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FREE VIBRATION ANALYSIS OF ELASTIC BEAMS USING HARMONIC DIFFERENTIAL QUADRATURE (HDQ)

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Abstract- Harmonic differential quadrature method is developed for the free vibration analysis of linear elastic beams. In the method of differential quadrature, partial space derivatives of a function appearing in a differential equation are approximated by means of a polynomial expressed as the weighted linear sum of the function values at a preselected grid of discrete points. The weighting coefficients are treated as the unknowns. Applying this concept to the governing differential equation of beam gives a set of linear simultaneous equations, which are solved for the unknown weighting coefficients by accounting for the boundary conditions. Beams of different support combinations such as clamped, simply supported, guided, and free are selected to demonstrate the accuracy of the method. Flexural vibration case is taken into consideration. First two frequencies are obtained in the applications. Numerical results are presented to illustrate the method and demonstrate its efficiency.
Keywords- Harmonic Differential Quadrature, Beams, Free Vibration, Numerical Analysis

1. INTRODUCTION

It is well known that, the analysis of engineering systems are included two main stage, such as construction of a mathematical model for a given physical phenomena and the solution of this mathematical equation. With the modern computer technology, various numerical methods were well developed and widely used to solve various kinds of engineering and science problems, which are described by the differential equations. This equations either linear or nonlinear and in most cases, their closed form solutions are extremely difficult to establish. As a result, approximate numerical methods have been widely used to solve such a differential equations which arise in almost all engineering disciplines. The most commonly used numerical methods for such applications are the finite element and finite difference, and most engineering problems can be solved by these methods to satisfactory accuracy if a proper and sufficient number of grid points are used. However, in a large number of practical applications where only reasonably accurate solutions at few specified physical coordinates are of interest, the finite element or finite difference method becomes inappropriate since they still require a large number of grid points and so large a computer capacity, especially in the cases of nonlinear problems where iteration becomes unavoidable [5,23]. Consequently, both CPU time and storage requirements are often considerable for the standard methods.

In seeking a more efficient numerical method which requires fewer grid points yet achieves acceptable accuracy, the method of differential quadrature (DQ), which is based on the assumptions that the partial derivatives of a function in one direction can be expressed as a linear combination of the function values at all mesh points along that direction, was introduced by Bellman et al. [1]. The method of differential quadrature circumvents the above difficulties by computing a moderately accurate solution from only a few points.

In this study, free vibration analysis of beams is investigated by using harmonic differential quadrature. The accuracy, efficiency and convenience of HDQ are demonstrated throughout the numerical examples.

2. DIFFERENTIAL QUADRATURE METHOD (DQM)

The idea of the differential quadrature method is to quickly compute the derivative of a function at any grid point within its bounded domain by estimating a weighted linear sum of values of the function at a small set of points related to the domain. As with other numerical analysis techniques, such as finite element or finite difference methods, the DQM also transforms the given differential equation into a set of analogous algebraic equations in terms of the unknown function values at the reselected sampling points in the field domain.

The problem areas in which the applications of differential quadrature method may be found in the available literature include static and dynamic structural mechanics and stability analysis of structures [16,17]. Recent works of Bert and associates, mainly on the vibration analysis of plates, have contributed significantly to the development of the DQM [3,4]. Authors applied the DQM to the stability, vibration and bending analysis of elastic bars [6,9], and plates [8] and other type structures [7,11]. It has been claimed that the DQM has the capability of producing highly accurate solutions with minimal computational effort. All this work has demonstrated that the application of the DQ methods leads to accurate results with less computational effort and that there is a potential that the method may be become an alternative to the conventional methods such as finite differences and finite element [10,12,13,17,18]. Therefore research on extension and application of the method becomes an important endeavor.

In the differential quadrature method, a partial derivative of a function with respect to a space variable at a discrete point is approximated as a weighted linear sum of the function values at all discrete points in the region of that variable. For simplicity, we consider a one-dimensional function $\Psi(x)$ in the [-1,1] domain, and N discrete points. Then the first derivatives at point i, at $x = x_i$ is given by

$$\Psi_x(x_i) = \left.\frac{\partial \Psi}{\partial x}\right|_{x = x_i} = \sum_{j=1}^{N} A_{ij}\ \Psi\ (x_j); \qquad i = 1,2,........,N \tag{1}$$

where x_j are the discrete points in the variable domain, $\Psi\ (x_j)$ are the function values at these points and A_{ij} are the weighting coefficients for the first order derivative attached to these function values. Bellman et al. [1,2] suggested two methods to determine the weighting coefficients. The first one is to let equation (1) be exact for the test functions

$$\Psi_k(x) = x^{k-1}, \qquad k = 1,2,...,N \tag{2}$$

which leads to a set of linear algebraic equations

$$(k-1)x_i^{k-2} = \sum_{j=1}^{N} A_{ij} x_j^{k-1}; \text{ for } i = 1,2,........,N \quad \text{and} \quad k = 1,2,........,N \tag{3}$$

which represents N sets of N linear algebraic equations. This equation system has a unique solution because its matrix is of Vandermonde form. This equation may be solved for the weighting coefficients analytically using the Hamming's method [22] or numerical method using the certain special algorithms for Vandermonde equations, such as the method of Bjorck and Pareyra [21]. As similar to the first order, the second order derivative can be written as

$$\Psi_{xx}(x_i)=\left.\frac{\partial^2\Psi}{\partial x^2}\right|_{x=x_i}=\sum_{j=1}^{N} B_{ij}\ \Psi\ (x_j);\quad i=1,2,\ldots\ldots,N \tag{4}$$

where the B_{ij} is the weighting coefficients for the second order derivative. Equation (4) also can be written

$$\Psi_{xx}(x_i)=\left.\frac{\partial^2\Psi}{\partial x^2}\right|_{x=x_i}=\sum_{j=1}^{N} A_{ij}\sum_{k=1}^{N} A_{jk}\,\Psi(x_k)\ ;\qquad i=1,2,\ldots\ldots,N \tag{5}$$

Again, the function given by equation (2) is used so that the second order derivative is

$$(k-1)(k-2)x_i^{k-3}=\sum_{j=1}^{N} B_{ij}x_j^{k-1} \tag{6}$$

this can be solved in the same manner as indicated for equation (3) above. The second method is also proposed by Bellman et al. [2] to obtain the weighting coefficients is similar to the first one with the exception that a different set of trial or test function is chosen for satisfying equation (1) exactly;

$$\Psi_k(x)=\frac{L_N(x)}{(x-x_k)L_N^{(1)}(x_k)},\qquad k=1,2,\ldots\ldots\ldots,N \tag{7}$$

where $L_N(x)$ is the Nth order Legendre polynomial and $L_N^{(1)}(x)$ the first order derivative of $L_N(x)$. N is the number of grid points as with the first one. However, it requires that x_k (k=1,2,....,N) have to be chosen to be roots of the shifted Legendre polynomial. This means that once number of grid points N is specified the roots of the shifted Legendre polynomial are given, thus the distribution of the grid points are fixed regardless of the physical problems being considered. By choosing x_k to be roots of the shifted Legendre polynomial and substituting equation (7) into equation (1), we obtained a direct simple algebraic expression for the weighting coefficients A_{ij}

$$A_{ij}=\frac{L'_N(x_i)}{(x_i-x_j)L'_N(x_j)}\qquad \text{for } i\neq j;\ \text{and } i,j=1,2,\ldots\ldots,N \tag{8}$$

$$A_{ii}=\frac{1-2x_i}{2x_i(x_i-1)}\qquad \text{for } i=j;\ \text{and } i,j=1,2,\ldots\ldots,N \tag{9}$$

In this second approach, the weighting coefficients that was defined equation (8) and (9) are easy to obtain without solving algebraic equations or having a singularity problem as with the first one.

3. HARMONIC DIFFERENTIAL QUADRATURE (HDQ)

A recently approach the original differential quadrature approximation called the HDQ has been proposed by Striz et al. [15]. Unlike the differential quadrature that uses the polynomial functions, such as Lagrange interpolated, and Legendre polynomials as the test functions, HDQ uses harmonic or trigonometric functions as the test functions. As the name of the test function suggested, this method is called the HDQ method. The harmonic test function $h_k(x)$ used in the HDQ method is defined as [14];

$$h_k(x)=\frac{\sin\frac{(x-x_0)\pi}{2}\cdots\cdots\sin\frac{(x-x_{k-1})\pi}{2}\sin\frac{(x-x_{k+1})\pi}{2}\cdots\cdots\sin\frac{(x-x_N)\pi}{2}}{\sin\frac{(x_k-x_0)\pi}{2}\cdots\cdots\sin\frac{(x_k-x_{k-1})\pi}{2}\sin\frac{(x_k-x_{k+1})\pi}{2}\cdots\cdots\sin\frac{(x_k-x_N)\pi}{2}}$$

for $k=0,1,2,\ldots,N$ (10)

According to the HDQ, the weighting coefficients of the first-order derivatives A_{ij} for $i \neq j$ can be obtained by using the following formula:

$$A_{ij}=\frac{(\pi/2)P(x_i)}{P(x_j)\sin[(x_i-x_j)/2]\pi}, \qquad i,j=1,2,3,\ldots,N \tag{11}$$

where

$$P(x_i)=\prod_{j=1, j\neq i}^{N}\sin\left(\frac{x_i-x_j}{2}\pi\right), \quad \text{for } j=1,2,3,\ldots,N$$

The weighting coefficients of the second-order derivatives B_{ij} for $i \neq j$ can be obtained using following formula:

$$B_{ij}=A_{ij}\left[2A_{ii}^{(1)}-\pi ctg\left(\frac{x_i-x_j}{2}\right)\pi\right], \qquad i,j=1,2,3,\ldots,N \tag{12}$$

The weighting coefficients of the first-order and second-order derivatives $A_{ij}^{(p)}$ for $i=j$ are given as

$$A_{ii}^{(p)}=-\sum_{j=1, j\neq i}^{N} A_{ij}^{(p)}, \; p=1 \text{ or } 2; \text{ and for } i=1,2,\ldots,N \tag{13}$$

The weighting coefficient of the fourth order derivative can be computed easily from B_{ij} by [14];

$$D_{ij}=\sum_{k=1}^{N} B_{ik}B_{kj} \tag{14}$$

The main advantage of HDQ over the differential quadrature is its ease of the computation of the weighting coefficients without any restriction on the choice of grid points. A factor decisive to the accuracy of the all type differential quadrature solution is the choice of the sampling or grid points. It should be mentioned that in the differential quadrature solutions, the sampling points in the various coordinate directions may be different in number as well as in their type. Two different types grid distribution are used in this study.

Type I: A natural, an often convenient, choice for sampling points is that of equally spaced point. This type sampling points are given as

$$x_i=\frac{i-1}{N-1}; \qquad i=1,2,\ldots,N \tag{15}$$

in the related direction. Another type grid points are known the Chebyshev-Gauss-Lobatto points (Type II) are proposed by Bert et al. [4] and given by;

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{2i-1}{N-1}\right)\pi\right]; \qquad i = 1,2,.....,N \tag{16}$$

4. NUMERICAL APPLICATION AND RESULTS

To verify the analytical formulation presented in the previous section; flexural vibration of beams is considered. Beams subjected to different boundary conditions are selected as the test examples to demonstrate the applicability and accuracy of the HDQ method. The governing differential equation of beam is presented. The present formulation is based on classical small deflection theory. Then, the harmonic differential quadrature method has been applied to this differential equation. Results are obtained for each case using various numbers of grid points. It is observed that the convergence of the method is very good. Reasonably accurate results can be achieved by using 7 and 9 grid points. The computational time on a standard PC (Pentium II having 64 RAM) is less than 2 sec for all the cases. Following, several test examples for different support conditions have been selected to demonstrate the convergence properties, accuracy and the simplicity in numerical implementation of the HDQ procedures. The numerical results for various type beam are tabulated (Table 1) and the comparison of the present results with the exact values available in the literature.

4.1. Flexural vibration of elastic beams

When the beam is vibrating transversely, the governing equation of motion is given by [20];

$$\frac{d^4 u}{d X^4} - \Omega^2 u = 0 \tag{17}$$

where $u = u(X)$ is the dimensionless mode function of the deflection, X is the dimensionless coordinate along the beam axis, and Ω is the dimensionless frequency. Differential quadrature form of equation (Eq. 17) is given by

$$\sum_{j=1}^{N} D_{ij} u_j - \Omega^2 u_i = 0 \qquad i = 3,4,......,N\text{-}2 \tag{18}$$

In this equation D_{ij} is the weighting coefficient of the fourth order derivative. Numerical applications have been done for a linearly elastic beam under five sets of different boundary conditions, namely clamped-simply supported (C-S), clamped-clamped (C-C), guided end- simply supported (G-S), clamped end-guided end (C-G), and cantilever beam (C-F). Following, these boundary conditions are given.

Clamped end (C) : $u= 0$ and $du/dX =0$ (19a)
Simply supported end (S): $u = 0$ and $d^2 u/dX^2 =0$ (19b)
Free end (F): $d^2u / dX^2 = 0$ and $d^3u / dX^3 = 0$ (19c)
Guided end (G): $du/dX = 0$ and $d^3u/dX^3 =0$ (19d)

Applying the differential quadrature approximation to the above equations at each discrete point on the grid, we obtain their DQ form. For clamped (C) support

$$u_i = 0 \text{ and } \sum_{j=1}^{N} A_{ij} u_j = 0 \qquad \text{for } i=1,2,.........,N \tag{20}$$

For (S) boundary condition

$$u_i = 0 \text{ and } \sum_{j=1}^{N} B_{ij} u_j = 0 \quad \text{for } i=1,2,\dots\dots\dots,N \tag{21}$$

For (F) boundary condition

$$\sum_{j=1}^{N} B_{ij} u_j = 0 \text{ and } \sum_{j=1}^{N} C_{ij} u_j = 0 \tag{22}$$

For (G) boundary conditions

$$\sum_{j=1}^{N} A_{ij} u_j = 0 \text{ and } \sum_{j=1}^{N} C_{ij} u_j = 0 \tag{23}$$

We have *(N-2)* equations from Eq. (18) and four equations from each beam support conditions. By rewriting them, one has the assembled form [8]

$$\begin{bmatrix} [S_{bb}] & [S_{bd}] \\ [S_{db}] & [S_{dd}] \end{bmatrix} \begin{Bmatrix} u_b \\ u_d \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \Omega^2 \{u_d\} \end{Bmatrix} \tag{24}$$

By some matrix operation, we obtains a typical standard eigenvalue equation

$$[S]\{u_d\} - \Omega^2 [I]\{u_d\} = 0 \tag{25}$$

where $[S] = [S_{dd}] - [S_{db}][S_{bb}]^{-1}[S_{bd}]$ and subscripts b and d indicate the grid points used for writing the quadrature analog of the boundary conditions and the governing differential equations of the system, Ω is the non-dimensional frequency. In order to simplify presentation S, C, F, and G represent simply- supported, clamped, free, and guided supports, respectively. The transversely vibration of elastic beams with various combinations of S, C, F, and Guided (G) boundary conditions had been investigated. The first two frequencies of flexural vibration of a C-C, C-F, C-G, G-S, and a C-S beam are given in Table 1, as obtained by the HDQ and DQ method. All results compare very well with the exact solutions [19]. The present results were obtained for HDQ using N=7 grid points. In all cases, the agreement between the compared results was found to be good. It is shown that in this table, HDQ results using seven (N=7) grid points is more than accurate than the DQ for nine (N=9) grid points. Reasonably accurate results can be achieved by using 7 grid points in HDQ for Type-II grid sampling in the related directions. Chebyshev-Gauss-Lobatto grid points (Type-II) have been shown to be consistently better than other choice under consideration.

Table 1 Comparison of non-dimensional frequencies ($\Omega_i^{2\,*}$) for flexural vibration

Support Conditions	Exact (Ref.19)		DQ (N=9) Type-II		HDQ (N=7) Type-I		HDQ (N=7) Type-II	
	Ω^2_1	Ω^2_2	Ω^2_1	Ω^2_2	Ω^2_1	Ω^2_2	Ω^2_1	Ω^2_2
C-C	4.730	7.853	4.682	7.846	4.688	7.850	4.728	7.853
C-F	1.875	4.694	1.870	4.711	1.873	4.691	1.875	4.694
C-G	2.365	5.498	2.361	5.620	2.360	5.500	2.364	5.498
G-S	1.571	4.712	1.568	4.699	1.568	4.708	1.571	4.711
C-S	3.927	7.069	3.920	7.114	3.923	7.101	3.926	7.070

$^{*}\ \Omega_i^2 = \rho A L^4 \omega_i^2 / EI$

5. CONCLUDING REMARKS

HDQ is recently proposed and there are only a few papers on this novel kind differential quadrature. A harmonic type differential quadrature method was introduced to study the free vibration elastic beams with various support conditions. The method of HDQ that was using the paper proposes a very simple algebraic formula to determine the connections weighting coefficients required by differential quadrature approximation without restricting the choice of mesh grids. The known boundary conditions are easily incorporated in the HDQ as well as the other type differential quadrature. The discretizing and programming procedures are straightforward and easy. In addition to this, choice of sampling grid points is one of the more important factors to obtain the accurate results. The authors believe that the HDQ method may be extended to the nonlinear problems in structural mechanics, including nonlinear static and dynamic response analysis.

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