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AN INFINITE PLATE WITH A CURVILINEAR HOLE AND FLOWING HEAT

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Abstract- In the present paper, we apply the complex variable method (Cauchy method) to derive exact expressions for the Goursat's functions for the boundary value problem of infinite plate weakened by a curvilinear hole. The hole is conformally mapped on the domain outside a unit circle by means of a general rational mapping function. Also the stress components, when an initial heat is uniformly flowing in the perpendicular direction of the hole, are obtained. Some applications are investigated. The interesting cases, when the shape of the hole takes different famous formulas are included as special cases. The work of many previous authors can be considered as special cases of this work.

Key Words- Infinite plate, Curvilinear hole, Cauchy method, Goursat's functions,

1.INTRODUCTION

Problems dealing with isotropic homogeneous perforated infinite plate have been investigated by many authors [1-10]. Some of them [6,7,10] used Laurent's theorem to express each complex potential as a power series, others [1,5,8] used complex variable method of Cauchy type integrals to express the complex potential in the Goursat's functions form.

Consider a thin infinite plate of thickness h with a curvilinear hole C where the origin lies inside the hole conformally mapped on the domain outside a unit circle γ by means of a rational mapping function $z = c\,w(\zeta)$, subject to the condition $z'(\zeta)$ does not vanish or become infinite outside a unit circle γ, $\zeta = e^{i\psi}$, $(0 \le \psi \le 2\pi)$, if a heat $\Theta = qy$, is flowing uniformly in the direction of the negative y-axis, where the increasing temperature Θ is assumed to be constant across the thickness of the plate i.e. $\Theta = \Theta(x, y)$, and q is the constant temperature gradient. Here, we take the x-axis to be the horizontal axis which is perpendicular to the y-axis. The uniform flow of heat is distributed by the presence of an insulated curvilinear hole C, and the heat equation satisfies the relation

a. $$\nabla^2\Theta = 0,\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

b. $$\frac{\partial\Theta}{\partial n} = 0, \text{ on the boundary } r = r_0, \tag{1}$$

where n is the unit vector perpendicular to the surface.

Neglecting the variation of the strain and the stress with respect to the thickness of the plate, the thermoelastic potential Φ satisfies the formula (see[9])

$$\nabla^2\Phi = (1+\upsilon)\alpha\Theta, \tag{2}$$

where α is a scalar presents the coefficient of the thermal expansion and υ is Poisson's ratio. Assume the faces of the plate are free of applied loads.

It is known that [1] the first and second boundary value problems in the plane theory of thermoelasticity are equivalent to finding two analytic functions, $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x + iy$. These functions must satisfy the boundary conditions

$$K\phi_1(t) - t\overline{\phi_1'(t)} - \overline{\psi_1(t)} = f(t), \tag{3}$$

where for the first boundary value problem $K = -1$, $f(t)$ is a given function of stresses, while for the second boundary value problem $K = k > 1$, $f(t)$ is a given function of the displacement, k is called the thermal conductivity of the material and t denoting the affix of a point on the boundary.

The formula (3) for the first and second boundary value problems, respectively, take the following form

$$\phi_1(t) + t\overline{\phi_1'(t)} + \overline{\psi_1(t)} = \frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y} + \frac{1}{2G}\int_0^s [iX(s) - Y(s)]ds + c, \tag{4}$$

$$k\phi_1(t) - t\overline{\phi_1'(t)} - \overline{\psi_1(t)} = u + iv - \frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y}, \tag{5}$$

where the applied stresses $X(s)$ and $Y(s)$ are prescribed on the boundary of the plane, s is the length measured form arbitrary point, u and v are the displacement components, G is the shear modulus, and Φ represents the thermoelastic potential function. Also, here the applied stresses $X(s)$ and $Y(s)$ must satisfy the following (see [9])

$$X(s) = \sigma_{xx}\frac{dy}{ds} - \sigma_{xy}\frac{dx}{ds}, \qquad Y(s) = \sigma_{yx}\frac{dy}{ds} - \sigma_{yy}\frac{dx}{ds},$$

where σ_{xx}, σ_{yy} and σ_{xy} are called the stress components and given by (see [9])

$$\sigma_{xx} + \sigma_{yy} = 4G\left[\phi'(z) + \overline{\phi'(z)} - \lambda\Theta\right],$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = 2G\left[\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} + 2i\frac{\partial^2\Phi}{\partial x\partial y}\right] - 4G\left[\overline{z\phi''(z)} + \overline{\psi'(z)}\right], \tag{6}$$

where $\lambda = \alpha\dfrac{1+\upsilon}{2}$ is the coefficient of heat transfer.

A very powerful method for solving the thermoelastic problems makes use of conformal mapping to reduce the problem for any given region whose boundary C satisfies certain regularity conditions, to corresponding problem for a region having a unit circle. In terms of the rational mapping function $z = c\,w(\zeta)$ where $c > 0$ and $w'(\zeta)$ does not vanish or becomes infinite for $|\zeta| > 1$, then the infinite region outside a close contour may be conformally mapped outside the unit circle γ.

The two complex potential functions $\phi_1(z)$ and $\psi_1(z)$ are written in the form (see[9])

$$\phi_1(z) = -\frac{S_x + iS_y}{2\pi(1+k)}\ln\zeta + c\Gamma\zeta + \phi(\zeta), \tag{7}$$

$$\psi_1(z) = \frac{k(S_x - iS_y)}{2\pi(1+k)}\ln\zeta + c\Gamma^*\zeta + \psi(\zeta), \tag{8}$$

where S_x, S_y are the components of the resultant vector of all external forces acting on the boundary Γ, Γ^* represent the stresses at infinity, generally complex, $\phi(\zeta)$, $\psi(\zeta)$ are single value analytic function within the region outside the unit circle γ and $\phi(\infty) = \psi(\infty) = 0$, which means that, $\phi(z)$ and $\psi(z)$ are homomorphic functions at infinity. It will be assumed that $\Gamma = \overline{\Gamma}$ and $S_x = S_y = 0$ for the first boundary value problem.

The rational mapping $z = c\,w(\zeta)$ maps the boundary C of the given region occupied by the middle plane of the plate in the z-plane onto the unit circle γ in the ζ-plane. Curvilinear coordinates (ρ,θ) are thus introduced into the z-plane which are the maps of the polar coordinates in the ζ-plane as given by $\zeta = \rho e^{i\theta}$.
Using $w(\zeta)$ in (3), we have

$$K\phi_1(c\,w(\zeta)) - \frac{w(\zeta)}{\overline{w'(\zeta)}}\overline{\phi_1'(c\,w(\zeta))} - \overline{\psi_1(c\,w(\zeta))} = f(c\,w(\zeta)). \tag{9}$$

Muskhelishvili [10] used the transformation $z = c\,w(\zeta) = c\left(\zeta + m\zeta^{-1}\right)$ in (9) when $K = -1$, for the first boundary value problem, and $K = k = \chi = \dfrac{\lambda + 3\mu}{\lambda + \mu} > 1$ for the second boundary value problem, λ, μ are called the Lame's constants, while χ is called Muskhelishvili constant, for solving the problem of stretching of an infinite plate weakened by an elliptic hole. This transformation conformally maps the infinite domain bounded internally by an ellipse onto the domain outside the unit circle $|\zeta| = 1$ in the ζ-plane. Also the application of the Hilbert problem for a stretched infinite plate weakened by a circular cut is discussed in [8].

The following rational mapping functions

$$z = c\frac{\zeta + m_1\zeta^{-1}}{1 - n_1\zeta^{-1}}, \qquad (c > 0,\ |n_1| < 1, \text{ see[5]}) \tag{10}$$

$$z = c\frac{\zeta + m_1\zeta^{-1} + m_2\zeta^{-2}}{1 - n_1\zeta^{-1}}, \qquad (c > 0,\ |n_1| < 1, \text{ see[1,4]}) \tag{11}$$

$$z = c\frac{\zeta + m_1\zeta^{-\ell}}{1 - n_1\zeta^{-\ell}}, \qquad (c > 0,\ |n_1| < 1,\ \ell = 1,\dots,p, \text{ see[3]}) \tag{12}$$

$$z = c\frac{\zeta + \sum_{i=1}^{3} m_1\zeta^{-i}}{1 - n_1\zeta^{-1}}, \qquad (c > 0,\ |n_1| < 1, \text{ see[2]}) \tag{13}$$

where $c > 0$, m's and n's are real parameters restricted such that $z'(\zeta)$ does not vanish or become infinite outside γ, are used by El-Sirafy and Abdou [5], Abdou [1], Abdou and Khar-Eldin [3] and Abdou and Badr [2] respectively in (9), to solve the first and second boundary value problems of the infinite plate with a curvilinear hole C in the same previous domain. Abdou and Salama [4] used the rational mapping (11) in Eq.(9) to obtain the stress components for the first and second boundary value problem in the thermoelastic infinite plate weakened by a curvilinear hole C.

In this paper, the complex variable method has been applied to obtain the two complex analytic potential functions, (Goursat's functions) $\phi(\zeta)$ and $\psi(\zeta)$, the three stress components σ_{xx}, σ_{yy} and σ_{xy} for the first and second boundary value problem in thermoelasticity of the same previous domain, for an infinite plate weakened by a curvilinear hole C conformally mapped outside a unite circle γ by the rational mapping function

$$z = c\,w(\zeta) = c\frac{\zeta + m_1\zeta^{-\ell} + m_2\zeta^{-2\ell}}{(1-n_1\zeta^{-\ell})(1-n_2\zeta^{-\ell})}, \qquad (c>0,\ n_1 \neq n_2,\ \ell = 1,\ldots,p) \tag{14}$$

when a heat $\Theta = qy$ is flowing uniformly in the negative direction of y-axis. The increasing temperature Θ is assumed to be constant across the thickness the plate. Here, in (14) m's and n's are real parameters restricted such that $w(\zeta)$ does not vanish or become infinite outside the unit circle γ. The interesting cases when the shape of the hole is an ellipse, hypotrochoidal a crescent or a cut having the shape of a circular arc is included as special ones. Holes corresponding to certain combination of the parameters m's and n's are sketched (see Figures 1-6). Some applications of the first and second boundary value problems of the infinite plate with a curvilinear hole having several poles are investigated.

2.THE METHOD OF SOLUTION

In view of the definition $z = x + iy = \rho e^{i\theta}$ and the rational mapping function of Eq.(13), the solution of the boundary value problem of Eq.(1) is given by

$$\Theta = q\left[\operatorname{Im} z + \frac{r_0^2 \sin^2\theta}{\operatorname{Im} z}\right]. \tag{15}$$

In determining the thermoelastic potential of Eq.(2), the uniform heat may be disregarded. So the formula (2) takes the form

$$\nabla^2\Phi = \alpha(1+\upsilon)\frac{r_0^2\sin^2\theta}{\operatorname{Im} z}. \tag{16}$$

Using the definition of $\nabla^2\Phi$ in polar coordinates and solving (16) in this domain, we have

$$\Phi(z,\bar{z}) = \frac{(1+\upsilon)r_0^2}{4}\operatorname{Im} z \ln(z\bar{z}). \tag{17}$$

Hence the value of Θ and Φ are completely determined.

The expression $\dfrac{w(\zeta^{-1})}{w'(\zeta)}$ can be written in the form

$$\frac{w(\zeta^{-1})}{w'(\zeta)} = \alpha(\zeta^{-1}) + \beta(\zeta), \tag{18}$$

where

$$\alpha(\zeta) = \frac{h_1}{\zeta^\ell - n_1} + \frac{h_2}{\zeta^\ell - n_2},$$

$$h_j = \frac{(n_j^{\upsilon+1} + m_1 n_j + m_2)(1-n_j^2)(1-n_1 n_2)^2}{(n_j - n_{j\pm1})\left[(1-n_j^2)(1-n_1 n_2)\gamma_j^{(1)} - \ell n_j \gamma_j^{(2)}(n_1 + n_2 - 2n_j n_{j+1})\right]},$$

$$\gamma_j^{(1)} = 1 - \ell m_1 n_j^\upsilon - 2\ell m_2 n_j^{\upsilon+1},$$

$$\gamma_j^{(2)} = 1 + m_1 n_j^\upsilon + m_2 n_j^{\upsilon+1},$$

$$(j = 1,2;\ \upsilon = 1 + \frac{1}{\ell}) \tag{19}$$

and $\beta(\zeta)$ is a regular function for $|\zeta| > 1$.

Using (7), (8) and (18), the boundary condition (9) can be written in the form

$$K\phi(\sigma) - \alpha(\sigma)\overline{\phi'(\sigma)} - \overline{\psi_*(\sigma)} = f_*(\sigma), \tag{20}$$

where $\sigma = e^{i\theta}$ denotes the value of ζ on the boundary of the unit circle γ, while

$$\psi_*(\zeta) = \psi(\zeta) + \beta(\zeta)\phi'(\zeta),$$

$$f_*(\zeta) = F(\zeta) - ck_1\Gamma\zeta + c\overline{\Gamma}^*\zeta^{-1} + N(\zeta)\left[\alpha(\zeta) + \overline{\beta(\zeta)}\right],$$

$$N(\zeta) = c\overline{\Gamma} - \frac{S_x - iS_y}{2\pi(1+k)}\zeta,$$

$$F(\zeta) = f(t) = f\big(c\, w(\zeta)\big). \tag{21}$$

Assume that the derivatives of $F(\sigma)$ must satisfy Hölder condition.

Our aim is to determine the functions $\phi(\zeta)$ and $\psi(\zeta)$ for the various boundary value problems, from (20). For this multiplying both sides of (20) by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma - \zeta}$ then integrating the result around the unit circle γ and evaluating the integrals thus formulated by residue theorems, one has

$$K\phi(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma - \zeta}d\sigma = c\overline{\Gamma}^*\zeta^{-1} + A(\zeta) + \frac{h_1}{\zeta - n_1^{\upsilon-1}}N(n_1^{\upsilon-1})$$
$$+ \frac{h_2}{\zeta - n_2^{\upsilon-1}}N(n_2^{\upsilon-1}), \qquad (\upsilon = 1 + \frac{1}{\ell}) \tag{22}$$

where

$$A(\zeta) = \frac{1}{2\pi i}\sum_{\eta=0}^{\infty}\zeta^{-\eta-1}\int_\gamma \sigma^\eta F(\sigma)d\sigma. \tag{23}$$

Using (18) in the integral term of (22) we assume

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma-\zeta}d\sigma=\frac{ch_1b_1}{n_1^{\upsilon-1}-\zeta}+\frac{ch_2b_2}{n_2^{\upsilon-1}-\zeta}, \tag{24}$$

where b's are complex constants to be determined.

Hence, we have

$$-K\phi(\zeta)=A(\zeta)-c\overline{\Gamma^*}\zeta^{-1}-\sum_{j=1}^{2}\frac{h_j\left(cb_j+N(n_j^{\upsilon-1})\right)}{h_j^{\upsilon-1}-\zeta} \tag{25}$$

Differentiating (25) with respect to ζ, and using the result of $\overline{\phi'(\sigma)}$ in (24), we obtain

$$cKb_j+cn_j^2\Gamma^*+d_jh_j\left[c\overline{b_j}+\overline{N(n_j^{\upsilon-1})}\right]=-\overline{A'(n_j)},\ (j=1,2). \tag{26}$$

The general solution of (26) is

$$b_j=\frac{KE_j-h_jd_j\overline{E_j}}{c(K^2-h_j^2d_j^2)},$$

where

$$E_j=-\overline{A'(n_j)}-cn_j^{2(\upsilon-1)}\Gamma^*-d_jh_j\overline{N(n_j^{\upsilon-1})},$$

and

$$d_j=n_j^{2(\upsilon-1)}\left(1-n_1^{\upsilon-1}n_2^{\upsilon-1}\right)^{-2}. \tag{27}$$

From the boundary condition (20), $\psi(\zeta)$ can be determined in the form

$$\begin{aligned}\psi(\zeta)=&\frac{cK\overline{\Gamma}}{\zeta}-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta)+\frac{h_1\zeta}{1-n_1^{\upsilon-1}\zeta}\phi_*(n_1^{\upsilon-1})\\&+\frac{h_2\zeta}{1-n_2^{\upsilon-1}\zeta}\phi_*(n_2^{\upsilon-1})+B(\zeta)-B,\end{aligned} \tag{28}$$

where

$$\phi_*(\zeta)=\phi'(\zeta)+\overline{N(\zeta)},$$

$$B(\zeta)=\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{F(\sigma)}}{\sigma-\zeta}d\sigma,$$

$$B=\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{F(\sigma)}}{\sigma}d\sigma. \tag{29}$$

Using (15), (17), (25) and (28) in (6) after some derivatives and algebraic relations, we have

$$\begin{aligned}\sigma_{xx}&=2G\left[-\gamma(z^2+4z\bar{z}+\bar{z}^2)\operatorname{Im}z+\operatorname{Re}\left(2\phi'(\zeta)-M(\zeta,\bar{\zeta})\right)\right],\\ \sigma_{yy}&=2G\left[\gamma(z^2+\bar{z}^2)\operatorname{Im}z+\operatorname{Re}\left(2\phi'(\zeta)+M(\zeta,\bar{\zeta})\right)\right],\\ \sigma_{xy}&=2G\left[\gamma\left(z\bar{z}-2(\operatorname{Im}z)^2\right)\operatorname{Re}z+\operatorname{Im}M(\zeta,\bar{\zeta})\right],\end{aligned} \tag{30}$$

where

$$M(\zeta,\bar{\zeta})=\left[c\,w(\zeta)-\frac{\overline{w(\zeta)}}{w'(\zeta)}\right]\overline{\phi''(\zeta)}+\frac{S_x-iS_y}{2\pi(1+k)}\frac{\overline{w(\zeta)}}{w'(\zeta)}+cK\Gamma\zeta^2$$

$$-\overline{\left(\frac{w(\zeta)}{w'(\zeta)}\right)'}\left[\overline{\phi'(\zeta)}+N(\zeta)\right]+\overline{B'(\zeta)}+\sum_{j=1}^{2}\frac{h_j\zeta^2}{(\zeta-n_j^{\upsilon-1})^2}\phi_*(n_j^{\upsilon-1}), \quad (31)$$

$$\gamma=\frac{(1+\upsilon)r_0^2}{4\left(z\bar{z}\right)^2}.$$

3.SPECIAL CASES

(i) For $m_2=n_2=0$, we have the rational mapping function

$$z=c\frac{\zeta+m\zeta^{-\ell}}{1-n\zeta^{-\ell}}, \quad \ell=1,2,\dots,p$$

and the results of (26), (28) are in agreement with the result of Abdou and Kar-Eldin[3].
(ii) For $m_2=n_2=n_1=0$, we have the rational mapping function

$$z=c(\zeta+m\zeta^{-\ell}), \quad (0\le m<\frac{1}{\ell})$$

the main reason of interest in this mapping is that the general shapes of the hypotrochoids are curvilinear polygons, for $\ell=1$, our basic functions (26), (28) agree with (82.4'), (82.5'), (83.10) and (83.11) of Muskhelishvili's result obtained for the elliptic hole [8]. For $\ell=2$, we have a curvilinear triangle, for $\ell=3$ curvilinear square, and hence approximate region of physical interest (see [6]).
(iii) For $m_1=m_2=n_2=0$, we have the transformation

$$z=\frac{c\zeta}{1-n\zeta^{-\ell}}, \quad (c>0,\ |n|\le\frac{1}{1+\ell})$$

which leads to a certain regular curvilinear polygon with ℓ sides and ℓ round vertices which become cusps when $|n|\le\frac{1}{1+\ell}$ (see Figures 2,3,4).
(iv) For $n_2=n_1=0$ we have the rational mapping function

$$z=c(\zeta+m_1\zeta^{-\ell}+m_2\zeta^{-2\ell}), \quad (0\le m_j\le 1,\ j=1,2;\ \ell=1,2,\dots,p)$$

the physical interest of this map comes from the following:

- A circle of radius C when $m_1=m_2=0$.
- An ellipse when $m_2=0,\ \ell=1$.
- A square with rounded corners with diagonals parallel to the x and y axis when $m_1=0,\ m_2=$ about $0.1,\ \ell=1$. The same square with its sides parallel to the axis's when $m_1=0,\ m_2=$ about -0.1.
- An ovaloid when $m_1=$ about $0.3,\ m_2=$ about -0.05 and $\ell=1$.

More information and applications on technology for these special cases are found in [6].
(v) For $m_2=n_2=0,\ \ell=1$, we have the rational mapping function

$$z = c\frac{\zeta + m\zeta^{-1}}{1 - n\zeta^{-1}}$$

and the results of (26), (28) are agree with the result of El-Sirafy and Abdou [5].

(vi) For $m_2 = n_2 = 0, \ell = 1$ and $m_1 = -1$ the boundary C degenerates into a circular cut and (26)-(28) reduce to

$$-K\phi(\zeta) = A(\zeta) - \frac{c\overline{\Gamma^*}}{\zeta} + \frac{1-n^2}{\zeta - n}(N + \frac{KE + n\overline{E}}{K^2 - n^4}), \tag{32}$$

$$\psi(\zeta) = B(\zeta) + \frac{ck\overline{\Gamma}}{\zeta} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) - \frac{(1-n^2)^2\zeta}{1 - n\zeta}\phi_*(n^{-1}) - B, \tag{33}$$

where

$$E = n^2(\overline{N} - c\Gamma^*) - \overline{A'(n^{-1})},$$

$$w(\zeta) = \frac{\zeta - \zeta^{-1}}{1 - n\zeta^{-1}}. \tag{34}$$

(vii) For $m_2 = n_2 = 0, \ell = 1$ and values of m near -1, the edge of the hole resembles the shape of a crescent.

(viii) For $m_2 = n_2 = 0, \ell = 1$ and $m = -n^2$ the hole is bounded by the circle $|z - nc| = c$ and the functions $\phi(\zeta)$ and $\psi(\zeta)$ becomes

$$-K\phi(\zeta) = A(\zeta) - \frac{c\overline{\Gamma^*}}{\zeta}, \tag{35}$$

$$\psi(\zeta) = B(\zeta) + \frac{ck\overline{\Gamma}}{\zeta}(n + \zeta^{-1})\phi_*(\zeta) - B. \tag{36}$$

(ix) For $n_2 = 0, \ell = 1$, we have the rational mapping function

$$z = c\frac{\zeta + m_1\zeta^{-1} + m_2\zeta^{-2}}{1 - n\zeta^{-1}}$$

and the values of $\phi(\zeta)$ and $\psi(\zeta)$ of (26), (28) respectively, in this case, agree with the work of Abdou [1].

4.EXAMPLES

1. Curvilinear hole for an infinite plate subjected to a uniform tensile stress and flowing heat:

For $K = -1$, $\Gamma = \frac{P}{4}$, $\Gamma^* = -\frac{P}{2}e^{-2i\theta}$, $0 \le \theta \le 2\pi$, $S_x = S_y = f = 0$ and $\Theta = qy$, we have an infinite plate stretched at infinity by the application of a uniform tensile stress of intensity P, making an angle Θ with the x-axis, and a flowing heat in the negative direction of y-axis. The plate weakened by a curvilinear hole C which is free from stress, and the two complex functions of (26) and (28) become

$$\phi(\zeta) = \frac{cP}{2}e^{2i\theta}\zeta^{-1} + \sum_{j=1}^{2}\frac{L_j^{(1)}}{n_j^{\upsilon-1} - \zeta},$$

$$L_j^{(1)} = \frac{Ph_j}{4}\left[\frac{1-2n_j^{2(\upsilon-1)}\cos 2\theta}{1-h_j d_j} + \frac{2n_j^{2(\upsilon-1)}\sin 2\theta}{1+h_j d_j}\right], \tag{37}$$

$$\psi(\zeta) = -\frac{cP}{4}\zeta^{-1} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) + \sum_{j=1}^{2}\frac{h_j\zeta}{1-n_j^{\upsilon-1}\zeta}\phi_*(n_j^{\upsilon-1}), \tag{38}$$

where

$$\phi_*(\zeta) = \phi'(\zeta) + \frac{cP}{4}.$$

The stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ can be obtained directly by using (37), (38) in (30).

2. Curvilinear hole having two poles and the edge of which is subject to a uniform pressure in the present of flowing heat:

For $K=-1$, $S_x = S_y = \Gamma = \Gamma^* = 0$, $\Theta = qy$ and $f(t) = Pt$ where P is a real constant. The formulas (26)-(28) becomes

$$\phi(\zeta) = \frac{cP(n_1^{1+\upsilon} + m_1 n_1 + m_2)}{(n_1^{\upsilon-1} - \zeta)(1-n_1 d_1)(n_1 - n_2)} + \frac{cP(n_2^{1+\upsilon} + m_1 n_2 + m_2)}{(n_2^{\upsilon-1} - \zeta)(1-h_2 d_2)(n_2 - n_1)}, \tag{39}$$

and

$$\begin{aligned}\psi(\zeta) = &-cP(n_1 + n_2 + \zeta^{-1}) - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi'(\zeta) + \frac{h_1\zeta}{1-n_1^{\upsilon-1}\zeta}\phi'(n_1^{1-\upsilon}) \\ &+ \frac{h_2\zeta}{1-n_2^{\upsilon-1}\zeta}\phi'(n_2^{1-\upsilon}).\end{aligned} \tag{40}$$

Hence (39)-(40) give the solution of the first boundary value problem when the edge of the hole is subjected to uniform pressure P. Putting in (39) and (40) $-iT$ instead of P, we have the first boundary value problem when the edge of the hole is subjected to a uniform tangential stress T. Using (30), the stress components, in this case, can be directly obtained.

3. The force acts on the center of the curvilinear hole:

In this case, it will be assumed that the stresses vanish at infinity. It is easily seen that the kernel does not rotate. In general, the kernel remains in its original position. Hence, one assumes $\Gamma = \Gamma^* = f(t) = 0$, $K = k$ and $\Theta = qy$, the Goursat's functions are

$$-k\phi(\zeta) = \frac{c}{2\pi(1+k)}\sum_{j=1}^{2}\frac{h_j n_j}{n_j^{\upsilon-1} - \zeta}\left[\frac{kh_j d_j(S_x - iS_y)}{c(k^2 - h_j^2 d_j^2)} - \left(1 + \frac{h_j^2 d_j^2}{c(k^2 - h_j^2 d_j^2)}\right)(S_x - iS_y)\right] \tag{41}$$

and

$$\psi(\zeta) = \frac{h_1\zeta}{1-n_1^{\upsilon-1}\zeta}\phi_*(n_1^{1-\upsilon}) + \frac{h_2\zeta}{1-n_2^{\upsilon-1}\zeta}\phi_*(n_2^{1-\upsilon}) - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta), \tag{42}$$

where

$$\phi_*(\zeta) = \phi'(\zeta) - \frac{S_x + iS_y}{2\pi(1+k)\zeta}. \tag{43}$$

Therefore, we have the solution of the second boundary value problem in the case when a force S_x, S_y acts on the center of the curvilinear kernel and when a heat is flowing in the negative direction of y-axis.

5.CONCLUSION

From the above results and discussions, the following may be concluded:

1. In the theory of two dimensional linear elasticity one of the most useful techniques for the solution of boundary value problems for awkwardly shaped region is to transform the region into one simpler shape.
2. The mapping function (14) maps the curvilinear hole C in the z-plane onto the domain outside a unit circle ζ-plane under the condition $w'(\zeta)$ does not vanish or becomes infinite outside γ.
3. The physical interest of the mapping (14) comes from its strong special cases which discussed here, moreover many new cases can be obtained according to the technology of the work, where the boundary value problems of the infinite plate with a curvilinear hole having finite poles are not discussed before.
4. The complex variable method (Cauchy method) is considered as one of the best method for solving the integro-differential equations (22) taken on closed contour γ, and obtained the two complex potential functions $\phi(z)$ and $\psi(z)$ directly.
5. Here, we assumed the conformal mapping of Eq.(14), which has two singular points, i.e. we can say that $z \to \infty$ at $\zeta = n_\ell, \ell = 1,2$. Also at infinity we can say that the function $\theta_{\text{inf}}(z,\bar{z})$ is equivalent to the term $\Theta_{\text{inf}} = qy$ and the thermoelasticity potential $\phi = (z,\bar{z})$ is equivalent, at infinity, to the value

$$\Phi_{\text{inf}}(z,\bar{z}) = \frac{(1+\upsilon)\alpha q}{8} z\bar{z}\,\text{Im}\, z\,.$$

So, we can say that the stress components, at infinity, are relative to the following equations

$$\sigma_{xx} + \sigma_{yy} + 2(1+\alpha)\Theta_{\text{inf}} = 8GA,$$

$$\sigma_{yy} - \sigma_{xx} + 4G\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{z}^2}\right]\Phi_{\text{inf}}(z,\bar{z}) = 8GB,$$

$$\sigma_{xy} + 4iG\left[\frac{\partial^2}{\partial \bar{z}^2} - \frac{\partial^2}{\partial z^2}\right]\Phi_{\text{inf}}(z,\bar{z}) = 8GC,$$

where the real constants A, B and C are related to the stress at infinity.

6. The complex function $\phi(z)$ is considered the solution of the integral equation with Cauchy kernel

$$\phi'(\zeta) + \frac{1}{2\pi i}\int_\gamma k(\zeta,\sigma)\overline{\phi'(\sigma)}d\sigma = A'(\zeta) - \lambda w'(\zeta),$$

where

$$\lambda = \frac{\overline{\phi'(0)}}{w'(0)}, \quad k(\zeta,\sigma) = \frac{w(\sigma) - w(\zeta) - (\sigma - \zeta)w'(\zeta)}{w'(\sigma)(\sigma-\zeta)^2},$$

and $w(\zeta)$ is given by (14).

7. This paper can be considered as a generalization of the work of the infinite plate with a curvilinear hole under certain conditions [1-10].

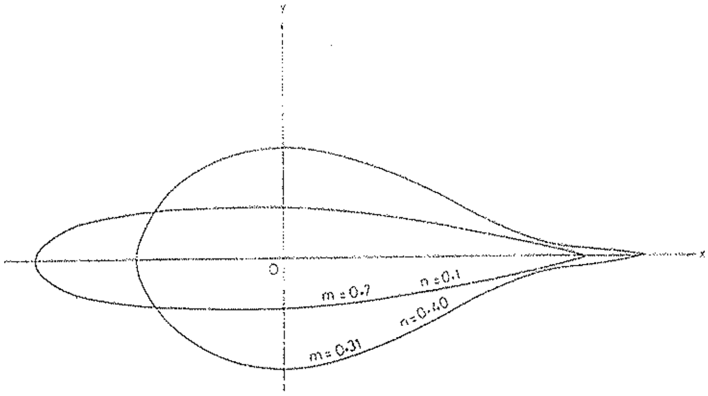

Fig.1: $z = c\dfrac{\zeta + m\zeta^{-1}}{1 - n\zeta^{-2}}$

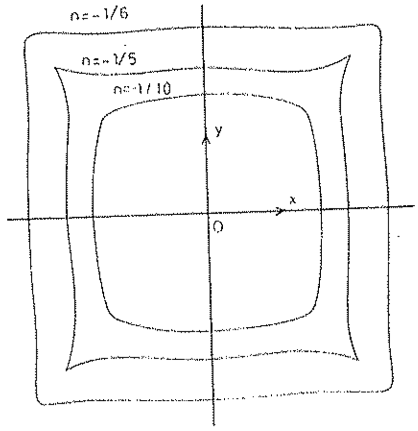

Fig.2: $z = c\dfrac{\zeta}{1 - n\zeta^{-4}}$

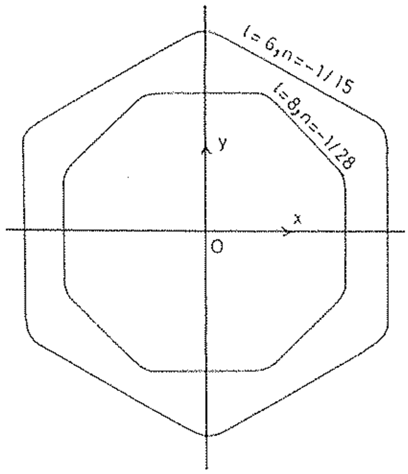

Fig.3: $z = c\dfrac{\zeta}{1 - n\zeta^{-\ell}}$, $\ell = 6,8$

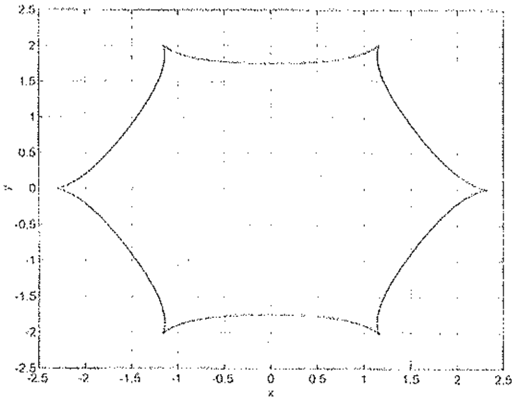

Fig.4: $z = c\dfrac{\zeta}{1 - \frac{1}{7}\zeta^{-7}}$

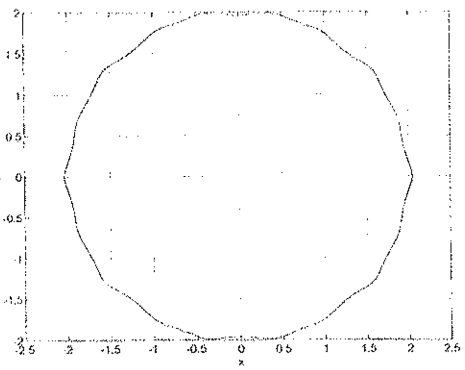

Fig.5: $z = c\dfrac{\zeta + 0.01\zeta^{-9}}{(1-0.1\zeta^{-9})(1+0.1\zeta^{-9})}$

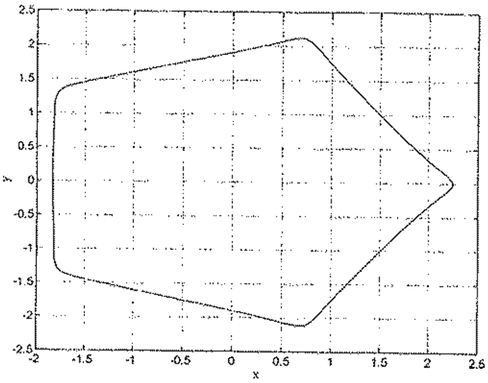

Fig.6: $z = c\dfrac{\zeta + 0.01\zeta^{-5}}{1-0.1\zeta^{-5}}$

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