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ON SISTER CELINE'S POLYNOMIALS OF SEVERAL VARIABLES

HSP Shrivastava
Department of Mathematics,Govt.Arts and Science PG College,
Ratlam (M.P.) – 457001, India
hspshrivastava@mantramail.com

Abstract-The aim of the present paper is to define Sister Celine's polynomials of two and more variables. We reduce the two variables Sister Celine's polynomials into many classical orthogonal polynomials and their product also such as – Jacobi, Gegenbauer, Legendre, Laguerre, Bessel and some discrete polynomials Bateman, Pasternak, Hahn, Krawtchouk, Meixner, Poisson-Charlier & others. Many integral representations and generating function relations are also established.

Keywords - Sister Celine's polynomials, Sister celine's polynomials of two and more variables, orthogonal polynomials, generalized Lauricella function of several variables.

1. INTRODUCTION

In 1947 Sister M. Celine Fasenmyer [7,8 ; see also 5, p.290] defined a polynomials known as Sister Celine's polynomials generated by

$$(1-t)^{-1}\,{}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};\frac{-4xt}{(1-t)^2}\right)=\sum_{n=0}^{\infty} f_n\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};x\right)t^n, \tag{1.1}$$

which yields

$$f_n(x)\equiv f_n\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};x\right)={}_{p+2}F_{q+2}\left(\begin{matrix}-n,n+1,a_1,\ldots,a_p\\ \frac{1}{2},1,b_1,\ldots,b_q\end{matrix};x\right). \tag{1.2}$$

Not much works has been done on this polynomials and one is unable to find many references of this polynomials in the mathematical literature. We have extended this polynomials to several variables in the next section of this note and defined the Sister Celine's polynomials of two and in general several variables interms of generalized Lauricella function of two and more variables defined by H.M. Srivastava and M.C. Daoust [11; see also 13, p.37], which is defined as :

$$F^{p:p_1;\ldots;p_r}_{q:q_1;\ldots;q_r}\left(\begin{matrix}(a_j:\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}:(c'_j,\gamma'_j)_{1,p_1};\ldots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r}\\ (b_j:\beta'_j,\ldots,\beta^{(r)}_j)_{1,q}:(d'_j,\delta'_j)_{1,q_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,q_r}\end{matrix};z_1,\ldots,z_r\right)$$

$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\Omega\,(m_1,\ldots,m_r)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!}, \tag{1.3}$$

where

$$\Omega\,(m_1,\ldots,m_r)=\frac{\prod_{j=1}^{p}(a_j)_{m_1\alpha'_j+\cdots+m_r\alpha^{(r)}_j}\prod_{j=1}^{p_1}(c'_j)_{m_1\gamma'_j}\cdots\prod_{j=1}^{p_r}(c^{(r)}_j)_{m_r\gamma^{(r)}_j}}{\prod_{j=1}^{q}(b_j)_{m_1\beta'_j+\cdots+m_r\beta^{(r)}_j}\prod_{j=1}^{q_1}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{q_r}(d^{(r)}_j)_{m_r\delta^{(r)}_j}}, \tag{1.4}$$

The parameters $a's,b's,c's,d's$ are complex numbers, the associated coefficients $\alpha's,\beta's,\gamma's,\delta's$ are real & positive and the variables $z_1,\ldots,z_r$ are complex.

For the convenience, when their is no ambiguity, we shall use following abbreviations :

The parameter $(a_j : \alpha'_j,\ldots,\alpha_j^{(r)})_{1,p}$ abbreviates the p-parameters array $(a_1 : \alpha'_1,\ldots,\alpha_1^{(r)}),\ldots,(a_p : \alpha'_p,\ldots,\alpha_p^{(r)})$ and similarly others.

$$a \equiv (a_j : \alpha'_j,\ldots,\alpha_j^{(r)})_{1,p} \quad ; \quad b \equiv (b_j : \beta'_j,\ldots,\beta_j^{(r)})_{1,q} \quad ;$$

$$c' \equiv (c_j,\gamma'_j)_{1,p_1};\ldots;c^r \equiv (c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \quad d' \equiv (d_j,\delta'_j)_{1,q_1};\ldots;d^r \equiv (d_j^{(r)},\delta_j^{(r)})_{1,q_r}$$

The multiple series (1.3) and it's special when r =2 converge absolutely [12 ; section 5 (p.157-158), section 3,4 (p.153-157); 3 section 3.7 ; 4, section 1.4] .

It is worth to mention here that

(i) For the sake of space in subsequent sections we have derived the results of Sister Celine's polynomials of two variables instead of several variables.

(ii) These results are new and if they are reduced into single variable, the results will also be believed to be new results.

(iii) When each of the positive real numbers $\alpha's,\beta's,\gamma's,\delta's$ is equated to unity , the generalized Lauricella series (1.3) reduces to a multiple Kampe-de-Feriet series.

2. SISTER CELINE'S POLYNOMIALS

In this section, we define Sister Celine's polynomials of two and r – variables interms of generalized Lauricella function of two and r – variables respectively. We also under the heading "Reductions" reduce the Sister Celine's polynomials of two variables into some well-known polynomials of two-variables such as Jacobi, Gegenbauer, Legendre, Hermite, Laguerre, Bessel and others. The Sister Celine's polynomials of r – variables is generalized by

$$\sum_{\substack{n_i=0\\(i=1,\ldots,r)}}^{\infty} f_{n_1,\ldots,n_r}\left(\begin{matrix}(a_j : \alpha'_j,\ldots,\alpha_j^{(r)})_{1,p} : (c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (b_j : \beta'_j,\ldots,\beta_j^{(r)})_{1,q} : (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix} ; z_1^{\lambda_1},\ldots,z_r^{\lambda_r}\right) t_1^{n_1}\ldots t_r^{n_r}$$

$$=\left[\prod_{i=1}^{r}(1-t_i)^{-1-c_i}\right]F^{p:p_1;\ldots;p_r}_{q:q_1;\ldots;q_r}\left(\begin{matrix}(a_j : \alpha'_j,\ldots,\alpha_j^{(r)})_{1,p} : (c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (b_j : \beta'_j,\ldots,\beta_j^{(r)})_{1,q} : (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix} ; \frac{(-4z_1t_1)^{\lambda_1}}{(1-t_1)^{2\lambda_1}},\ldots,\frac{(-4z_rt_r)^{\lambda_r}}{(1-t_r)^{2\lambda_r}}\right), \quad (2.1)$$

which gives

$$f_{n_1,\ldots,n_r}\left(z_1^{\lambda_1},\ldots,z_r^{\lambda_r}\right) \equiv f_{n_1,\ldots,n_r}\left(\begin{matrix}(a_j : \alpha'_j,\ldots,\alpha_j^{(r)})_{1,p} : (c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (b_j : \beta'_j,\ldots,\beta_j^{(r)})_{1,q} : (d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix} ; z_1^{\lambda_1},\ldots,z_r^{\lambda_r}\right)$$

$$=\prod_{i=1}^{r}\frac{(1+c_i)_{n_i}}{(n_i)!}F^{p:p_1+2;\ldots;p_r+2}_{q:q_1+2;\ldots;q_r+2}\left(\begin{matrix}(a_j : \alpha'_j,\ldots,\alpha_j^{(r)})_{1,p} : (-n_1,\lambda_1),(1+n_1+c_1+,\lambda_1)(c'_j,\gamma'_j)_{1,p_1};\ldots; \\ (b_j : \beta'_j,\ldots,\beta_j^{(r)})_{1,q} : \left(\frac{1+c_1}{2},\lambda_1\right),\left(\frac{2+c_1}{2},\lambda_1\right)(d'_j,\delta'_j)_{1,q_1};\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}(-n_r,\lambda_r),\ (1+n_r+c_r,\lambda_r),(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ \left(\frac{1+c_r}{2},\lambda_r\right),\ \left(\frac{2+c_r}{2},\lambda_r\right),(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix} ; z_1^{\lambda_1},\ldots,z_r^{\lambda_r}\right), \quad (2.2)$$

In (2.1),(2.2), if we take r = 2 and replace $n_1,n_2,\lambda_1,\lambda_2,z_1,z_2,t_1,t_2,c_1,c_2$ respectively by $n,m,\lambda,\mu,x,y,t,h,c,d$; we get the Sister Celine's polynomials of two variables as follows :

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{n,m}\left(\begin{matrix}(a_j : \alpha'_j,\alpha''_j)_{1,p} : (c'_j,\gamma'_j)_{1,p_1};(c''_j,\gamma''_j)_{1,p_2} \\ (b_j : \beta'_j,\beta''_j)_{1,q} : (d'_j,\delta'_j)_{1,q_1};(d''_j,\delta''_j)_{1,q_2}\end{matrix} ; x^{\lambda},y^{\mu}\right) t^n h^m$$

$$= \frac{(1-t)^{-1-c}}{(1-h)^{1+d}} F^{p:p_1;p_2}_{q:q_1;q_2}\left(\begin{matrix}(a_j:\alpha'_j,\alpha''_j)_{1,p}:(c'_j,\gamma'_j)_{1,p_1};(c''_j,\gamma''_j)_{1,p_2}\\(b_j:\beta'_j,\beta''_j)_{1,q}:(d'_j,\delta'_j)_{1,q_1};(d''_j,\delta''_j)_{1,q_2}\end{matrix};\frac{(-4xt)^{\lambda}}{(1-t)^{2\lambda}},\frac{(-4yh)^{\mu}}{(1-h)^{2\mu}}\right), \tag{2.3}$$

and

$$f_{n,m}(x^{\lambda},y^{\mu}) \equiv f_{n,m}\left(\begin{matrix}(a_j:\alpha'_j,\alpha''_j)_{1,p}:(c'_j,\gamma'_j)_{1,p_1};(c''_j,\gamma''_j)_{1,p_2}\\(b_j:\beta'_j,\beta''_j)_{1,q}:(d'_j,\delta'_j)_{1,q_1};(d''_j,\delta''_j)_{1,q_2}\end{matrix};x^{\lambda},y^{\mu}\right)$$

$$= \frac{(1+c)_n(1+d)_m}{n!m!} F^{p:p_1+2;p_2+2}_{q:q_1+2;q_2+2}\left(\begin{matrix}(a_j:\alpha'_j,\alpha''_j)_{1,p}:(-n,\lambda),(n+c+1,\lambda),(c'_j,\gamma'_j)_{1,p_1};\\(b_j:\beta'_j,\beta''_j)_{1,q}:\left(\frac{1+c}{2},\lambda\right),\left(\frac{2+c}{2},\lambda\right),(d'_j,\delta'_j)_{1,q_1};\end{matrix}\right.$$

$$\left.\begin{matrix}(-m,\mu),(m+d+1,\mu),(c''_j,\gamma''_j)_{1,p_2}\\\left(\frac{1+d}{2},\mu\right),\left(\frac{2+d}{2},\mu\right),(d'_j,\delta'_j)_{1,q_2}\end{matrix};x^{\lambda},y^{\mu}\right), \tag{2.4}$$

Where $\lambda_1,\ldots,\lambda_r$ λ,μ are positive real numbers and $c_1,\ldots,c_r,c,d$ are complex numbers. The variables are complex.

Proof of (2.4) : Taking (2.3), expressing involved generalized Lauricella function of two-variables on RHS in terms of series, collecting the powers of $(1-t),(1-h)$,using the binomial expansion, the relations [5,eq.(3) p.23 ,exp. 8 p.32 , Lemma 5 p.22] and finally comparing the coefficients of $t^n h^m$,we obtain the required result in view of relation (1.3) .

Similarly the result (2.2) can easily be established .

Reductions :

(a) In (2.4), replacing $\alpha's,\beta's,\gamma's,\delta's,\lambda,\mu$ by unity and $c=d=0$ with putting the following, the Sister Celine's polynomials of two variables reduced into :

(i) $m=0,p=q=0,$ we obtain the well-known Sister Celine's polynomials of single variable (1.1) and (1.2).

(ii)

$p=1,a_1=-n;q=0;p_1=3,c'_1=1+\alpha+\beta+n,c'_2=\frac{1}{2},c'_3=1;q_1=3,d'_1=1+\alpha,d'_2=-n,d'_3=n+1;$

$p_2=3,c''_1=1+\alpha'+\beta'+n,c''_2=\frac{1}{2},c''_3=1;q_2=3,d''_1=1+\alpha',d''_2=-m,d''_3=m+1;$

$x\to\frac{1-x}{2},y\to\frac{1-y}{2}$;

we obtain Jacobi polynomials of two variables [9]. :

$$f_{n,m}\left(\begin{matrix}(-n:1,1):(1+\alpha+\beta+n,1),\left(\frac{1}{2},1\right),(1,1);(1+\alpha'+\beta'+n,1),\left(\frac{1}{2},1\right),(1,1)\\ -:\quad(1+\alpha,1),(-n,1),(n+1,1);(1+\alpha',1),(-m,1),(m+1,1)\end{matrix};\frac{1-x}{2},\frac{1-y}{2}\right)=\frac{(n!)^2}{(1+\alpha)_n(1+\alpha')_n}P_n^{(\alpha,\beta;\alpha',\beta')}(x,y), \tag{2.5}$$

(iii) $p=0;q=0;p_1=3,c'_1=1+\alpha+\beta+n,c'_2=\frac{1}{2},c'_3=1;q_1=2,d'_1=n+1,d'_2=1+\alpha;$

$p_2=3;c''_1=1+\alpha'+\beta'+m,c''_2=\frac{1}{2},c''_3=1;q_2=2,d''_1=m+1,d''_2=1+\alpha';x\to\frac{1-x}{2},y\to\frac{1-y}{2}$; we get the product of two Jacobi polynomials of single variables [5].

$$f_{n,m}\left(\begin{matrix}-:(1+\alpha+\beta+n,1),\left(\frac{1}{2},1\right),(1,1);(1+\alpha'+\beta'+m,1),\left(\frac{1}{2},1\right),(1,1)\\ -:\quad(1+\alpha,1),(n+1,1);(1+\alpha',1),(m+1,1)\end{matrix};\frac{1-x}{2},\frac{1-y}{2}\right)=\frac{n!m!}{(1+\alpha)_n(1+\alpha')_n}P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha',\beta')}(y), \tag{2.6}$$

(iv) Same substitutions as in (ii) with
$\alpha=\beta=\alpha'=\beta'=\nu-\frac{1}{2};\alpha=\beta=\alpha'=\beta'=0;\alpha=\beta=\alpha'=\beta'=-\frac{1}{2};\ \alpha=\beta=\alpha'=\beta'=\frac{1}{2}$, we can easily obtain two variables polynomials – Gegenbauer, Legendre, Chebyshev of first kind and second kind [9] .

(v) $p=1,a_1=-n;q=0;p_1=3,c_1'=1+\alpha+n,c_2'=\frac{1}{2},c_3'=1;q_1=2,d'=-n,d_2''=n+1;p_2=3,c_1''=1+\beta+m,$
$c_2''=\frac{1}{2},c_3''=1;q_2=2,d_1''=-m,d_2''=m+1;x\to\left(-\frac{x}{2}\right),y\to\left(-\frac{y}{2}\right)$ and
$p=0;q=0;p_1=3,c_1'=1+\alpha+n$
$c_2'=\frac{1}{2},c_3'=1;q_1=1,d_1'=n+1;p_2=3,c_1''=1+\beta+m,c_2''=\frac{1}{2},c_3''=1;q_2=1,d_1''=m+1;x\to\left(-\frac{x}{2}\right),y\to\left(-\frac{y}{2}\right)$
,
we get respectively Bessel polynomials of two variables [10] and product of Bessel polynomials of single variable [5].

$$f_{n,m}\left(\begin{matrix}(-n:1,1):(1+\alpha+n,1),\left(\frac{1}{2},1\right),(1,1);(1+\beta+m,1),\left(\frac{1}{2},1\right),(1,1)\\ —:(-n,1),(n+1,1);(-m,1);(1,1),(m+1,1)\end{matrix};-\frac{x}{2},-\frac{y}{2}\right)=y_{n,m}^{(\alpha,\beta)}(x,y)\ , \tag{2.7}$$

$$f_{n,m}\left(\begin{matrix}—:(1+\alpha+n,1),\left(\frac{1}{2},1\right),(1,1);(1+\beta+m,1),\left(\frac{1}{2},1\right),(1,1)\\ —:\quad(n+1,1);\quad(m+1,1)\end{matrix};-\frac{x}{2},-\frac{y}{2}\right)=y_n^{(\alpha)}(x).y_m^{(\beta)}(y)\ , \tag{2.8}$$

(vi) $p=1,a_1=-n;q=0;p_1=2,c_1'=\frac{1}{2},c_2'=1;q_1=3,d_1'=-n,d_2'=n+1,d_3'=1+\alpha,;p_2=2,c_1''=\frac{1}{2},c_2''=1;$
$q_2=3,d_1''=-m,d_2''=m+1,d_3''=1+\alpha'$ and
$p=0;q=0;p_1=2,c_1'=\frac{1}{2},c_2'=1;q_1=2,d_1'=n+1,d_2'=1+\alpha;$
$p_2=2,c_1''=\frac{1}{2},c_2''=1;q_2=2,d_1''=m+1,d_2''=1+\alpha'$; we obtain respectively Laguerre polynomials of two variables [9] and product of Laguerre polynomials of single variable [5] :

$$f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1)\quad;\left(\frac{1}{2},1\right),(1,1)\quad;\\ —:(-n,1),(n+1,1),(1+\alpha,1);(-m,1),(m+1,1),(1+\alpha',1);\end{matrix}\ x,y\right)=\frac{(n!)^2}{(1+\alpha)_n(1+\alpha')_n}.L_n^{(\alpha,\alpha')}(x,y)\ , \tag{2.9}$$

$$f_{n,m}\left(\begin{matrix}—:\left(\frac{1}{2},1\right),(1,1)\quad;\left(\frac{1}{2},1\right),(1,1)\\ —:(n+1,1),(1+\alpha,1);(m+1,1),(1+\alpha',1)\end{matrix};x,y\right)=\frac{n!m!}{(1+\alpha)_n(1+\alpha')_m}.L_n^{(\alpha)}(x)L_m^{(\alpha')}(y)\ , \tag{2.10}$$

(vii)
$p=1,a_1=-n;q=0;p_1=3,c_1'=\frac{1}{2},c_2'=1,c_3'=-n+\frac{1}{2},q_1=2,d_1'=-n,d_2'=n+1,p_2=3,c_1''=\frac{1}{2},c_2''=1,$
$c_3''=-m+\frac{1}{2};q_2=2,d_1''=-m,d_2''=m+1;x\to\left(-\frac{1}{x^2}\right),y\to\left(-\frac{1}{y^2}\right)$ and
$p=0;q=0;p_1=4,c_1'=\frac{1}{2},c_2'=1,$
$c_3'=-\frac{n}{2},c_4'=\frac{-n+1}{2};q_1=2,d_1'=-n,d_2'=n+1;p_2=4,c_1''=\frac{1}{2},c_2''=1;c_3''=-\frac{m}{2},c_4''=\frac{-m+1}{2};q_2=2,d_1''=-m,$
$d_2''=m+1;x\to\left(-\frac{1}{x^2}\right),y\to\left(-\frac{1}{y^2}\right)$ we get respectively Hermite polynomials of two variables [9] and product of Hermite polynomials of single variables [5] :

$$f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1),(-n+\frac{1}{2},1);\left(\frac{1}{2},1\right),(1,1),\left(-m+\frac{1}{2},1\right);-\frac{1}{x^2},-\frac{1}{y^2}\\ -:(-n,1),(n+1,1)\qquad;(-m,1),(m+1,1)\end{matrix}\right)=H_{2n,2m}(x,y)\,, \tag{2.11}$$

$$f_{n,m}\left(\begin{matrix}-:\left(\frac{1}{2},1\right),(1,1),\left(-\frac{n}{2},1\right)\left(\frac{-n+1}{2},1\right);\left(\frac{1}{2},1\right)\left(-\frac{m}{2},1\right)\left(\frac{-m+1}{2},1\right);-\frac{1}{x^2},-\frac{1}{y^2}\\ -:(-n,1),(n+1,1)\qquad;(-m,1),(m+1,1)\end{matrix}\right)=(2x)^{-n}(2y)^{-m}H_n(x)H_m(y)\,, \tag{2.12}$$

(viii) $p=1,a_1=-n;q=0;p_1=3,c_1'=\frac{1}{2},c_2'=1,c_3'=2v_1+n;q_1=4,d_1'=-n,d_2'=n+1,d_3'=v_1+\frac{1}{2},d_4'=b_1+1;$ $p_2=3,c_1''=\frac{1}{2},c_2''=1,c_3''=2v_2+m;q_2=4,d_1''=-m,d_2''=m+1,d_3''=v_2+\frac{1}{2},d_4''=b_2+1$ and $p=0;q=0;$ $p_1=3,c_1'=\frac{1}{2},c_2'=1,c_3'=2v_1+n;q_1=3,d_1'=v_1+\frac{1}{2},d_2'=b_1+1,d_3'=n+1;p_2=3,c_1''=\frac{1}{2},c_2''=1,c_3''=2v_2+m;$ $q_2=3,d_1''=v_2+\frac{1}{2},d_2''=b_2+1,d_3''=n+1;$ we get respectively generalized Bateman's polynomials of two variables and product of generalized Bateman's polynomials of single variables [5, eq (9), p.286] :

$$f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1),(2v_1+n,1)\qquad;\left(\frac{1}{2},1\right),(1,1),(2v_2+m,1)\\ -:(-n,1),(n+1,1),\left(v_1+\frac{1}{2},1\right),(b_1+1,1);(-m,1),(m+1,1),\left(v_2+\frac{1}{2},1\right),(b_2+1,1)\end{matrix};x,y\right)=Z_{n,m}(x,y)\,, \tag{2.13}$$

$$f_{n,m}\left(\begin{matrix}-:(2v_1+n,1),(\frac{1}{2},1),(1,1)\qquad;(2v_2+m,1)(\frac{1}{2},1),(1,1)\\ -:(v_1+\frac{1}{2},1),(b_1+1,1),(n+1,1);(v_2+\frac{1}{2},1),(b_2+1,1),(m+1,1)\end{matrix};x,y\right)=Z_n(x)Z_m(y)\,, \tag{2.14}$$

It is obvious that these generalized Bateman's polynomials reduces to simple Bateman's polynomial [5, eq. (2), p.285] by putting $v_1=v_2=\frac{1}{2},b_1=b_2=0$.

(ix) $p=1,a_1=-n;q=0;p_1=2,c_1'=\frac{1}{2},c_2'=\xi_1;q_1=2,d_1'=-n,d_2'=\lambda_1;p_2=2,c_1''=\frac{1}{2},c_2''=\xi_2;q_2=2,$ $d_1''=-m,,d_2''=\lambda_2;x=v_1,y=v_2$ and $p=0,q=0;p_1=2,c_1'=\frac{1}{2},c_2'=\xi_1;q_1=1,d_1'=\lambda_1;p_2=2,c_1''=\frac{1}{2},$ $c_2''=\xi_2;q_2=1,d_1''=\lambda_2;x=v_1,y=v_2;$ we obtain respectively Rice's polynomials of two variables and product of Rice's polynomials of single variables [5, eq(1), p.287] :

$$f_{n,m}\left(\begin{matrix}(-n:1,1);(\frac{1}{2},1),(\xi_1,1);(\frac{1}{2},1),(\xi_2,1)\\ -:(-n,1),(\lambda_1,1);(-m,1),(\lambda_2,1)\end{matrix};v_1,v_2\right)=H_{n,m}(\xi_1,\lambda_1,v_1).H_{n,m}(\xi_2,\lambda_2,v_2)\,, \tag{2.15}$$

$$f_{n,m}(v_1,v_2)=f_{n,m}\left(\begin{matrix}-:(\frac{1}{2},1),(\xi_1,1);(\frac{1}{2},1),(\xi_2,1)\\ -:(\lambda_1,1)\qquad;(\lambda_2,1)\end{matrix};v_1,v_2\right)=H_n(\xi_1,\lambda_1,v_1).H_m(\xi_2,\lambda_2,v_2)\,, \tag{2.16}$$

(b) Now, Here we shall discuss few discrete polynomials – Hahn, Krawtchouk, Mixner, Poisson – Charlier, and Pasternak's polynomials. First we define these polynomials for two variables then show that their limiting cases with the use for relation $\lim\limits_{z\to\infty}\frac{(z)_r}{z^r}=1$ is Celine's polynomials of two variables

Hahn Polynomials

$$Q_{n,m}(x;\alpha,\beta,N;y,\alpha',\beta',N')=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(-n)_{m_1+m_2}(1+\alpha+\beta+n)_{m_1}(-x)_{m_1}(1+\alpha'+\beta'+m)_{m_2}(-y)_{m_2}}{m_1!\,m_2!(1+\alpha)_{m_1}(-N)_{m_1}(1+\alpha')_{m_2}(-N')_{m_2}}\,,$$

It reduces to Hahn polynomials of Single variable [6, p.541], when y=0 i.e.

$$Q_n(x;\alpha,\beta,N)={}_3F_2\left(\begin{matrix}-n,1+\alpha+\beta+n,-x\\1+\alpha,-N\end{matrix};1\right),$$

$$\lim_{N,N'\to\infty} Q_{n,m}(xN;\alpha,\beta,N:yN';\alpha',\beta',N')$$

$$=f_{n,m}\left(\begin{matrix}(-n:1,1):(1+\alpha+\beta+n,1),\left(\frac{1}{2},1\right),(1,1);(1+\alpha'+\beta'+m,1),\left(\frac{1}{2},1\right),(1,1)\\ —:\quad(1+\alpha,1),(-n,1),(n+1,1);(1+\alpha'+,1),(-m,1),(m+1,1)\end{matrix};x,y\right) \qquad (2.17)$$

Krawtchouk Polynomials

$$K_n(x;p,N:y;p',N')=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(-n)_{m_1+m_2}(-x)_{m_1}(-y)_{m_2}\left(\frac{1}{p}\right)^{m_1}\left(\frac{1}{p'}\right)^{m_2}}{m_1!\ m_2!\ (-N)_{m_1}(-N')_{m_2}},$$

It reduces to Krawtchouk polynomials of single variables [6, p.54] , when y = 0.

$$K_n(x;p,n)={}_2F_1\left(\begin{matrix}-n,-x\\-n\end{matrix};\frac{1}{p}\right),$$

$$\lim_{N,N'\to\infty} K_n(Nx;p,N:N'y;p',N')=f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1)\quad;\left(\frac{1}{2},1\right),(1,1)\\ —:(-n,1),(n+1,1);(-m,1),(m+1,1)\end{matrix};\frac{x}{p},\frac{y}{p'}\right), \qquad (2.18)$$

Meixner Polynomials

$$M_n(x;\beta,c:y;\beta',c')=(\beta)_n(\beta')_n\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(-n)_{m_1+m_2}(-x)_{m_1}(-y)_{m_2}}{m_1!\ m_2!\ (-\beta)_{m_1}(-\beta')_{m_2}}\left(1-\frac{1}{c}\right)^{m_1}\left(1-\frac{1}{c'}\right)^{m_2},$$

It reduces to Meixner polynomials of single variable [6; p.542] , when y = 0

$$M_n(x;\beta,c)=(\beta)_n\ {}_2F_1\left(\begin{matrix}-n,-x\\\beta\end{matrix};1-\frac{1}{c}\right),$$

$$\lim_{c,c''\to\infty} M_n(-cx;\beta,\tfrac{c}{c-1}:-c'y;\beta',\tfrac{c'}{c'-1})$$

$$=(\beta)_n(\beta'_n)f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1);\left(\frac{1}{2},1\right),(1,1)\\ —:(-n,1),(n+1,1),(\beta,1);(-m,1),(m+1,1),(\beta',1)\end{matrix};x,y\right), \qquad (2.19)$$

Poisson-Charlier Polynomials

$$C_n(x;\alpha:y;\alpha')=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(-n)_{m_1+m_2}(-x)_{m_1}(-y)_{m_2}}{m_1!\ m_2!}\left(-\tfrac{1}{\alpha}\right)^{m_1}\left(-\tfrac{1}{\alpha'}\right)^{m_2},$$

It reduces to Poisson-Charlier Polynomials of single variables [6, p.542] , when y = 0

$$C_n(x;\alpha)={}_2F_0\left(\begin{matrix}-n,-x\\—\end{matrix};-\frac{1}{\alpha}\right),$$

$$\lim_{\alpha,\alpha'\to\infty} C_n(\alpha x;\tfrac{1}{\alpha}:\alpha'y;\tfrac{1}{\alpha'})=f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1)\quad;\left(\frac{1}{2},1\right),(1,1)\\ —:(-n,1),(n+1,1);(-m,1),(m+1,1)\end{matrix};1,1\right), \qquad (2.20)$$

Pasternak Polynomials

$$F_{n,m}^{\lambda,\lambda'}(z,z')=\sum_{m_1=0}\sum_{m_2=0}\frac{(-n)_{m_1+m_2}(n+1)_{m_1}(m+1)_{m_2}\left(\frac{z+1+\lambda}{2}\right)_{m_1}\left(\frac{z'+1+\lambda'}{2}\right)_{m_2}}{m_1!\ m_2!\ (1)_{m_1}(1)_{m_2}(1+\lambda)_{m_1}(1+\lambda')_{m_2}},$$

It reduces to Pasternak's polynomials of single variable [5]. when $z'+1+\lambda'=0$.

$$F_n^{\lambda}(z) = {}_3F_2\left(\begin{matrix} -n, n+1, \frac{z+1+\lambda}{2} \\ 1, 1+\lambda \end{matrix} ;1\right),$$

$$\lim_{\lambda,\lambda'\to\infty} F_{n,m}^{\lambda z,\lambda' z'}(z,z') = f_{n,m}\left(\begin{matrix} (-n:1,1):\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right) \\ -:(-n,1);(-m,1) \end{matrix} ;\frac{1}{2},\frac{1}{2}\right), \tag{2.21}$$

It is worth to note here that Pasternak's polynomial is a generalization of Bateman's polynomial $F_n(z)$ [5,eq (1), P.289]. or we can say when $\lambda=0$ the Pasternak's polynomials reduces to Bateman's polynomials $F_n(z)$.

3. INTEGRAL REPRESENTION

Here, we shall establish few integral representations of Sister Celine's polynomials of two variables defined by equation (2.4) of proceeding section with $\lambda=\mu=1, c=d=0$. One the these, integral representations (3.1) is used in next section in obtaining the generating functions.

$$f_{n,m}\left(\begin{matrix} a:c',(1+n+\lambda_1,1);c'',(1+m+\lambda_2,1) \\ b:d' \quad ;d'' \end{matrix};x,y\right)$$

$$= \frac{(\mu_1)^{1+n+\lambda_1}(\mu_2)^{1+m+\lambda_2}}{\Gamma(1+n+\lambda_1)\,\Gamma(1+m+\lambda_2)}\int_0^\infty\int_0^\infty t^{n+\lambda_1}h^{m+\lambda_2}\exp(-\mu_1 t-\mu_2 h)\, f_{n,m}(\mu_1 xt,\mu_2 yh)\,dt\,dh, \tag{3.1}$$

$$f_{n,m}(x,y) = \frac{(\mu_1)^{1+n+\lambda_1}(\mu_2)^{1+m+\lambda_2}}{\Gamma(1+n+\lambda_1)\,\Gamma(1+m+\lambda_2)}\int_0^\infty\int_0^\infty t^{n+\lambda_1}h^{m+\lambda_2}\exp(-\mu_1 t-\mu_2 h)$$

$$\times f_{n,m}\left(\begin{matrix} a: \quad c'; \quad c'' \\ b:(1+n+\lambda_1,1),d';(1+m+\lambda_2,1),d'' \end{matrix};\mu_1 xt,\mu_2 yh\right)dt\,dh, \tag{3.2}$$

$$f_{n,m}(t,h) = \frac{t^{-\lambda_1-\mu_1-1}h^{-\lambda_2-\mu_2-1}}{B(\lambda_1+1,\mu_1+1)\,B(\lambda_2+1,\mu_2+1)}\int_0^t\int_0^h x^{\lambda_1}y^{\lambda_2}(t-x)^{\mu_1}(h-y)^{\mu_2}$$

$$\times f_{n,m}\left(\begin{matrix} a: \quad c'; \quad c''; \\ b:(1+\lambda_1,1),d';\ (\lambda_2+1,1),d''; \end{matrix}\ x,y\right)dx\,dy, \tag{3.3}$$

$$f_{n,m}(1,1) = \frac{(2t)^{1+\lambda_1+\mu_1}(2h)^{1+\lambda_2+\mu_2}}{B(1+\lambda_1,1+\mu_1)\,B(1+\lambda_2,1+\mu_2)}\int_{-t}^t\int_{-h}^h (t+x)^{\lambda_1}(t-x)^{\mu_1}(h+y)^{\lambda_2}(h-y)^{\mu_2}$$

$$\times f_{n,m}\left(\begin{matrix} a:(\lambda_1+\mu_1+2,1),c';(\lambda_2+\mu_2+2,1),c''; \\ b: \quad (\lambda_1+1,1),d'; \quad (\lambda_2+1,1),d''; \end{matrix}\ \frac{t+x}{2t},\frac{h+y}{2h}\right)dy, \tag{3.4}$$

$$f_{n,m}(4x,4y) = \frac{1}{B(\lambda_1,\mu_1)\,B(\lambda_2,\mu_2)}\int_0^1\int_0^1 t^{\lambda_1-1}h^{\lambda_2-1}(1-t)^{\mu_1-1}(1-h)^{\mu_2-1}$$

$$\times f_{n,m}\left(\begin{matrix} a:(\lambda_1+\mu_1,r_1+s_1),c';(\lambda_2+\mu_2,r_2+s_2),c''; \\ b:(\lambda_1,r_1),(\mu_1,s_1),d';(\lambda_2,r_2),(\mu_2,s_2),d''; \end{matrix}\ 4xt^{r_1}(1-t)^{s_1},4yh^{r_2}(1-h)^{s_2}\right)dt\,dh, \tag{3.5}$$

$$f_{n,m}(x,y) = \frac{(-1)^{n+m+1}}{4\sin\alpha\pi\,\sin\beta\pi\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+m)}\int_\infty^{(0+)}\int_\infty^{(0+)}(-t)^{\alpha+n}(-h)^{\beta+m}\exp(-t-h)$$

$$\times f_{n,m}\left(\begin{matrix} a: & c'; & c'; \\ b:(1+\alpha+n,1),d';(1+\beta+m,1),d''; \end{matrix} xt, yt\right) dt\, dh\ , \tag{3.6}$$

$$f_{n,m}(x,y) = \frac{(-1)^{n+m+1} \sin\alpha\pi \ \sin\beta\pi \ \Gamma(1+\alpha+n)\ \Gamma(1+\beta+m)}{(\pi\sqrt{2})^4} \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} t^{\alpha+n} h^{\beta+m} \exp(t+h)$$

$$\times f_{n,m}\left(\begin{matrix} a:(1+\alpha+n,1),c';(1+\beta+m,1),c''; \\ b: \quad d'; \quad d''; \end{matrix} -xt,-yt\right) dt\, dh\ , \tag{3.7}$$

provided all the above results exist. The result (3.6) and (3.7) are contour integrals and (3.7) is Schlafli's type contour integral.

Taking RHS of (3.1), expressing Celine's polynomials of two variables interms of series, the definition (2.4) with $\lambda = \mu = 1, c = d = 0$; changing the order of summation and integration (which obviously justificable) and evaluating the inner integrals with the help of a known relation [2; eq (5), p.1]. Finally once again using the definition (2.4), we get the required result (3.1).

Similarly the results (3.2) to (3.5) can easily be established .

Proceeding as above and using relations [2;eq(4) p.14 , eq(2) p.13], we obtain the results (3.6) & (3.7) .

Reductions :

We reduce some results from above relations ,which are all new results.

(i) In (3.2), putting $\lambda_1 = -n-\frac{1}{2}, \lambda_2 = -m-\frac{1}{2}, \mu_1 = \mu_2 = 1$.

In(3.4),taking $r_1 = s_1 = r_2 = s_2 = 1; r_1 = s_1 = r_2 = s_2 = 1, \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1; s_1 = s_2 = 0, \mu_1 = \mu_2 = 1,$ $4x = z_1, 4y = z_2; s_1 = s_2 = 0, \mu_1 = \mu_1 = \lambda_1 = \lambda_2 = 1, 4x = z_1, 4y = z_2$; we get the following integral representations of Celine's polynomials of two-variables :

$$f_{n,m}(x,y) = \frac{\sqrt{\mu_1 \mu_2}}{\pi} \int_0^{\infty} \int_0^{\infty} (th)^{-\frac{1}{2}} e^{-t-h} f_{n,m}\left(\begin{matrix} a:c' \quad ;c'' \\ b:\left(\frac{1}{2},1\right),d';\left(\frac{1}{2},1\right),d''; \end{matrix} xt, yt\right) dtdh\ , \tag{3.8}$$

$$f_{n,m}(x,y) = \frac{1}{B(\lambda_1,\mu_1)\ B(\lambda_2,\mu_2)} \int_0^1 \int_0^1 t^{\lambda_1-1} h^{\lambda_2-1} (1-t)^{\mu_1-1} (1-h)^{\mu_2-1}$$

$$\times f_{n,m}\left(\begin{matrix} a:c',\left(\frac{\lambda_1+\mu_1}{2},1\right),\left(\frac{\lambda_2+\mu_2}{2},1\right);c'',\left(\frac{\lambda_1+\mu_1}{2},1\right),\left(\frac{\lambda_2+\mu_2}{2},1\right); \\ b:d',(\lambda_1,1),(\mu_1,1) \quad ;d'',(\lambda_2,1),(\mu_2,1) \quad ; \end{matrix} 4xt(1-t),4yh(1-h)\right) dtdh\ , \tag{3.9}$$

$$f_{n,m}(x,y) = \int_0^1 \int_0^1 f_{n,m}\left(\begin{matrix} a:c',\left(\frac{3}{2},1\right);c'',\left(\frac{3}{2},1\right); \\ b:d',(1,1);d'',(1,1) \end{matrix} 4xt(1-t),4yh(1-h)\right) dtdh\ , \tag{3.10}$$

$$f_{n,m}(z_1,z_2) = (\lambda_1 \lambda_2) \int_0^1 \int_0^1 t^{\lambda_1-1} t^{\lambda_2-1} f_{n,m}\left(\begin{matrix} a:c',(\lambda_1+1,1);c'',(\lambda_2+1,1); \\ b:d',(\lambda_1,1) \quad ;d'',(\lambda_2,1) \quad ; \end{matrix} z_1 t, z_2 h\right) dtdh\ , \tag{3.11}$$

$$f_{n,m}(z_1,z_2) = \int_0^1 \int_0^1 f_{n,m}\left(\begin{matrix} a:c',(2,1);c'',(2,1); \\ b:d',(1,1);d'',(1,1); \end{matrix} z_1 t, z_2 h\right) dtdh\ , \tag{3.12}$$

(ii) In(3.1),taking

$p=1, a_1=-n; q=0; p_1=3, c_1'=1, c_2'=1, c_3'=1+\alpha+n; q_1=2, d_1'=-n, d_2'=n+1; p_2=3, c_1''=\frac{1}{2}, c_2''=1,$
$c_3''=1+\beta=m; q_2=2, d_1''=-m, d_2''=1+m$ and $p=q=0; p_1=3, c_1'=\frac{1}{2}, c_2'=1, c_3'=1+\alpha+n; q_1=1, d_1'=1+n;$
$p_2=3, c_1''=\frac{1}{2}, c_2''=1, c_3''=1+\beta+m; q_2=1, d_1''=1+n$

the Celine's polynomials involved, we get the following relations :

$$f_{n,m}\left(\begin{matrix}(-n:1,1):(1+\alpha+n,1),(\tfrac{1}{2},1),(1,1);(1+\beta+m,1),(\tfrac{1}{2},1),(1,1); \\ —:(-n,1),(n+1,1) \qquad ;(-m,1),(m+1,1) \qquad ;\end{matrix} -x-y\right)$$

$$=\frac{(-1)^{1+n+m}}{4\sin\alpha\pi\ \sin\beta\pi\ \ \Gamma(1+\alpha+n)\ \Gamma(1+\beta+m)}\int_{\infty}^{(0+)}\int_{\infty}^{(0+)}(-t)^{\alpha+n}(-h)^{\beta+m}\exp(-t-h)\,(1+xt+yh)^n\,dtdh\ , \quad (3.13)$$

$$f_{n,m}\left(\begin{matrix}—:(1+\alpha+\mathrm{n},1),(\tfrac{1}{2},1),(1,1);(1+\beta+m,1),\left(\frac{1}{2},1\right),(1,1); \\ —:(n+1,1) \qquad ;(m+1,1) \qquad ;\end{matrix} -x-y\right)$$

$$=\frac{4}{4\sin\alpha\pi\ \ \sin\beta\pi\ \ \Gamma(1+\alpha+n)\ \Gamma(1+\beta+m)}\times\int_{\infty}^{(0+)}\int_{\infty}^{(0+)}(-t)^{\alpha+n}(-h)^{\beta+m}\exp(-t-h)(1+xt)^n(1+yh)^m\,dtdh\ , \quad (3.14)$$

(iii) In(3.7),putting

$p=1, a_1=-n; q=0; p_1=2, c_1'=\frac{1}{2}, c_2'=1; q_1=3, d_1'=-n, d_2'=n+1, d_3'=1+\alpha+n;$
$p_2=2, c_1''=\frac{1}{2}, c_2''=1; q_2=3, d_1''=-m, d_2''=m+1, d_3''=1+\beta+m$ and $p=q=0; p_1=2, c_1'=\frac{1}{2}, c_1''=1; q_1=2,$
$d_1'=n+1, d_2'=1+\alpha+n; p_2=2, c_1''=\frac{1}{2}, c_2''=1; q_2=2, d_1''=m+1, d_2''=1+\beta+m$ in the Celine's polynomials involved, we can obtain another new results

4. GENERATING FUNCTIONS

In this section, we shall establish some single and double generating function relation of Celine's polynomials interms of exponential & binomial expansion, Celine's polynomials. Also few generating function are proved with the help of integral (3.1) established earlier in section 3, interms of Jacobi; Laguerre polynomials and Bessel functions. Further we reduce generating function interms of discrete polynomials such as Hahn, Krawtchouk and Meixner and trigonometric function.

$$(1-2yt)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1-2yt}}\right]^{\beta}\left[1-\frac{xt}{1+\sqrt{1-2yt}}\right]^{-\alpha}\exp\left(\frac{2t}{1+\sqrt{1-2yt}}\right)$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{n!}f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1),(\alpha,1);\left(\frac{1}{2},1\right),(1,1),(\beta+n+1,1); \\ —:(-n,1),(n+1,1) \quad ;(-n,1),(n+1,1) \qquad ;\end{matrix} -\frac{x}{2},-\frac{y}{2}\right), \quad (4.1)$$

$$\left[1-\frac{xt}{2+yt}\right]^{-\alpha}\left(1+\frac{yt}{2}\right)^{\beta}\exp\left(\frac{2t}{2+yt}\right)$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{n!}f_{n,m}\left(\begin{matrix}(-n:1,1):(\tfrac{1}{2},1),(1,1),(\alpha,1);(\tfrac{1}{2},1),(1,1),(1-n+\beta,1); \\ —:(-n,1),(n+1,1) \quad ;(-n,1),(n+1,1) \qquad ;\end{matrix} -\frac{x}{2},-\frac{y}{2}\right), \quad (4.2)$$

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-\lambda)^{k_1}(-\mu)^{k_2}f_{n,m}\left(\begin{matrix}a:c',(1+k_1,1);c'',(1+k_2,1); \\ b:d',(1,1) \quad ;d'',(1,1) \quad ;\end{matrix} x,y\right)=\frac{1}{(1+\lambda)}f_{n,m}\left(\frac{x}{1+\lambda},\frac{y}{1+\mu}\right), \quad (4.3)$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-n)_{r_1+r_2}(1+\alpha_1+\beta_1+n)_{r_2}(1+\alpha_2+\beta_2+n)_{r_1}}{r_1!\,r_2!}f_{n,m}\left(\begin{matrix}a:c',(1+\alpha_1+r_2,1);c'',(1+\alpha_2+r_1,1);\\ b:d' \qquad ;d'' \qquad ;\end{matrix}x,y\right)$$

$$=\frac{(\mu_1)^{1+\alpha_1}(\mu_2)^{1+\alpha_2}(n!)^2}{\Gamma(1+\alpha_1+n)\,\Gamma(1+\alpha_2+n)}\int_0^{\infty}\int_0^{\infty}t^{\alpha_1}h^{\alpha_2}\exp(-\mu_1 t-\mu_2 h)f_{n,m}(\mu_1 xt,\mu_2 yh)\times P_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(1-2\mu_1 t,1-2\mu_2 h)\,dtdh, \tag{4.4}$$

$$\lim_{\beta_1,\beta_2\to\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-n)_{r_1+r_2}(1+\alpha_1+\beta_1+n)_{r_2}(1+\alpha_2+\beta_2+n)_{r_1}}{r_1!\,r_2!\,(\beta_1)^{r_2}(\beta_2)^{r_1}}$$

$$\times f_{n,m}\left(\begin{matrix}a:c',(1+\alpha_1+r_2,1);c'',(1+\alpha_2+r_1,1);\\ b:d',(1,1) \qquad ;d'',(1,1) \qquad ;\end{matrix}x,y\right)$$

$$=\frac{(\mu_1)^{1+\alpha_1}(\mu_2)^{1+\alpha_2}(n!)^2}{\Gamma(1+\alpha_1+n)\,\Gamma(1+\alpha_2+n)}\int_0^{\infty}\int_0^{\infty}t^{\alpha_1}h^{\alpha_2}\exp(-\mu_1 t-\mu_2 h)f_{n,m}(\mu_1 xt,\mu_2 yh)\,L_n^{(\alpha_1,\alpha_2)}(\mu_1 t,\mu_2 h)\,dtdh, \tag{4.5}$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-\lambda)^{r_1}(-\mu)^{r_2}}{r_1!\,r_2!}f_{n,m}\left(\begin{matrix}a:c',(1+n+r_1,1);c'',(1+m+r_2,1);\\ b:d' \qquad ;d'' \qquad ;\end{matrix}x,y\right)$$

$$=(\mu_1)^{1+n}(\mu_2)^{1+m}\int_0^{\infty}\int_0^{\infty}t^n h^m\exp(-\mu_1 t-\mu_2 h)f_{n,m}(\mu_1 xt,\mu_2 yt)J_{n_1,m}\left(2\sqrt{\lambda\,\mu_1 t},2\sqrt{\mu\,\mu_2 t}\right)dtdh, \tag{4.6}$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-n)_{r_1+r_2}(1+\alpha_1+\beta_1+n)_{r_2}(1+\alpha_2+\beta_2+n)_{r_1}}{r_1!\,r_2!}f_{n,m}\left(\begin{matrix}a:c',(1+\alpha_1+r_2,1);c'',(1+\alpha_2+r_1,1);\\ b:d' \qquad ;d'' \qquad ;\end{matrix}x,y\right)$$

$$=\lim_{N_1,N_2\to\infty}\frac{(\mu_1)^{1+\alpha_1}(\mu_2)^{1+\alpha_2}}{\alpha_1!\;\alpha_2!}\int_0^{\infty}\int_0^{\infty}t^{\alpha_1}h^{\alpha_2}\exp(-\mu_1 t-\mu_2 h)$$

$$\times\, f_{n,m}(\mu_1 xt,\mu_2 yh)\;Q_n(\mu_1 tN_1;\alpha_1,\beta_1,N_1:\mu_2 hN_2;\alpha_2,\beta_2,N_2)\;dtdh, \tag{4.7}$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-n)_{r_1+r_2}(-x)_{r_2}(-y)_{r_1}}{r_1!\,r_2!}f_{n,m}\left(\begin{matrix}a:c',(r_2-N_1,1);c'',(r_1-N_2,1)\\ b:d' \qquad ;d''\end{matrix}x,y\right)$$

$$=\frac{(\mu_1)^{-N_1}(\mu_2)^{-N_2}}{\Gamma(-N_1)\,\Gamma(-N_2)}\int_0^{\infty}\int_0^{\infty}(p_1)^{N_1-1}(p_2)^{N_2-1}\exp\left(-\frac{\mu_1}{p_1}-\frac{\mu_2}{p_2}\right)f_{n,m}\left(\frac{\mu_1 x}{p_1},\frac{\mu_2 y}{p_2}\right)\times K_n\left(\frac{\mu_1}{p_1};p_1,N_1:\frac{\mu_2}{p_2};p_2,N_2\right)dp_1\;dp_2, \tag{4.8}$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-n)_{r_1+r_2}(-x)_{r_2}(-y)_{r_1}}{r_1!\,r_2!}f_{n,m}\left(\begin{matrix}a:c',(\beta_1+r_2,1);c'',(\beta_2+r_1,1)\\ b:d' \qquad ;d''\end{matrix}x,y\right)$$

$$=\frac{1}{(\mu_1,\mu_2)\,\Gamma(\beta_1+n)\,\Gamma(\beta_2+n)}\int_0^1\int_0^1(c_1c_2)^{-2}\left(1-\frac{1}{c_1}\right)^{\beta_1}\left(1-\frac{1}{c_2}\right)^{\beta_2}\exp\left(-2+\frac{1}{c_1}+\frac{1}{c_2}\right)$$

$$\times f_{n,m}\left(x\left(1-\tfrac{1}{c_1}\right),y\left(1-\tfrac{1}{c_2}\right)\right)M_n(x:\beta_1,c_1:y;\beta_2,c_2)\,dc_1\;dc_2, \tag{4.9}$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-\lambda)^{r_1}(-\mu)^{r_2}}{r_1!\,r_2!}f_{1/2,1/2}\left(\begin{matrix}a:c',\left(\frac{3}{2}+r_1,1\right);c'',\left(\frac{3}{2}+r_1,1\right);x,y\\ b:d' \qquad ;d''\end{matrix}\right)$$

$$=\frac{\mu_1\mu_2}{\pi\sqrt{\lambda\,\mu}}\int_0^{\infty}\int_0^{\infty}\exp(-\mu_1 t-\mu_2 t)f_{1/2,1/2}(\mu_1 xt,\mu_2 yh)\sin 2\sqrt{\lambda\,\mu_1 t}\;\sin 2\sqrt{\lambda\,\mu_2 h}\;dtdh, \tag{4.10}$$

Taking LHS of (4.1), using binomial ,exponential expansion, a known relation

[5, exp (10), p.70] and finally expressing the inner series in terms of Celine's polynomials of two variable (2.4) with $\lambda=\mu=1, c=d=0$, we get the required result (4.1). Similarly the result (4.2) can be established without using the above relation .
Taking LHS of(4.3), in view of definition (2.4) with $\lambda=\mu=1, c=d=0$; and using on the factors the relation , $(1+k)_r = \dfrac{\Gamma(1+k+r)}{\Gamma(1+k)} = \dfrac{(1+r)_k\, r!}{k!}$,
now using binomial expansion on inner series and finally once again using the definition (2.4) with $\lambda=\mu=1, c=d=0$ the result (4.3) is obtained.
In (3.1) replacing λ_1, λ_2 respectively by $(\alpha_1+r_2-n),(\alpha_2+r_1-m)$ and multiplying both the sides by :

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-n)_{r_1+r_2}(1+\alpha_1+\beta_1+n)_{r_2}(1+\alpha_2+\beta_2+n)_{r_1}}{r_1!\, r_2!}$$

on RHS, changing the order of summation & integration and using the definition of Jacobi polynomials of two variables defined by the author [9;eq(2),p.62], we obtained the desired result (4.4).

In(4.4),putting $\alpha_1=\alpha_2=\beta_1=\beta_2=\nu-\frac{1}{2}; \alpha_1=\alpha_2=\beta_1=\beta_2=0; \alpha_1=\alpha_2=\beta_1=\beta_2=-\frac{1}{2};$ $\alpha_1=\alpha_2=\beta_1=\beta_2=\frac{1}{2}$, we respectively can get the result involving polynomials Gegnbauer, Legendre,Chebyshev I & II kind [9;eq(28),(30),(33),p.68-69].
Similarly the result (4.5) to (4.9) can be obtained.
In result (4.6), putting $n=m=\frac{1}{2}$, we get the result (4.14) involving product of trigonometric functions.

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