

Structured Optical Toroidal Vortices with Rotational Symmetry

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Abstract: Toroidal vortices, as intriguing topological structures, play a fundamental role across a wide range of physical fields. In this study, we theoretically propose a family of structured optical toroidal vortices as generalized forms of toroidal vortices in paraxial continuous wave beams. These structured optical toroidal vortices exhibit unique rotational symmetry while preserving the topological properties of standard toroidal vortices. The three-dimensional topological structures demonstrate l -fold rotational symmetry, which is closely related to the topological charges. Structured toroidal vortices introduce additional topological invariants within the toroidal light field. These topological light fields hold significant potential applications in the synthesis of complex topological structure and optical information encoding.

Keywords: toroidal vortices; topological structure; optical singularity; knot

1. Introduction

Toroidal vortices, also referred to as vortex rings or vortex loops, are prevalent topological structures in air or water [1]. They have a ring-shaped structure with a vortex core forming a closed loop. Typical examples are smoke rings, bubble rings created by dolphins, and vortex rings formed in drop splashing [2]. The controlled generation and manipulation of toroidal vortices have garnered significant attention across various physical fields, including fluid dynamics and Bose–Einstein condensates [3–5]. Research on toroidal vortices has significantly advanced topological studies, particularly in knot theory [6]. Topologically, toroidal vortices are indispensable members of the knot family. Knot physics lies at the heart of current topological physics, ranging from liquid crystals [7], magnetic field [8], optics [9,10], and acoustic fields to quantum fields [11,12]. These topological physical fields have demonstrated important applications in interactions with matter and promoted modern technologies [13,14].

In optics, structured light fields with multiple controllable degrees of freedom both in spatial and temporal domains provide a versatile platform for the study of optical topologies [15,16]. In recent years, the realization of optical toroidal vortices has garnered notable interest [17], especially scalar toroidal vortices and vector toroidal pulses [18,19]. Scalar toroidal vortices are constructed by stretching a spatiotemporal vortex and transforming it to toroidal phase vortices with conformal mapping [18]. A characteristic of a vector toroidal pulse is that electric or magnetic field lines form a toroidal surface [19]. Recently, hybrid electromagnetic toroidal vortices have been proposed that integrate scalar and vector characteristics [20]. Furthermore, the generation of toroidal vortices within



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propagating paraxial beams has been demonstrated [21], providing a foundation for extending various optical topologies, including scalar Hopf structures and vector skyrmionic hopfions [22–24]. The fundamental structure of these toroidal topologies typically involves a circular core within a specific two-dimensional plane. Three-dimensional (3D) toroidal vortices with chiral symmetry breaking have been proposed [25], which have a non-closed topological structure.

In this work, we introduce the concept of structured optical toroidal vortices in continuous wave laser beams. The structured optical toroidal vortices maintain the topological properties of the standard toroidal vortices and form closed 3D singularity structures. The 3D topological structures exhibit l -fold rotation symmetry associated with the topological charges. Furthermore, the topological structure of the structured toroidal vortices is also determined by the signs of the topological charges.

2. Mathematical Description of Toroidal Vortices

We consider the construction of vortex topologies in monochromatic and paraxial continuous wave laser beams. Considering that the vortices are complex zeros in the scalar field, the investigation of knotted vortices originates from the topological analysis of the high-dimensional complex space. Non-intuitive topological properties of a high-dimensional complex space can be observed in a lower dimensional space with the aid of stereographic projection. A set of projection functions established between complex space (u, v) and 3D real space (x, y, z) can be expressed as follows [9,26]:

$$\begin{aligned} u &= \frac{x^2 + y^2 + z^2 - 1 + 2iz}{x^2 + y^2 + z^2 + 1} \\ v &= \frac{2(x + iy)}{x^2 + y^2 + z^2 + 1} \end{aligned} \quad (1)$$

By analyzing the zero set of stereographic projection, it can be found that toroidal vortices and the vortex line are hidden in the projection functions u and v , respectively. As shown in Figure 1a, the function u has a standard toroidal amplitude contour, and the toroidal vortex is a standard circle, which is located only in the $z = 0$ plane. This means that the standard toroidal vortices are completely rotationally symmetrical. Based on the standard toroidal vortices, we consider structured toroidal vortices with fixed l -fold rotational symmetry. Mathematically, the parametric equation for the core of standard toroidal vortices is $[\cos\alpha, \sin\alpha, 0]$, where α is the azimuthal angle. The parametric equation for the structured toroidal vortices can be rewritten as $[\cos\alpha, \sin\alpha, \sin l\alpha]$, with l as the variable. Figure 1b,c illustrate the concept of structured toroidal vortices and the phase structure around their core. The amplitude contour is no longer a standard torus structure, while the phase structure still retains the characteristics of toroidal vortices.

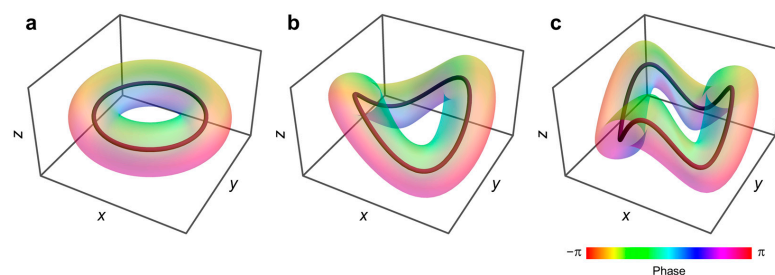


Figure 1. Standard toroidal vortices and structured toroidal vortices. (a) Toroidal amplitude contour and 3D phase structure of standard toroidal vortices. The singularity of toroidal vortices is shown by a red loop. (b,c) 3D phase structure of structured toroidal vortices; the singularity loops are saddle-shaped and shown as trajectories associated with z . The topological structure exhibits 2-fold rotational symmetry for (b) and 3-fold rotational symmetry for (c).

3. Results

3.1. Optical Paraxial Toroidal Vortices

Based on the theoretical model, we first consider the formation of standard optical toroidal vortices. In the projection function u , the topological feature of the toroidal vortices can be extracted as $f = x^2 + y^2 - 1$. Although the polynomial does not satisfy the paraxial wave equation, it agrees with the paraxial modified polynomial in the $z = 0$ plane. After using Gaussian factor to constrain the amplitude distribution of the polynomial, the resulting field $F(x, y, w)$ becomes the target field of the $z = 0$ plane; w is the waist of Gaussian factor. In order to obtain the 3D topological light field, the target field F can be further decomposed into a physically realizable solution by using Laguerre–Gaussian modes [27], indicated by LG(l, p). The azimuthal mode number is denoted by l , which also represents the topological charge of the mode, and radial mode number is denoted by p . In this case, the standard optical toroidal vortices can be expressed as $\Phi = c_0 \text{LG}(0, 0) + c_1 \text{LG}(0, 1)$; c_0 and c_1 are the mode coefficients of the two LG modes, respectively, and determine the radius of the toroidal vortices. The Gouy phase in the LG modes leads the light field to propagate in 3D space and form topological structures.

The advantage of mode decomposition is that it allows us to use the mode components to control the topology formed by the light field. In the light field of the standard toroidal vortices, the topological charge of both components is zero. Recalling the formation of the optical vortex knots [10], the mode components can be decomposed into two groups with zero and non-zero topological charges, which can be considered the perturbation component and vortex component. In this case, the standard toroidal vortices can also be regarded as a special perturbation component. With the aid of vortex component, the topological light field can be further expressed as

$$\Psi = \sum_{p=0}^1 c_p \text{LG}(0, p) + c_2 \text{LG}(l, 0). \quad (2)$$

The vortex component LG($l, 0$) provides a new degree of freedom for controlling toroidal vortices. In this two-component model, the topological properties of the light field are determined by the proportion of the two components. In other words, when the perturbation component is dominant, the topological structure of the light field tends to be standard toroidal vortices. On the contrary, it tends to be a high-order vortex; the infinite extension of the vortex line causes the light field to form a non-closed singularity structure. Therefore, in order to construct the structured toroidal vortices, the proportion of the perturbation component should be greater than the vortex component.

Without loss in generality, the parameter c_0 is set to 1, so the parameter c_1 determines the radius of the toroidal vortices; the parameter c_2 needs to be less than 1 to reduce the proportion of the vortex component. As an example, the parameter is specified as $c_0 = 1$, $c_1 = -4$, $c_2 = 0.8$, and $l = 3$ to illustrate the topological properties of the structured toroidal vortices. According to Equation (2), the numerically simulated amplitude profile of the topological light field in the x - y plane is shown in Figure 2a. Due to the influence of the vortex component, the darkness in the beam central is no longer the standard circle, and the beam is in the shape of three lobes. Furthermore, the topological configuration of the light field needs to be extracted from the phase profiles. In general, high-order vortices are unstable under the perturbation; they will split into multiple single-charged vortices. When the perturbation component is a standard toroidal vortex rather than a simple plane wave, the interference between the perturbation component and the vortex component determines the resulting number of singularities. As shown in Figure 2d, six independent singularities are generated in the $z = 0$ plane. These singularities rise from the inner and

outer regions of standard toroidal vortices, which together form the structured toroidal vortices in 3D space. By extracting the intersection of zero isosurfaces of the real and imaginary parts of the 3D light field, the singularity structure of the structured toroidal vortices is visualized in Figure 2b,c. The closed and saddle-shaped singularity trajectories indicate the formation of structured toroidal vortices in a paraxial beam.

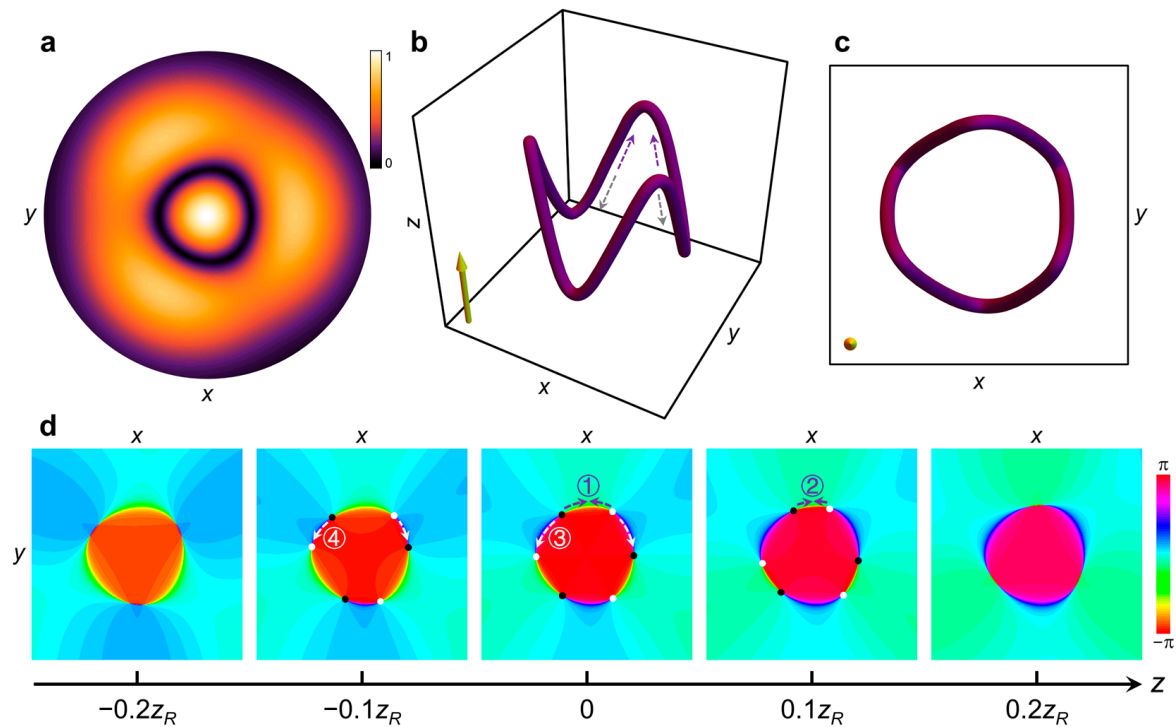


Figure 2. Construction of optical structured toroidal vortices. (a) Amplitude profile of the paraxial beam in $z = 0$ plane. (b) Visualization of the 3D topological structure of structured toroidal vortices; the yellow arrow marks the direction of propagation. (c) Top view of the 3D topological structure, which indicates the topology is a closed loop. (d) 2D phase profiles at different longitudinal positions (z), and z_R is the Rayleigh length. Vortices with opposite topological charges are marked by black and white points. Numbers 1–4 and dotted arrows mark the motion of vortices along the z axis.

3.2. Optical Structured Toroidal Vortices with l -Fold Symmetry

As shown in Figure 2d, along the positive direction of the z -axis ($z > 0$), pairs of vortices with opposite topological charges will approach each other and eventually annihilate, as indicated by the numerical labels (1, 2) and the purple arrows. Along the negative direction of the z -axis ($z < 0$), these two vortices move away from each other, as indicated by the numerical labels (3, 4) and the white arrows. In 3D space, the evolution trend of these two singularities is shown by the dotted line in Figure 2b; they form a singularity line with one end closed. The structured toroidal vortices with 3-fold symmetry are formed by three pairs of singularities with the same propagation characteristics.

Since the standard toroidal vortices are completely rotationally symmetrical, in essence, the rotational symmetry of the structured toroidal vortices is caused by the vortex component in Equation (2). The amplitude of high-order vortex component LG ($l, 0$) is completely rotationally symmetrical, but it has an azimuthally changed phase with a period of $2\pi/l$. Therefore, l pairs of singularities also appear periodically along the azimuthal direction. The structured toroidal vortices can overlap with themselves by rotating $2\pi/l$ about the z axis, which can be achieved by adding additional phase factors to the vortex component. According to Equation (2), we consider the structured toroidal vortices corresponding to different high-order topological charges. Figure 3a–d show the amplitude profiles of the

light fields in the $z = 0$ plane, which exhibit a multilobe structure associated with topological charges. The 3D vortex topologies with the same waist are visualized in Figure 3e–h. With the increase in the topological charge, the propagation distance of the topological structure decreases, which is due to the Gouy phase associated with the topological charge in LG ($l, 0$) mode.

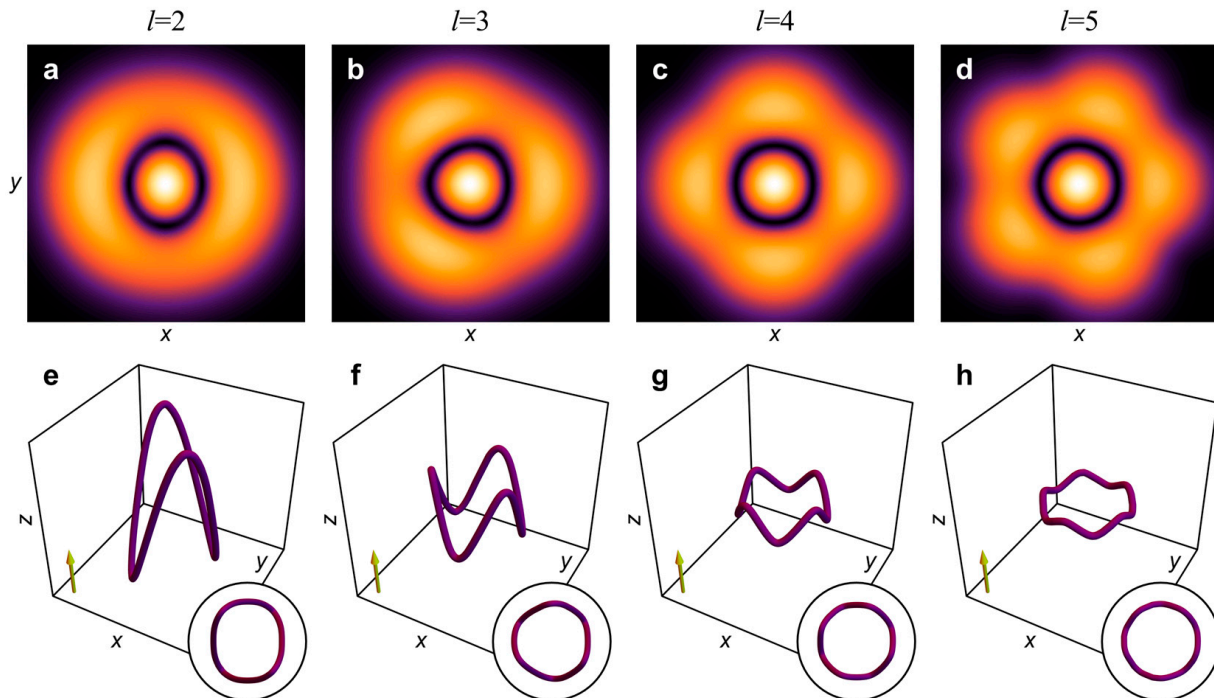


Figure 3. Structured toroidal vortices associated with topological charge. (a–d) Amplitude profiles of structured toroidal vortices in $z = 0$ plane with different integer topological charges from $l = 2$ to 5. (e–h) Visualization of the 3D topological structures with the top view in the corner.

3.3. Optical Structured Toroidal Vortices with Opposite Topological Charge

Having constructed the optical structured toroidal vortices with a positive integer topological charge, we further consider the effect of the opposite topological charge on the structured toroidal vortices. The sign of topological charge determines the direction of the phase increase, and it also affects the vortex formed by the interference of perturbation and vortex components of Equation (2). Figure 4a,b show the phase profiles of the light field in the $z = 0$ plane corresponding to the topological charge of $+3$ and -3 , respectively. It can be seen that the singularity position is same, but the topological charges of the vortices are opposite. The opposite topological charge will further affect the propagation of singularities in 3D space and the resulting topological structure. As shown in Figure 4c,d, when the topological charge is positive, pairs of vortices are connected along the positive direction of the z -axis ($z > 0$); when the topological charge is negative, they connect along the negative direction of the z -axis ($z < 0$). Figure 4e shows the two topologies that intersect in the $z = 0$ plane. It should be noted that these two topological structures cannot be completely overlapped by rotation. The pairs of vortices are formed from the inner and outer regions of standard toroidal vortices, and are therefore asymmetrical in space.

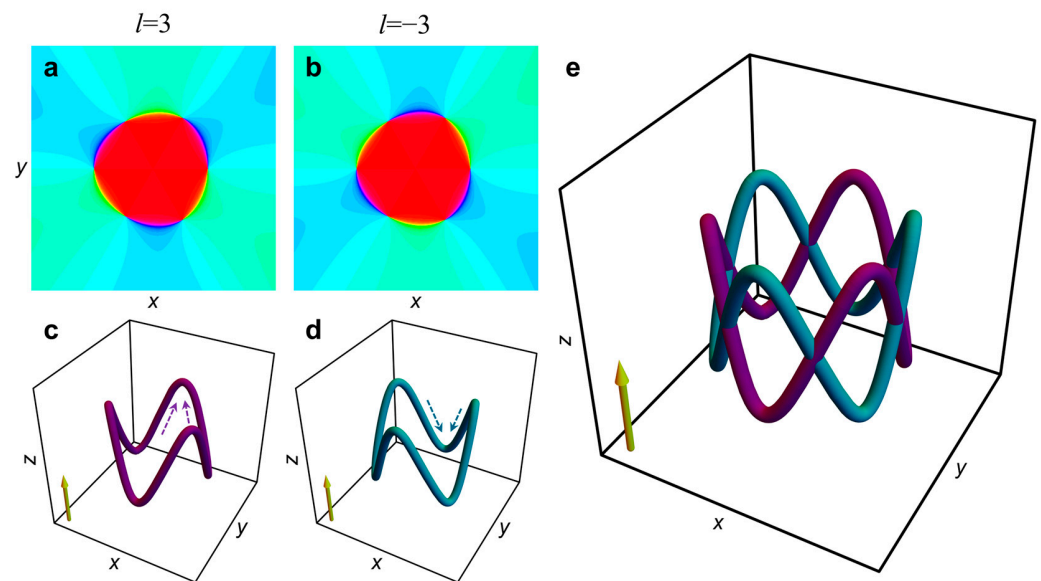


Figure 4. Structured toroidal vortices with opposite topological charge. (a,b) Phase profiles of the light field in $z = 0$ plane with topological charge of 3 and -3 . (c,d) Visualizations of the corresponding 3D topological structures. (e) Two topologies intersect in the $z = 0$ plane.

4. Discussion

The proposed structured toroidal vortices possess two distinct topological invariants: the topological charge of the toroidal vortices and the topological charge of the vortex component in Equation (2). The topological charge of toroidal vortices is 1, and the realization of toroidal vortices with high-order topological charge remains experimentally challenging due to their inherent instability [28]. In comparison, the topological charge of the vortex component in Equation (2) is easier to control. Therefore, the proposed topological light fields with high-order topological invariants offer promising applications in information encoding schemes [14]. Furthermore, the structured toroidal vortex expands the types of toroidal vortices, providing fundamental topological units for synthesizing more complex scalar and vector topological structures [29]. These complex fields can be experimentally realized using existing spatial light modulator-based beam shaping techniques. The complex amplitude of the designed paraxial light fields can be accurately modulated by a phase-only pattern, while the vortex topological structures are extracted from the darkness of the beam.

5. Conclusions

In conclusion, we propose a class of structured toroidal vortices within paraxial beams, which can be considered the more general forms of standard toroidal vortices. The structured toroidal vortices preserve the topological properties of the standard toroidal vortices while forming 3D singularity structures. The structured toroidal vortices can be constructed by linear superposition of three Laguerre–Gaussian modes, where the additional vortex mode introduces a new degree of freedom. The topological structure of the light field is controlled by the topological charge of the vortex mode. Furthermore, the structured toroidal vortices have l -fold rotation symmetry related to the topological charges, and the topological configuration also depends on the sign of the topological charge. The proposed scheme plays an active role in the synthesis of complex high-dimensional vortex topologies.

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