photonics

MDPI

Article

Phase Transitions in Quasi-Hermitian Quantum Models at Exceptional Points of Order Four

Miloslav Znojil [1,2,3,4]

[1] The Czech Academy of Sciences, Nuclear Physics Institute, Řež 292, 250 68 Husinec, Czech Republic; znojil@ujf.cas.cz
[2] Department of Physics, Faculty of Science, University of Hradec Králové, Rokitanského 62, 500 03 Hradec Králové, Czech Republic
[3] School for Data Science and Computational Thinking, Stellenbosch University, Stellenbosch 7600, South Africa
[4] Institute of System Science, Durban University of Technology, Durban 4000, South Africa

Abstract

Phase transition in quantum mechanics is interpreted as an evolution, at the end of which, typically, a parameter-dependent and Hermitizable Hamiltonian $H(g)$ loses its observability. In the language of mathematics, such a "quantum catastrophe" occurs at an exceptional point of order N (EPN). Although the Hamiltonian $H(g)$ itself becomes unphysical in the limit of $g \to g^{EPN}$, it is shown that it can play the role of an unperturbed operator in an innovative perturbation-approximation analysis of the vicinity of the EPN singularity. As long as such an analysis is elementary at $N \leq 3$ and purely numerical at $N \geq 5$, we pick up $N = 4$ and demonstrate that for an arbitrary quantum system, the specific (i.e., already sufficiently phenomenologically rich) EP4 degeneracy becomes accessible via a unitary evolution process. This process is shown realizable inside a parametric domain $\mathcal{D}_{\text{physical}}$, the boundaries of which are determined, near g^{EP4}, non-numerically. Possible relevance of such a mathematical result in the context of non-Hermitian photonics is emphasized.

Keywords: quantum mechanics using quasi-Hermitian observables; non-Hermitian but Hermitizable *alias* quasi-Hermitian Hamiltonians; unitary quantum phase transitions at exceptional points; specific fourth-order exceptional points (EP4s); relevance of non-numerical study of EP4s in photonics

1. Introduction

At present, a deep and productive relevance of an originally purely mathematical concept of exceptional point (EP, [1]) in photonics is well known [2]. From a historical perspective (cf., e.g., the introductory chapter in dedicated monograph [3]), one could really localize several independent roots of the related successful theoretical as well as experimental developments.

During their first stage, for example, a key driving force originated from the relativistic quantum field theory, in the framework of which the challenges emerged due to the discoveries of divergences of many innocent-looking perturbation expansions. Soon, these phenomena found some of their very natural though not quite expected mathematical explanations precisely in the existence of the EP values of certain relevant phenomenological parameters (cf., e.g., [4,5]).

The currently popular idea of the use of the EPs in photonics only emerged when several authors realized that there must exist an intimate connection between the Schrödinger equation of quantum mechanics (in which the EPs already found their applications) and the

Check for updates

Received: 3 February 2026
Revised: 18 February 2026
Accepted: 25 February 2026
Published: 26 February 2026

Copyright: © 2026 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

classical Maxwell equations (especially when considered in the so-called paraxial approximation, cf., e.g., [6]). In any case, the subsequent progress was truly impressive (cf., e.g., the recent review [7]), based mainly on a wide range of mathematics shared by the quantum theory and photonics. Incidentally, this also motivated our present paper in which we are going to restrict our attention to several rather interesting interdisciplinary aspects of the shared mathematics using, in most cases, just the representative language of the so-called quasi-Hermitian quantum models in which the EPs manifest themselves as the instants of a specific class of the so-called quantum phase transitions.

The task of description of a quantum phase transition is, for several reasons, difficult. First of all, one has to leave the comparably comfortable formalism of classical mechanics. After the quantization, the numbers representing the classical observable quantities like, e.g., a (possibly time-dependent) point-particle momentum $p = p(t)$ must be replaced by operators. In particular, the quantized energy E, conserved or not, has to be treated as an eigenvalue of a preselected energy-representing Hamiltonian H, etc. Thus, a more or less purely geometric nature of the classical phase-transition theories (cf., e.g., [8]) is lost.

Secondly, all of the operators $\Lambda(t)$ representing a relevant quantum observable have to be diagonalizable and, moreover, they are usually chosen or constructed as self-adjoint in a suitable (and, mostly, infinite-dimensional) Hilbert space of states $\mathcal{H}_{\text{physical}}$. In such a traditional quantum model-building setup, the very existence of a genuine "change of phase" (i.e., basically, of a loss of the reality of the spectrum of at least one of the observables) seems to contradict the well known tendency of eigenvalues of any self-adjoint operator to avoid any phase-transition-mimicking merger. Just an apparent "repulsion of levels" is encountered in many experiments.

On the level of abstract quantum theory, both of the latter obstacles can be circumvented when one resorts to the less usual formulation of quantum mechanics called quasi-Hermitian quantum mechanics (QHQM, cf., one of its oldest reviews [9] for a comprehensive introduction). The resolution of at least some of the quantum phase-transition puzzles is offered there by the possibility of the realization of the phase-transition-mimicking mergers of the eigenvalues as well as of the states at a singularity called, by mathematicians, exceptional point (EP, [1]). By definition, this means that in the singular, manifestly unphysical EP limit, the overall spectrum of an observable under consideration remains real but, at the same time, it degenerates and ceases to admit standard interpretation. Such a way of thinking about quantum phase-transition processes will also be accepted in our present paper.

We will start from the observation that in the majority of the existing constructions of models admitting an EP-related phase transition, the authors prefer the systems characterized by the observability of the mere parameter-dependent bound-state energy levels $E_n(g)$ with $n = 0, 1, \ldots$. For the sake of definiteness, moreover, just the models admitting the most elementary EP mergers of order two (EP2) are usually considered. In the language of mathematics, this means that the "first nontrivial" form of the dynamical scenario is only being described, in which the loss of the diagonalizability of the Hamiltonian $H(g)$ in the limit of $g \to g^{(EP2)}$ is restricted to a two-dimensional subspace of the whole Hilbert space. In other words, the EP2-related merger of energies $E_{n_j}(g)$ as well as of the related bound-state eigenvectors $|\psi_{n_j}(g)\rangle$ only involves a single pair of states with subscripts $j = 1$ and $j = 2$.

Even the EP2-related studies need not be trivial. In the non-Hermitian physics framework admitting complex energies (i.e., resonances—cf., e.g., an introductory outline of the underlying theory in monograph [10]), one of the oldest and, at the same time, one of the best known illustrative examples of the EP2 mergers was found, many years ago, by Bender with Wu [4]. In their methodically motivated considerations concerning the general

relativistic quantum field theory, they paid attention just to a zero-dimensional version of this theory. More specifically, they considered the quantum-mechanical anharmonic-oscillator Hamiltonian $H(g) = p^2 + x^2 + g\,x^4$ with $x \in (-\infty, \infty)$, and they managed to localize its EP singularities. All of them appeared to lie off the real axis, i.e., at the complex values of $g = g_{m,n}^{(EP2)}$ numbered by the two integers $m, n = 0, 1, \ldots$. Although all of them belonged just to the simplest, second-order category, their discovery was important: these degeneracies became known as the Bender–Wu singularities [5].

Out of quantum physics, some authors also speak about non-Hermitian degeneracies [11–13]. Beyond all of their most elementary EP = EP2 models, quantum or non-quantum (see also their most recent samples in [14–16]), there appeared studies of certain more complicated, viz., EP3-supporting phase-transition models [17,18]. The point was that the authors of the latter generalizations (in which the degeneracy involves a triplet of states) could make use of the exact solvability of the related secular equations (i.e., of the mere algebraic polynomial equations of the third order in the energy) in terms of the well known Cardano formulae.

The existence of such a form of solvability also inspired our present paper. We imagined that the next EP4-supporting family of models is in fact the only remaining class of the special quantum systems enjoying the advantage of being non-numerical. Indeed, the related algebraic polynomial secular equations (of the fourth order) are in fact the last ones solvable in closed form. Unfortunately, in contrast to Cardano formulae, the closed formulae for the quartic-polynomial roots can hardly be considered user-friendly. For this reason, the EP4-supporting quantum phase-transition models have only marginally and not too deeply been considered in the literature [19].

In our present paper, we decided to fill the gap. The presentation of our results will be preceded by Section 2, in which we will outline, briefly, an overall correspondence between the exceptional points and the quantum phase transitions. Next, in Section 3, we will restrict our attention to the case of the so-called closed quantum systems in which, by definition, the evolution remains unitary. We will explain there that the quantum phase transitions may exist even under such a constraint, provided only that one makes use of the above-mentioned QHQM description of the dynamics. For the sake of a maximal completeness of the paper, a few basic features of the QHQM reformulation of the standard quantum theory are also added there.

Our main message will finally be presented in Sections 4 and 5, in which we will formulate the problem and describe its solution. The basic idea of our approach will lie in a reduction of the general EP4-related phase-transition problem to its analysis in a small vicinity of the EP4 degeneracy. This will enable us to construct its sub-vicinity $\mathcal{D}_{\text{physical}}$ (i.e., a unitarity-compatible subdomain of the full space of the available dynamical parameters) and to localize its boundaries using a suitable form of mathematical perturbation theory.

These results are then discussed and summarized in Section 6.

2. Exceptional Points

2.1. Unitary- and Non-Unitary-Evolution Scenarios

In some applications of quantum mechanics, it need not be easy to distinguish between the systems characterized by the unitary and non-unitary forms of evolution. During the study of the former scenarios (with the models describing the so-called closed quantum systems), the main emphasis is usually put on a completeness and on a maximal theoretical consistency of the picture [20]. In contrast, the admissibility of a manifestly non-unitary dynamics finds its strongest support in experimental physics [21,22]. One feels more inclined there to admit that we only rarely manage to separate the system from its ubiquitous

but only partially understood environment. Thus, it makes sense to speak about the open quantum systems in such a case [10].

In both of the latter phenomenologically non-equivalent situations, one can encounter the phenomena in which it is not too easy to separate the unitary and non-unitary aspects of the evolution. As an example, we could recall the models in which some, but not all, of the operators representing an observable (i.e., say, a parameter-dependent Hamiltonian $H(g)$) will lose their diagonalizability and, hence, the status of a quantum observable. For an explicit illustration of such a rather involved scenario the authors of review paper [9] recalled the traditional two-parametric Lipkin–Meshkov–Glick multiparticle model in a generalized form characterized by a non-Hermitian Hamiltonian operator (cf., equation Nr. (3.1) in *loc. cit.*). Manifestly, these authors demonstrated that their model really admitted several rather realistic forms of a genuine quantum phase transition.

A few basic qualitative features of the example may be found summarized in the phase diagram Nr. 1 of *loc. cit*. In this diagram, indeed, only some subdomains of the space of the parameters appeared to admit a closed-system interpretation of the dynamics of the system, with some of their boundaries being, precisely, the EP singularities mimicking certain rather realistic forms of the quantum phase transitions.

2.2. Non-Hermitian Degeneracies

In a more general, model-independent manner, by definition, the EP singularities will be, in general, of order N (EPN, [1]). At $g \neq g^{(EPN)}$, before we move to the EPN extreme and to the phenomenon of the $N-$tuple degeneracy, we may assume that this degeneracy is going to involve an $N-$plet of non-degenerate energy-representing eigenvalues $E_n(g)$ as well as, simultaneously, the related eigenstates $|\psi_n(g)\rangle$ of the Hamiltonian $H(g)$ with $n = n_1, n_2, \ldots, n_N$. Once we assume that both of these $N-$plets happen to merge in the EPN limit, we may write

$$\lim_{g\to g^{(EPN)}} E_{n_j}(g) = E^{(EPN)}\,, \qquad \lim_{g\to g^{(EPN)}} |\psi_{n_j}(g)\rangle = |\psi^{(EPN)}\rangle\,, \quad j = 1, 2, \ldots, N\,. \tag{1}$$

Under certain standard mathematical assumptions [1], there will also exist a vicinity of the critical parameter such that $g \neq g^{(EPN)}$, inside which the operator $H(g)$ itself remains diagonalizable and tractable as a quantum Hamiltonian. In the limit $g \to g^{(EPN)}$ (and after the restriction of our interest to the mere relevant, $N-$dimensional subspace of the Hilbert space), in contrast, the conventional diagonalizability, i.e., the solvability of the conventional Schrödinger equation

$$H^{(N)}(g)|\psi_n(g)\rangle = E_n(g)\,|\psi_n(g)\rangle\,, \quad n = 0, 1, \ldots, N \tag{2}$$

becomes replaced, in the same subspace, by the solvability of another, unphysical N-by-N-matrix equation

$$H^{(N)}(g^{(EPN)})\,\mathcal{U} = \mathcal{U}\,J^{(N)}(E^{(EPN)})\,. \tag{3}$$

Its solution $\mathcal{U} = \mathcal{U}^{(EPN)}$ is called transition matrix, while the symbol

$$J^{(N)}(x) = \begin{bmatrix} x & 1 & 0 & \ldots & 0 \\ 0 & x & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & x & 1 \\ 0 & \ldots & 0 & 0 & x \end{bmatrix}$$

stands for the so-called Jordan-matrix function of the degenerate, unphysical energy $x = E^{(EPN)}$.

2.3. Phase Transitions

The construction of the N-by-N-matrix solution $\mathcal{U}^{(EPN)}$ of the canonical, Schrödinger-equation-resembling Equation (3) is important in applications. It opens, in particular, the way towards a decisive simplification of the matrix form of every physical Hamiltonian $H(g)$ in which the parameter g is different from $g^{(EPN)}$ but still lying not too far from this critical value. Thus, for the sake of definiteness, we assume that the absolute value of the difference $\lambda = g - g^{(EPN)}$ remains small. In such a situation, we have to reconsider Schrödinger Equation (2), treating it as a $\lambda-$dependent eigenvalue problem

$$H^{(N)}(g^{(EPN)}+\lambda)|\psi_n(g^{(EPN)}+\lambda)\rangle = E_n(g^{(EPN)}+\lambda)\,|\psi_n(g^{(EPN)}+\lambda)\rangle\,, \quad n=0,1,\ldots,N \tag{4}$$

i.e., as a rather unusual but mathematically well motivated perturbation of its manifestly unphysical EPN limit.

At a sufficiently small λ, we now have to use perturbation theory, say, in its QHQM-adapted form of Ref. [23], proceeding along the lines as described, in detail, e.g., in papers [24,25]. Naturally, even when we restrict the range of the parameter λ to a small vicinity of zero, we can still speak about the phase-transition process controlled by the variable g reaching its loss-of-the-observability extreme in the EPN limit of $g \to g^{(EPN)}$.

During the implementation of perturbation theory, we reveal that our knowledge of the solutions $\mathcal{U} = \mathcal{U}^{(EPN)}$ of the unperturbed and $g-$independent Schrödinger Equation (3) proves particularly useful because it enables us to simplify the perturbation-approximation construction via a replacement of any initial perturbed, $g-$dependent Hamiltonian $H(g)$ by its isospectral $\lambda-$dependent avatar

$$P(\lambda) = \mathcal{U}^{-1}\,H(g^{(EPN)}+\lambda)\,\mathcal{U} = J^{(N)}(E^{(EPN)}) + \text{a perturbation}\,. \tag{5}$$

The point is that at the sufficiently small λs, the new matrix $P(\lambda)$ remains dominated by its unperturbed Jordan-matrix component so that the systematic construction of the perturbed bound-state solutions becomes perceivably facilitated.

3. Closed Systems Admitting Phase Transitions

We saw above that the description of the phase transition, i.e., of the passage of the quantum system in question through its EPN singularity, may acquire an explicit perturbation-approximation form when we employ a parameter λ which would be properly time-dependent. Marginally, let us add that the perturbed energy spectra near EPNs are, in general, complex. In the literature, nevertheless, their study found a persuasive mathematical motivation, first of all, in the abstract perturbation theory [1]. Soon, nevertheless, the EPN-related mathematics found also important applications in physics. Nevertheless, as long as all of the energies near an EP singularity are, in general, complex, one must interpret the corresponding states as resonances, with the non-Hermitian system in question acquiring the status of the so-called open quantum system.

In the case of the other, unitary, closed quantum systems (for which the states are stable of course), one has to add a few new conditions to guarantee the reality of the energies. This is a fairly complicated task, a fulfillment of which will be discussed in the rest of this paper.

3.1. Probabilistic Interpretation of Quasi-Hermitian Observables

In the majority of textbooks on quantum mechanics, the authors pay attention, first of all, to the unitary-evolution processes and to their rather detailed description in a specific

representation called Schrödinger picture [20]. In such a very traditional setting, the time-evolution of the state-representing wave functions $|\psi\succ$ in a suitable physical Hilbert space (say, $\mathcal{L}$) has to be generated by a conventional diagonalizable and self-adjoint Hamiltonian operator (say, $\mathfrak{h} = \mathfrak{h}^\dagger$).

A transition from the latter, conventional formulation of quantum theory to the non-Hermitian QHQM formalism can be traced back to the papers by Dyson [26]. Empirically, he revealed that for the variational-calculation purposes, the conventional textbook form $\mathfrak{h}$ of the Hamiltonian need not be optimal, and that a truly significant enhancement of the efficiency of the calculations can be achieved via a suitable isospectral but non-unitary preconditioning of this operator,

$$\mathfrak{h} \ \to \ H = \Omega^{-1}\,\mathfrak{h}\,\Omega\,, \quad \Omega^\dagger\Omega = \Theta \neq I\,. \tag{6}$$

From the point of view of the prediction of the results of experimental measurements, both of the operators $\mathfrak{h}$ and H describe the same bound states (i.e., the same closed and unitary quantum system). Thus, the Dyson-proposed preference of the use of H is to be understood as just an *a posteriori* observation of a purely technical aspect of his specific model.

In a fully abstract theoretical setting, the mapping (6) can be inverted. Still, one can easily verify that the Hermiticity of $\mathfrak{h}$ in $\mathcal{L}$ is formally equivalent, for non-Hermitian $H \neq H^\dagger$, to its quasi-Hermiticity property

$$H^\dagger\,\Theta = \Theta\,H \tag{7}$$

alias Θ—quasi-Hermiticity property valid in the corresponding Hilbert space, say, $\mathcal{H}_{\rm mathematical}$.

Admissibly [26,27], the latter space need not coincide with $\mathcal{L}$ so that, due to the non-Hermiticity of H, the space $\mathcal{H}_{\rm mathematical}$ itself is unphysical, i.e., it is not equivalent to the above-mentioned QHQM Hilbert space $\mathcal{H}_{\rm physical}$. Fortunately, both of these spaces can be chosen as sharing their ket-vector elements $|\psi\rangle$. Then, the only difference involves the respective inner products. Once we accept the conventional bra-ket denotation $\langle\psi_1|\psi_2\rangle$ of the inner product in $\mathcal{H}_{\rm mathematical}$, we may succeed in representing $\mathcal{H}_{\rm physical}$ in the same (but, by assumption, user-friendlier) representation space $\mathcal{H}_{\rm mathematical}$. For this purpose, it is sufficient to change the inner product and use the following metric-mediated "correct physical product" overlap

$$\langle\psi_1|\Theta|\psi_2\rangle\,. \tag{8}$$

The latter values, incidentally, coincide with the conventional textbook inner products $\prec\psi_1|\psi_2\succ$ in $\mathcal{L}$: To see this, it is sufficient to recall Equation (6) and to conclude that $|\psi\succ = \Omega\,|\psi\rangle$ (cf., e.g., [27] for the reasons and merits of such a slightly generalized version of the Dirac's notation convention).

For an efficient explicit description of an EP-related phase transition in a closed quantum system, it may prove sufficient to work with a suitable non-Hermitian and parameter-dependent Hamiltonian operator $H = H(g)$ with real spectrum admitting an EPN singularity at $g = g^{(EPN)}$. One only has to keep in mind that the existence of the Dyson-map correspondence (6) as well as of the related correct physical inner-product metric Θ is "fragile" [28,29]. In the limit of $g \ \to \ g^{(EPN)}$, by definition, both of these operators would have to cease to exist.

In order to make the theory internally consistent and complete, our task becomes twofold. For a given $H(g)$, we have to demonstrate, firstly, the existence of the EPN singularity. Secondly, we have to demonstrate the existence of an unfolding of this singularity

during which the spectrum remains all real. Both of these tasks are fairly nontrivial and requiring, in general, a purely numerical treatment.

3.2. Exactly Solvable Benchmark Models

For the closed quantum systems admitting the second-order (i.e., $N = 2$) and the third-order (i.e., $N = 3$) non-Hermitian EPN degeneracies, the realization of both of the two above-mentioned tasks (viz., of the EPN localization and of the specification of a path of a unitary unfolding of the singularity) can already be found in the literature. In fact, the true difficulties only occur at $N = 4$, which is still solvable exactly and non-numerically, in principle at least. Beyond this case, all of the $N \geq 5$ EPN models can only be treated, on a suitable approximation level, numerically.

Even at the arbitrarily large Ns, there are, of course, solvable-model exceptions [29]. In Ref. [30], for example, the readers may find a commented reference to a certain fairly realistic many-body Bose–Hubbard (BH) quantum system as proposed and studied, in a broader phenomenological context, by Graefe et al. [31]. Out of the sequence of the underlying N by N Hamiltonian matrices of a variable dimension N, we picked up, in *loc. cit.*, the strictly quasi-Hermitian subset

$$H^{(2)}_{(BH)}(g) = \begin{bmatrix} -ig & 1 \\ 1 & ig \end{bmatrix}, \quad H^{(3)}_{(BH)}(g) = \begin{bmatrix} -2\,ig & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 2\,ig \end{bmatrix}, \quad \ldots \tag{9}$$

having real spectra if and only if $|g| \leq 1$. We could treat this system as closed (i.e., unitary) at $|g| < 1$. Moreover, we showed, constructively, that due to a certain immanent symmetry, the specific models (9) happen to admit, at any N, a quantum phase transition in the two EPN limits of $g \to \pm 1$. The choice of $|g| < 1$ forms then a corridor of passage from the standard unitary dynamical regime to the corresponding, phase-transition-representing EPN singularity.

Among the most important features of the exactly solvable quasi-Hermitian quantum models as sampled by Equation (9), one can count, first of all, the existence of the closed formulae for the real bound-state energies—in [30], these formulae were displayed and shown valid at all of the physical couplings $g < g^{(EPN)}$. Another specific merit of the Bose–Hubbard-like toy-model family of Equation (9) has to be seen in the availability of the exact N-by-N transition-matrix solutions of the unperturbed Schrödinger Equation (3), say, at $g = g^{(EPN)} = 1$,

$$\mathcal{U}^{(2)}_{(BH)} = \begin{bmatrix} -i & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{U}^{(3)}_{(BH)} = \begin{bmatrix} -2 & -2\,i & 1 \\ -2\,i\sqrt{2} & \sqrt{2} & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad \ldots . \tag{10}$$

Indeed, as we already emphasized above, the knowledge of $\mathcal{U}$ in the "unphysical", auxiliary EPN limit of $g \to g^{(EPN)}$ is a key to the conversion (5) of the "realistic", perturbed physical Hamiltonian $H(g)$ with $g \neq g^{(EPN)}$ to its representation $P(\lambda)$ which is more user-friendly.

Even beyond the class of solvable models, the representation $P(\lambda)$ of any preselected perturbed Hamiltonian is, in some sense, universal and maximally economical because it allows us to ignore all of the inessential parameters which just remain encoded in $\mathcal{U}$, and which do not enter directly the simplified canonical Hamiltonian $P(\lambda)$.

4. Fourth-Order Exceptional Points

A realization of the fall of a quantum system into its EPN singularity can be studied via Schrödinger Equation (4) at a small λ. A basic technical tool of its perturbation-expansion solution lies then in the specification of unperturbed basis, the role of which is to be played by the N by N matrix solution $\mathcal{U}$ of a rather anomalous unperturbed Schrödinger Equation (3).

For our illustrative sequence of Hamiltonian matrices (9), in particular, matrices $\mathcal{U}^{(N)}$ were obtained, in [30], in the form of sequence (10), with its individual N-by-N-array elements defined, at all N, in terms of certain compact algebraic (i.e., full-precision) expressions. Now, we are going to turn attention to the more general classes of, potentially, more realistic quantum models admitting, in general, just an approximative treatment.

4.1. The Choice of $N = 4$

All of the solvable-model results as mentioned in Section 3 enhanced our interest in the possibility of an analysis of the EPN-generated quantum phase transitions in a model-independent but still analytic and manifestly non-numerical manner. This led us immediately to the choice of $N = 4$. Obviously, besides the related, purely mathematical challenge given by its last-solvable-model status, we also imagined that the older, physics-motivated studies of the quasi-Hermitian quantum systems exhibiting the EPN non-Hermitian degeneracies with $N \leq 3$ still could not cover a wealth of possible phenomena emerging at the larger Ns.

Naturally, all of the theoretical as well as experimental physicists active in the field were very well aware of the fact that the costs of a move beyond the "natural boundary" of $N = 3$ could be high. In our eyes, nevertheless, such a move appeared to be really very well motivated by its undeniable relevance in the applied quantum physics concerning both of its open- and closed-system subcategories. Moreover, we found a partial resolution of the conflict between the appeal and feasibility in the fact that the mathematics needed in the former, open-system subcategory remains elementary. Thus, we decided to reduce the task and pay attention to the study of the mere EPN-related closed systems with $N = 4$.

Marginally, we also felt encouraged by the rather formal observation that the EP4-supporting quantum systems offer and form, in fact, a unique "missing link" between the algebraic and numerical tractability of the non-Hermitian but potentially unitary phase-transition processes. In our present paper, we intend to show, therefore, that their realization remains analytic and exact not only for the well known simplest scenarios with the EPN orders $N = 2$ and $N = 3$, but also in the more realistic but also more sophisticated and less frequently studied systems with $N = 4$.

4.2. Six-Parametric Canonical Hamiltonian $P^{(N)}(\lambda)$ at $N = 4$

The choice of $N = 4$ is special since the generic input Hamiltonian $H^{(4)}(g)$ (i.e., not an exactly solvable one) will still be assigned just a purely numerical form of $\mathcal{U}$ in applications. The reason is that in spite of the existence of the closed formulae, their use remains utterly impractical due to their overcomplicated structure. Nevertheless, the possibility of having just an approximate, numerical form of the transition matrix $\mathcal{U}$ still opens the way towards an efficient replacement of the perturbed Schrödinger Equation (2) or (4) by a simplified, much more user-friendly alternative.

Beyond the very special class of the exactly solvable models, the construction of $\mathcal{U} = \mathcal{U}^{(N)}$ becomes numerical. Even then, there remain good reasons for transition (5) from any preselected phenomenological input Hamiltonian $H^{(N)}(g)$ to its isospectral version $P^{(N)}(\lambda)$. Especially at the larger $N \geq 5$, the diagonalization of $H^{(N)}(g)$ be-

comes replaced by the diagonalization of $P^{(N)}(\lambda)$ which can be performed, near the EPN singularity, perturbatively.

In the $P(\lambda) \leftrightarrow H(g)$ correspondence of Equation (5) with $N \geq 5$, one must really start working with the numerical form of the transition matrices. In practice, the use of their numerical form could prove recommendable even at $N = 4$, in spite of the fact that in this case, the closed formulae also do exist. One can make a choice between the exact and approximate $\mathcal{U}^{(N)}$ but, in both cases, one has to guarantee that the spectrum of the resulting simplified perturbed Schrödinger equation

$$P^{(N)}(\lambda)|\phi_n(\lambda)\rangle = E_n(g^{(EP4)} + \lambda)\,|\phi_n(\lambda)\rangle\,, \quad n = 0, 1, 2, \ldots, N \tag{11}$$

remains real, especially in a small vicinity of EPN, i.e., in the vanishing-perturbation limit of $\lambda \to 0$. Thus, the proofs of these properties form one of our key mathematical tasks since the related guarantee of the unitarity of evolution forms the very core of the description of the related quantum phase transition.

Whenever we decide to study the phenomenon in a closed quantum system which is characterized by its N-by-N matrix Hamiltonian $H^{(N)}(g)$ possessing, in a physical domain of $g \in \mathcal{D}$, a real and non-degenerate spectrum of energies $\{E_n(g)\}$, we have to guarantee, first of all, that this model really does exhibit an EPN degeneracy in the limit of $g \to g^{(EPN)}$. Secondly, it also makes sense to require that the value of $g^{(EPN)}$ is an element of the boundary of the physical domain of parameters, i.e., $g^{(EPN)} \in \partial(\mathcal{D})$. Under these assumptions, we may always use the standard tools of linear algebra and construct the "unperturbed-basis" solution $\mathcal{U}^{(N)}$ of Equation (3).

As a consequence, we will be able to perform the transformation (5) yielding our perturbed Schrödinger Equation (4) in its equivalent canonical form (11). As long as we decide to restrict our attention to the quantum systems which are quasi-Hermitian, we have to keep in mind that their evolution, which is non-unitary in the mathematically preferable but manifestly unphysical representation space $\mathcal{H}_{\text{mathematical}}$, will have to be, simultaneously, unitary in the "correct space" $\mathcal{H}_{\text{physical}}$.

In a way explained in Ref. [19], this leads to the necessity of a severe restriction imposed upon the class of the admissible, spectral-reality-supporting perturbations in Equation (5). At $N = 4$, for an enhancement of the transparency of such a procedure, we will shift the origin on the real line of the bound-state energies, set $\eta = E - E^{(EP4)}$ and get $\eta^{(EP4)} = 0$. We will also shift the parameter g yielding $g^{(EP4)} = 0$. Finally, we will re-scale the measure λ of the smallness of perturbations (see [25] for the reasons). As a consequence, our Equation (11) (i.e., the simplified perturbed Schrödinger equation living in an $N-$dimensional EPN-related subspace of $\mathcal{H}_{\text{mathematical}}$ with $N = 4$) will read

$$P^{(4)}(\lambda)|\phi_n(\lambda)\rangle = \eta_n(\lambda)\,|\phi_n(\lambda)\rangle\,, \quad n = 0, 1, 2, 3\,. \tag{12}$$

Here, Hamiltonian $P^{(4)}(\lambda)$ is still defined by a properly adapted special case of Equation (5) at $N = 4$. This means that such a matrix contains 16 matrix elements which carry a complete information about the dynamics of the system in question at small $\lambda \neq 0$.

At this stage of developments, several questions have to be addressed in explicit manner. The first and the most important one concerns the phenomenological and descriptive role of matrix $P^{(4)}(\lambda)$ in experiments including, first of all, non-Hermitian (although, not necessarily, gain/loss) photonics. Indeed, such a matrix can only be perceived as an optimal representation of dynamics in a not too easily specified vicinity of the underlying EP4 degeneracy of the system.

Even a brief answer to such a question would require a return to Equation (5) and, in particular, to the original form of Hamiltonian $H(g^{(EP4)} + \lambda)$. In it, expectedly, one still

possesses a clear and explicit knowledge of the physical meaning of the parameter(s) g. Only the isospectral one-to-one mapping $H(g^{(EP4)}+\lambda) \to P^{(4)}(\lambda)$ enables us to clarify the criteria of the smallness of perturbation. At the same time, the connection of the latter matrix with the experimentally relevant parameters only remains encoded in the purely formal, optimal-basis-determining transition matrix $\mathcal{U}$.

The second important phenomenology-oriented question is related to the above-emphasized distinction between the real and non-real (i.e., partially or completely complex) spectra. A return to the need of a better understanding of this distinction would re-open not only the problem of a consequent separation of the closed quantum systems from the open quantum systems, but also the separation between the quantum and classical physics in general. Plus, last but not least, of the further desirable contribution to the search for mutual parallels between the Schrödinger equation for quantum systems and its very close formal parallel of Maxwell equations in paraxial approximation in photonics.

In the latter setting of photonics, the mathematics using non-Hermitian generators of evolution which was originally developed in quantum-physics framework found a truly productive domain of innovative applications [7]. Many of the authors paid also exquisite attention to the study of the evolution dynamics near the exceptional-point regime. Having all of these practical needs in mind, we still felt forced to narrow the scope of the present paper. Using the perturbation-motivated replacement of a given $g-$dependent Hamiltonian $H(g)$ by its avatar $P(\lambda)$, we decided to analyze, first of all, the problem of the smallness of the perturbations in $P(\lambda)$. Due to the domination of this matrix by an unperturbed (as well as unphysical) Jordan-matrix limit, we decided to put the main emphasis upon the fact that what seems to be most desirable is an amendment of insight in the perturbative solutions.

With a broader context being clarified, let us now return again to the narrower research field in which the spectrum is required real. According to the general analysis of Equation (12) as performed in [25], we know that near the EP4 singularity, the reality or non-reality of the spectrum is only influenced by the matrix elements of $P^{(4)}(\lambda)$ which are localized below the main diagonal. Without any loss of generality, therefore, it is sufficient to study, for our present purposes (of a guarantee of the reality of the spectrum near EP4), just the following six-parametric and properly scaled perturbed-Jordan-matrix Hamiltonian

$$P^{(4)}_{(a,b,c,x,y,z)}(\lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda^2 z & 0 & 1 & 0 \\ \lambda^3 x & \lambda^2 y & 0 & 1 \\ \lambda^4 a & \lambda^3 b & \lambda^2 c & 0 \end{bmatrix}. \tag{13}$$

We are now going to demonstrate that such a reduced-matrix subfamily of our canonical non-Hermitian Hamiltonians can be interpreted as a source of an exhaustive qualitative classification of spectra of the phenomenological input-information-carrying Hamiltonians $H^{(4)}(g)$, connected with its canonical isospectral avatar of Equation (13) by the transition-matrix-mediated correspondence of Equation (5).

5. Conditions of Unitarity

Under the above-mentioned methodical guidance by paper [25], it makes sense to reparametrize the eigenvalues $\eta = \eta(\lambda)$ of $H^{(4)}(g)$ *alias* $P^{(4)}(\lambda)$ of Schrödinger Equation (12) as follows,

$$\eta(\lambda) = \lambda\, E(\lambda)\,.$$

We are now prepared to prove the existence of a unitarity-compatible (i.e., of the reality-of-spectrum supporting *alias* stable-bound-states-supporting) vicinity $\mathcal{D}_{physical}$ of the EP4 singularity in the space of parameters.

5.1. Secular Equation

The real four-by-four-matrix form (13) of our perturbed canonical Hamiltonian is asymmetric. During a decrease of the absolute value of parameter λ to zero, this parameter measures the smallness of perturbation corrections. It is not too surprising that this quantity enters the Hamiltonian matrix in a nonlinear manner which weights the influence of its individual matrix elements, ordering them as proportional to their distance from the main diagonal.

Most easily, the latter consequence of the general theory can be very easily verified when we write down our $N = 4$ secular equation determining the quadruplet of the energies $\eta = \lambda E$,

$$S(E) = \det \begin{bmatrix} -\lambda E & 1 & 0 & 0 \\ \lambda^2 z & -\lambda E & 1 & 0 \\ \lambda^3 x & \lambda^2 y & -\lambda E & 1 \\ \lambda^4 a & \lambda^3 b & \lambda^2 c & -\lambda E \end{bmatrix} = 0\,. \tag{14}$$

The construction of the solutions becomes simplified when we reparametrize the lowest row of the elements of the perturbed Hamiltonian matrix as follows,

$$c = c(\gamma) = \gamma - y - z\,, \quad b = b(\beta) = \beta - x \quad a = a(\alpha) = \alpha - z(y - \gamma) - z^2\,.$$

This leads to a maximally compact four-parametric characteristic polynomial

$$\eta^4 - \lambda^2 \gamma\, \eta^2 - \lambda^3 \beta\, \eta - \lambda^4 \alpha$$

which specifies the complete set of the three $\lambda-$dependent energy roots $\eta = \eta(\lambda)$, real or complex. Alternatively, in terms of a rescaled energy E, we obtain the simpler, three-parametric secular equation

$$S(E) = E^4 - \gamma\, E^2 - \beta\, E - \alpha = 0\,. \tag{15}$$

It determines the quadruplet of the rescaled-energy roots $E = E_n(\alpha, \beta, \gamma)$, $n = 0, 1, 2, 3$ which may be real or complex, and which remain all $\lambda-$independent. Thus, the experimental bound-state or resonant-state energies $\eta_n(\lambda)$ will be the linear functions of parameter λ. Near the EP4 degeneracy, this yields the unfolding perturbed energies, real or complex, in explicit form.

5.2. Corridor of Unitary Access to the Degeneracy

As we repeatedly emphasized, our interest in the dynamics of quantum systems near their EP4 degeneracies is mainly restricted to the quasi-Hermitian systems which satisfy constraint (7) and which possess a strictly real energy spectrum. As long as such a spectrum is given by the roots of secular equation $S(E) = 0$, we only have to recall its explicit three-parametric $N = 4$ form (15). On these grounds, we finally have to determine the boundaries of the physical domain $\mathcal{D}$.

5.2.1. Construction

The requirement of the unitarity of the evolution in the perturbed dynamical regime with physical $\lambda \neq 0$ will only be satisfied for the triplet of parameters $\alpha = \alpha_{physical}$, $\beta = \beta_{physical}$ and $\gamma = \gamma_{physical}$ belonging to a real and open three-dimensional physical domain,

$$\{\alpha_{physical}, \beta_{physical}, \gamma_{physical}\} \in \mathcal{D} = \mathcal{D}_{physical}\,. \tag{16}$$

The elementary form of the secular Equation (15) will now be used to localize the boundaries $\partial\mathcal{D}$. This means that inside $\mathcal{D}$, we may require the non-degeneracy and reality of the quadruplet of the bound-state energy roots. Let us denote and order these numbers as follows,

$$E^{[4]}_{--} < E^{[4]}_{-} < E^{[4]}_{+} < E^{[4]}_{++}\,. \tag{17}$$

In the light of such a requirement, the graph of our secular polynomial $S(E)$ of Equation (15) has to have the two negative minima (say, at an ordered pair of certain real values of $E = E^{[3]}_{\pm}$ to be specified later) and one positive maximum (in the middle, at another real $E = E^{[3]}_{0}$). At these extremes, therefore, it is necessary and sufficient to have

$$S(E^{[3]}_{-}) < 0\,, \quad S(E^{[3]}_{0}) > 0\,, \quad S(E^{[3]}_{+}) < 0\,. \tag{18}$$

The ordered triplet

$$E^{[3]}_{-} < E^{[3]}_{0} < E^{[3]}_{+} \tag{19}$$

of the coordinates of these extremes coincides with the roots of the third-order polynomial

$$S'(E) = 4\,E^3 - 2\,\gamma\,E - \beta\,. \tag{20}$$

The mutual relationship between the local extremes (18) of the secular polynomial $S(x)$ of Equation (15) and the zeros (19) of its derivative $S'(x)$ of Equation (20) is a very core of our present considerations. For its explicit illustration, let us pick up a sample set of parameters

$$\{\alpha, \beta, \gamma\} = \{-24, -10, 15\} \tag{21}$$

which yield the specific secular polynomial $S(x)$, and which have to be tested whether they truly lie safely inside $\mathcal{D}_{physical}$. As long as both the secular polynomial $S(x) = x^4 - 15\,x^2 + 10\,x + 24$ and its derivative $S'(x) = 4\,x^3 - 30\,x + 10$ are extremely elementary, the simplest version of the test can be just numerical and graphical. Its affirmative result can immediately be read off Figure 1, in which we see that all of the four roots of $S(x)$ (marked, in the picture, by the upper quadruplet of small circles) are real, and that its maxima really occur at the roots of the other polynomial $S'(x)$ (for the sake of the clarity of the picture, both of the graphs are mutually shifted and rescaled, i.e., displayed in arbitrary units).

During the subsequent continued rigorous analysis, an analogous argumentation can be applied: the graph of function $S'(E)$ has to have one maximum (say, at some smaller real value of $E = E^{[2]}_{-}$) and one minimum (at a larger real value of $E = E^{[2]}_{+}$). At these extremes, it is necessary and sufficient to have

$$S'(E^{[2]}_{-}) > 0\,, \quad S'(E^{[2]}_{+}) < 0\,. \tag{22}$$

In a complete analogy with the preceding step, the coordinates of the latter extremes are easily identified with the roots of polynomial

$$S''(E) = 12\,E^2 - 2\,\gamma\,. \tag{23}$$

Thus, the necessary and sufficient reality of these roots can be easily guaranteed via a trivial reparametrization (i.e., via the positivity) of

$$\gamma_{physical} = 6\,\kappa^2\,, \quad \kappa > 0\,. \tag{24}$$

This yields the closed formula for $E_{\mp}^{[2]} = \mp\kappa$ and, ultimately, this also converts the pair of inequalities (22) into the following explicit constraint imposed upon the admissible physical values of parameter β,

$$-8\,\kappa^3 < \beta_{physical} < 8\,\kappa^3\,. \tag{25}$$

What remains for us to guarantee is the validity of relations (18).

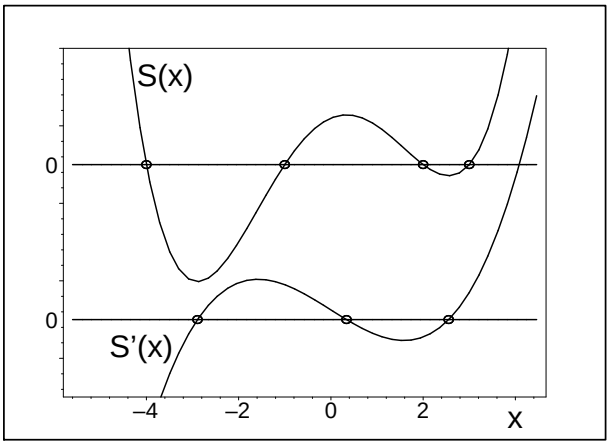


Figure 1. The roots of the toy-model $S(x)$ (small circles in the upper graph) and of $S'(x)$ (small circles in the shifted, lower graph).

This is the task which requires the evaluation of the triplet of the roots (19) of the cubic polynomial (20). The strictly algebraic and exhaustive exact answer (i.e., an explicit specification of the admissible range of $\alpha_{physical}$) can be provided in terms of the well known Cardano formulae. For our present purposes, nevertheless, we intend to skip the display of these formulae. In order to provide a more intuitive insight in the structure of the boundary of the physical domain $\mathcal{D}$, we will use a less formal alternative approach.

5.2.2. EP4-Unfolding Paths with Vanishing β

In order to simplify the analysis, let us first pick up a special, sample value of

$$\beta = \beta_{physical}^{(sample)} = 0\,.$$

This will enable us to deduce the related, most elementary sample forms of

$$E_{\mp}^{[3](sample)} = \mp\sqrt{3}\,\kappa\,, \quad E_{0}^{[3](sample)} = 0\,, \tag{26}$$

i.e., of the coordinates to be inserted in Equation (18). From its middle item, we get the upper bound while the other two inequality items Equation (18) give the same lower-bound constraint. Thus, we arrive at the following special, $\beta = 0$ boundaries,

$$0 > \alpha_{physical}^{(sample)} > -9\,\kappa^4\,. \tag{27}$$

The domain is fully determined. In fact, the most important consequence of this result is that it implies that in a sufficiently small vicinity of the EP4 degeneracy, the physical quasi-Hermiticty domain $\mathcal{D}$ is not empty because even its "central", $\beta = 0$ subdomain

$\mathcal{D}^{(sample)}_{physical}$ is non-empty. There exists a unitary-evolution path of possible fall of the system into an EP4-mediated quantum phase transition.

5.2.3. Small but Non-Vanishing Choice of β

After a small change (i.e., say, growth) of

$$\beta = \beta^{(perturbed)}_{physical} = \delta^2\,\kappa^3\,, \quad \delta \ll 1 \tag{28}$$

we get, at the small δs,

$$E_0^{[3](perturbed)} \approx -\kappa\,\delta^2/12\,, \quad E_{\mp}^{[3](perturbed)} \approx \mp\kappa\,\sqrt{3 \mp \sqrt{3}\,\delta^2/12}\,,$$

i.e., we detect the decrease of $E_0^{[3](perturbed)}$ and the growth of $E_-^{[3](perturbed)}$ (plus the growth of $E_+^{[3](perturbed)}$ which is, due to our assumption (28), irrelevant). This implies the growth of the lower bound for

$$\alpha^{(perturbed)}_{physical}/\kappa^4 \gtrapprox -9 + \sqrt{3}\,\delta^2$$

Also, the upper bound of the same quantity becomes modified by perturbation (28),

$$\alpha^{(perturbed)}_{physical}/\kappa^4 \lessapprox \delta^4/24\,.$$

What can be deduced is the fact that the interval of the admissible αs remains non-empty, and that it shrinks and moves to the right.

5.2.4. The Limit of a Maximal Size of β

Out of the two maximal-size choices of β (representing, strictly speaking, just an auxiliary, limiting, EP-degenerate case), let us pick up, for illustration, just the positive one,

$$\beta = \beta^{(extreme)}_{physical} = 8\,\kappa^3\,.$$

This choice implies the following values of the ultimate bounds imposed upon the following triplet of roots,

$$E_-^{[3](extreme)} = E_0^{[3](extreme)} = -\kappa\,, \quad E_+^{[3](extreme)} = 2\,\kappa\,. \tag{29}$$

Finally, we obtain the following unique limiting value of

$$\alpha = \alpha^{(extreme)}_{physical} = 3\,\kappa^4\,,$$

confirming both of the above-indicated tendencies of the movement of the interval of the admissible αs to the right. The process of its shrinking ends up with the ultimate zero length of the interval, and its position is found to have moved to the right from zero, yielding the ultimate positive values of all of the eligible αs in this extreme.

6. Discussion

Half a century ago, the notion of exceptional points only served the needs of the perturbation theory of linear operators [1], but it did not last too long before it also found multiple other applications in quantum physics [4,5,32], especially in the context of the so called open and/or resonant quantum systems [10]. Moreover, the notion of EPs also appeared ubiquitous in several other areas of physics including even the non-quantum ones [2,6,7,11,12,33].

Several open theoretical questions are currently emerging in this setting. A remarkable progress in their understanding has been achieved, in particular, after people recognized that the unitarity of the evolution (considered, traditionally, robust and guaranteed by the Stone theorem by which the unitarity of the evolution follows from the "obligatory" Hermiticity of the Hamiltonian [34]) is in fact conditional. It has been revealed, indeed, that it may make sense to work with more Hilbert spaces or, in some cases, even with just two or more eligible inner products [35].

Currently, it is widely accepted that the traditional requirement of the Hermiticity of the operators of observables can be replaced by a less restrictive mathematical requirement of their quasi-Hermiticity *alias* Hermitizability [3]. Several formal advantages of such an amendment of the formalism of quantum mechanics (cf., e.g., its older comprehensive reviews [9,36,37]) are well recognized at present: they can even be well sampled in Buslaev's and Grecchi's fairly old study of certain quantum anharmonic oscillators [38], etc.

One of the decisive practical advantages of working with the non-Hermitian but Hermitizable (*alias* quasi-Hermitian) operators and Hamiltonians can be seen, in similar examples, in the possibility of an optimal representation of a quantum system in two (or in three [27], or even in more than three [39]) different Hilbert spaces. The information about dynamics as carried, traditionally, just by the Hamiltonian $H(g)$ is then shared and distributed between the Hamiltonian and another operator Θ called physical inner-product metric.

One of the most important consequences of the new flexibility introduced via the unusual $\Theta \neq I$ appeared to make multiple quantum models conceptually closer to its classical partners. Such an enhancement of the contacts with classical physics also inspired new forms of an insight in the genuine quantum structures. In our present paper, we addressed, in this sense, the emergence of singularities which found, in their classical forms, their appropriate mathematical description in the language of geometry: Thom's popular catastrophe theory [8] could be recalled as one of the fairly universal mathematical formalisms of such a type.

In the context of quantum theory, the latter form of inspiration is of a comparatively new date. One of the reasons is that the study of the quantum singular behavior is much more complicated, for several reasons ranging from the underlying Hilbert-space-based theory up to the probabilistic nature of its experimental tests. Fortunately, the recent growth of popularity of certain non-standard forms of the Hilbert space (in which, say, the operator representing a quantum observable need not be self-adjoint [3,9,26,36,37]) also opened a number of innovative approaches to the formulation of the concept of a quantum singularity.

Some of the related and most successful models of a singularity in a quantum system (involving, typically, a quantum phase transition [30]) were based on the use of the Kato's [1] concept of exceptional point $g^{(EP)}$. In this sense, the results described in our present paper belong precisely to the latter context. With our project being inspired by certain technical obstacles as encountered in the literature (and, in particular, in paper [16]), we imagined that precisely due to the existence of these technical obstacles, the progress in the related applied physics is comparatively slow: typically, most authors were circumventing the technical challenges and feelt forced to restrict their analyses of quantum phase transitions to the EPN models with $N = 2$ or $N = 3$. This was, in fact, the main reason why we redirected here the detailed technical analysis to the general class of quantum systems with $N = 4$.

https://doi.org/10.3390/photonics13030224

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The author declares no conflicts of interests.

References

1. Kato, T. *Perturbation Theory for Linear Operators*; Springer: Berlin/Heidelberg, Germany, 1966.
2. Özdemir, Ş.; Rotter, S.; Nori, F.; Yang, L. Parity-time symmetry and exceptional points in photonics. *Nat. Mater.* **2019**, *18*, 783. [CrossRef]
3. Bagarello, F.; Gazeau, J.-P.; Szafraniec, F.; Znojil, M. (Eds.) *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects*; Wiley: Hoboken, NJ, USA, 2015.
4. Bender, C.M.; Wu, T.T. Anharmonic Oscillator. *Phys. Rev.* **1969**, *184*, 1231–1260. [CrossRef]
5. Alvarez, G. Bender-Wu branch points in the cubic oscillator. *J. Phys. A Math. Gen.* **1995**, *28*, 4589–4598. [CrossRef]
6. Rüter, C.E.; Makris, K.G.; El-Ganainy, R.; Christodoulides, D.N.; Segev, M.; Kip, D. Observation of parity-time symmetry in optics. *Nat. Phys.* **2010**, *6*, 192–195. [CrossRef]
7. Christodoulides, D.; Yang, J.-K. (Eds.) *Parity-Time Symmetry and Its Applications*; Springer: Singapore, 2018.
8. Arnold, V.I. *Catastrophe Theory*; Springer: Berlin/Heidelberg, Germany, 1992.
9. Scholtz, F.G.; Geyer, H.B.; Hahne, F.J.W. Quasi-Hermitian Operators in Quantum Mechanics and the Variational Principle. *Ann. Phys.* **1992**, *213*, 74–101. [CrossRef]
10. Moiseyev, N. *Non-Hermitian Quantum Mechanics*; Cambridge University Press: Cambridge, UK, 2011.
11. Berry, M.V. Physics of Nonhermitian Degeneracies. *Czech. J. Phys.* **2004**, *54*, 1039–1047. [CrossRef]
12. Heiss, W.D. Exceptional points—Their universal occurrence and their physical significance. *Czech. J. Phys.* **2004**, *54*, 1091–1100. [CrossRef]
13. Heiss, W.D. The physics of exceptional points. *J. Phys. A Math. Theor.* **2012**, *45*, 444016. [CrossRef]
14. Znojil, M. Quantum phase transitions in nonhermitian harmonic oscillator. *Sci. Rep.* **2020**, *10*, 18523. [CrossRef]
15. Bagchi, B.; Ghosh, R.; Sen, S. Exceptional point in a coupled Swanson system. *Europhys. Lett.* **2022**, *137*, 50004. [CrossRef]
16. Sinha, A.; Ghosh, A.; Bagchi, B. Exceptional points and quantum phase transition in a fermionic extension of the Swanson oscillator. *Phys. Scr.* **2024**, *99*, 105534. [CrossRef]
17. Znojil, M. A return to observability near exceptional points in a schematic PT-symmetric model. *Phys. Lett. B* **2007**, *647*, 225–230. [CrossRef]
18. Guria, C.; Zhong, Q.; Özdemir, S.K.; Patil, Y.S.S.; El-Ganainy, R.; Harris, J.G.E. Resolving the topology of encircling multiple exceptional points. *Nat. Commun.* **2024**, *15*, 1369. [CrossRef]
19. Znojil, M. Unitarity corridors to exceptional points. *Phys. Rev. A* **2019**, *100*, 032124. [CrossRef]
20. Messiah, A. *Quantum Mechanics*; North Holland: Amsterdam, The Netherlands, 1961.
21. Rotter, I. A non-Hermitian Hamilton operator and the physics of open quantum systems. *J. Phys. A Math. Theor.* **2009**, *42*, 153001. [CrossRef]
22. Li, J.-M.; Harter, A.K.; Liu, J.; de Melo, L.; Joglekar, Y.N.; Luo, L. Observation of parity-time symmetry breaking transitions in a dissipative Floquet system of ultracold atoms. *Nat. Comm.* **2019**, *10*, 855. [CrossRef] [PubMed]
23. Znojil, M. Theory of Response to Perturbations in Non-Hermitian Systems Using Five-Hilbert-Space Reformulation of Unitary Quantum Mechanics. *Entropy* **2020**, *22*, 80. [CrossRef] [PubMed]
24. Znojil, M. Admissible perturbations and false instabilities in PT-symmetric quantum systems. *Phys. Rev. A* **2018**, *97*, 032114. [CrossRef]
25. Znojil, M. Perturbation Theory Near Degenerate Exceptional Points. *Symmetry* **2020**, *12*, 1309. [CrossRef]
26. Dyson, F.J. General theory of spin-wave interactions. *Phys. Rev.* **1956**, *102*, 1217–1230. [CrossRef]
27. Znojil, M. Three-Hilbert-space formulation of Quantum Mechanics. *Symm. Integ. Geom. Meth. Appl.* **2009**, *5*, 001. [CrossRef]
28. Znojil, M. Fragile $\mathcal{PT}$-symmetry in a solvable model. *J. Math. Phys.* **2004**, *45*, 4418–4430. [CrossRef]
29. Znojil, M. Quantum catastrophes: A case study. *J. Phys. A Math. Theor.* **2012**, *45*, 444036. [CrossRef]
30. Znojil, M. Passage through exceptional point: Case study. *Proc. Roy. Soc. A Math. Phys. Eng. Sci.* **2020**, *476*, 20190831. [CrossRef]
31. Graefe, E.M.; Günther, U.; Korsch, H.J.; Niederle, A.E. A non-Hermitian $\mathcal{PT}$-symmetric Bose-Hubbard model: Eigenvalue rings from unfolding higher-order exceptional points. *J. Phys. A Math. Theor.* **2008**, *41*, 255206. [CrossRef]
32. Bender, C.M.; Boettcher, S. Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry. *Phys. Rev. Lett.* **1998**, *80*, 5243. [CrossRef]
33. Zezyulin, D.A.; Kartashov, Y.V.; Konotop, V.V. Metastable two-component solitons near an exceptional point. *Phys. Rev. A* **2021**, *104*, 023504. [CrossRef]
34. Stone, M.H. On one-parameter unitary groups in Hilbert space. *Ann. Math.* **1932**, **33**, 643–648. [CrossRef]

35. Znojil, M. Hybrid form of quantum theory with non-Hermitian Hamiltonians. *Phys. Lett. A* **2023**, *457*, 128556. [CrossRef]
36. Bender, C.M. Making sense of non-Hermitian Hamiltonians. *Rep. Prog. Phys.* **2007**, *70*, 947–1118. [CrossRef]
37. Mostafazadeh, A. Pseudo-Hermitian Representation of Quantum Mechanics. *Int. J. Geom. Meth. Mod. Phys.* **2010**, *7*, 1191–1306. [CrossRef]
38. Buslaev, V.; Grecchi, V. Equivalence of unstable anharmonic oscillators and double wells. *J. Phys. A Math. Gen.* **1993**, *26*, 5541–5549. [CrossRef]
39. Znojil, M. Systematics of quasi-Hermitian representations of non-Hermitian quantum models. *Ann. Phys.* **2023**, *448*, 169198. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.