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Geometric Interpretation and General Classification of Three-Dimensional Polarization States through the Intrinsic Stokes Parameters

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Abstract: In contrast with what happens for two-dimensional polarization states, defined as those whose electric field fluctuates in a fixed plane, which can readily be represented by means of the Poincaré sphere, the complete description of general three-dimensional polarization states involves nine measurable parameters, called the generalized Stokes parameters, so that the generalized Poincaré object takes the complicated form of an eight-dimensional quadric hypersurface. In this work, the geometric representation of general polarization states, described by means of a simple polarization object constituted by the combination of an ellipsoid and a vector, is interpreted in terms of the intrinsic Stokes parameters, which allows for a complete and systematic classification of polarization states in terms of meaningful rotationally invariant descriptors.

1. Introduction

The Poincaré sphere [1] provides a simple and meaningful representation of those polarization states whose electric field fluctuates in a fixed plane (2D states). Despite the interest of such a particular type of polarization states, which is commonly applied in many problems involving paraxial fields and is characterized through the conventional four Stokes parameters [2], or (equivalently) by means of the 2×2 polarization matrix (or coherency matrix) [3–8], the description of a general polarization state involves nine generalized Stokes parameters [9–22] instead of the conventional four ones, and therefore their geometric representation through a generalized Poincaré sphere is determined by an eight-dimensional object, which does not admit a simple geometric and physical interpretation.

The relevance of the pioneering work of Soleilllet [4], who for the first time introduced a matrix structure that is equivalent to polarization matrices of 3D states has been discussed and emphasized by Arteaga and Nichols [23].

In the case of Mueller matrices, which represent linear transformations of 2D polarization states, appropriate geometric representations have been introduced by means characteristic ellipsoids [24–32], thus avoiding the problem of interpreting the extremely complicated 16-dimensional quadric object defined from the 16 elements of Mueller matrices.

In this work, the geometric representation introduced by Dennis [33] for general three-dimensional (3D) polarization states is studied and interpreted in terms of the intrinsic Stokes parameters [21] and other meaningful descriptors that are invariant under rotations of the reference frame [17–22,33–45]. The classification introduced in Refs. [20,46] is improved and completed in the light of the recent approaches on nonregularity [39,42,45,47,48], polarimetric dimension [41], spin of a polarization state [43,47,49] and interpretation of sets of orthogonal 3D polarization states [44].

The intrinsic geometric representation of a polarization state is determined by the *polarization density object*, which is constituted by the combination of an ellipsoid (the

polarization density ellipsoid) and a vector (the spin density vector). Apart from the scale parameter given by the intensity of the state, the shape of the polarization density ellipsoid, the magnitude of the spin density vector and its relative orientation with respect to the symmetry axes of the polarization density ellipsoid describe completely the intrinsic properties of the polarization state, while the spatial orientation of the polarization density object involves the three additional angular parameters required for its representation with respect to the reference frame considered.

That is, the complete information on a polarization state can be parameterized through the following set of nine parameters (whose definitions are summarized in Section 2): (1) the intensity, I, which plays the geometric role of a scale factor that determines the size of the polarization object; (2) the degree of linear polarization, P_l and the degree of directionality, P_d, which determine the shape of the polarization density ellipsoid; (3) the spin density vector $[\hat{\mathbf{n}} \equiv (\hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})^T]$ (three real parameters) associated with the polarization state, whose magnitude and relative orientation with respect to the polarization density ellipsoid are fixed for each polarization state, and (4) the three orientation angles (ϕ, θ, φ) of the polarization density ellipsoid with respect to the Cartesian reference frame XYZ considered [20].

It should be noted that the parameters describing a polarization state are defined for a given point **r** in space, and therefore they do not carry direct information on the direction of propagation of the wave at that point. Thus, the name degree of directionality used for P_d refers to the stability of the plane containing the polarization ellipse, and not to the direction of propagation.

Similarly to what happens with other intrinsic quantities in physics (such as, for instance, the tangential and normal components of the acceleration vector in kinematics) which give a natural a meaningful view of the physical quantities involved, the polarization object provides direct geometric representation of the intrinsic Stokes parameters $(I, P_l, P_d, \hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})$ [20,21] (note that, even though the definition of the intrinsic Stokes parameters involves $P_d/\sqrt{3}$ instead of P_d, we omit the coefficient $1/\sqrt{3}$ for brevity and simplicity).

The contents of this paper are organized as follows: Section 2 contains a summary of the concepts and notations that are necessary to make the required developments in further sections; the definition and discussion of the concept of polarization object is introduced in Section 3; Section 4 includes a classification of polarization states based on the peculiar geometric features of the polarization object, and Section 5 is devoted to the conclusions.

2. Materials and Methods

All the second-order polarization properties of an electromagnetic wave, at a given point **r** in space, are embodied in the three-dimensional polarization matrix (or coherency matrix) **R**, whose mathematical structure is that of a 3×3 positive semidefinite Hermitian matrix determined by the second-order moments of the analytic signals $\varepsilon_i(t)$ $(i = 1, 2, 3)$ (complex random variables, assumed stationary, at least in wide sense) associated with the three fluctuating Cartesian components (referenced with respect to a laboratory reference frame XYZ) of the electric field vector at point **r**:

$$\mathbf{R} \equiv \begin{pmatrix} \langle \varepsilon_1(t)\, \varepsilon_1^*(t) \rangle & \langle \varepsilon_1(t)\, \varepsilon_2^*(t) \rangle & \langle \varepsilon_1(t)\, \varepsilon_3^*(t) \rangle \\ \langle \varepsilon_2(t)\, \varepsilon_1^*(t) \rangle & \langle \varepsilon_2(t)\, \varepsilon_2^*(t) \rangle & \langle \varepsilon_2(t)\, \varepsilon_3^*(t) \rangle \\ \langle \varepsilon_3(t)\, \varepsilon_1^*(t) \rangle & \langle \varepsilon_3(t)\, \varepsilon_2^*(t) \rangle & \langle \varepsilon_3(t)\, \varepsilon_3^*(t) \rangle \end{pmatrix}, \tag{1}$$

where the superscript * indicates complex conjugate, while the brackets $\langle \rangle$ stand for time averaging over the measurement time. The analytic signal vector is defined as $\boldsymbol{\varepsilon}(t) = [\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t)]^T$, so that **R** can be expressed as $\mathbf{R} = \langle \boldsymbol{\varepsilon}(t) \otimes \boldsymbol{\varepsilon}^\dagger(t) \rangle$, where $\otimes$ represents Kronecker product and the superscript † stands for conjugate transpose.

When the fluctuations of $\varepsilon_i(t)$ $(i = 1, 2, 3)$ have Gaussian probability density functions, their second-order moments (the elements of **R**) characterize completely the statistical properties, so that the higher-order moments do not add complementary information, and

R fully characterizes the polarization state. In the most general case, **R** only characterizes the second order polarization properties.

The above indicated features are intimately linked to the fact, from its very definition, **R** is Hermitian and positive semidefinite, and therefore **R** has the mathematical structure of a covariance matrix of the three random complex variables $[\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t)]$. In fact, the diagonal elements of **R** are the respective (real and nonnegative) variances $\langle \varepsilon_i(t) \varepsilon_i^*(t) \rangle$ of $\varepsilon_i(t)$, while the off-diagonal elements of **R** are the (complex-valued) covariances determined by the respective quantities $\langle \varepsilon_i(t) \varepsilon_j^*(t) \rangle$.

Since subsequent analyses involve a large number of peculiar quantities, vectors and matrices, these, together with the corresponding references, are summarized in Table A1.

It has been proven that **R** can always be expressed as [20,21,33]:

$$\mathbf{R} = \mathbf{Q}\,\mathbf{R}_O\mathbf{Q}^T,$$

$$\mathbf{R}_O = I \begin{pmatrix} \hat{a}_1 & -i\,\hat{n}_{O3}/2 & i\,\hat{n}_{O2}/2 \\ i\,\hat{n}_{O3}/2 & \hat{a}_2 & -i\,\hat{n}_{O1}/2 \\ -i\,\hat{n}_{O2}/2 & i\,\hat{n}_{O1}/2 & \hat{a}_2 \end{pmatrix}, \quad \begin{bmatrix} \hat{a}_i, \hat{n}_{Oi} \in \mathbb{R} \;\; (i = 1,2,3) \\ \hat{a}_1 \geq \hat{a}_2 \geq \hat{a}_3 \geq 0 \\ \hat{a}_1 + \hat{a}_2 + \hat{a}_3 = 1 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{Q}^T = \mathbf{Q}^{-1} \\ \det\mathbf{Q} = +1 \end{bmatrix}, \tag{2}$$

where $I = \mathrm{tr}\mathbf{R}$ is the intensity and **Q** is the orthogonal matrix that diagonalizes the real part of **R**, so that the polarization matrix of the state under consideration takes the form **R** when the field variables are represented with respect to the given reference frame XYZ, and takes the form $\mathbf{R}_O$ when represented with respect to the intrinsic reference frame $X_OY_OZ_O$, which is characterized by the fact that the real part of $\mathbf{R}_O$ is a diagonal matrix, $\mathrm{Re}\mathbf{R}_O = I\mathrm{diag}\,(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ (with $\hat{a}_1 + \hat{a}_2 + \hat{a}_3 = 1$) and where the convention $\hat{a}_1 \geq \hat{a}_2 \geq \hat{a}_3$ is taken without loss of generality [note that since **R** is positive semidefinite, then necessarily $\hat{a}_i \geq 0\,(i = 1,2,3)$].

From a statistical point of view, the diagonal elements $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ of the intrinsic polarization density matrix $\hat{\mathbf{R}}_O \equiv \mathbf{R}_O/I$ are dimensionless and represent the intensity-normalized variances of the field variables (referenced with respect to $X_OY_OZ_O$), while the off-diagonal components represent the respective intensity-normalized covariances.

The principal variances $\hat{a}_i$ can be expressed as follows in terms of the degree of linear polarization, $P_l = \hat{a}_2 - \hat{a}_1$, and the degree of directionality, $P_d = 1 - 3\hat{a}_3$, [20,21]:

$$\hat{a}_1 = \frac{2 + P_d + 3P_l}{6}, \quad \hat{a}_2 = \frac{2 + P_d - 3P_l}{6}, \quad \hat{a}_3 = \frac{2(1 - P_d)}{6}, \tag{3}$$

so that the intrinsic polarization matrix $\mathbf{R}_O$ takes the form:

$$\mathbf{R}_O = \frac{I}{2} \begin{pmatrix} \frac{2+P_d+3P_l}{3} & -i\,\hat{n}_{O3} & i\,\hat{n}_{O2} \\ i\,\hat{n}_{O3} & \frac{2+P_d-3P_l}{3} & -i\,\hat{n}_{O1} \\ -i\,\hat{n}_{O2} & i\,\hat{n}_{O1} & \frac{2(1-P_d)}{3} \end{pmatrix}, \quad \begin{bmatrix} 0 \leq P_l \leq P_d \leq 1 \\ P_c \equiv \sqrt{\hat{n}_{O1}^2 + \hat{n}_{O2}^2 + \hat{n}_{O3}^2} \leq 1 \end{bmatrix}, \tag{4}$$

where the set of six rotationally invariant parameters $(I, P_l, P_d, \hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})$ constitute the intrinsic Stokes parameters of the state **R**. The intrinsic representation of the spin density vector of **R** is given by $\hat{\mathbf{n}}_O \equiv (\hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})^T$, which can also be parameterized through its magnitude $|\hat{\mathbf{n}}_O| \equiv P_c$ and its orientation angles (ϕ_n, θ_n) with respect to the axes $X_OY_OZ_O$ along which lie the principal intensities $a_1 \equiv I\hat{a}_1$, $a_2 \equiv I\hat{a}_2$ and $a_3 \equiv I\hat{a}_3$, respectively (Figure 1), i.e., $\hat{n}_{O1} = |\hat{\mathbf{n}}| \sin\theta_n \cos\phi_n$, $\hat{n}_{O2} = |\hat{\mathbf{n}}| \sin\theta_n \sin\phi_n$ and $\hat{n}_{O3} = |\hat{\mathbf{n}}| \cos\theta_n$.
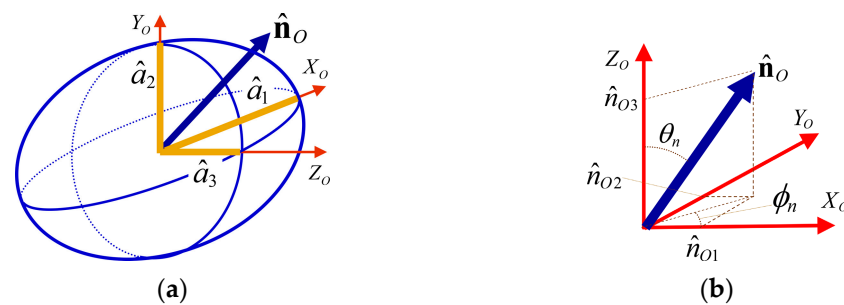
Figure 1. (**a**) Leaving aside the intensity, the polarization state is fully characterized by its polarization density object, composed of the polarization density ellipsoid E (with semiaxes $\hat{a}_1 \geq \hat{a}_2 \geq \hat{a}_3$), together with the spin density vector $\hat{\mathbf{n}}_O$. (**b**) $\hat{\mathbf{n}}_O$ is determined by its magnitude $|\hat{\mathbf{n}}_O| \equiv P_c$ and its orientation angles ϕ_n, θ_n with respect to the symmetry axes $X_O Y_O Z_O$ of E. The polarization object can be expressed in terms of the intrinsic Stokes parameters $(P_l, P_d, \hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})$ and therefore it has tensor character in the sense that it is invariant with respect to changes of the reference frame. When the polarization density object is referenced with respect to an arbitrary coordinate system, the orientation angles (ϕ, θ, φ) of the polarization density ellipsoid should be considered as additional parameters.

Note that even though the spin density vector is dimensionless (and therefore it has no dimensions of angular momentum) it is a proper descriptor of the spin anisotropy of the polarization state to which it corresponds [47]. Furthermore, as with $\mathbf{R}$ and $\mathbf{R}_O$, $\hat{\mathbf{n}}_O$ is relative to a specific point $\mathbf{r}$ in space and the term *density* in this context indicates that it describes the intensity-normalized version $\hat{\mathbf{n}}_O \equiv \mathbf{n}_O / I$ of the spin vector $\mathbf{n}_O$ associated with the given state [43,47].

The degree of polarimetric purity (or degree of polarization) P_{3D} [40] of $\mathbf{R}$ is related to intrinsic Stokes parameters through the following weighted square average of the components of purity (P_l, P_c, P_d) of $\mathbf{R}$ [38]:

$$P_{3D} = \sqrt{\frac{3}{4}\left(P_l^2 + P_c^2\right) + \frac{1}{4}P_d^2}, \tag{5}$$

where the quantity $P_e \equiv \sqrt{P_l^2 + P_c^2}$ that summarizes the combined contributions of P_l and P_c to the overall purity P_{3D} is called the degree of elliptical purity [39]. It is also worth recalling that P_{3D} can also be expressed as the following equivalent weighted quadratic average [41,43]:

$$P_{3D} = \sqrt{d^2 + \frac{3}{4}P_c^2}, \qquad \left[d \equiv \frac{1}{2}\sqrt{3P_l^2 + P_d^2}\right], \tag{6}$$

where d represents the degree of intensity anisotropy [41,43] of the polarization state, while the degree of circular polarization $P_c = |\hat{\mathbf{n}}_O|$ gives a measure of the degree of spin anisotropy [43].

Moreover, the orthogonal matrix $\mathbf{Q}$ of the rotation transformation $\mathbf{R} = \mathbf{Q}\,\mathbf{R}_O\mathbf{Q}^T$ depends on three angular parameters, and can be parameterized as follows in terms of the overall azimuth and elevation angles, ϕ and θ, respectively, of the axis Z of the reference frame XYZ and the azimuth φ of the XY axes about the direction Z, all referenced with respect to the intrinsic reference frame $X_O Y_O Z_O$:

$$\mathbf{Q}_O = \begin{pmatrix} c_\phi & s_\phi & 0 \\ -s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{pmatrix} \begin{pmatrix} c_\varphi & s_\varphi & 0 \\ -s_\varphi & c_\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_\theta\, c_\phi\, c_\varphi - s_\phi\, s_\varphi & c_\theta\, c_\phi\, s_\varphi + s_\phi\, c_\varphi & c_\phi s_\theta \\ -c_\theta\, s_\phi\, c_\varphi - c_\phi\, s_\varphi & -c_\theta\, s_\phi\, s_\varphi + c_\phi\, c_\varphi & -s_\phi s_\theta \\ -s_\theta\, c_\varphi & -s_\theta\, s_\varphi & c_\theta \end{pmatrix} \tag{7}$$

$$[s_x \equiv \sin x, \ c_x \equiv \cos x]$$

Another relevant concept that will be used in the classification of polarization states in Section 4 is the degree of nonregularity [39,42] whose definition relies on the characteristic decomposition of **R** [34,35]:

$$\mathbf{R} = P_1\, I\, \hat{\mathbf{R}}_p + (P_2 - P_1)I\, \hat{\mathbf{R}}_m + (1 - P_2)I\, \hat{\mathbf{R}}_{u-3D},$$
$$\left[\hat{\mathbf{R}}_p \equiv \mathbf{U}\text{diag}(1,0,0)\mathbf{U}^\dagger, \quad \hat{\mathbf{R}}_m \equiv \tfrac{1}{2}\mathbf{U}\text{diag}(1,1,0)\mathbf{U}^\dagger, \quad \hat{\mathbf{R}}_{u-3D} \equiv \tfrac{1}{3}\text{diag}(1,1,1)\right], \quad (8)$$

where **U** is the unitary matrix that diagonalizes **R**, while P_1 and P_2 are the indices of polarimetric purity (IPP) defined as [36,37]:

$$P_1 \equiv \hat{\lambda}_2 - \hat{\lambda}_1, \qquad P_2 \equiv 1 - 3\hat{\lambda}_3, \qquad (\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 = 1), \qquad (9)$$

$(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$ being the eigenvalues of the polarization density matrix $\hat{\mathbf{R}} \equiv \mathbf{R}/I$, taken in decreasing order $(\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \hat{\lambda}_3)$, so that the IPP satisfy the peculiar nested inequalities $0 \leq P_1 \leq P_1 \leq 1$ and fully determine the quantitative structure of polarimetric randomness of **R** [39]. Regarding the intensity-normalized components of the characteristic decomposition, $\hat{\mathbf{R}}_p$ is a pure (totally polarized) state, the discriminating state $\hat{\mathbf{R}}_m$ is given by an equiprobable incoherent mixture of the two eigenstates of **R** with largest eigenvalues $(\hat{\lambda}_1, \hat{\lambda}_2)$ and the unpolarized component $\hat{\mathbf{R}}_{u-3D}$ lacks both intensity and spin anisotropies [43]. The pure component $\hat{\mathbf{R}}_p$ has a well-defined polarization ellipse that evolves in a fixed plane. Except for the particular case that $\hat{\mathbf{R}}_m$ is regular (see below), the electric field of the discriminating component does not fluctuate in a fixed plane. Both polarization plane and shape of the polarization ellipse of the unpolarized component $\hat{\mathbf{R}}_{u-3D}$ evolve fully randomly, so that $\hat{\mathbf{R}}_{u-3D}$ completely lacks anisotropy and is proportional to the identity matrix.

Regular states are defined as those for which either $P_1 = P_2$ (in which case $\hat{\mathbf{R}}_m$ does not take place in the characteristic decomposition) or $\hat{\mathbf{R}}_m$ is a real-valued matrix, in which case $\hat{\mathbf{R}}_m$ lacks spin and takes the form of a 2D-unpolarized state $\hat{\mathbf{R}}_m = \hat{\mathbf{R}}_{u-2D} = (1/2)\,\text{diag}(1,1,0)$. Thus, $\hat{\mathbf{R}}_{u-2D}$ is a particular limiting situation of the general case where $\text{Im}\hat{\mathbf{R}}_m \neq 0$ and $0 \leq P_c\,(\hat{\mathbf{R}}_m) \leq 1/2$ [42,48]. In general, $P_1 \leq P_e$ and $P_2 \geq P_d$, where the equality $P_1 = P_e$ (which implies $P_2 = P_d$ and vice versa) is a characteristic and peculiar property of regular states. That is to say, **R** represents a nonregular state if and only if $P_1 < P_e$. The degree of nonregularity P_N of **R** is defined as follows by any of the following expressions in terms of the components of purity of $\hat{\mathbf{R}}_m$ [42]:

$$P_N = 4(P_2 - P_1)P_l(\mathbf{R}_m) = (P_2 - P_1)\left[1 - \sqrt{4P_c^2(\mathbf{R}_m)}\right] = (4/3)(P_2 - P_1)[1 - P_d(\mathbf{R}_m)], \quad (10)$$

$P_N = 0$ if and only if **R** corresponds to a regular state. States with maximal nonregularity, $P_N = 1$, are called perfect nonregular states, and necessarily satisfy $P_c\,(\hat{\mathbf{R}}_m) = 1/2$ while they are equivalent of an equiprobable incoherent mixture of a circularly polarized state and a linearly polarized state whose electric field vibrates in a direction orthogonal to the plane containing the polarization circle of the circular component [42].

3. Results

3.1. The Polarization Object

The nondimensional quantities $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ can be considered as the semiaxes of the polarization density ellipsoid, denoted by E, or inertia ellipsoid [33], associated with $\hat{\mathbf{R}}_O$, and leaving aside the intensity (I), which plays the role of a scale factor, E is determined by the intrinsic Stokes parameters P_l and P_d related to the intensity anisotropies [43]. The geometric parameterization is completed with the intrinsic representation $\hat{\mathbf{n}}_O \equiv (\hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})^T$ of the spin density vector of the state.

Therefore, from Equations (2) and (4) it follows that a general three-dimensional polarization state admits a simple geometric representation through its associated polarization density object, constituted by the combination of E and $\hat{\mathbf{n}}_O$, in such a manner that the

center of E and the origin of $\hat{\mathbf{n}}_O$ coincide with the point $\mathbf{r}$ to which the polarization state corresponds. Thus, both the magnitude $|\hat{\mathbf{n}}_O| = P_c$ of $\hat{\mathbf{n}}_O$ and its relative orientation with respect to E (determined by the intrinsic components $\hat{n}_{O1}$, $\hat{n}_{O2}$ and $\hat{n}_{O3}$) are fixed, in such a manner that the polarization density object has a fixed shape, while its orientation with respect to a given reference frame XYZ involves the three angles associated with the rotation from the intrinsic axes $X_O Y_O Z_O$ to XYZ.

As with the Poincaré sphere, the polarization density object has been defined through intensity-normalized quantities and therefore, it is the intensity-normalized version of the polarization object composed of the intensity ellipsoid E_I, with semiaxes (a_1, a_2, a_3), and the spin vector $\mathbf{n}_O \equiv I \, \hat{\mathbf{n}}_O$.

The polarization density object is represented in Figure 1 for a single point $\mathbf{r}$ and referenced with respect to $X_O Y_O Z_O$.

The expressions of the semiaxes of the polarization density ellipsoid in terms of P_l and P_d, allows for a direct analysis of certain important features of the polarization density object. When $P_d = 1$, the electric field of the state fluctuates in a fixed plane Π (2D state) and the polarization density ellipsoid becomes an ellipse, while $\hat{\mathbf{n}}_O$ is orthogonal to Π. Moreover, $P_d = 1$ implies $P_N = 1$, which means that all 2D states are regular states. A subclass of 2D states is that constituted by pure states (Figure 2), which are characterized by $P_{3D} = 1$, or equivalently $1 = P_d = P_2 = P_1 = P_e$), and include the limiting particular cases of linearly polarized states ($P_l = 1$, E degenerates into a simple straight segment, and therefore $P_c = 0$), and circularly polarized states ($P_c = 1$ and E takes the form of a circle). It should be stressed that pure states as well as any kind of 2D states (characterized by $P_d = 1$), constitute a particular subclass of 3D states.
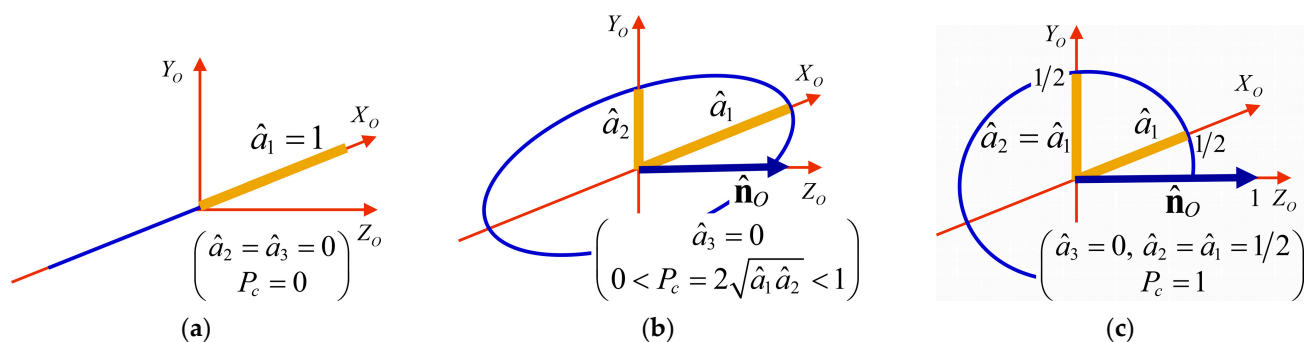

Figure 2. Polarization density object of a pure state ($P_e = 1$): linearly polarized state ($P_l = 1$) (**a**); elliptically polarized state ($0 < P_c < 1$) (**b**), and circularly polarized state ($P_c = 1$) (**c**). The spin density vector of pure states is orthogonal to the variance ellipse and is zero in the case of linearly polarized states.

Thus, despite the obvious fact that polarization states are realized in the three-dimensional physical space, states with $P_d < 1$ are called genuine 3D states (which may be regular or not) and are characterized by the fact that the three semiaxes of the associated polarization density ellipsoid are nonzero.

3.2. Classification of Three-Dimensional Polarization States Based on the Polarization Object

The concept of polarization object, together with other polarization descriptors such as the CP, the IPP, P_{3D} and P_N allows for a meaningful classification of both 2D states and genuine 3D states, where each category can also be interpreted in terms of the corresponding characteristic decomposition.

For the sake of clarity, the classification is performed through a pair of tables that correspond to 2D states (Table 1), and genuine 3D states (Table 2).

Table 1. Classification of 2D states ($P_d = 1$).

| $P_d = 1$ (2D states) $\Rightarrow P_2 = 1, P_1 = P_e, P_N = 0, P_{3D} \geq 1/2$				
$P_e = 1$ (pure states)			$P_e < 1$ (2D mixed states)	
$\mathbf{R} = \mathbf{R}_p$			$\mathbf{R} = P_1 \mathbf{R}_p + (1 - P_1)\mathbf{R}_{u-2D}$	
$P_l = 1$ $\updownarrow$ $P_c = 0$	$0 < P_l < 1$ $\updownarrow$ $0 < P_c < 1$	$P_l = 0$ $\updownarrow$ $P_c = 1$	$0 \leq P_e < 1$	$P_1 = 0$ $\mathbf{R} = \mathbf{R}_{u-2D}$
Linear	Elliptical	Circular	Partially polarized	Unpolarized
Indep. parameters: I, ϕ, θ, φ	Indep. parameters: $I, P_c, \phi, \theta, \varphi$	Indep. parameters: I, ϕ, θ	Indep. parameters: $I, P_c, P_l, \phi, \theta, \varphi$	Indep. parameters: I, ϕ, θ
Principal variances: $0 = \hat{a}_3 = \hat{a}_2, \ \hat{a}_1 = 1$	Principal variances: $0 = \hat{a}_3, \ 4\hat{a}_1\hat{a}_2 < P_c^2$	Principal variances: $0 = \hat{a}_3, \ \hat{a}_2 = \hat{a}_1 = 1/2$	Principal variances: $0 = \hat{a}_3 < \hat{a}_2 < \hat{a}_1$	Principal variances: $0 = \hat{a}_3, \ \hat{a}_2 = \hat{a}_1 = 1/2$
Spin density vector: $\hat{\mathbf{n}}_O = 0$	Spin density vector: $0 \neq \hat{\mathbf{n}}_O, \ \hat{\mathbf{n}}_O \perp \Pi$	Spin density vector: $|\hat{\mathbf{n}}_O| = 1, \ \hat{\mathbf{n}}_O \perp \Pi$	Spin density vector: $|\hat{\mathbf{n}}_O| < 1, \ \hat{\mathbf{n}}_O \perp \Pi$	Spin density vector: $|\hat{\mathbf{n}}_O| = 0, \ \hat{\mathbf{n}}_O \perp \Pi$
Polarization object: Figure 2a	Polarization object: Figure 2b	Polarization object: Figure 2c	Polarization object: Figure 3a	Polarization object: Figure 3b
Characteristic decomposition: $\mathbf{R} = \mathbf{R}_p$			Characteristic decomp.: $\mathbf{R} = P_1 \mathbf{R}_p +$ $(1 - P_1)\mathbf{R}_{u-2D}$ $(P_1 > 0)$ Figure 4a	Characteristic decomp.: $\mathbf{R} = \mathbf{R}_{u-2D}$ $(P_1 = 0)$ Figure 4b

(a)　　　　　　　　　**(b)**
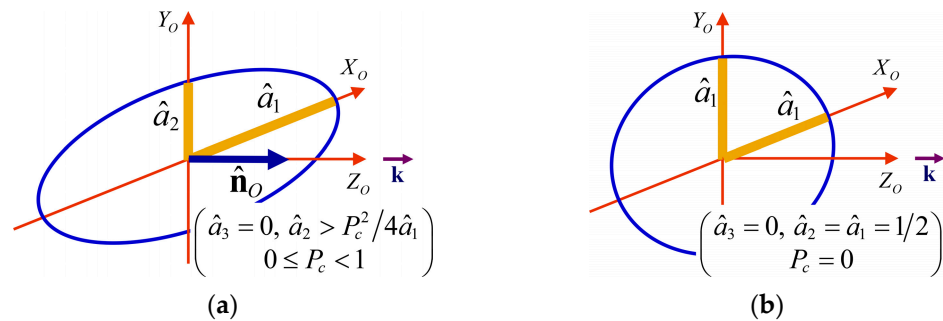
Figure 3. (**a**) Polarization object of a 2D mixed state ($P_d = 1, P_e = P_1 < 1$). (**b**) When $P_e = 0$, the state is itself a regular discriminating state $\mathbf{R}_{u-2D} = I\,\hat{\mathbf{R}}_{u-2D}$ (**b**).
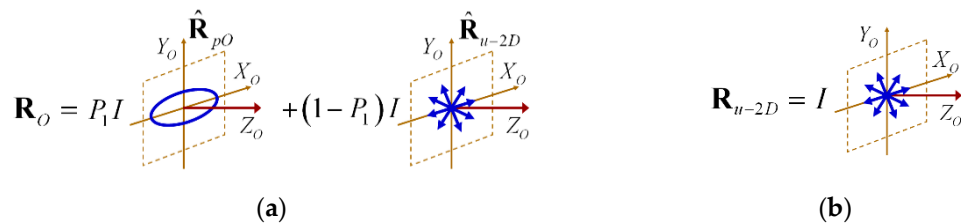

(a)　　　　　　　　　**(b)**

Figure 4. Intrinsic representation of the characteristic decomposition of a mixed 2D state ($P_d = 1, P_e = P_1 < 1$). (**a**) In general, $\mathbf{R}_O$ it is given by the incoherent superposition of a pure state $\mathbf{R}_{pO} = I\,\hat{\mathbf{R}}_{pO}$ and a 2D unpolarized state $\mathbf{R}_{u-2D} = I\,\hat{\mathbf{R}}_{u-2D}$ whose polarization planes coincide [Gil 2014a]. (**b**) When $P_e = 0$, then the pure component vanishes and the state is itself 2D unpolarized state (the electric field fluctuates fully randomly in a fixed plane. $X_O Y_O$).

Table 2. Classification of genuine 3D states $P_d < 1$.

$P_d < 1$ (genuine 3D states)	
Regular 3D mixed states $(P_N = 0)$	Nonregular 3D mixed states
$P_N = 0 \Leftrightarrow P_1 < P_e \Leftrightarrow P_2 > P_d$	$0 < P_N \leq 1 \Leftrightarrow P_1 < P_e \Leftrightarrow P_2 > P_d$
Independent parameters: $I, P_l, P_c, P_d, \phi, \theta, \varphi$	Independent parameters: $I, P_l, P_d, P_c, \phi_n, \theta_n, \phi, \theta, \varphi$
Principal variances: $0 < \hat{a}_3 \leq \hat{a}_2 \leq \hat{a}_1$	Principal variances: $0 < \hat{a}_3 \leq \hat{a}_2 \leq \hat{a}_1$
Spin density vector: $0 \leq |\hat{\mathbf{n}}_O|$ $(\hat{\mathbf{n}}_O \| Z_O)$	Spin density vector:
	$0 < |\hat{\mathbf{n}}_O|,\ \ 0 < P_c\,(\mathbf{R}_m) \leq 1/2,\ \ |\hat{n}_{O3}| \neq P_c$
Polarization object: Figure 5a	Polarization object: Figure 5b
Characteristic decomposition: Figure 6	Characteristic decomposition: Figure 7
$\mathbf{R} = P_1\,\mathbf{R}_p + (P_2 - P_1)\mathbf{R}_{u-2D} + (1 - P_2)\mathbf{R}_{u-3D}$	$\mathbf{R} = P_1\,\mathbf{R}_p + (P_2 - P_1)\mathbf{R}_m + (1 - P_2)\mathbf{R}_{u-3D}$ $(\mathbf{R}_m \neq \mathbf{R}_{u-2D})$

(a)

(b)

Figure 5. Polarization density object of genuine 3D states $(P_d < 1$). (**a**) The 3D state is regular when the spin density vector $\hat{\mathbf{n}}_O$ is orthogonal to the plane $X_O Y_O$. (**b**) the state it is nonregular if and only if $\hat{\mathbf{n}}_O$ is not orthogonal to $X_O Y_O$.
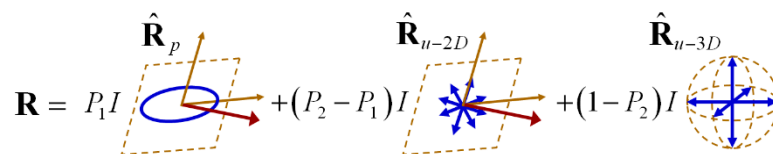

Figure 6. Characteristic decomposition of a regular genuine 3D state. The discriminating component is a 2D unpolarized state $\hat{\mathbf{R}}_{u-2D}$ whose polarization plane coincides with that of the pure component $\hat{\mathbf{R}}_{pO}$ and the 3D unpolarized state has nonzero contribution $(P_d = P_2 < 1$) [20].
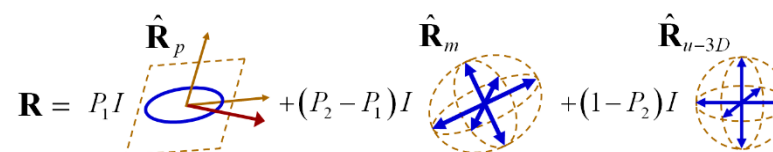

Figure 7. Characteristic decomposition of a nonregular 3D state. The discriminating component $\hat{\mathbf{R}}_m$ is itself nonregular.

2D mixed states (or 2D partially polarized states) are characterized by $1 = P_d = P_2$ and $P_1 = P_e < 1$. The characteristic decomposition of a 2D mixed state consists of a combination of a pure state $\mathbf{R}_p$ and a 2D partially polarized state $\mathbf{R}_{u-2D}$, both components sharing a common polarization plane. The case of a regular discriminating state $(P_2 = P_d = 1, P_1 = 0)$ corresponds precisely to $\mathbf{R}_{u-2D}$ (Figure 3b).

The types of genuine 3D states are summarized in Table 2, whose columns, from left to right, are devoted to the cases of (a) regular genuine 3D states $(P_d < 1, P_N = 0)$, whose discriminating component is a 2D unpolarized state, and (b) nonregular states $(P_N > 0)$. The polarization planes of the eigenstates $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ associated with the respective eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of $\hat{\mathbf{R}}$ (which coincide with those of $\hat{\mathbf{R}}_O$, and are taken so as to satisfy $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \hat{\lambda}_3$) are denoted by Π_1 and Π_2, and they only coincide for regular states.

Due to the critical role played by the discriminating states for the interpretation of polarization states [47,48], Table 3 summarizes the main features of (from left to right) (a) regular discriminating states; (b) partially nonregular discriminating states, and (c) perfect nonregular states.

Table 3. Classification of discriminating states $(P_2 = 1, P_1 = 0)$.

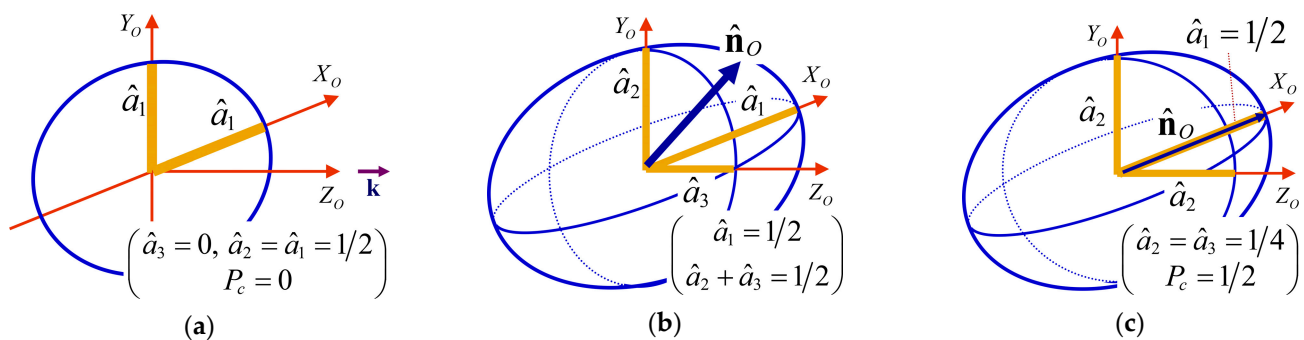| $P_2 = 1, P_1 = 0$ (discriminating states, $\mathbf{R} = \mathbf{R}_m$)		
$P_N = 0$ $\Updownarrow$ $P_d = P_2 = 1$ $(P_c = P_l = 0, P_d = 1)$	$0 < P_N < 1$ $\Updownarrow$ $P_1 < P_e \Leftrightarrow P_2 > P_d$ $(0 < P_c < 1/2, 0 < P_l < 1/4, P_d > 1/4)$	$P_N = 1$ $\Updownarrow$ $P_1 = 0, P_2 = 1$ $P_l = P_d = 1/4, P_c = 1/2$
2D unpolarized state	Nonregular discriminating state	Perfect nonregular state
Independent parameters: I, ϕ, θ	Independent parameters: $I, \phi, \theta, \varphi, P_c$	Independent parameters: I, ϕ, θ
Principal variances: $\hat{a}_3 = 0, \ \hat{a}_2 = \hat{a}_1 = 1/2$	Principal variances: $0 < \hat{a}_3 < \hat{a}_2 <$ $\hat{a}_1 = 1/2, \quad \hat{a}_2 + \hat{a}_3 = 1/2$	Principal variances: $\hat{a}_3 = \hat{a}_2 = 1/4, \ \hat{a}_1 = 1/2$
Spin density vector: $\hat{\mathbf{n}} = 0$	Spin density vector: $0 < |\hat{\mathbf{n}}_O| < 1/2$	Spin density vector: $|\hat{\mathbf{n}}_O| = 1/2 \quad (\hat{\mathbf{n}}_O \| X_O)$
Polarization object: Figure 8a	Polarization object: Figure 8b	Polarization object: Figure 8c

Figure 8. Polarization density object of a discriminating state $\mathbf{R}_m(P_1 = 0, \ P_2 = 1)$: (**a**) zero spin corresponds uniquely to the regular case $P_c(\mathbf{R}_m) = 0 \Leftrightarrow \mathbf{R}_m = \mathbf{R}_{u-2D}$; (**b**) $\mathbf{R}_m$ is nonregular if and only if $P_c(\mathbf{R}_m) > 0$, and (**c**) $\mathbf{R}_m$ is perfect nonregular if and only if $P_c(\mathbf{R}_m) = 1/2$.

Regular discriminating states have the simple form $\mathbf{R}_{u-2D}$ and are built by equiprobable incoherent compositions of arbitrary pairs of mutually orthogonal states with a common polarization plane, as for instance two linearly polarized states whose electric fields vibrate along two orthogonal directions embedded in the polarization plane.

Nonregular discriminating states and are built by equiprobable incoherent compositions of certain pairs of mutually orthogonal states with different polarization planes, including the combination of a linearly polarized state and an elliptically polarized state whose polarization planes are mutually orthogonal.

The case of perfect nonregular states corresponds to the limiting situation equivalent to an equiprobable incoherent mixture of a linearly polarized state and a circularly polarized state whose polarization planes are orthogonal.

4. Discussion

The mere qualitative properties of the polarization density object are sufficient to identify certain properties of the polarization state, while other features are linked to quantitative aspects. For instance, $\mathbf{R}$ corresponds to a 2D state if and only if at least one of the semiaxes of E is zero $(P_d = 1)$; $\mathbf{R}$ corresponds to a nonregular state if and only if $\hat{\mathbf{n}}_O$ is not parallel to Z_O. Moreover, $\mathbf{R}$ corresponds to a pure state if and only if the (quantitative)

condition $P_e = 1$ is satisfied (recall that and $P_e^2 \equiv P_l^2 + P_c^2$ and $P_c^2 \equiv |\hat{\mathbf{n}}_O|^2$). From the sole inspection of the polarization density object, the classification presented in Tables 1–3 can be synthesized as follows:

- $P_d = 1 \, (\hat{a}_3 = 0) \Leftrightarrow \mathbf{R}$ is a 2D state $\Leftrightarrow$ the polarization density ellipsoid is an ellipse.
  - ○ $P_e = 0 \Leftrightarrow \mathbf{R}$ is a 2D mixed state.
    - ■ $P_e < 1 \Leftrightarrow \mathbf{R}$ is a 2D unpolarized state (i.e., $\mathbf{R}$ is a 2D discriminating state), $\mathbf{R} = \mathbf{R}_{u-2D} \, (\hat{a}_2 = \hat{a}_1$ and $\hat{\mathbf{n}}_O = 0)$.
  - ○ $P_e = 1 \Leftrightarrow \mathbf{R}$ is a pure state, $\hat{a}_1 \, \hat{a}_2 = P_c^2 / 4$.
    - ■ $P_l = 1 \Leftrightarrow \mathbf{R}$ is a linearly polarized pure state, $\hat{a}_1 = 1 \; (\Rightarrow P_c = 0)$.
    - ■ $0 < P_l < 1 \Leftrightarrow \mathbf{R}$ is an elliptically polarized pure state, $0 < \hat{a}_2 < \hat{a}_1$, with $0 < P_c^2 = 1 - P_l^2 < 1$.
    - ■ $P_l = 0 \Leftrightarrow \mathbf{R}$ is a circularly polarized pure state, $\hat{a}_2 = \hat{a}_1 = 1/2$ with $P_c = 1$.
- $P_d < 1 \, (\hat{a}_3 > 0) \Leftrightarrow \mathbf{R}$ is a genuine 3D state.
  - ○ $\hat{\mathbf{n}}_O = 0$ or $\hat{\mathbf{n}}_O \parallel Z_O \Leftrightarrow \mathbf{R}$ is a regular genuine 3D state.
  - ○ $\hat{\mathbf{n}}_O \neq 0$ and $\hat{\mathbf{n}}_O$ not parallel to $Z_O \Leftrightarrow \mathbf{R}$ is a nonregular state.
  - ○ $P_l = P_d = 0 \Leftrightarrow$ The polarization density ellipsoid E of $\mathbf{R}$ is a sphere ($\hat{a}_3 = \hat{a}_2 = \hat{a}_1$, full intensity isotropy, $d = 0$)
    - ■ $P_c > 0 \Leftrightarrow \hat{\mathbf{n}}_O \neq 0$ (full intensity isotropy, $d = 0$, with nonzero spin).
    - ■ $P_c = 0 \Leftrightarrow \mathbf{R}$ is a 3D unpolarized state, $\mathbf{R} = \mathbf{R}_{u-3D}$ (full intensity and spin isotropy).
  - ○ $P_d + 3P_l = 1 \Leftrightarrow \mathbf{R}$ is a 3D discriminating state, $\mathbf{R} = \mathbf{R}_m \, (\hat{a}_2 + \hat{a}_3 = \hat{a}_1 = 1/2$, $0 < P_c \leq 1/2$ and $\hat{\mathbf{n}}_O$ not parallel to $Z_O)$.
    - ■ $P_c = 1/2 \Leftrightarrow \mathbf{R}$ is a perfect nonregular state ($\hat{\mathbf{n}}_O \parallel X_O$, $P_l = P_d = 1/4$)

5. Conclusions

Given a polarization matrix $\mathbf{R}$, its associated intrinsic form $\mathbf{R}_O$ is obtained through the rotation transformation (in the real space) that diagonalizes the real part of $\mathbf{R}$. The three (real) diagonal elements of $\mathbf{R}_O$, together with the three (pure imaginary) off-diagonal elements determine biunivocally the six intrinsic Stokes parameters $(I, P_l, P_d, \hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})$ of the state, which have a direct physical interpretation, namely the intensity, I, the degree of linear polarization, P_l, the degree of directionality, P_d (which is an objective measure of the degree of stability of the fluctuating polarization plane containing the polarization ellipse) and the three intrinsic components $(\hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})$ of the spin density vector $\hat{\mathbf{n}}_O$ of the state.

Consequently, any polarization state is fully characterized through its associated intrinsic Stokes parameters, which are invariant under rotation transformations of the laboratory reference frame XYZ, together with a three-dimensional rotation (determined by three angular parameters, ϕ, θ, φ, that depend on the specific spatial orientation of XYZ). As with the conventional four Stokes parameters characterizing 2D states, these quantities have phenomenological nature, and therefore, are always measurable.

The diagonal elements $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ of the intrinsic polarization density matrix $\hat{\mathbf{R}}_O = \mathbf{R}_O / I$, (with the convention $\hat{a}_1 \geq \hat{a}_2 \geq \hat{a}_3$, taken without loss of generality) are called the principal variances (because of their statistical nature) and constitute the semiaxes of an ellipsoid (polarization density ellipsoid, denoted by E), which in turn determines the directions of the respective axes $X_O Y_O Z_O$ of the intrinsic reference frame of the state.

Furthermore, the principal variances can be readily expressed in terms of the pair of intrinsic Stokes parameters P_l and P_d [see Equation (3)]. Leaving aside the intensity, I, which plays the geometric role a scale factor, the polarization density object is defined as the composition of the polarization density ellipsoid (which depends on P_l and P_d) and $\hat{\mathbf{n}}_O$, so that the relative orientation of $\hat{\mathbf{n}}_O$ with respect to $X_O Y_O Z_O$ is fixed and therefore the shape and features of the polarization density object are rotationally invariant.

The approach presented solves the problem of representing geometrically, in a simple and meaningful manner, all polarization states (three-dimensional, in general). The geometric features of the polarization density object are determined by the intrinsic Stokes parameters, which allows for a complete and systematic classification of polarization states.

Appendix A

Table A1. Quantities and structures relative to three-dimensional polarization states.

Structure or Quantity	Definition	Properties	Physical Meaning
Intensity	$I = \mathrm{tr}\mathbf{R} = \sum_{i=1}^{3} \langle \varepsilon_i(t)\,\varepsilon_i^*(t) \rangle$	Invariant under rotation and under the action of birefringent devices	Averaged power of the electromagnetic wave at point $\mathbf{r}$
Polarization matrix	$\mathbf{R}$	Hermitian positive semidefinite	Provides complete information on second-order polarization properties
Polarization density matrix	$\hat{\mathbf{R}} = \mathbf{R}/I$	Hermitian positive semidefinite, with $\mathrm{tr}\hat{\mathbf{R}} = 1$	Intensity-normalized polarization matrix. Formally equivalent to a density matrix
Eigenvalues of $\hat{\mathbf{R}}$	$\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$	$\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 = 1$ $\hat{\lambda}_3 \le \hat{\lambda}_2 \le \hat{\lambda}_2$	Relative weights of the spectral incoherent components of $\hat{\mathbf{R}}$ [20] $\hat{\mathbf{R}} = \hat{\lambda}_1\hat{\mathbf{R}}_{p1} + \hat{\lambda}_2\hat{\mathbf{R}}_{p2} + \hat{\lambda}_3\hat{\mathbf{R}}_{p3}$ $\hat{\mathbf{R}}_{pi} \equiv \hat{\mathbf{u}}_i \otimes \hat{\mathbf{u}}_i^\dagger$ ($\hat{\mathbf{u}}_i$: eigenvectors of $\hat{\mathbf{R}}$)
Indices of polarimetric purity (IPP)	$P_1 \equiv \hat{\lambda}_2 - \hat{\lambda}_1, \quad P_2 \equiv 1 - 3\hat{\lambda}_3$	$0 \le P_1 \le P_2 \le 1$	The IPP provide a complete quantitative characterization of the structure of polarimetric purity [36,37]
Intrinsic polarization matrix	$\mathbf{R}_O = I \begin{pmatrix} \hat{a}_1 & -i\,\hat{n}_{O3}/2 & i\,\hat{n}_{O2}/2 \\ i\,\hat{n}_{O3}/2 & \hat{a}_2 & -i\,\hat{n}_{O1}/2 \\ -i\,\hat{n}_{O2}/2 & i\,\hat{n}_{O1}/2 & \hat{a}_3 \end{pmatrix}$	Intrinsic representation of the polarization state. Principal variances: $\hat{a}_1, \hat{a}_2, \hat{a}_3$ Spin density vector: $\hat{\mathbf{n}}_O \equiv \mathbf{n}_O/I \equiv (\hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})^T$	Represents the same state as R, but referenced with respect to the corresponding intrinsic reference frame. The off-diagonal elements are pure imaginary because $\mathbf{R}_O$ is defined through the diagonalization of the real part of $\mathbf{R}$ [20,33].
Principal variances of $\hat{\mathbf{R}}_O$	$\hat{a}_1, \hat{a}_2, \hat{a}_3$	$\begin{bmatrix} \hat{a}_1 \ge \hat{a}_2 \ge \hat{a}_3 \ge 1 \\ \hat{a}_1 + \hat{a}_2 + \hat{a}_3 = 1 \end{bmatrix}$	Semiaxes of the polarization density ellipsoid [20,33].
Spin vector	$\mathbf{n}_O \equiv (n_{O1}, n_{O2}, n_{O3})^T$	$1 \ge \frac{n_{O1}^2}{4\hat{a}_2\,\hat{a}_3} + \frac{n_{O2}^2}{4\hat{a}_1\,\hat{a}_3} + \frac{n_{O3}^2}{4\hat{a}_1\,\hat{a}_2}$	Spin vector of the state, with dimensions of intensity [33,47]
Spin density vector	$\hat{\mathbf{n}}_O \equiv \mathbf{n}_O/I \equiv (\hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})^T$	$1 \ge \frac{\hat{n}_{O1}^2}{4\hat{a}_2\,\hat{a}_3} + \frac{\hat{n}_{O2}^2}{4\hat{a}_1\,\hat{a}_3} + \frac{\hat{n}_{O3}^2}{4\hat{a}_1\,\hat{a}_2}$	Spin density vector of the state (nondimensional) [20,33]
Spin density	$|\hat{\mathbf{n}}_O| = |\mathbf{n}_O|/I$	$|\hat{\mathbf{n}}_O| = P_c$	Absolute value of the spin density vector. Is a measure of the degree of circular polarization P_c of the state
Polarization object	Intensity ellipsoid E_I, with semiaxes a_1, a_2, a_3 and spin vector $\mathbf{n}_O$	Rigid composition of E_I and $\mathbf{n}_O$ The orientation angles (ϕ_n, θ_n) of $\mathbf{n}_O$ with respect to the symmetry axes $X_O Y_O Z_O$ of E_I are fixed and invariant	Determines geometrically all intrinsic properties of the state.
Polarization density object	polarization density ellipsoid E, with semiaxes $\hat{a}_1, \hat{a}_2, \hat{a}_3$ $\hat{a}_1 = \frac{2+P_d+3P_l}{6}, \ \hat{a}_2 = \frac{2+P_d-3P_l}{6}$ $\hat{a}_3 = \frac{2(1-P_d)}{6}$ and spin density vector $\hat{\mathbf{n}}_O$	Rigid composition of E and $\hat{\mathbf{n}}_O$ The orientation angles (ϕ_n, θ_n) of $\hat{\mathbf{n}}_O$ with respect to the symmetry axes $X_O Y_O Z_O$ of E_I are fixed and invariant	Determines geometrically all intrinsic properties of the state, but I, as with the Poincaré sphere of 2D polarization states

Table A1. *Cont.*

Structure or Quantity	Definition	Properties	Physical Meaning
Orientation angles of the polarization object	ϕ, θ, φ	ϕ, θ, φ determine the rotation from the intrinsic reference frame axes $X_O Y_O Z_O$ to an arbitrary one.	The angles that allow for representing the polarization object with respect to a given reference frame
Degree of linear polarization	$P_l = \hat{a}_2 - \hat{a}_1$	$0 \le P_l \le P_d \le 1$	An objective measure of how close to a linearly polarized state the state is [20,21]
Degree of circular polarization	$P_c = \|\hat{\mathbf{n}}_O\|$	$P_c^2 + P_l^2 \le 1$	An objective of how close to a circularly polarized state the state is [20,21]
Degree of directionality	$P_d = 1 - 3\hat{a}_3$	$0 \le P_l \le P_d \le 1$	An objective measure of the degree of stability of the plane containing the fluctuating polarization ellipse. Equivalently, a measure of the closeness of the 3D state to a 2D one [20,21]
Intrinsic Stokes parameters	$I, P_l, P_d/\sqrt{3}, \hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3}$	$\frac{3}{4}\left(P_l^2 + P_c^2\right) + \frac{1}{4}P_d^2 \le 1$ $\left(P_c^2 = \hat{n}_{O1}^2 + \hat{n}_{O2}^2 + \hat{n}_{O3}^2\right)$	Intrinsic measurable quantities. Have phenomenological nature: They are always well defined, regardless of the underlying microscopic model considered [20,21]
Dimensionality index, d and polarimetric dimension, D_I.	$d \equiv \sqrt{\frac{3P_l^2 + P_d^2}{2}}, \quad D_I \equiv 3 - 2d$	$0 \le d \le 1, \quad 1 \le D_I \le 3$	Determine the effective dimensions taking place in the state. $D_I = 1$: linearly polarized; $D_I \le 2$, 2D state; $2 < D_I \le 3$, 3D state [41]
Degree of polarimetric purity	$P_{3D} = \sqrt{\frac{3}{4}\left(P_l^2 + P_c^2\right) + \frac{1}{4}P_d^2}$ $P_{3D} = \sqrt{\frac{3}{4}P_1^2 + \frac{1}{4}P_2^2} = \sqrt{d^2 + \frac{3}{4}P_c^2}$	$0 \le P_{3D} \le 1$	An objective measure of how close to a pure state $\mathbf{R}$ is. It is determined by: (1) IPP contributions; (2) CP contributions and (3) Intensity and spin anisotropies [38,43]
Complete parameterization of $\mathbf{R}$	$(I, P_l, P_d, \hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3}, \varphi, \theta, \phi)$	Determine completely the polarization object and its spatial orientation with respect to the laboratory reference frame	Complete information carried by $\mathbf{R}$ in terms of nine meaningful quantities: the six intrinsic Stokes parameters and the three orientation angles of the polarization density object [20,21]
Characteristic decomposition	$\mathbf{R} = P_1 I \hat{\mathbf{R}}_p$ $+(P_2 - P_1)I\,\hat{\mathbf{R}}_m + (1 - P_2)I\,\hat{\mathbf{R}}_{u-3D}$	R is polarimetrically equivalent to an incoherent composition of pure state $\mathbf{R}_p$, a discriminating state $\mathbf{R}_m$ and a unpolarized state $\mathbf{R}_{u-3D}$	In the case of 2D states becomes the well known decomposition into an incoherent combination of a pure state and a 2D unpolarized state $\mathbf{R}_{u-2D}$ [39]
Discriminating component (in its own intrinsic representation)	$\hat{\mathbf{R}}_{mO} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\chi & i\cos\chi\sin\chi \\ 0 & -i\cos\chi\sin\chi & \sin^2\chi \end{pmatrix}$	This is the general form of a discriminating state, when referenced with respect to its own intrinsic reference frame [42]	In general, the discriminating component is different from $\mathbf{R}_{u-2D}$. When $\mathbf{R}_m \ne \mathbf{R}_{u-2D}$, then $\mathbf{R}_m$ is a 3D state ($P_d < 1$) and is said to be nonregular. Nonregular discriminating states exhibit nonzero spin, nonzero degree of linear polarization [42]
Degree of nonregularity	$P_N = (P_2 - P_1)\left[1 - \sqrt{4P_c^2(\mathbf{R}_m)}\right]$	$0 \le P_N \le 1$	An objective measure of the distance of the state to a regular state [42]

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