



Article Mueller-Polarimetry of Barley Leaves II: Mueller Matrix Decompositions

Sergey Savenkov^{1,*}, Yevgen Oberemok¹, Ivan Kolomiets¹, Ranjan Muttiah² and Roman Kurylenko¹

- ¹ Faculty of Radio Physics, Electronics and Computer Systems, Taras Shevchenko National University of Kyiv, 01601 Kyiv, Ukraine; oberemok@knu.ua (Y.O.)
- ² Civil Engineering Department, University of Texas at Arlington, Arlington, TX 76019, USA
- * Correspondence: sns@univ.kiev.ua; Tel.: +380-673188313

Abstract: This paper highlights the application of decomposition methods in Mueller polarimetry for the discrimination of three groups of barley leaf samples from *Hordeum vulgare*: *Chlorina* mutant, *Chlorina* etiolated mutant and *Cesaer* varieties in the visible wavelength at $\lambda = 632.8$ nm. To obtain the anisotropic and depolarizing properties of the samples under study, the additive and multiplicative decompositions of experimental Mueller matrices were used. We show how a rich set of anisotropy and depolarization parameters obtained from decompositions can be used as effective observables for the discrimination between different varieties of the same plant species.

Keywords: depolarization; Mueller matrix; additive and multiplicative decomposition of the Mueller matrix

1. Introduction

In this paper, we proceed with the analysis of the experimental Mueller matrices for three groups of common barley leaf samples (Hordeum vulgare)-Chlorina mutant, *Chlorina* etiolated mutant and *Cesaer* varieties—started in the papers [1,2]. The main goal of this paper is the same as in our previous papers [1,2], i.e., whether these groups of barley leaves can be discriminated based on the obtained polarimetric data in visible wavelength (λ = 632.8 nm). However, based on the results presented in [1,2], the methods of this paper are different. Indeed, in [2], based on the analysis of the output degree of polarization *DoP* [3] and so-called single value depolarization metrics, which can be obtained directly from the elements of the Mueller matrices, i.e., the average degree of polarization AverageDoP [4], the depolarization index DI(M) [5], Q(M)-metric [6,7] and $R(\mathbf{M})$ [8], we show that all the samples under study are characterized by high and different output depolarization. Furthermore, all groups of samples are characterized by anisotropic depolarization, i.e., the output polarization degree depends significantly on the input polarizations; see Table 1 in [2]. This result fully corresponds to the promise of modern polarimetry [9–14], namely, the depolarization in remote sensing of biological media is an important source for understanding the interaction of polarized light with botanical scenes ranging from individual leaves to plants and canopy.

Note that the single-value depolarization metrics representing averaged information on the depolarization properties of the studied samples describe the dependence of the output depolarization on the observation angle. However, they do not allow one to conclude whether the output depolarization depends on the input polarization. In [2], we observed that the anisotropy of depolarization occurred by analyzing the output degree of polarization *DoP* as a function of the ellipticity ε_{inp} and azimuth θ_{inp} of the input polarizations. Another interesting result observed in [2] for all three groups of leaves (see Tables 4 and 5 in [2]), in both forward and backward scattering, is dichroism when there is a dependency of output intensity on the ellipticity ε_{inp} and azimuth θ_{inp} of the input polarizations. Obviously, these effects require further detailed analysis, which can



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Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). only be carried out using the matrix description of anisotropy and depolarization of the studied samples.

Therefore, in this paper for analysis of the anisotropy and depolarizing properties of the samples under study and thereby solving the main problem of this and the accompanying paper [2], specifically, the discrimination of three groups of barley leaves under study, we use the additive and multiplicative decompositions of experimental Mueller matrices and some depolarization metrics that can be deduced from these decompositions.

In addition, the matrix decomposition methods that we use in this paper to analyze the experimental Mueller matrices give us, as they say, a synergistic effect. Indeed, as one can see the results presented in this paper are an important contribution to the analysis of the inverse problem of polarimetry and highlight the ambiguity in solving the inverse problem of polarimetry in two important aspects: (i) the existence of various decompositions of arbitrary depolarizing Mueller matrices, including various decompositions of deterministic Mueller matrices; and (ii) existence of different Mueller matrices of depolarizers.

All information about the method of the Mueller matrix measurement, which was carried out in this experiment, the geometry of the experiments, the description of the samples under study and the results of measuring the Muller matrix elements can be found in [1,2]. This paper is organized as follows: in Section 2, we present a brief outline of the depolarizing Mueller matrix additive and multiplicative decompositions and the matrix description of depolarization; results and discussion of our experiments and calculations are given in Section 3; and conclusions are presented in Section 4.

2. The Mueller Matrix Decompositions

The first single-value depolarization metric proposed in polarimetry, except for the degree of polarization *DoP* [3], was deduced from the concept of the Cloude coherency matrix [15,16]. The concept of coherency matrix is now widely used in various fields of modern polarimetry [3,17]. It is particularly useful when one is interested in additive (parallel) decompositions of the arbitrary depolarizing Mueller matrix, i.e., when a depolarizing Mueller matrix can be represented by a convex sum of non-depolarizing (pure) Mueller matrices. In this decomposition, the non-depolarizing Mueller matrix contains one pure component, whereas depolarizing Mueller matrices contain two or more (up to four) pure components.

The Cloude coherency matrix **J** is derived from the corresponding Mueller matrix elements m_{ij} as follows:

$$\begin{split} j_{11} &= 1/4(m_{11} + m_{22} + m_{33} + m_{44}), \ j_{22} &= 1/4(m_{11} + m_{22} - m_{33} - m_{44}), \\ j_{33} &= 1/4(m_{11} - m_{22} + m_{33} - m_{44}), \ j_{44} &= 1/4(m_{11} - m_{22} - m_{33} + m_{44}), \\ j_{14} &= 1/4(m_{14} - i m_{23} + i m_{32} + m_{41}), \ j_{23} &= 1/4(i m_{14} + m_{23} + m_{32} - i m_{41}), \\ j_{32} &= 1/4(-i m_{14} + m_{23} + m_{32} + i m_{41}), \ j_{41} &= 1/4(m_{14} + i m_{23} - i m_{32} + m_{41}), \\ j_{12} &= 1/4(m_{12} + m_{21} - i m_{34} + i m_{43}), \ j_{21} &= 1/4(m_{12} + m_{21} + i m_{34} - i m_{43}), \\ j_{34} &= 1/4(i m_{12} - i m_{21} + m_{34} + m_{43}), \ j_{43} &= 1/4(-i m_{12} + i m_{21} + m_{34} + m_{43}), \\ j_{13} &= 1/4(m_{13} + m_{31} + i m_{24} - i m_{42}), \ j_{31} &= 1/4(m_{13} - i m_{31} + m_{24} + m_{42}), \\ j_{24} &= 1/4(-i m_{13} + i m_{31} + m_{24} + m_{42}), \ j_{42} &= 1/4(i m_{13} - i m_{31} + m_{24} + m_{42}). \end{split}$$

It can be seen that coherency matrix **J** is positive semidefinite Hermitian and, hence, has always four real eigenvalues. This defines a requirement for the Mueller matrix **M** to be physically realizable, i.e., the coherency matrix **J** associated with **M** should have all four non-negative eigenvalues [18].

For the average characterization of depolarization for a given Mueller matrix, the following single value metric, called Cloude entropy, can be used:

$$H = \sum_{i=1}^{4} -P_i \log_4 P_i,$$
 (2)

where

$$P_i = \frac{\lambda_i}{\sum\limits_j \lambda_j},\tag{3}$$

 λ_i are the eigenvalues of coherency matrix J from Equation (1).

The entropy is bounded according to $0 \le H \le 1$. For a sample without depolarization: H = 0 and $\lambda_1 \ne 0$, $\lambda_{i\ne 1} = 0$. For totally depolarizing samples: H = 1 and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. When H < 0.5 and H > 0.5 samples are weakly and strongly depolarizing, respectively.

Given eigenvalues λ_i of the coherency matrix **J**, the initial arbitrary depolarizing Mueller matrix can be represented through additive Cloude decomposition as

$$\mathbf{M} = \sum_{k=1}^{4} \lambda_k \mathbf{M}_k,\tag{4}$$

where \mathbf{M}_k are the non-depolarizing (pure) Mueller matrices derivable from corresponding Jones matrices \mathbf{T}_i [3,19,20]. The Jones matrices \mathbf{T}_i corresponding to pure Muller matrices \mathbf{M}_k in Equation (4) can in turn be derived as

$$t_{11}^{(k)} = \Psi_1^{(k)} + \Psi_2^{(k)}, t_{12}^{(k)} = \Psi_3^{(k)} - i\Psi_4^{(k)} t_{21}^{(k)} = \Psi_3^{(k)} + i\Psi_4^{(k)}, t_{22}^{(k)} = \Psi_1^{(k)} - \Psi_2^{(k)}$$
(5)

where $\Psi^{(k)} = \begin{pmatrix} \Psi_1 & \Psi_2 & \Psi_3 & \Psi_4 \end{pmatrix}_k^T$ is *k*-th eigenvector of the coherence matrix **J**.

We can gain a better understanding by rewriting Equation (4) in the following form [3]:

$$\mathbf{M} = \lambda_1 \mathbf{M}_1 + \sum_{k=2}^4 \lambda_k \mathbf{M}_k = \mathbf{M}_0 + \Delta \mathbf{M},$$
(6)

which allows for a straightforward physical interpretation. Namely, the part M_0 is the pure estimation of M that generally contains seven independent parameters and ΔM is the depolarizing Mueller matrix containing up to nine independent parameters [20] characterizing depolarization. Thus, in the general case, we have 16 independent parameters that completely characterize the anisotropic and depolarizing properties of the sample under study. In this way, the Cloude additive decomposition determines both the anisotropy and depolarization properties of an arbitrary depolarizing sample. The former is described by the non-depolarizing Mueller matrix M_0 in Equation (6) and the latter by depolarizing Mueller matrix ΔM and/or by entropy *H* in Equation (2).

Apparently, the very first version of the multiplicative decomposition of the depolarizing Mueller matrices, called polar decomposition, was suggested in [21,22]. According to [21,22] the polar decomposition of an arbitrary depolarizing Mueller matrix can be represented as follows:

$$\mathbf{M} = \mathbf{M}_{\Delta} \mathbf{M}_{R} \mathbf{M}_{D},\tag{7}$$

where \mathbf{M}_R and \mathbf{M}_D are the Mueller matrices of elliptical retarder and diattenuator (retarder and diattenuator polar forms), respectively; \mathbf{M}_Δ is a depolarizer. The Mueller matrices of elliptical diattenuator and retarder are pure and characterized by orthogonal eigenpolarizations and generally by seven independent parameters: each of the matrices \mathbf{M}_R and \mathbf{M}_D by three independent parameters and the total intensity. Depolarizer Mueller matrix \mathbf{M}_Δ contains up to nine independent parameters. Evidently, all four matrices \mathbf{M}_i in Equation (4) can be represented as a multiplicative decomposition $\mathbf{M}_R\mathbf{M}_D$.

The Mueller matrix of retarder polar form M_R (using notation from Lu and Chipman [22]) is given by

where $\overrightarrow{0}$ is the 3 × 1 zero vector; $\begin{pmatrix} 1 & a_1 & a_2 & a_3 \end{pmatrix}^T = \begin{pmatrix} 1 & R^T \end{pmatrix}^T$ is the normalized Stokes vector for the fast axis of \mathbf{M}_R ; δ_{ij} is the Kronecker delta; ε_{ijk} is the Levi–Civita permutation symbol; \mathbf{m}_R is the 3 × 3 submatrix of \mathbf{M}_R obtained by striking out the first row; and the first column of \mathbf{M}_R and R is the birefringence given by

$$R = \arccos\left(\frac{1}{2}\operatorname{Tr} \mathbf{M}_{R} - 1\right),$$

$$a_{i} = \frac{1}{2\sin R} \sum_{j,k=1}^{3} \varepsilon_{ijk}(\mathbf{m}_{R})_{jk}.$$
(9)

The Mueller matrix of diattenuator polar form M_D is as follows:

$$\mathbf{M}_{D} = T_{u} \begin{pmatrix} \mathbf{1} & \overrightarrow{\mathbf{D}} \\ \overrightarrow{\mathbf{D}} & \mathbf{m}_{P} \end{pmatrix},$$

$$(10)$$

$$\sqrt{1 - D^{2}} \mathbf{I} + \begin{pmatrix} \mathbf{1} & \sqrt{1 - D^{2}} \end{pmatrix} \widehat{\mathbf{D}} \widehat{\mathbf{D}}^{T}$$

$$\mathbf{m}_D = \sqrt{1 - D^2} \mathbf{I} + \left(1 - \sqrt{1 - D^2}\right) \mathbf{D} \mathbf{D}^{\mathsf{T}},$$

where **I** is the 3 × 3 identity matrix; $\overrightarrow{\mathbf{D}} = \overrightarrow{\mathbf{D}}/|\overrightarrow{\mathbf{D}}|$ is the unit vector in the direction of the diattenuation vector $\overrightarrow{\mathbf{D}}$; T_u is the transmittance for input unpolarized light.

The value of diattenuation can be obtained as

$$D = \sqrt{m_{12}^2 + m_{13}^2 + m_{14}^2}.$$
 (11)

In the general case, the explicit form of the retarder polar form is nothing more than the first partial Jones equivalence theorem [23]. As for the diattenuator polar form, the matter is more complicated. On the one hand, we have diattenuator polar forms proposed in [22,24]. However, in these variants of the diattenuator polar forms, there is no explicit information about the value of the linear and circular amplitude anisotropy. In this sense, following Shurcliff's classification [25], both of these options are transcendent. A variant of the diattenuator polar form in which information about the linear (magnitude and orientation) and circular (magnitude) amplitude anisotropy is explicitly present was proposed in [26,27].

The Mueller matrix of depolarizer M_{Δ} is as follows:

$$\mathbf{M}_{\Delta} = \begin{pmatrix} \mathbf{I} & \stackrel{\rightarrow T}{\mathbf{0}} \\ \stackrel{\rightarrow}{\mathbf{P}_{\Delta}} & \mathbf{m}_{\Delta} \end{pmatrix}, \tag{12}$$

where $\overrightarrow{\mathbf{P}}_{\Delta}$ is the polarizance vector and \mathbf{m}_{Δ} is a 3 × 3 symmetric matrix; $\overrightarrow{\mathbf{0}}^{T}$ is a 1 × 3 zero vector; the superscript T stands for vector or matrix transposition.

The averaged depolarization capability of the depolarizer \mathbf{M}_{Δ} can be determined by the metric

$$\Delta = 1 - \frac{|tr(\mathbf{m}_{\Delta})|}{3}, \quad 0 \le \Delta \le 1,$$
(13)

which is called the depolarization power.

Note, like the matrix $\Delta \mathbf{M}$ found in Equation (6), the matrix \mathbf{M}_{Δ} Equation (12) generally contains nine independent depolarization parameters. The polarizance vector $\overrightarrow{\mathbf{P}_{\Delta}}$ contains three parameters and symmetric submatrix \mathbf{m}_{Δ} with six parameters.

Due to the non-commutativity of the matrices \mathbf{M}_{Δ} , \mathbf{M}_R and \mathbf{M}_D , the decomposition Equation (7) is not unique. An important discussion of the ambiguity of the polar decomposition and, hence, the inverse problem of polarimetry due to the non-commutativity of matrices \mathbf{M}_{Δ} , \mathbf{M}_R and \mathbf{M}_D can be found in [3]. In this paper, we use the decomposition form given by Equation (7).

Another multiplicative decomposition of the arbitrary depolarizing Muller matrix, which can be used to characterize the anisotropy and depolarizing properties of the sample described by the Mueller matrix **M**, was proposed in [28]:

$$\mathbf{M} = \mathbf{M}_{D2}\mathbf{M}_{R2}\mathbf{M}_{dep}\mathbf{M}_{R1}^{T}\mathbf{M}_{D1},$$
(14)

where \mathbf{M}_{D1} , \mathbf{M}_{D2} and \mathbf{M}_{R1} , \mathbf{M}_{R2} are four non-depolarizing Mueller matrices, respectively, two elliptical diattenuators and two retarders; \mathbf{M}_{dev} is the Mueller matrix of diagonal depolarizer

$$\mathbf{M}_{dep} = \begin{pmatrix} m_{11}^{dep} & 0 & 0 & 0\\ 0 & m_{22}^{dep} & 0 & 0\\ 0 & 0 & m_{33}^{dep} & 0\\ 0 & 0 & 0 & m_{44}^{dep} \end{pmatrix}.$$
 (15)

It can be seen that each of the matrices representing elliptical diattenuators \mathbf{M}_{D1} , \mathbf{M}_{D2} and retarders \mathbf{M}_{R1} , \mathbf{M}_{R2} contain three parameters. Thus, the total number of "pure" independent parameters in decomposition Equation (14) generally is 12. The matrix of the depolarizer Equation (15) contains 4 parameters. If the diagonal elements of the matrix Equation (15) are $m_{22}^{dep} = m_{33}^{dep} = m_{44}^{dep} < m_{11}^{dep}$, then this is the Mueller matrix of an isotropic partial depolarizer containing one depolarization parameter. If the diagonal elements are $m_{22}^{dep} \neq m_{33}^{dep} \neq m_{44}^{dep}$, then this is the Mueller matrix of the anisotropic depolarizer. Indeed, in this case, the output polarization degree differs by different input polarizations. In principle, under certain conditions, the depolarizer matrix \mathbf{M}_{dep} in Equation (14) can be non-diagonal. However, in our case, as discussed below, these conditions were not realized. Additional discussion can be found in [3,28].

When analyzing the Mueller matrices of depolarizing samples, along with obtaining an explicit form of the depolarizer Muller matrices appearing in the additive and multiplicative decompositions considered above, the analysis of the anisotropic properties of the samples under study is of interest as well. This is due to the fact that the anisotropy parameters themselves represent additional effective observables for distinguishing between samples. Furthermore, the presence of amplitude anisotropy can significantly affect the depolarization of the input light by the sample.

In this paper, we analyze the anisotropic properties using the "pure" part:

$$\mathbf{M} = \mathbf{M}_R \mathbf{M}_D, \tag{16}$$

of the polar decomposition Equation (7), Equations (9) and (11) for (i) the Mueller matrices \mathbf{M}_0 corresponding to the largest eigenvalue of the coherency matrix Equation (1) in the Cloude additive decomposition Equation (6); (ii) the non-depolarizing part of the polar decomposition Equation (7); and (iii) the non-depolarizing parts of the symmetric decomposition Equation (14).

For completeness of the anisotropy analysis of the samples under study, we also use the generalized equivalence theorem [29], which is a natural generalization of two partial Jones equivalence theorems [22], and represents an arbitrary non-depolarizing Muller matrix in the form

$$\mathbf{M}^{CP}\mathbf{M}^{LP} \ \mathbf{M}^{CA} \ \mathbf{M}^{LA}, \tag{17}$$

$$\theta = 1/2 \arctan\left(\frac{m_{13}}{m_{12}}\right),$$

$$P = \frac{(m_{11} - m_{12} \cos(2\theta) - m_{13} \sin(2\theta))^2}{m_{11}^2 - (m_{12} \cos(2\theta) - m_{13} \sin(2\theta))^2},$$

$$R = \frac{m'_{11} \pm \sqrt{(m'_{11})^2 - (m'_{14})^2}}{m'_{14}},$$

$$\alpha = \frac{1}{2} \operatorname{arctg}\left(\frac{m''_{42}}{-m''_{43}}\right)$$

$$\delta = \operatorname{arctg}\left(\frac{m''_{42}}{m''_{44} \sin(2\alpha)}\right)$$

$$\phi = \operatorname{arctg}\left(\frac{m''_{34} m''_{42} - m''_{24} m''_{43}}{m''_{34} m''_{43} + m''_{24} m''_{42}}\right)$$
(18)

where

$$\mathbf{M}' = \mathbf{M} \left(\mathbf{M}^{LA}\right)^{-1}$$

$$\mathbf{M}'' = \mathbf{M}' \left(\mathbf{M}^{CA}\right)^{-1}.$$
 (19)

3. Results and Discussion

Before proceeding to the analysis of the results obtained for the convenience of reading the paper, similar to what was carried out in [2], we note the following regarding the presentation of the results below. All figures below depict the dependences of the matrix elements, depolarization metrics and anisotropy parameters presented in Section 2 on the observation angles. Therefore, the following unified legend is adopted throughout the text: group (a), *Chlorina* mutant, which was grown under ordinary lighting conditions; group (b), *Chlorina* mutant, whose plants were etiolated (left in the dark) during growth; and group (c), *Cesaer varieties*. In addition, in order not to overwhelm the figures, we did not label the abscissa axis every time, while the ordinate axes are properly indicated throughout the text. All figures in the text below do not have error bars because the standard deviations in each case were comparable to the plotted symbols and less than 2%.

3.1. Depolarizers in Different Decompositions

Figures 1–3 show matrix elements of the depolarizers Δ **M**, Equation (6), **M**_{Δ}, Equation (7) and **M**_{*dep*}, Equation (14), for forward (i) and backward (ii) scattering versus observation angle, respectively.

The largest number of non-zero matrix elements for forward scattering is observed for the Cloude depolarizer $\Delta \mathbf{M}$ Equation (6). This is the only depolarizer in which, for forward scattering, in addition to diagonal elements m_{ii} , the elements m_{34} and m_{43} are non-zero. Matrix elements m_{34} and m_{43} make it possible to distinguish between groups of samples (c) and pair (a) and (b). However, these elements do not allow one to distinguish between groups of samples (a) and (b).

Diagonal matrix elements m_{ii} of depolarizers \mathbf{M}_{Δ} , Equation (7), and \mathbf{M}_{dep} , Equation (14), behave quite similarly. For observation angles less than 45 degrees, these elements make it possible to distinguish all three groups of samples under study. However, the element m_{22} that describes the depolarization for input vertical and horizontal linear polarizations is the most effective observable for all three groups of samples for all forward scattering observation angles.



Figure 1. The matrix elements of the depolarizer Δ **M**, Equation (6), for forward (i) and backward (ii) scattering versus observation angle.



Figure 2. The matrix elements of the depolarizer M_{Δ} , Equation (7), for forward (i) and backward (ii) scattering versus observation angle.

The behavior of the diagonal matrix elements of the depolarizer $\Delta \mathbf{M}$ differs noticeably from that discussed above. In particular, the element m_{33} shows minimal separability of the samples. This element allows one to distinguish only groups (a) and (c) for observation angles less than 30 degrees. The element m_{22} allows distinguishing effectively between a group of samples (c) and a pair of (a) and (b), which differ minimally. The element m_{44} shows stronger separation for all three groups of samples.



Figure 3. The matrix elements of the depolarizer \mathbf{M}_{dep} , Equation (14), for forward (i) and backward (ii) scattering versus observation angle.

For backscattering, a completely different pattern is observed. Obviously, the most informative is the Cloude depolarizer Equation (6). In addition to diagonal elements, the matrix elements m_{12} , m_{21} , m_{34} and m_{43} are non-zero. Matrix elements m_{12} and m_{21} do not allow for identification between the groups. The matrix elements m_{34} and m_{43} identify groups of studied samples for observation angles 110–140 degrees.

In general, the dependence of the diagonal matrix elements of depolarizers \mathbf{M}_{Δ} Equation (7) and \mathbf{M}_{dep} Equation (14) on observation angle is less than that for forward scattering. All three diagonal elements of the depolarizers Equations (7) and (14) make it possible to identify a group of samples (b) and pair (a) and (c) for all observation angles. Samples from the pair (a) and (c) can be reliably identified based on the element m_{22} , m_{33} for observation angles 130–165 degrees and the element m_{44} for observation angles 150–165 degrees. For observation angles smaller than 150 degrees, element m_{44} does not have the ability to distinguish between groups (a) and (c).

From Figure 1ii, the dependence of the diagonal elements of the Cloude depolarizer $\Delta \mathbf{M}$ Equation (6) on the observation angle for backscattering is qualitatively different. Indeed, for matrix elements m_{33} and m_{44} , there is an intersection by observation angle for a group of samples (b) and a pair of (a) and (c), which is uncharacteristic for both depolarizers \mathbf{M}_{Δ} and \mathbf{M}_{dep} and depolarization metrics given in Equations (2) and (13). Moreover, based on these matrix elements, groups (a) and (c) for all backscatter observation angles are indistinguishable. However, both on the basis of the elements m_{33} and m_{44} and the element m_{22} , a group of samples (b) is effectively identified. Obviously, the most informative among the diagonal elements of the Cloude depolarizer $\Delta \mathbf{M}$ for backscattering is the element m_{22} for observation angles higher than 130 degrees.

All of the above features of the depolarizer Mueller matrices Δ **M**, Equation (6), **M**_{Δ}, Equation (7), and **M**_{*dep*}, Equation (14), necessitate a more detailed analysis which was carried out in Section 3.3.

Figures 4 and 5 show the Cloude entropy *H* Equation (2) and the depolarization power Δ Equation (13) versus the observation angle for both experimental geometries. In these figures, group (b), the etiolated mutant variety under restricted lighting, is clearly distinguishable from (a) and (c) in both scattering directions. The most likely cause for this is the altered stacked thylakoid membrane structure in the etiolated mutant variety (*Chlorina* mutants differ from wild barley variety due to anomalous thylakoid stacking). Electron transfer during photosynthesis occurs over a chemical potential across thylakoid membranes.



Figure 4. Dependences of Cloude entropy *H* for forward (i) and backward (ii) scattering versus observation angle.



Figure 5. Dependence of the depolarization power Δ for forward (i) and backward (ii) scattering versus observation angle.

As noted in Section 2, the Cloude entropy H and the depolarization power Δ represent the averaged depolarization properties of the studied groups of samples and in this sense, they are similar to the single value depolarization metrics considered in the accompanying

paper [2], with the only difference being that to obtain them it is necessary to accomplish the decompositions Equations (6) and (7) of the experimental Mueller matrices, respectively.

A comparison of the Cloude entropy H and the depolarization power Δ with the most effective observable $Q(\mathbf{M})$ -metric among depolarization metrics [2] shows that their behavior is largely similar. However, it should be noted that the depolarization power Δ does not allow one to distinguish between groups of samples (a) and (c) over the entire observation angle range of forward scattering. For backscatter, the range of observation angles for which all three groups of samples can be effectively distinguished is limited to 130–160 degrees.

3.2. "Pure" Anisotropy in Different Decompositions

Of particular interest in solving the main problems of this paper is the analysis of deterministic Mueller matrices obtained in the decompositions presented in Section 2. This interest is based on the inequality of the diagonal elements and non-zero values of the non-diagonal matrix elements m_{12} , m_{21} , m_{23} , m_{32} , m_{34} and m_{43} (see Figures A1 and A2 in [2]), and, as a consequence, the dependence of the output intensity on the input polarizations (see Tables 4 and 5 in [2]). To analyze the information contained in deterministic Mueller matrices, we use "pure" decompositions Equations (16) and (17). As was shown in Section 2, when using the decomposition Equation (16), phase and amplitude anisotropy is characterized by the retardation Equation (9) and diattenuation Equation (11), respectively. These parameters, as they are defined in Equations (9) and (11), are general identifiers of amplitude and phase anisotropy. In this context, these parameters are presented below. When using decomposition Equation (17), linear and circular phase and amplitude anisotropy is characterized, respectively, by φ (the value of circular phase anisotropy), R (the value of circular amplitude anisotropy), δ (the value of linear phase anisotropy), P (the value of linear amplitude anisotropy) and azimuths α , θ of linear phase and amplitude anisotropy Equation (18).

Figures 6–12 show the anisotropy parameters obtained by the analysis of deterministic Mueller matrices in decompositions Equations (6), (7) and (14) for forward (i) and backward (ii) scattering versus observation angle obtained from "pure" decompositions Equations (16) and (17).

For forward and backward scattering, as can be seen, both decompositions Equations (16) and (17) give somewhat similar results. Except for forward scattering, the orientations of retardance and linear phase (δ) anisotropy differ by approximately 90 degrees. The retardance azimuth for group (a) at observation angles less than 45 degrees is uninformative due to the zero retardance. The highest values of retardance and linear phase anisotropy are observed for group (c) but circular amplitude (*R*) and phase (φ) anisotropy are absent. It is noteworthy that changes in the value of linear amplitude (*P*) anisotropy are somewhat greater than changes in diattenuation (*D*), depending on the observation angle. As for the orientation of diattenuation and linear anisotropy (θ), they differ, as in the case of phase anisotropy by approximately 90 degrees. The latter makes it possible for forward scattering to confidently identify groups of samples (b) and (c) based on linear amplitude anisotropy.

Figures 6 and 7 show that the changes in diattenuation, retardance, linear amplitude and phase anisotropy on observation angle for backward scattering are significantly greater than those for forward scattering. Both retardation and linear phase anisotropy make it possible to identify all three groups of studied samples for observation angles of 110–140 degrees and groups (a) and (c) in almost the entire range of observation angles Retardance via decomposition Equation (16) shows nearly quarter wave (90 degrees) phase shift as the limits of the backscattering angles are approached (Figure 6ii). While retardance in the forward scattering direction is not as pronounced in the backscattering direction, group (b) stands out with a preferential direction for the azimuth of retardance with respect to the observation angle. Diattenuation, representing preferential absorption of polarization states, changes with backscatter observation angles while azimuth of diattenuation is discernable as the larger forward scattering angles are approached. Circular amplitude anisotropy is absent, as it is in the case of forward scattering. Of interest is the non-zero value of the circular phase anisotropy for groups (b) and (c), which is apparently due to the non-zero values of the matrix elements m_{23} and m_{32} , see Figure A2 in [2]. Linear amplitude and circular phase anisotropy for backscattering allows distinguishing groups of samples (b) and (c) for observation angles of 110–165 and 110–140 degrees, respectively.



Figure 6. Anisotropy parameters as function of observation angles for the pure Mueller matrix Equation (6) obtained from decomposition Equation (16) for (i) forward and (ii) backward scattering.

The next two Figures 8 and 9 present the results of the analysis, which are similar to that presented above for pure Mueller matrices obtained in the decomposition Equation (6), and for pure Mueller matrices obtained from the decomposition Equation (7).

It can be seen that in this case there are certain analogies with data presented in Figures 6 and 7. However, significant differences are also observed. In particular, for forward scattering the diattenuation is non-zero, although very insignificant. There is no circular amplitude anisotropy (R). Linear amplitude anisotropy (P) has a larger range of changes than diattenuation. Based on linear amplitude anisotropy, it is possible to distinguish groups of samples (a) and (b) in the range of observation angles from 0 to 40 degrees.

Phase anisotropy behaves almost similarly for both decompositions Equations (6) and (7), taking into account the change in orientation for group (a) and groups (b) and (c) by 90 degrees. Thus, retardance and linear phase anisotropy (δ) are effective observers allowing one to distinguish all three groups of samples over the entire range of observation angles. It is noteworthy that the linear phase anisotropy (δ) for a group of samples (a) in the range of 10–35 degrees is absent. In general, the dependence of linear phase anisotropy on the observation angle for all three groups of samples is insignificant, increasing for large observation angles. Circular phase anisotropy is absent over the entire range of forward scattering.



Figure 7. Anisotropy parameters as function of observation angles for the pure Mueller matrix Equation (6) obtained from decomposition Equation (17) for (i) forward and (ii) backward scattering.



Figure 8. Anisotropy parameters as function of observation angles for the pure Mueller matrix Equation (7) obtained from decomposition Equation (16) for (i) forward and (ii) backward scattering.

As was the case for Figures 6 and 7, backscattering is characterized by observable dependence of the amplitude and phase anisotropy on the observation angle. It is interesting that the "rate" of change in the amplitude and phase anisotropy has an inverse character: the groups of samples (c) and (b) are characterized by the highest and lowest rates of change in amplitude anisotropy. At the same time, for phase anisotropy the opposite pattern is observed, i.e., the groups of samples (b) and (c) are characterized by the highest and lowest rates, respectively. Both amplitude and phase anisotropy for backscatter are effective identifiers for all three groups of samples in the observation angles of 110–150 degrees.

Circular amplitude anisotropy (*R*) is absent over the entire range of observation angles. As in Figures 6 and 7, circular phase anisotropy (φ) is of particular interest. For group (a), it is on average close to zero, while for groups (b) and (c) at backscattering angles of 110–140 degrees, right (φ_1) and left circular phase (φ_2) anisotropy of approximately 5 degrees is observed, respectively.

Figure 10 shows the results of the analysis for pure left and right Mueller matrices obtained in the symmetry decomposition Equation (14) on the basis of pure decomposition Equation (16). Figures 11 and 12 show the same for pure decomposition Equation (17).

For forward scattering, the right deterministic part of the symmetric decomposition shows minor non-zero diattenuation and retardation (approximately 3 degrees) for all three groups of samples for the entire range of observation angles, while the "pure" decomposition Equation (17) shows a larger change in linear amplitude anisotropy and significant dependence of linear phase (δ) anisotropy by observation angle. In this case, a circular phase (φ) anisotropy is observed near 90 degrees, which depends little on the observation angle. Amplitude anisotropy for both "pure" decompositions does not allow one to distinguish between groups of samples under study. Linear phase anisotropy distinguishes groups (b) and (c) for almost all observation angles. Groups of samples (a) and (c) can be distinguished for observation angles greater than 30 degrees. Circular phase anisotropy makes it possible to confidently distinguish groups of samples (b) and pairs (a) and (c) for observation angles from 0 to 40 degrees.



Figure 9. Anisotropy parameters as function of observation angles for the pure Mueller matrix Equation (7) obtained from decomposition Equation (17) for (i) forward and (ii) backward scattering.



Figure 10. Anisotropy parameters as function of observation angles for the pure Mueller matrices Equation (14) obtained from decomposition Equation (16) for (i) forward and (ii) backward scattering.

For the left deterministic part of decomposition Equation (14), we have in the scope of decomposition Equation (16) a complete absence of diattenuation and the same value of retardation as for the right part. That is, decomposition Equation (16) for forward scatter has no deterministic observables that can discriminate groups of leaf samples.

For backscattering, there is again no dependence of retardation and a more noticeable dependence of diattenuation for all three groups of samples by observation angle for the right and left parts of the decomposition. Diattenuation and linear amplitude anisotropy of the right side makes it possible to distinguish groups of samples (b) and pair of (a) and (c) in almost the entire range of observation angles, while for the left side of decomposition, it was the groups of samples (a) and (c) and the group of samples (b). It is interesting that the linear phase anisotropy of the right part allows one to distinguish between group (b) and the pair (a) and (c), while the left part in the observation angles of 100–135 degrees can distinguish all three groups of the studied samples.



Figure 11. Anisotropy parameters as function of observation angles for the right pure deterministic side of symmetric decomposition Equation (16) for (i) forward and (ii) backward scattering.



Figure 12. Anisotropy parameters as function of observation angles for the left pure deterministic side of symmetric decomposition Equation (16) for (i) forward and (ii) backward scattering.

3.3. Anisotropy of Depolarization

In Section 3.1, we obtained explicit forms for the Mueller matrices of depolarizers ΔM , M_{Δ} and M_{dep} for three decompositions Equations (6), (7) and (14). In addition, in Section 3.1, we noted that depolarizer Mueller matrices, describing the dependence of depolarization on the input polarizations (anisotropy of depolarization), also impact the dependence of the output polarizations on the input polarizations, i.e., the depolarizer Mueller matrices could contain information about the "deterministic" anisotropy of the corresponding depolarizers. Next, we address the "deterministic" anisotropy of depolarizers in more detail.

The additive decomposition Equation (6) for the Mueller matrices of the depolarizer M_{Δ} , Equations (7) and (12) can be written as:

$$\mathbf{M}_{\Delta} = \mathbf{M}_{0}^{\Delta} + \Delta \mathbf{M}^{\Delta}, \tag{20}$$

for all three groups of samples for forward and backward scattering.

Figures 13 and 14 represent the anisotropic parameters of pure Mueller matrices \mathbf{M}_0^{Δ} of depolarizer \mathbf{M}_{Δ} , Equation (20), obtained from two "pure" decompositions Equations (16) and (17) as a function of observation angle.



Figure 13. The diattenuation and retardance as functions of observation angle for deterministic Mueller matrices \mathbf{M}_0^{Δ} Equation (20) obtained from "pure" decompositions Equation (16) for (i) forward and (ii) backward scattering.

Next, the multiplicative decomposition Equation (7) for the Mueller matrices of the depolarizer ΔM , Equation (6) can be written as:

$$\Delta \mathbf{M} = \mathbf{M}^{\Delta}_{\Lambda} \mathbf{M}^{\Delta}_{R} \mathbf{M}^{\Delta}_{D}, \tag{21}$$

for all three groups of samples in forward and backward scattering directions.

Figures 15 and 16 show anisotropic parameters for the pure part $\mathbf{M}_{R}^{\Delta}\mathbf{M}_{D}^{\Delta}$ of the depolarizer $\Delta \mathbf{M}$ in the decomposition Equation (21), which were obtained on the basis of two pure decompositions from Equations (16) and (17).



Figure 14. Anisotropy parameters as function of observation angle for the deterministic Mueller matrices \mathbf{M}_0^{Δ} Equation (20) obtained from "pure" decompositions Equation (17) for (i) forward and (ii) backward scattering.





Figure 15. Diattenuation and retardance as function of observation angle for the deterministic Mueller matrices $\mathbf{M}_{R}^{\Delta}\mathbf{M}_{D}^{\Delta}$ Equation (21) obtained from "pure" decompositions Equation (16) for (i) forward and (ii) backward scattering.

Another interesting point about the depolarizer $\Delta \mathbf{M}$ is its further analysis in the context of additive decomposition from Equation (6). Obviously, in this case, we have an additive decomposition with three non-zero eigenvalues of the coherence matrix Equation (1):

$$\Delta \mathbf{M} = \lambda_2 \mathbf{M}_2 + \sum_{k=3}^4 \lambda_k \mathbf{M}_k = \Delta \mathbf{M}_0 + \Delta \Delta \mathbf{M}.$$
 (22)

The Mueller matrix ΔM_0 , corresponding to the largest of the remaining three eigenvalues of the coherence matrix Equation (1), can obviously be considered as pure estimation of ΔM . Figures 17 and 18 show the anisotropic parameters of the pure Mueller matrix ΔM_0 , which were again obtained on the basis of two pure decompositions from Equations (16) and (17).

For forward scattering, the retardance, linear (δ) and circular phase (φ) anisotropy are absent. Only a very small amplitude anisotropy is observed with some increase for observation angles greater than 45 degrees. For retardance, linear and circular phase anisotropy, a similar situation is also observed for backward scattering. There is a significant difference in amplitude anisotropy. The minimum values of amplitude anisotropy are observed for observation angles close to the exact backscattering direction. As the observation angle decreases, the amplitude anisotropy increases significantly. Moreover, for all observation angles, the group of samples (b) is characterized by the maximum, and the group of samples (a) by the minimum values of amplitude anisotropy. It is noteworthy that the range of changes in linear amplitude anisotropy is greater than that of diattenuation.



Figure 16. Anisotropy parameters as function of observation angle for the deterministic Mueller matrices $\mathbf{M}_{R}^{\Delta} \mathbf{M}_{D}^{\Delta}$ Equation (21) obtained from "pure" decomposition Equation (17) for (i) forward and (ii) backward scattering.



Figure 17. Diattenuation and retardance as function of observation angle for the deterministic Mueller matrices $\Delta \mathbf{M}_0$ Equation (22) obtained from "pure" decomposition Equation (16) for (i) forward and (ii) backward scattering.

Unlike the previous case, for forward scattering, both amplitude and phase anisotropy are observed for all three groups of samples. For group (c), diattenuation and linear amplitude anisotropy are very small. The minimum and maximum values of retardance and linear phase anisotropy are experienced by groups (a) and (c). For groups (a) and (b), circular amplitude anisotropy is observed; for group (c), it is absent. For groups (a) and (b), there is no circular phase anisotropy. For group (c), it is approximately 10 degrees and does not depend on the observation angle.

For backscattering, both amplitude and phase anisotropy are also observed. The minimum values of amplitude and phase anisotropy are observed at exact backscatter for all groups of samples, which increase noticeably with decreasing observation angle. The minimum and maximum values of amplitude anisotropy for all observation angles are characteristic of groups (b) and (c), respectively, and vice versa for phase anisotropy. Groups (a) and (b) are characterized by a small circular amplitude anisotropy for observation angles greater than 140 degrees. For group (c), as in the case of forward scattering, there is no circular amplitude anisotropy. As for circular phase anisotropy, for groups (a), it was as in the case of forward scattering, and for (c), it is absent. For group (c), it is approximately 5–7 degrees with a tendency to slightly increase with increasing observation angle.

There are no diattenuation (*D*), linear (*P*) and circular (*R*) amplitude anisotropy for all three groups for forward scattering. This resembles the case of the matrix \mathbf{M}_0^{Δ} Equation (20), Figures 13i and 14i. For groups (a) and (b), there is also no retardance, linear and circular phase anisotropy. For group (c), the value and orientation of retardance are approximately 160–165 and 90 degrees, respectively, and the value, the orientation of the linear and value of the circular phase anisotropy are 160–165, 175–180 and 175–180 degrees, respectively.



Figure 18. Anisotropy parameters as function of observation angle for the deterministic Mueller matrices $\Delta \mathbf{M}_0$ Equation (22) obtained from "pure" decomposition Equation (17) for (i) forward and (ii) backward scattering.

In this case, to interpret the observed value of circular phase anisotropy, we note that from Figure 18i, the values of retardance and linear phase anisotropy are very close to half-wave. As known, the Mueller matrix for the half-wave linear phase plate $\mathbf{M}^{LP}(180^0, \alpha_1)$ and the Muller matrix of the first partial Jones theorem [23] for the half-wave linear and circular anisotropy, i.e., $\mathbf{M}^{LP}(180^0, \alpha_2)\mathbf{M}^{CP}(\varphi)$, are structurally similar

/1	0	0	0 \		/1	0	0	0 \					
0	$\cos(4\alpha_1)$	$\sin(4\alpha_1)$	0	and	and	and	and	0	$\cos(4\alpha_2 - 2\phi) \sin(4\alpha_2)$	$\sin(4\alpha_2 - 2\phi)$	0		(22)
0	$\sin(4\alpha_1)$	$-\cos(4\alpha_1)$	0		0	$\sin(4\alpha_2 - 2\phi)$	$-\cos(4\alpha_2-2\phi)$	0	,	(23)			
0/	0	0	-1/		$\setminus 0$	0	0	-1/					

and, hence, cannot clearly be distinguished without involving additional information about the sample under study.

For backscattering, non-zero diattenuation and linear amplitude anisotropy are observed. The dependence of these quantities on the observation angle is very similar, i.e., a monotonic increase with deviation from the exact backscatter. Note that the range of changes in diattenuation and linear amplitude anisotropy for all three groups of samples is different. There is no circular amplitude anisotropy.

The character and ranges of the dependence of retardance and linear phase anisotropy on observation angles are approximately the same. Noteworthy is the inverse dependence of these values on the observation angle for groups (a) and (c) and group (b). For group (b), at angles close to the exact backscatter 150–170 degrees, circular phase anisotropy of approximately 5 to 10 degrees is also observed.

4. Conclusions

The objective of this paper was to determine whether three different groups of barley leaf samples (*Hordeum vulgare*), *Chlorina* mutant, *Chlorina* etiolated mutant and *Cesaer* varieties, can be discriminated in the visible wavelength ($\lambda = 632.8$ nm) using Mueller matrix polarimetry. Barley leaves with different internal structures from mutation or by illumination during the growth are an interesting testbed for Mueller polarimetry on biological scenes. We used the additive Cloude decomposition Equation (6), the multiplicative Lu-Chipman decomposition Equation (7) and Ossikovski symmetric decomposition Equation (14) to derive depolarization and anisotropy parameters from the measured Mueller matrices. The main result of this paper is proof that the Mueller matrix polarimetry provides a wide range of effective observables for detailed discrimination of the studied groups of samples at one wavelength. In Sections 3.1 and 3.2, we presented the detailed results for the relevant anisotropic and depolarization observables.

It was seen that the depolarizer Mueller matrices $\Delta \mathbf{M}$, \mathbf{M}_{Δ} , \mathbf{M}_{dep} for forward and backward scattering obtained in the decompositions Equations (6), (7) and (14) have a priori different structures and, therefore, generally contain a different number of parameters characterizing depolarization, i.e., generally contain different information about depolarization.

The groups of studied botanic samples, the internal structure of the samples and the conditions for their growth characterized by the higher and lower depolarization are determined. The matrix elements of depolarizers and observation angles, which are the most effective observables for the studied groups of samples, are determined as well. We have demonstrated that the dependence of depolarization on the observation angle for backward scattering is much stronger than for forward scattering. For both forward and backward scattering, the group of samples (b) is characterized by the greatest depolarization, and group (c) by the smallest. This reflects the internal features (thylakoid stacking) within group (b).

To characterize the depolarizing properties of the studied samples on the basis of the single value depolarization metrics, the results presented in Section 3.1 demonstrate that the Cloude entropy Equation (2) and depolarization power Equation (13) do not provide any additional information in comparison with metrics obtained directly from the initial experimental Mueller matrices [2]. When these metrics are compared to discriminate

between groups of samples under study, Cloude entropy demonstrated a higher level of discrimination, especially for backscatter.

Section 3.2 shows that different decompositions generally yield different pure Mueller matrices, i.e., there is an ambiguity of the inverse problem due to the matrix non-commutativity. However, discussion of this issue is beyond the scope of this paper. In this paper, we limit ourselves to studying and comparing the anisotropy of samples based on two multiplicative decompositions Equations (16) and (17). It was seen that the structure of these decompositions, in consideration of the multiplication order of the Mueller matrices describing the amplitude and phase anisotropy, is the same. This is of additional interest to the results presented in Section 3.2.

In Section 3.3, we were interested not so much in the possibility of discrimination (although the latter, apparently, is being fully implemented), but in the analysis of the information contained in the depolarizer Mueller matrices and demonstrated on the barley leaves. The results presented in Section 3.3 are a further development of the concept of the reduced Mueller matrix [30].

The effectiveness of anisotropic and depolarization observables demonstrated in this paper to characterize the groups of barley leaves with different internal structures achieved either due to mutation or by illumination during the growth are important to clarify the features of the polarized light passage through the surface and thickness of a leaf.

The results obtained in this paper clearly confirm and further develop the previous conclusion (see, for example [31–34]) that the decomposition used cannot be chosen a priori unambiguously. The optimal choice of decomposition in each case is determined by the properties of the sample under study. In this case, additive Cloude decomposition is preferable.

Importantly, the analysis of the anisotropic and depolarization properties of barley leaves in forward and backward scattering contributes to understanding the ambiguity problem of the Mueller matrix inverse problem [3]. In this case, we are dealing with the ambiguity that has received the least attention in the polarimetric bibliography to date, namely, the one associated with different decompositions.

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