Mathematical & Computational Applications, Vol. 9, No. 2, pp. 299-302, 2004

ESTIMATION OF EFFICIENCY AND INFINITESIMALS IN DATA ENVELOPMENT ANALYSIS

G. R. Jahanshahloo
Department of Mathematics, Teacher Training University, Tehran, Iran.
jahanshahlo@yahoo.com

F. Hosseinzadeh Lotfi and M. Zohrehbandian
Department of Mathematics, Science & Research Branch,
Islamic Azad University, Tehran, Iran.
hosseinzadeh_lotfi@yahoo.com

Abstract- The role of the non-Archimedean construct ε in the DEA models is clarified. It is established that the multiplier (envelopment) forms of DEA models can be infeasible (unbounded). Sufficient conditions are established for feasibility (bounded ness) of multiplier (envelopment) forms of DEA models and a method of securing a global assurance interval for ε is provided.

Key Words- Data Envelopment Analysis, Efficiency, non-Archimedean.

1.INTRODUCTION

Data Envelopment Analysis (DEA), introduced by Charnes et al.[3], is a method of evaluating relative performance of a group of similar units, called Decision Making Unit (DMU). DMUs are essentially perform the same task using similar multiple inputs to produce similar multiple outputs.

DEA gives a measure of efficiency, which is essentially defined as a ratio of weighted outputs to weighted inputs. The computation of a weighted ratio requires a set of weights to be defined and this can be not easy. Charnes et al.'s idea is to define the efficiency measure by assigning to each unit the most favorable weights.

The traditional CCR models, as introduced by Charnes et al.[3] are fractional linear programs that can easily be formulated and solved as linear program. Consider n DMUs, each of which consume varying amount of m inputs in the production of s outputs. Suppose x_{ij} denotes the amount consumed of the i-th input measure and y_{rj} denotes the amount produced of the r-th output measure by the j-th decision making unit. Let us define

u_r: Weight assigned to output r (r=1,2,...,s)

v_i: Weight assigned to input i (i=1,2,...,m)

Then the CCR model in u and v is formulated as follows:

$$CCR_p) \qquad \text{Max} \quad \sum_{r=1}^{s} u_r y_{rp}$$

$$\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad j=1,2,\ldots,n$$

$$\sum_{i=1}^{m} v_i x_{ip} = 1 \qquad (1)$$
$$u_r \geq \varepsilon \qquad r=1,2,\ldots,s$$
$$v_i \geq \varepsilon \qquad i=1,2,\ldots,m$$

The dual problem will also be used afterwards:

$$CCR_D) \qquad \text{Min} \quad \theta_p - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} - \theta_p x_{ip} + s_i^- = 0 \qquad i=1,2,\ldots,m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp} \qquad r=1,2,\ldots,s$$
$$\lambda_j \geq 0 \qquad j=1,2,\ldots,n \qquad (2)$$
$$s_r^+ \geq 0 \qquad r=1,2,\ldots,s$$
$$s_i^- \geq 0 \qquad i=1,2,\ldots,m$$

Where ε is a convenient small number that prevents the weights from vanishing (formally, ε should be seen as a non-Archimedean infinitesimal constant; on this subject see Charnes et al.[2]). Potential errors in (and in some cases, the impossibility of) solving standard DEA models are clarified. It is tempting to represent $\varepsilon>0$ by a small real number such as $\varepsilon=10^{-6}$. However, this is not advisable. It can lead to erroneous results and the situation may be worsened by replacing $\varepsilon=10^{-6}$ by even smaller values, see Ali and Seiford[1].

Therefore in most DEA computer codes, it is not necessary to explicitly assign a value to $\varepsilon>0$. Instead, this is taken care of operationally by using a two-stage computation, which may be formalized as follows.

Stage 1 accords priority to θ_p^*=Min θ_p, subject to (2). The later are dealt with in stage 2 by incorporating this value of θ_p^* instead of θ_p in (2) and by means of objective function Max $\{ \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \}$.

Our purpose in this paper is the correction of Ali and Seiford's technique, to provide a valid global assurance interval for ε.

2.ASCERTAINMENT OF A GLOBAL ASSURANCE INTERVAL FOR ε

Ali and Seiford[1], apply a technique to produce a global assurance interval $[0,\delta]$, where $\delta = 1/Max_{1\leq j\leq n}\{\sum_{i=1}^{m} x_{ij}\}$. But this interval is not a valid interval for ε. In this section, by means of correction of Ali and Seiford's technique, a global assurance interval for non-Archimedean infinitesimal (ε) is provided.

2.1. Value of ε

Definition Interval $[0,\delta_p]$ is called assurance interval associated to DMU_p, if for each $\varepsilon \in [0,\delta_p]$, model CCR_P in evaluating DMU_p is bounded (or as its counterpart, model CCR_D in evaluating DMU_p is feasible).

Definition Interval $[0,\delta]$ is called global assurance interval, if for each DMU_j (j=1,2,...,n) and for each $\varepsilon \in [0,\delta]$, model CCR_P in evaluating DMU_j is bounded (or as its counterpart, model CCR_D in evaluating DMU_j is feasible).

Theorem Suppose M=$Max_{1\le j\le n}\{Min_{1\le i\le m}\{x_{ip}/x_{ij} : x_{ij} \neq 0\}\}$. Interval $[0,\delta_p]$ is an assurance interval associated to DMU_p, if $\delta_p = 1/\{\sum_{i=1}^m x_{ip} + M\sum_{j=1}^n \sum_{r=1}^s y_{rj}\}$.

Proof We shall prove that for each $\varepsilon \in [0,\delta_p]$, model CCR_P in evaluating DMU_p is bounded. $\theta_p = 1$, $s^- = s^+ = 0$, $\lambda = e_p = (0,...,0,1,0,...,0)$, where 1 appears in position p, is a feasible solution and due to non-negativity of parameters λ_j, x_{ij} and y_{rj}, we have $\theta_p^* > 0$. Therefore, if model CCR_P is unbounded then some s_i^- or s_r^+ must be made arbitrarily large. Furthermore, since by increasing s_i^- or s_r^+, value of θ_p is also increased, we can concentrate on enhancing θ_p by one unit and compute maximum alterations in s_i^- , s_r^+ and objective function.

When θ_p increases by one unit ($\Delta\theta_p$=1), for feasibility of the input and output constraints, the following situations must be hold.

$\forall i;\ 1 \le i \le m$: $\Delta s_i^- \le x_{ip}$, therefore $\sum_{i=1}^m \Delta s_i^- \le \sum_{i=1}^m x_{ip}$.

$\forall j;\ 1 \le j \le n$: $\Delta\lambda_j \le Min_{1\le i\le m}\{x_{ip}/x_{ij} : x_{ij} \neq 0\}$.

And if $M = Max_{1\le j\le n}\{Min_{1\le i\le m}\{x_{ip}/x_{ij} : x_{ij} \neq 0\}\}$, then

$\forall r;\ 1 \le r \le s$: $\Delta s_r^+ \le \sum_{j=1}^n y_{rj}\Delta\lambda_j \le M\sum_{j=1}^n y_{rj}$, therefore $\sum_{r=1}^s \Delta s_r^+ \le M\sum_{j=1}^n \sum_{r=1}^s y_{rj}$.

The change in the value of the objective function is given by:

$\Delta Z = 1 - \varepsilon(\sum_{i=1}^m \Delta s_i^- + \sum_{r=1}^s s_r^+) \ge 1 - \varepsilon(\sum_{i=1}^m x_{ip} + M\sum_{j=1}^n \sum_{r=1}^s y_{rj})$.

To complete the proof, value of δ_p is determined in such a way that alterations in objective function be positive. In other words, it is sufficient to determine δ_p in such a way that the model CCR_P in evaluating DMU_p has not any improvement recession direction. Therefore, it is sufficient to have $\varepsilon \le \delta_p = 1/\{\sum_{i=1}^m x_{ip} + M\sum_{j=1}^n \sum_{r=1}^s y_{rj}\}$ ■

Corollary Interval $[0,\delta]$ is a global assurance interval, where $\delta = Min\{\delta_1,\delta_2,...,\delta_n\}$.

Corollary Suppose M= $\underset{1\le j<k\le n}{Max}\{\underset{1\le i\le m}{Min}\{\frac{x_{ik}}{x_{ij}} : x_{ij} \neq 0\}, \frac{1}{\underset{1\le i\le m}{Max}\{x_{ik}/x_{ij} : x_{ij} \neq 0\}}\}$. Interval $[0,\delta]$ is a global assurance interval, if $\delta = 1/\{Max_{1\le j\le n}\{\sum_{i=1}^m x_{ij}\} + M\sum_{j=1}^n \sum_{r=1}^s y_{rj}\}$.

Proof For each $p \in \{1,2,\ldots,n\}$, let $\Delta\theta_p = 1$, then for feasibility of the input and output constraints, the following situations must be hold.

$\forall i;\ 1 \le i \le m:\ \Delta s_i^- \le x_{ip}$, therefore $\sum_{i=1}^m \Delta s_i^- \le \sum_{i=1}^m x_{ip} \le Max_{1\le j\le n}\{\sum_{i=1}^m x_{ij}\}$.

$\forall j;\ 1 \le j \le n:\ \Delta\lambda_j \le Min_{1\le i\le m}\{x_{ip}/x_{ij} : x_{ij} \ne 0\}$.

And if $M = Max_{1\le j<k\le n}\{Min_{1\le i\le m}\{x_{ik}/x_{ij} : x_{ij} \ne 0\}, 1/Max_{1\le i\le m}\{x_{ik}/x_{ij} : x_{ij} \ne 0\}\}$, then

$\forall r;\ 1 \le r \le s:\ \Delta s_r^+ \le \sum_{j=1}^n y_{rj}\Delta\lambda_j \le M\sum_{j=1}^n y_{rj}$, therefore $\sum_{r=1}^s \Delta s_r^+ \le M\sum_{j=1}^n \sum_{r=1}^s y_{rj}$.

For each p, the change in the value of the objective function is given by:

$\Delta Z = 1 - \varepsilon(\sum_{i=1}^m \Delta s_i^- + \sum_{r=1}^s s_r^+) \ge 1 - \varepsilon(Max_{1\le j\le n}\{\sum_{i=1}^m x_{ij}\} + M\sum_{j=1}^n \sum_{r=1}^s y_{rj})$.

Therefore, it is sufficient to have $\varepsilon \le \delta = 1/\{Max_{1\le j\le n}\{\sum_{i=1}^m x_{ij}\} + M\sum_{j=1}^n \sum_{r=1}^s y_{rj}\}$ ■

2.2. Numerical Example

Consider the following DMUs:

DMU	1	2	3
Input	1.0	0.5	2.0
Output	1.0	1.0	2.0

By means of Ali and Seiford's boundary of ε, it is sufficient that ε be less than $1/2$, but with $\varepsilon = 3/8 \le 1/2$, multiplier form of CCR model in evaluating DMU_3 is infeasible. By means of our proposed boundary, interval $[0, 1/18]$ is an assurance interval associated to DMU_3.

3. CONCLUSION

The above discussion is by no means esoteric. As ε is below the threshold value, while finite objective are obtained in envelopment model, the obtained value of the objective can overshoot optimality. In general, the effect of reducing ε is that obtained efficiency scores increase because of the effect of the slack term is diminished. Therefore, results are sensitive to the specific value for ε, even if this value be in assurance interval.

5. REFERENCES

1. A.I. Ali and L.M. Seiford, Computational Accuracy and Infinitesimals in Data Envelopment Analysis, *INFOR* **31**, 290-297, 1993.

2. A. Charnes, W.W. Cooper, A.Y. Lewin and L.M. Seiford, *Data Envelopment Analysis: Theory, Methodology, and Application*, Kluwer, Boston., 1994.

3. A. Charnes, W.W. Cooper and E. Rhodes, Measuring the Efficiency of Decision Making Units, *European Journal of Operational Research* **2**, 429-444, 1978.

Mathematical & Computational Applications, Vol. 9, No. 2, pp. 303-308, 2004

A NOTE ON THE NORMS OF THE GCD MATRIX

Ramazan Türkmen & Durmuş Bozkurt
Department of Mathematics, Selcuk University
42031, Campus, Konya-TURKEY
rturkmen@selcuk.edu.tr

Abstract- Let $S=\{1, 2,..., n\}$ be a set of positive integers. The $n\times n$ matrix $[S]=(i, j)$, where $s_{ij}=(x_i, x_j)$ the greatest common divisor of x_i and x_j, is called the greatest common divisor GCD matrix on S. In this study, we have obtained some bounds of norms of this matrix. In addition, we have obtained upper bounds of norms of the almost Hilbert-Smith GCD matrix is defined

$$(S)=\left[\frac{(i,j)}{ij}\right]_{i,j=1}^{n}.$$

Keywords- Matrix Norm, Unitarily Invariant Norm, GCD matrix, Hadamard Product, Singular Values, Positive Definite

1. INTRODUCTION

Let $S=\{x_1, x_2, \ldots, x_n\}$ be a set of positive integers. The matrix (S) having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j entry is called the greatest common divisor GCD matrix on S. The study of GCD matrices was initiated by Beslin and Ligh [3]. They have shown that every GCD matrix is positive definite, and, in fact, is the product of a specified matrix and transpose.

Let A be an $m\times n$ matrix. Then Euclidean norm, ℓ_p norm and spectral norm of the matrix A are defined by

$$\|A\|_F=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2},$$

$$\|A\|_p=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\right)^{1/p}, 1\le p<\infty$$

and

$$\|A\|_2=\sqrt{\max_i|\lambda_i(A^*A)|}=\sigma_1(A)$$

respectively where A^* is the conjugate transpose of the matrix A.

A function ψ is called polygamma function if

$$\psi(x)=\frac{d}{dx}\{\log[\Gamma(x)]\}$$

where

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt .$$

The function $\psi(m,x)$ have the property:

$$\lim_{n\to\infty}\psi(a,n+b)=0 \tag{1.1}$$

where $a>0$ and b are any numbers and n is positive integer.

Denote the space of m-by-n complex matrices by $M_{m,n}$ and set $M_n \equiv M_{n,n}$. The Hadamard (entry-wise) product of $A=(a_{ij})$, $B=(b_{ij})\in M_{m,n}$ is defined by

$$A\circ B\equiv(a_{ij}b_{ij})\in M_{m,n}.$$

For any $A\in M_{m,n}$, we denote by $c_1(A),c_2(A),\ldots,c_n(A)\geq 0$ (the Euclidean lengths of the n columns of A), listed in descending order, and by $r_1(A),r_2(A),\ldots,r_n(A)\geq 0$ (the Euclidean lengths of the n rows of A), similarly ordered. The singular values of $A\in M_{m,n}$, which we shall always exhibit in descending order

$$\sigma_1(A),\sigma_2(A),\ldots,\sigma_n(A)\geq 0$$

are the non-negative square roots of the eigenvalues of AA^* as well as the non-negative square roots of the n largest eigenvalues of AA^*. Let $\sum(A)\equiv(\sigma_{ij})\in M_{m,n}$ have $\sigma_{ii}(A)\equiv\sigma_i(A)$ for i=1,2,...,n and all others for $\sigma_{ij}\equiv 0$ for $i\neq j$. We know that A has a singular value decomposition $A=V\sum(A)W^*$, in which $V\in M_m$ and $W\in M_n$ are unitary [4]. For any $x\in\partial^n$ and $A\in M_n$, if $x^*Ax\geq 0$ then A matrix is called positive semidefinite (if $x^*Ax>0$ then the matrix A is called positive definite).

A norm $\|.\|$ on $M_{m,n}$ is unitarily invariant if $\|A\|=\|UAV\|$ for all $A\in M_{m,n}$ and all unitary $U\in M_m$ and $V\in M_n$. For any $A\in M_{m,n}$, spectral norms and Euclidean norm is A matrix are both unitarily invariant norms because two norm is connected singular values. Also, spectral norm is induced matrix norm or operator matrix norm.

Definition 1. 1 [5] If $n\geq 1$ the Euler totient $\phi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n.

Definition 1. 2 [3] A set S of positive integers is said to be factor-closed (FC) is whenever x_i is in S and d divides x_i, then d is in S.

The above definition is due to J. J. Malone.

Definition 1. 3 [3] $S=\{x_1,x_2,\ldots,x_n\}$ be a set of positive integers. The $n\times n$ matrix $[S]=(s_{ij})$ where $(s_{ij})=(x_i,x_j)$, the greatest common divisor of x_i and x_j, is called the greatest common divisor (GCD) matrix on S.

It is clear that any set of positive integers is contained in an FC set. The following theorem describes the structure of GCD matrices.

Theorem 1. 1 [3] $S=\{x_1,x_2,\ldots,x_n\}$, be a set of positive integers. Then the GCD matrix [*S*] is the product of an $n\times m$ matrix *A* and the $m\times n$ matrix A^*, where the nonzero entries of *A* are of the form $\sqrt{\phi(d)}$ for some *d* in an FC set that contains *S*, and $\phi(x)$ is Euler's totient function.

Proof. Suppose $D=\{d_1,d_2,\ldots,d_m\}$ is an FC set containing *S*. Let the matrix *A*=(a_{ij}) be defined as follows:

$$a_{ij}=e_{ij}(\lambda_j)^{1/2} \tag{1.2}$$

where

$$e_{ij}=\begin{cases}1, & d_j \mid x_j\\ 0, & otherwise\end{cases} \tag{1.3}$$

and $\lambda_j=\phi(d_j)$. Hence *A* is $n\times m$ and A^T is $m\times n$. Furthermore,

$$\begin{aligned}(AA^T)&=\sum_{k=1}^{m}a_{ik}a_{jk}\\ &=\sum_{\substack{d_k|x_i\\ d_k|x_j}}\sqrt{\phi(d_k)\phi(d_k)}\\ &=\sum_{d_k|(x_i,x_j)}\phi(d_k)\\ &=(x_i,x_j)\\ &=s_{ij}.\end{aligned}$$

Thus [*S*]= AA^T.

Theorem 1. 2 [4] Let $A,B\in M_n$. Then

$$\sum_{i=1}^{k}\sigma_i(A\circ B)\le\begin{cases}\sum_{i=1}^{k}r_i(A)c_i(B)\\ \sum_{i=1}^{k}c_i(A)r_i(B)\end{cases},\quad k=1,2,\ldots,n \tag{1.4}$$

Proof. Because of the Hadamard product is commutative, the two inequalities in (1.4) are equivalent; we verify the upper one. We first note the case *k*=1. Let $\|.\|_F$ denotes the Euclidean norm on ∂^n, let *A*=(a_{ij}) and *B*=(b_{ij}) and let *x*=(x_i)$\in\partial^n$ be given unit vector. Then

$$\|(A\circ B)x\|_F^2=\sum_{i=1}^{n}\left|\sum_{j=1}^{n}a_{ij}b_{ij}x_j\right|^2\le\sum_{i=1}^{n}\left[\sum_{j=1}^{n}|a_{ij}|^2\right]\left[\sum_{j=1}^{n}|b_{ij}x_j|^2\right]$$
$$\le r_1(A)^2\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2|x_j|^2=r_1(A)^2\sum_{j=1}^{n}|x_j|^2\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|^2$$
$$\le r_1(A)^2c_1(B)^2\sum_{j=1}^{n}|x_j|^2=r_1(A)^2c_1(B)^2.$$

Since $\sigma_1(A\circ B)=\max\{\|(A\circ B)x\|_F:\|x\|_F=1\}$, the desired bound have been obtained.

Theorem 1. 3 [4] Let m, n be two positive integers, q=min$\{m, n\}$ and A, $B\in M_{m,n}$. Then

$$\sum_{i=1}^{q}\sigma_i(A\circ B)\le\sum_{i=1}^{q}r_i(X)c_i(Y)\sigma_i(B)$$

such that $A=X^*Y$ where $X\in M_{r,m}$ and $Y\in M_{r,n}$.

2. MAIN RESULTS

In this section, we have obtained some upper bounds for norms of defined matrices on the set S.

Theorem 2.1 Let $S=\{1,2,.., n\}$ be a set of positive integer. If S is factor-closed, then for the upper bound for Euclidean norm of GCD matrix $[S]$ defined on S by

$$\|[S]\|_F\le\sum_{k=1}^{n}\left[\frac{n}{k}\right]\phi(k)=\frac{n(n+1)}{2}.$$

Proof. From [3], the GCD matrix $[S]$ is written $[S]=AA^T$. Firstly, we evaluate Euclidean norm of A. In the case n is prime number, the matrix A can be written of the form

$$A=\begin{bmatrix}\sqrt{\phi(1)}&0&0&\cdots&0\\\sqrt{\phi(1)}&\sqrt{\phi(2)}&0&\cdots&0\\\sqrt{\phi(1)}&0&\sqrt{\phi(3)}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&0\\\sqrt{\phi(1)}&0&0&\cdots&\sqrt{\phi(n)}\end{bmatrix}.$$

Then the Euclidean norm of the matrix A is

$$\|A\|_F^2=\sum_{j=1}^{n}\left(\sum_{j|i}\phi(j)\right),\ j=1,2,\ldots,n.$$

The Euclidean norms of the matrices A and A^T are equal. From properties of product of matrix norm, for Euclidean norm of $[S]$ matrix

$$\|[S]\|_F = \|AA^T\| \le \|A\|_F \|A^T\|_F \le \sum_{j=1}^{n}\left(\sum_{j|i}\phi(j)\right)$$
$$= \sum_{k=1}^{n}\left[\frac{n}{k}\right]\phi(k) = \frac{n(n+1)}{2}$$

is valid where ϕ, Euler's totient function and $[.]$ is the greatest integer function.

Similarly, from structure of GCD matrix, Altınışık has defined almost Hilbert-Smith matrix on the set $S=\{1, 2,..., n\}$ in [2] such that

$$(S) = \left[\frac{(i,j)}{ij}\right]_{i,j=1}^{n}. \tag{2.1}$$

We consider the matrix S as Hadamard product two matrices. In addition, we have obtained an upper bounds for Euclidean and spectral norms of this matrix.

Theorem 2.2 $\|.\|$ is unitarily invariant norm. Then, for the matrix (S) be as in (2.1)

$$\|(S)\|_2 \le n\left(\psi(1,n+1) + \frac{\pi^2}{6}\right)$$

where (S) is the GCD matrix on set of S so that S is factor-closed as in Theorem 2.1.

Proof. At first, let $(S)=[S]\circ K$ and $[S]=AA^T$ then from [4]

$$\|(S)\|_2 \le c_1(A)c_1(A)\sigma_1(K) = c_1^2(A)\sigma_1(K)$$

where $[S]=[(i,j)]=AA^T$ and $K=\left[\frac{1}{ij}\right]$ respectively. $c_1(A)$ is the biggest Euclidean lengths of columns of the matrix A and $\sigma_1(K)$ is the biggest singular value of the matrix K as follows defined respectively

$$c_1(A) = \max_j \sqrt{\sum_{i=1}^{n}|a_{ij}|^2} = \sqrt{\sum_{j=1}^{n}\phi(1)} = \sqrt{n}$$

and characteristic polynomial of matrix K is to be

$$\Delta_K(\lambda) = \lambda^n - \text{trace}(K)\lambda^{n-1}.$$

The roots of this polynomial are $x_i = 0$, $i=1,2,...,n-1$ and $x_n=\text{trace}(K)$. Also, for any nonsingular matrix A

$$\|A\|_F = (\text{trace}(A^*A))^{1/2} = \left(\sum_{i=1}^{n}\lambda_i(A^*A)\right)^{1/2} = \left(\sum_{i=1}^{n}\sigma_i^2(A)\right)^{1/2}.$$

The symmetric matrix K has only one nonzero eigenvalue and we obtain

$$\|K\|_F = (\text{trace}(K^2))^{1/2} = \text{trace}(K) = (\lambda_1(K^2))^{1/2} = (\sigma_1^2(K))^{1/2} = \sigma_1(K)$$

and

$$\sigma_1(K) = \|K\|_2 = \text{trace}(K) = \sum_{k=1}^{n} \frac{1}{k^2} = \psi(1, n+1) + \frac{\pi^2}{6}.$$

Thus, we have obtained upper bounds for spectral norm of the (S) matrix as follows

$$\|(S)\|_2 \le c_1^2(A)\sigma_1(K) \le n\left(\psi(1, n+1) + \frac{\pi^2}{6}\right).$$

Corollary 2.1 Let the matrix (S) be as in (2.1). Then

$$\|(S)\|_F \le \frac{\pi^2}{6}\frac{n(n+1)}{2} = \frac{n\pi^2(n+1)}{12}$$

is valid where $\|.\|_F$ is the Euclidean norm.

Proof. For any matrix norm, $\|A \circ B\| \le \|A\|\|B\|$. Proof of Theorem is very simple obtained by this norm inequality.

REFERENCES

1. Roger A. Horn and Charles R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.
2. Ercan Altınışık, unpublished PhD. thesis, Selcuk University, 1991.
3. Beslin Scott and Ligh Steve, Greatest common divisor matrices, Linear Algebra and Its Applications, **118**, 69-76, 1989.
4. T. Ando, Roger A. Horn and Charles R. Johnson, the singuler values of a Hadamard Product: A basic inequality, Linear and Multilinear Algebra, **21**, 345-365, 1987.
5. Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.