) with Equations (14) and (15), respectively, it is easy to see that the superposition of wave and particle behaviors builds quantum chaos condition $F = 1/4, 3/4$ when

$$C_p = C_w = \frac{1}{\sqrt{2}} \quad (16)$$

since

$$|\psi\rangle_{wp}^{\phi=0} = \frac{1}{\sqrt{2}}|H, 0\rangle^{\phi=0} + \frac{1}{2}|H, 0\rangle^{\phi=\pi/4} + \frac{i}{2}|0, V\rangle^{\phi=\pi/4} \quad (17)$$

reproduce the probability $P_T(H) = 3/4$ with $F = 1/4$, and:

$$|\psi\rangle_{wp}^{\phi=\pi/2} = \frac{i}{\sqrt{2}}|0, V\rangle^{\phi=\pi/2} + \frac{1}{2}|H, 0\rangle^{\phi=\pi/4} + \frac{i}{2}|0, V\rangle^{\phi=\pi/4} \quad (18)$$

for $P_T(H) = 1/4$ with $F = 3/4$.

In other words, we interpret quantum chaos behavior ($F = 1/4, 3/4$) when a single photon behaves half of the time as a particle and half of the time as a wave. Moreover, information entropy measures the degree of disorder or randomness in the system. Since our system is binary, we used Shannon's entropy, with the notation $p_i(\phi)$, where $i = 1, 0$, the probabilities of each photon being detected or not, for some polarization angle ϕ . Figure 6 shows the behavior of three quantities: the Shannon's entropy, $S = -\sum_i p_i(\phi) \log(p_i(\phi))$, the complexity $C = 4S(S - 1)$, and regularity $1 - S$ [25,26]. An intuitive definition of complexity may be associated with a composite systems with interacting components, where the balance between regularity and disorder comes from emergent interactions. The complexity invoked here has to do with an 'optimal' mix of regularity and disorder. The experimental behavior of this balance shows two peaks of

complexity. There is no mathematical expression as yet that relates complexity and chaos, and we can only say that the two quantum chaos behaviors at $F = 1/4, 3/4$ can be associated with the two maxima of complexity $C = 1$ appearing here at $S = 1/2$, while we obtain $S = 0$ for the particle-like behavior and $S = 1$ for the wave-like behavior. A superposition of both behaviors, wave and particle, is found here when $S \approx 0.81$, which corresponds to quantum chaos.

Figure 6. Experimental results for Shannon's information S (blue triangles), complexity $C = 4S(1 - S)$ (red dots), and regularity $1 - S$ (black squares), as a function of the probability of transmission.

7. Conclusions

We have shown that it is possible to obtain time series of single photons with different statistics, in particular, chaotic statistics. This was tested using the Fano factor and the power spectrum method criteria. In addition, we obtain a complexity maxima near each quantum chaos probability region, as a signature of their relation. Most importantly, we find that quantum chaos can be interpreted as the balanced superposition of particle and wave states of single photons. As a consequence, a single photon can be thought to behave in a quantum chaotic way and characterized by $1/f$ or pink noise for $F = 3/4$ ($P_T = 1/4$) and noise f for $F = 1/4$ ($P_T = 3/4$). In the same way, maxima of complexity appear for $S = 1/2$ as an 'optimal' balance between order and disorder. We have also shown that both signals are complementary to each other using photon-counting signals since their root mean square deviation around the counting average is the same for $F = 1/4, 3/4$, which entails $\langle (\Delta \hat{n})^2 \rangle = 3/16$. Finally, we believe that the chaotic time series of superposed states of single photons discussed in this paper may have interesting applications in quantum cryptography, for which pulsed lasers would be needed.

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