A NOTE ON THE NORMS OF THE GCD MATRIX

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Abstract- Let $S=\{1, 2, \dots, n\}$ be a set of positive integers. The $n \times n$ matrix $[S]=(s_{ij})$, where $s_{ij}=(x_i, x_j)$ the greatest common divisor of x_i and x_j , is called the greatest common divisor GCD matrix on S . In this study, we have obtained some bounds of norms of this matrix. In addition, we have obtained upper bounds of norms of the almost Hilbert-Smith GCD matrix is defined

$$(S) = \left[\frac{(i, j)}{ij} \right]_{i, j=1}^n.$$

Keywords- Matrix Norm, Unitarily Invariant Norm, GCD matrix, Hadamard Product, Singular Values, Positive Definite

1. INTRODUCTION

Let $S=\{x_1, x_2, \dots, x_n\}$ be a set of positive integers. The matrix (S) having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j entry is called the greatest common divisor GCD matrix on S . The study of GCD matrices was initiated by Beslin and Ligh [3]. They have shown that every GCD matrix is positive definite, and, in fact, is the product of a specified matrix and transpose.

Let A be an $m \times n$ matrix. Then Euclidean norm, ℓ_p norm and spectral norm of the matrix A are defined by

$$\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2},$$

$$\|A\|_p = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^p \right)^{1/p}, 1 \leq p < \infty$$

and

$$\|A\|_2 = \sqrt{\max_i |\lambda_i(A^*A)|} = \sigma_1(A)$$

respectively where A^* is the conjugate transpose of the matrix A .

A function ψ is called polygamma function if

$$\psi(x) = \frac{d}{dx} \{\log[\Gamma(x)]\}$$

where

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

The function $\psi(m, x)$ have the property:

$$\lim_{n \rightarrow \infty} \psi(a, n + b) = 0 \quad (1.1)$$

where $a > 0$ and b are any numbers and n is positive integer.

Denote the space of m -by- n complex matrices by $M_{m,n}$ and set $M_n \equiv M_{n,n}$. The Hadamard (entry-wise) product of $A = (a_{ij})$, $B = (b_{ij}) \in M_{m,n}$ is defined by

$$A \circ B \equiv (a_{ij} b_{ij}) \in M_{m,n}.$$

For any $A \in M_{m,n}$, we denote by $c_1(A), c_2(A), \dots, c_n(A) \geq 0$ (the Euclidean lengths of the n columns of A), listed in descending order, and by $r_1(A), r_2(A), \dots, r_n(A) \geq 0$ (the Euclidean lengths of the n rows of A), similarly ordered. The singular values of $A \in M_{m,n}$, which we shall always exhibit in descending order

$$\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A) \geq 0$$

are the non-negative square roots of the eigenvalues of AA^* as well as the non-negative square roots of the n largest eigenvalues of AA^* . Let $\Sigma(A) \equiv (\sigma_{ij}) \in M_{m,n}$ have $\sigma_{ii}(A) \equiv \sigma_i(A)$ for $i=1, 2, \dots, n$ and all others for $\sigma_{ij} \equiv 0$ for $i \neq j$. We know that A has a singular value decomposition $A = V \Sigma(A) W^*$, in which $V \in M_m$ and $W \in M_n$ are unitary [4]. For any $x \in \partial^n$ and $A \in M_n$, if $x^* A x \geq 0$ then A matrix is called positive semidefinite (if $x^* A x > 0$ then the matrix A is called positive definite).

A norm $\|\cdot\|$ on $M_{m,n}$ is unitarily invariant if $\|A\| = \|UAV\|$ for all $A \in M_{m,n}$ and all unitary $U \in M_m$ and $V \in M_n$. For any $A \in M_{m,n}$, spectral norms and Euclidean norm is A matrix are both unitarily invariant norms because two norm is connected singular values. Also, spectral norm is induced matrix norm or operator matrix norm.

Definition 1. 1 [5] If $n \geq 1$ the Euler totient $\phi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n .

Definition 1. 2 [3] A set S of positive integers is said to be factor-closed (FC) is whenever x_i is in S and d divides x_i , then d is in S .

The above definition is due to J. J. Malone.

Definition 1. 3 [3] $S = \{x_1, x_2, \dots, x_n\}$ be a set of positive integers. The $n \times n$ matrix $[S] = (s_{ij})$ where $(s_{ij}) = (x_i, x_j)$, the greatest common divisor of x_i and x_j , is called the greatest common divisor (GCD) matrix on S .

It is clear that any set of positive integers is contained in an FC set. The following theorem describes the structure of GCD matrices.

Theorem 1.1 [3] $S = \{x_1, x_2, \dots, x_n\}$, be a set of positive integers. Then the GCD matrix $[S]$ is the product of an $n \times m$ matrix A and the $m \times n$ matrix A^* , where the nonzero entries of A are of the form $\sqrt{\phi(d)}$ for some d in an FC set that contains S , and $\phi(x)$ is Euler's totient function.

Proof. Suppose $D = \{d_1, d_2, \dots, d_m\}$ is an FC set containing S . Let the matrix $A = (a_{ij})$ be defined as follows:

$$a_{ij} = e_{ij}(\lambda_j)^{1/2} \quad (1.2)$$

where

$$e_{ij} = \begin{cases} 1, & d_j \mid x_j \\ 0, & \text{otherwise} \end{cases} \quad (1.3)$$

and $\lambda_j = \phi(d_j)$. Hence A is $n \times m$ and A^T is $m \times n$. Furthermore,

$$\begin{aligned} (AA^T) &= \sum_{k=1}^m a_{ik} a_{jk} \\ &= \sum_{\substack{d_k \mid x_i \\ d_k \mid x_j}} \sqrt{\phi(d_k) \phi(d_k)} \\ &= \sum_{d_k \mid (x_i, x_j)} \phi(d_k) \\ &= (x_i, x_j) \\ &= s_{ij}. \end{aligned}$$

Thus $[S] = AA^T$.

Theorem 1.2 [4] Let $A, B \in M_n$. Then

$$\sum_{i=1}^k \sigma_i(A \circ B) \leq \begin{cases} \sum_{i=1}^k r_i(A) c_i(B) \\ \sum_{i=1}^k c_i(A) r_i(B) \end{cases}, \quad k=1, 2, \dots, n \quad (1.4).$$

Proof. Because of the Hadamard product is commutative, the two inequalities in (1.4) are equivalent; we verify the upper one. We first note the case $k=1$. Let $\|\cdot\|_F$ denotes the Euclidean norm on ∂^n , let $A = (a_{ij})$ and $B = (b_{ij})$ and let $x = (x_i) \in \partial^n$ be given unit vector. Then

$$\begin{aligned}
\|(A \circ B)x\|_F^2 &= \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} b_{ij} x_j \right|^2 \leq \sum_{i=1}^n \left[\sum_{j=1}^n |a_{ij}|^2 \right] \left[\sum_{j=1}^n |b_{ij} x_j|^2 \right] \\
&\leq r_1(A)^2 \sum_{i=1}^n \sum_{j=1}^n |b_{ij}|^2 |x_j|^2 = r_1(A)^2 \sum_{j=1}^n |x_j|^2 \sum_{i=1}^n |b_{ij}|^2 \\
&\leq r_1(A)^2 c_1(B)^2 \sum_{j=1}^n |x_j|^2 = r_1(A)^2 c_1(B)^2.
\end{aligned}$$

Since $\sigma_1(A \circ B) = \max \{ \|(A \circ B)x\|_F : \|x\|_F = 1 \}$, the desired bound have been obtained.

Theorem 1.3 [4] Let m, n be two positive integers, $q = \min\{m, n\}$ and $A, B \in M_{m,n}$. Then

$$\sum_{i=1}^q \sigma_i(A \circ B) \leq \sum_{i=1}^q r_i(X) c_i(Y) \sigma_i(B)$$

such that $A = X^* Y$ where $X \in M_{r,m}$ and $Y \in M_{r,n}$.

2. MAIN RESULTS

In this section, we have obtained some upper bounds for norms of defined matrices on the set S .

Theorem 2.1 Let $S = \{1, 2, \dots, n\}$ be a set of positive integer. If S is factor-closed, then for the upper bound for Euclidean norm of GCD matrix $[S]$ defined on S by

$$\|[S]\|_F \leq \sum_{k=1}^n \left[\frac{n}{k} \right] \phi(k) = \frac{n(n+1)}{2}.$$

Proof. From [3], the GCD matrix $[S]$ is written $[S] = AA^T$. Firstly, we evaluate Euclidean norm of A . In the case n is prime number, the matrix A can be written of the form

$$A = \begin{bmatrix} \sqrt{\phi(1)} & 0 & 0 & \cdots & 0 \\ \sqrt{\phi(1)} & \sqrt{\phi(2)} & 0 & \cdots & 0 \\ \sqrt{\phi(1)} & 0 & \sqrt{\phi(3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \sqrt{\phi(1)} & 0 & 0 & \cdots & \sqrt{\phi(n)} \end{bmatrix}.$$

Then the Euclidean norm of the matrix A is

$$\|A\|_F^2 = \sum_{j=1}^n \left(\sum_{j|i} \phi(j) \right), \quad j=1, 2, \dots, n.$$

The Euclidean norms of the matrices A and A^T are equal. From properties of product of matrix norm, for Euclidean norm of $[S]$ matrix

$$\begin{aligned}\| [S] \|_F &= \| AA^T \| \leq \| A \|_F \| A^T \|_F \leq \sum_{j=1}^n \left(\sum_{j|i} \phi(j) \right) \\ &= \sum_{k=1}^n \left[\frac{n}{k} \right] \phi(k) = \frac{n(n+1)}{2}\end{aligned}$$

is valid where ϕ , Euler's totient function and $[.]$ is the greatest integer function.

Similarly, from structure of GCD matrix, Altınışık has defined almost Hilbert-Smith matrix on the set $S = \{1, 2, \dots, n\}$ in [2] such that

$$(S) = \left[\frac{(i, j)}{ij} \right]_{i, j=1}^n. \quad (2.1)$$

We consider the matrix S as Hadamard product two matrices. In addition, we have obtained an upper bounds for Euclidean and spectral norms of this matrix.

Theorem 2.2 $\|.\|$ is unitarily invariant norm. Then, for the matrix (S) be as in (2.1)

$$\| (S) \|_2 \leq n \left(\psi(1, n+1) + \frac{\pi^2}{6} \right)$$

where (S) is the GCD matrix on set of S so that S is factor-closed as in Theorem 2.1.

Proof. At first, let $(S) = [S] \circ K$ and $[S] = AA^T$ then from [4]

$$\| (S) \|_2 \leq c_1(A) c_1(A) \sigma_1(K) = c_1^2(A) \sigma_1(K)$$

where $[S] = [(i, j)] = AA^T$ and $K = \left[\frac{1}{ij} \right]$ respectively. $c_1(A)$ is the biggest Euclidean lengths of columns of the matrix A and $\sigma_1(K)$ is the biggest singular value of the matrix K as follows defined respectively

$$c_1(A) = \max_j \sqrt{\sum_{i=1}^n |a_{ij}|^2} = \sqrt{\sum_{j=1}^n \phi(1)} = \sqrt{n}$$

and characteristic polynomial of matrix K is to be

$$\Delta_K(\lambda) = \lambda^n - \text{trace}(K) \lambda^{n-1}.$$

The roots of this polynomial are $x_i = 0$, $i=1, 2, \dots, n-1$ and $x_n = \text{trace}(K)$. Also, for any nonsingular matrix A

$$\| A \|_F = (\text{trace}(A^* A))^{1/2} = \left(\sum_{i=1}^n \lambda_i(A^* A) \right)^{1/2} = \left(\sum_{i=1}^n \sigma_i^2(A) \right)^{1/2}.$$

The symmetric matrix K has only one nonzero eigenvalue and we obtain

$$\| K \|_F = (\text{trace}(K^2))^{1/2} = \text{trace}(K) = (\lambda_1(K^2))^{1/2} = (\sigma_1^2(K))^{1/2} = \sigma_1(K)$$

and

$$\sigma_1(K) = \|K\|_2 = \text{trace}(K) = \sum_{k=1}^n \frac{1}{k^2} = \psi(1, n+1) + \frac{\pi^2}{6}.$$

Thus, we have obtained upper bounds for spectral norm of the (S) matrix as follows

$$\|(S)\|_2 \leq c_1^2(A) \sigma_1(K) \leq n \left(\psi(1, n+1) + \frac{\pi^2}{6} \right).$$

Corollary 2.1 Let the matrix (S) be as in (2.1). Then

$$\|(S)\|_F \leq \frac{\pi^2}{6} \frac{n(n+1)}{2} = \frac{n\pi^2(n+1)}{12}$$

is valid where $\|\cdot\|_F$ is the Euclidean norm.

Proof. For any matrix norm, $\|A \circ B\| \leq \|A\| \|B\|$. Proof of Theorem is very simple obtained by this norm inequality.

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