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ON A CLASS OF NONLINEAR SINGULAR INTEGRAL EQUATIONS WITH CARLEMAN SHIFT IN GENERALIZED HOLDER SPACES

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Abstract-The paper concerns with the investigation of a class of nonlinear singular integral equations with Carleman shift. Existence results are given by application of Schauder's fixed-point theorem to this type of equations in generalized Holder space.

Keywords-Nonlinear singular integral equations, Carleman shift, Schauder fixed point theorem.

INTRODUCTION

Many authors (see for instance, [1,2,5,6,11,15,16,17]) have studied nonlinear singular integral equations (NSIE). The theory of singular integral equations (SIE) with shift has been developed in the works of [3,7,8,9,14] and others. The Noether theory of such equations is developed for a closed and open contour. Detailed exposition of the subject and its history can be found in monographs [10,12].

Now, we consider the following NSIE with shift:

$$(Tu)(t)=a(t)u(t)+b(t)u(\alpha(t))+\frac{c(t)}{\pi i}\int_L\frac{u(\tau)}{\tau-t}d\tau+\frac{d(t)}{\pi i}\int_L\frac{u(\tau)}{\tau-\alpha(t)}d\tau+ \tag{0.1}$$
$$+\int_L K_T(t,\tau)u(\tau)d\tau=\lambda_1\int_L\frac{F(t,\tau,u(\tau))}{\tau-t}d\tau+\lambda_2\int_L\frac{G(\alpha(t),\tau,u(\tau))}{\tau-\alpha(t)}d\tau$$

where L is a simple smooth closed Lyapunov contour, $\alpha(t)$ is the Carleman's shift of L, the coefficients $a(t), b(t), c(t)$ and $d(t)$ satisfy Holder's condition on L, $u(t)$ is the unknown function, λ_1 and λ_2 are numerical parameters and the homeomorphism $\alpha: L \to L$ is the shift changing orientation and satisfying Carleman condition:

$$\alpha(\alpha(t))=\alpha_2(t)=t;\quad t\in L\ ,$$

whose derivative $\alpha'(t)\neq 0$ satisfies the usual Holder condition:

$$|\alpha'(t_1)-\alpha'(t_2)|\le A_\alpha|t_1-t_2|^\mu, t_1,\, t_2\in L,$$

where $\mu\in(0,1)$ and A_α is a constant.

If the right hand side of equation (0.1) doesn't depend on $u(t)$, then we obtain the well-known linear singular integral equation that already considered in [12]. Existence results for equation (0.1) with the shift preserving orientation in a generalized Holder space $H_L(\omega)$, have been established in [14]. In this paper we shall prove the existence of the solutions of equation (0.1), in the case of Carleman shift changing the orientation in the space $H_L(\omega)$.

This paper consists of four sections. In section 1, we shall introduce basic assumptions and notations, which are necessary for our work. In section 2, we study the invariance of the generalized Holder space for the singular integral operator with shift. In section 3, we study the Nother property for singular integral functional operator

(SIO) in case of an orientation changing Carleman shift. Finally, we prove the existence of the solutions of equation (0.1) in the space $H_L(\omega)$.

1.BASIC ASSUMPTIONS AND NOTATIONS

Let us introduce some definitions, which will be used in this paper.

Definition 1.1 [5,14]. Let $\omega(\delta)$ be a function defined on $(0,l\,]$, where l is the length of the curve L, such that it satisfies the following conditions:

1) $\omega(\delta)$ is a modulus of continuity,

2) $\sup\limits_{\delta>0}\dfrac{1}{\omega(\delta)}\int\limits_0^\delta \dfrac{\omega(s)}{s}ds = I_\omega < \infty$,

3) $\sup\limits_{\delta>0}\dfrac{\delta}{\omega(\delta)}\int\limits_\delta^l \dfrac{\omega(s)}{s^2}ds = J_\omega < \infty.$

Definition 1.2 [14]. We denote by $c(L)$ the space of all continuous functions $u(t)$ defined on L with the norm

$$\|u\|_{c(l)} = \max_{t\in L}|u(t)|.$$

b) Let the generalized Holder space $H_L(\omega)$ be defined as follows

$$H_L(\omega)=\left\{u\in c(L): H_L^\omega(u)=\sup_{t_1,t_2\in L}\frac{|u(t_1)-u(t_2)|}{\omega(|t_1-t_2|)}<\infty\right\},$$

where, $u\in H_L(\omega)$, the norm is defined by:

$$\|u\|_{H_L(\omega)}=\|u\|_{c(L)}+\sup_{t_1,t_2\in L}\frac{|u(t_1)-u(t_2)|}{\omega(|t_1-t_2|)}.$$

Definition 1.3 [14]. Let R and K be positive numbers and the function $\omega(t)$ satisfies definition 1.1. We say that the function $u(t)$ in $H_L(\omega)$, $t\in L$, belongs to the class $H_L^{R,K}(\omega)$ if

1) $|u(t)|\le R,\ t\in L,$
2) $|u(t_1)-u(t_2)|\le K\omega(|t_1-t_2|).$

We denote to the nonlinear singular integral operators in the right hand side of equation (0.1) by the notations

$$(\Lambda_F u)(t)=\lambda_1\int\limits_L\frac{F(t,\tau,u(\tau))}{\tau-t}d\tau, \qquad (\Lambda_G u)(t)=\lambda_2\int\limits_L\frac{G(t,\tau,u(\tau))}{\tau-t}d\tau.$$

Hence by definition of shift operator W and singular integral operator S the NSIE with shift (0.1) can be written in the following operator form

$$Tu = au + bWu + cSu + dWSu + Q_T u = \Lambda_F u + W\Lambda_G u, \tag{1.1}$$

where

$$Wu(t)=u(\alpha(t)), \qquad Su(t)=\frac{1}{\pi i}\int\frac{u(\tau)}{\tau-t}d\tau,$$

and

$$(Q_T u)(t)=\int\limits_L K_T(t,\tau)u(\tau)d\tau,$$

and assume that $K_T(t,\tau)$ takes the form:

$$K_T(t,\tau)=\frac{B_T(t,\tau)}{(\tau-t)^{\gamma}}, \qquad 0<\gamma<1$$

such that the function $B_T(t,\tau)$ and its derivatives are continuous functions defined on $L\times L$.

2.THE INVARIANCE OF THE GENERALIZED HOLDER SPACE FOR THE LINEAR SINGULAR OPERATOR

In this section we introduce the following auxiliary results.

Lemma 2.1 [14]. The shift operator W maps $H_L^{R,K}(\omega)$ into $H_L^{R,K\left(\|\alpha'\|_c+1\right)}(\omega)$.

Lemma 2.2 [14]. Let $u(t)$ be a function in $H_L^{R,K}(\omega)$, then $(Su)(t)$ belongs to the class $H_L^{R',K'}(\omega)$, where R',K' are some positive constants.

Lemma 2.3 [13]. The operator $Q_T:H_L(\omega)\to H_L(\omega)$ with weakly singular kernel $K_T(t,\tau)$ is compact operator.

Theorem 2.1 [14]. Let $a(t),b(t),c(t),d(t)$ be continuously differentiable functions on the contour L, and conditions of Lemmas 2.1, 2.2 and 2.3 are satisfied. Then the operator

$$T=aI+bW+cS+dWS+Q_T,$$

maps $H_L(\omega)$ into itself.

Now, we study the nonlinear singular operators Λ_F and Λ_G where the functions F and G satisfy the following conditions:

$$|F(t_1,\tau_1,u_1)-F(t_1,\tau_2,u_2)|\le A_F\omega^*(|t_1-t_2|)+B_F\omega(|\tau_1-\tau_2|)+C_F|u_1-u_2| \tag{2.1}$$

$$|G(t_1,\tau_1,u_1)-G(t_1,\tau_2,u_2)|\le A_G\omega^*(|t_1-t_2|)+B_G\omega(|\tau_1-\tau_2|)+C_G|u_1-u_2| \tag{2.2}$$

where $A_F, A_G, B_F, B_G, C_F \; and \; C_G$ are positive constants, ω,ω^* satisfy definition 1.1 such that

$$\omega^*(\delta)\ln(1/\delta)\le\tilde{c}\,\omega(\delta), \qquad \tilde{c}>0. \tag{2.3}$$

Theorem 2.2. Suppose that the functions $F(t,\tau,u(\tau))$ and $G(t,\tau,u(\tau))$ satisfy the conditions (2.1)- (2.3) respectively. Then for every $u\in H_L^{R,K}(\omega)$ we have

$$1)\ (\Lambda_F u)(t)\in H_L^{R_F,K_F}(\omega) \qquad 2)\ (\Lambda_G u)(t)\in H_L^{R_G,K_G}(\omega),$$

where the constants R_G, R_F, K_G and K_F depend on R, K, λ_1,λ_2.

Proof. Putting,

$$g(t,\tau)=F(t,\tau,u(\tau)), \quad f(t,\tau)=G(t,\tau,u(\tau)),$$

hence, the conditions (2.1) and (2.2) can be rewritten as follows

$$|g(t_1,\tau_1)-g(t_2,\tau_2)|\le A_g\omega^*(|t_1-t_2|)+B_g\omega(|\tau_1-\tau_2|) \tag{2.4}$$

$$|f(t_1,\tau_1,)-f(t_2,\tau_2)|\le A_f\omega^*(|t_1-t_2|)+B_g\omega(|\tau_1-\tau_2|) \tag{2.5}$$

where, $A_g=A_F$, $A_f=A_G$, $B_g=B_F+KC_F$, $B_f=B_G+KC_G$.

Now, putting

$$\bar{f}(t)=\int_L \frac{g(t,\tau)}{\tau-t}d\tau,\ M_g=\max_{(t,\tau)\in L\times L}|g(t,\tau)|,\ M_f=\max_{(t,\tau)\in L\times L}|f(t,\tau)|$$

Firstly, from the inequality (2.4) and (2.5), we estimate:

$$\left|\bar{f}(t)\right|\leq\left|\int_L \frac{g(t,\tau)-g(t,t)}{\tau-t}d\tau\right|+|g(t,t)|\left|\int_L \frac{d\tau}{\tau-t}\right|\leq$$

$$\leq m^* B_g I_\omega \omega(l)+\pi M_g\ ,$$

where

$$|d\tau|\leq m^* dr,$$

m^* is a positive constant [4].

Secondly, we estimate $\left|\bar{f}(t_1)-\bar{f}(t_2)\right|$ as follows:

Suppose $|t_1-t_2|<\sigma_0$, fix an arbitrary number n, $1<n<\sigma_0/|t_1-t_2|$. Draw a circle of radius $\sigma=n|t_1-t_2|$ centered at the point t_1. This circle intersects L at two points ε_1 and ε_2. The part of L lying within this circle is denoted by $\varepsilon_1\varepsilon_2$.

$$\left|\bar{f}(t_1)-\bar{f}(t_2)\right|=\left|\int_L \frac{g(t_1,\tau)-g(t_1,t_1)}{\tau-t_1}d\tau+g(t_1,t_1)\int_L\frac{d\tau}{\tau-t_1}-\int_L\frac{g(t_2,\tau)-g(t_2,t_2)}{\tau-t_1}d\tau-g(t_2,t_2)\int_L\frac{d\tau}{\tau-t_2}\right|.$$

Therefore, we get

$$\left|\bar{f}(t_1)-\bar{f}(t_2)\right|\leq\left|\int_{\varepsilon_1\varepsilon_2}\frac{g(t_1,\tau)-g(t_1,t_1)}{\tau-t_1}d\tau\right|+\left|\int_{\varepsilon_1\varepsilon_2}\frac{g(t_2,\tau)-g(t_2,t_2)}{\tau-t_2}d\tau\right|+\left|\int_{L\setminus\varepsilon_1\varepsilon_2}\frac{g(t_1,\tau)-g(t_2,\tau)}{\tau-t_1}d\tau\right|+$$

$$+\left|\int_{L\setminus\varepsilon_1\varepsilon_2}\frac{(t_1-t_2)(g(t_2,\tau)-g(t_2,t_2))}{(\tau-t_1)((\tau-t_2))}d\tau\right|+\left|(g(t_2,t_2)-g(t_1,t_1))\left[\int_{L\setminus\varepsilon_1\varepsilon_2}\frac{d\tau}{\tau-t_1}-\pi i\right]\right|$$

$$=I_1+I_2+I_3+I_4+I_5\,.$$

Now, we estimate each term of the right hand side of the above inequality:

$$I_1\leq\int_{\varepsilon_1\varepsilon_2}\frac{|g(t_1,\tau)-g(t_1,t_1)|}{|\tau-t_1|}|d\tau|\leq 2m^*B_g\int_0^\sigma\frac{\omega(r)}{r}dr\leq\tilde{M}_1 I_\omega\omega(|t_1-t_2|),$$

where

$$\tilde{M}_1=2m^*B_g(n+1)\cdot$$

Similarly, we have

$$I_2\leq\tilde{M}_2 I_\omega\omega(|t_1-t_2|),$$

$$I_3\leq A_g\int_{L\setminus\varepsilon_1\varepsilon_2}\frac{\omega^*(|t_1-t_2|)}{|\tau-t_1|}|d\tau|\leq m^*A_g\ln(l/\delta)\omega^*(|t_1-t_2|)\leq\tilde{M}_3\omega(|t_1-t_2|),$$

where

$$\tilde{M}_3=m^*A_g\tilde{c}\,,$$

$$I_4\leq B_g|t_1-t_2|\int_{L\setminus\varepsilon_1\varepsilon_2}\frac{\omega(|\tau-t_2|)}{|\tau-t_1||\tau-t_2|}|d\tau|\leq\tilde{M}_4 J_\omega\omega(|t_1-t_2|),$$

where,

$$\tilde{M}_4 = \left(\frac{n+1}{n}\right)^2 m^* B_g .$$

Also

$$I_5 \le \left|g(t_2,t_2) - g(t_1,t_1)\right| \left| \int\limits_{L\setminus\varepsilon_1\varepsilon_2} \frac{d\tau}{\tau - t_1} - \pi i \right| \le \tilde{M}_5 \left[A_g \tilde{c}(\ln 1/\delta)^{-1} + B_g\right] \omega(|t_2 - t_1|),$$

where,

$$\tilde{M}_5 = \pi + M_1, \qquad M_1 = \left| \int\limits_{L\setminus\varepsilon_1\varepsilon_2} \frac{d\tau}{\tau - t_1} \right| .$$

Thus,

$$\left|\bar{f}(t_1) - \bar{f}(t_2)\right| \le \left[(\tilde{M}_1 + \tilde{M}_2) I_\omega + \tilde{M}_3 + \tilde{M}_4 J_\omega + \tilde{M}_5 \left(A_g \tilde{c} (\ln 1/\delta)^{-1} + B_g\right)\right] \omega(|t_1 - t_2|)$$

where $\tilde{M}_i$ $, i = 1-5$ are positive constants.

Setting

$$M_{\bar{f}} = \left\{ (\tilde{M}_1 + \tilde{M}_2) I_\omega + \tilde{M}_3 + \tilde{M}_4 J_\omega + \tilde{M}_5 \left(A_g \tilde{c} (\ln 1/\delta)^{-1} + B_g \right) \right\},$$

$$R_F = |\lambda_1| \left(m^* B_g I_\omega \omega(l) + \pi M_g \right),$$

and

$$K_F = \Lambda_1 M_f$$

Then, we have

$$(\Lambda_F u)(t) \in H_L^{R_F, K_F}(\omega).$$

Analogously,

$$(\Lambda_G u)(t) \in H_L^{R_G, K_G}(\omega) .$$

Thus, the theorem is proved.

3.NOETHER PROPERTY FOR SIO WITH SHIFT

In this section, we study the criterion of Noetherity and index formula for singular integral functional operator with a shift changing the orientation.

Lemma 3.1. Let the coefficients $a(t), b(t), c(t)$ and $d(t)$ belong to $c(L)$, and the shift $\alpha(t)$ change the orientation of a closed contour L. Then the operator

$$T = a(t)I + b(t)W + c(t)S + d(t)WS + Q_T : H_L(\omega) \to H_L(\omega)$$

is Noetherian if the following two reduced operators T_1 and T_2 of the operator T are Noetherian,

$$T_1 = \tilde{a}(t)I + \tilde{b}(t)W^2 + \tilde{c}(t)S + \tilde{d}(t)W^2 S, \tag{3.1}$$

$$T_2 = \tilde{a}(\alpha_{-1}(t))I + \tilde{b}(t)W^2 + \tilde{c}(\alpha_{-1}(t))S + \tilde{d}(t)W^2 S, \tag{3.2}$$

where

$$\begin{aligned} \tilde{a}(t) &= -a(t)a(\alpha(t)) + c(t)c(\alpha(t)), \\ \tilde{b}(t) &= b(t)b(\alpha(t)) - d(t)d(\alpha(t)), \\ \tilde{c}(t) &= a(t)c(\alpha(t)) - a(\alpha(t))c(t), \\ \tilde{d}(t) &= b(t)d(\alpha(t)) - b(\alpha(t))d(t). \end{aligned} \tag{3.3}$$

Proof. We introduce the associated operators of the operator T as follows:

$$T_3 = -a(\alpha(t))I + b(t)W + c(\alpha(t))S + d(t)WS,$$

and

$$T_4 = -a(\alpha_{-1}(t))I + b(t)W + c(\alpha_{-1}(t))S + d(t)WS.$$

From [9,10], taking into account that

$$a(t)S \simeq Sa(t)I, \quad S^2 = I \text{ and } WS = -SW$$

we obtain the relation

$$T_3T \simeq (-a(t)a(\alpha(t)) + c(t)c(\alpha(t)))I + (b(t)b(\alpha(t)) - d(t)d(\alpha(t)))W^2 +$$
$$+ (a(t)c(\alpha(t)) - a(\alpha(t))c(t))S + (b(t)d(\alpha(t)) - b(\alpha(t))d(t))W^2S$$

Consequently,

$$T_3T \simeq T_1. \tag{3.4}$$

Using the equality

$$Wu(v(t)) = u(Wv(t)),$$

where $u(t), v(t) \in H_L(\omega)$, which leads to the following relation

$$TT_4 \simeq (-a(t)a(\alpha_{-1}(t)) + c(t)c(\alpha_{-1}(t)))I + (b(t)b(\alpha(t)) - d(t)d(\alpha(t)))W^2 +$$
$$+ (a(\alpha_{-}(t))c(t) - a(t)c(\alpha_{-1}(t)))S + (b(t)d(\alpha(t)) - b(\alpha(t))d(t))W^2S,$$

hence

$$TT_4 \simeq T_2. \tag{3.5}$$

Since the operators T_1 and T_2 are supposed to be Noetherian, then due to Atkinson's theorem [10], we get the initial operator T is also Noetherian.Thus Lemma 3.1 is proved.

Now, we introduce the following notation:

$$\begin{aligned} x(t) &= (a(t) - c(t))(a(\alpha(t)) + c(\alpha(t))), \\ y(t) &= (b(t) + d(t))(b(\alpha(t)) - d(\alpha(t))), \\ x_1(t) &= (a(t) + c(t))(a(\alpha(t)) - c(\alpha(t))), \\ y_1(t) &= (b(t) - d(t))(b(\alpha(t)) + d(\alpha(t))). \end{aligned} \tag{3.6}$$

Hence, the reduced operators take the form:

$$T_1 = (-x_1(t)I + y_1(t)W^2)P_+ + (-x(t)I + y(t)W^2)P_-,$$
$$T_2 = (-x_1(\alpha_{-1}(t))I + y_1(t)W^2)P_+ + (-x(\alpha_{-1}(t))I + y(t)W^2)P_-.$$

Where $P_\pm = \frac{1}{2}(I \pm S)$ are complementary projection operators.

Theorem 3.1. Let $a(t), b(t), c(t)$ and $d(t)$ belong to $c(L)$ and $\alpha(t)$ be a Carleman shift changes the orientation of a closed contour L. Then the operator

$$T = a(t)I + b(t)W + c(t)S + d(t)WS + Q_T : H_L(\omega) \to H_L(\omega)$$

with $W^2 = I$ is Noetherian if and only if

$$\Delta(t) = (a(t) - c(t))(a(\alpha(t)) + c(\alpha(t))) - (b(t) + d(t))(b(\alpha(t)) - d(\alpha(t))) \neq 0$$

is fulfilled. The index of the Noetherian operator T is determined by the formula

$$indT = \frac{1}{2\pi}(\arg \Delta(t))_L$$

4.EXISTENCE OF THE SOLUTIONS OF NSIE WITH SHIFT IN THE SPACE $H_L(\omega)$.

In this section, we investigate the existence theorem of the solution of NSIE with shift (0.1) in the generalized Holder space $H_L(\omega)$.
It is well-known (see [4]) that for the operator

$$T = a(t)I + b(t)W + c(t)S + d(t)WS + Q_T,$$

there exists a regular operator of the form

$$\tilde{T} = a(\alpha(t))I - b(t)W - c(\alpha(t))S - d(t)WS,$$

which maps $H_L^{R,K}(\omega)$ into $H_L^{R_{\tilde{T}},K_{\tilde{T}}}(\omega)$ where $R_{\tilde{T}}, K_{\tilde{T}}$ are some positive constants.
Now, we have

$$T\tilde{T} = \tilde{m}I + \tilde{n}S + Q_{T\tilde{T}},$$

where

$$\tilde{m} = a(t)a(\alpha(t)) - b(t)b(\alpha(t)) - c(t)c(\alpha(t)) + d(t)d(\alpha(t)),$$
$$\tilde{n} = -a(t)c(\alpha(t)) + c(t)a(\alpha(t)) - b(t)d(\alpha(t)) + d(t)b(\alpha(t)),$$
$$Q_{T\tilde{T}} = \sum_{i=1}^{4} Q_i + Q_T\tilde{T},$$

and

$$Q_1 = (c(t)S + d(t)WS)(Sc(\alpha(t)) - c(\alpha(t))S),$$
$$Q_2 = (-d(t)WS - c(t)S)(Sd(t)W + d(t)WS),$$
$$Q_3 = (c(t) + d((t))W)(Sa(\alpha(t)) - a(\alpha(t))S),$$
$$Q_4 = (-c(t) - d(t)W)(b(t)WS + Sb(t)W).$$

Hence, we can rewrite the equation (1.1) in the form:

$$\tilde{m}u + \tilde{n}Su + Q_{T\tilde{T}}u = \Lambda_F \tilde{T}u + W\Lambda_G \tilde{T}u. \tag{4.1}$$

Lemma 4.1 [10,13]. The operator $SW + WS$ is an operator with weakly singular kernel.
Lemma 4.2 [10,13]. When the function $a(t)$ is continuous, $t \in L$, then the operator $aS - SaI$ is compact.

Hence, From Lemma 4.1 and 4.2 we deduce that the operators $Q_i\,(i = 1-4)$ are compact operators. Consequently, according to Lemma 2.3 the operator $Q_{T\tilde{T}}$ is compact.

Therefore, our aim is to find the solution of equation (4.1). Putting

$$\Delta_1(t) = \Delta(t) = \tilde{m}(t) - \tilde{n}(t), \qquad \Delta_2(t) = \Delta(\alpha(t)) = \tilde{m}(t) + \tilde{n}(t),$$

and assume that $\Delta_1(t) \neq 0$, $\Delta_2(t) \neq 0$ on L, hence

$$\chi_{T\tilde{T}} = ind \frac{\Delta(t)}{\Delta(\alpha(t))}$$

Let. $\chi_{T\tilde{T}} \geq 0$, then by Vekua-Carleman's method, [4], the equation (4.1) can be reduced to Fredholm's integral equation for this aim we consider the sectionally holomorphic function

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{u(\tau)}{\tau - z} d\tau \tag{4.2}$$

where

$$u(t)=\Phi^{+}(t)-\Phi^{-}(t) \tag{4.3}$$

and

$$\frac{1}{\pi i}\int_L \frac{u(\tau)}{\tau-t}d\tau=\Phi^{+}(t)+\Phi^{-}(t). \tag{4.4}$$

where, the dominant equation of equation (4.1) has been written as follows

$$\tilde{m}u+\tilde{n}Su=\Lambda_F\tilde{T}u+W\Lambda_G\tilde{T}u. \tag{4.5}$$

From the theory of singular integral equations, [4,13], the solution of equation (4.5) can be given as follows

$$\begin{aligned}u(t)=\tilde{m}(t)[(\Lambda_F\tilde{T}u)(t)+(W\Lambda_G\tilde{T}u)(t)]-\\ -\frac{\tilde{n}(t)\tilde{Z}(t)}{\pi i}\int_L\frac{(\Lambda_F\tilde{T}u)(\tau)+(W\Lambda_G\tilde{T}u)(\tau)}{\tilde{Z}(\tau)(\tau-t)}d\tau+\\ +\tilde{n}(t)\tilde{Z}(t)P_{\chi_{T\tilde{T}}-1}(t)=\tilde{h}(t).\end{aligned}$$

where

$$\tilde{Z}(t)=X^{+}(t)\Delta(\alpha(t))=X^{-}(t)\Delta(t)=e^{\varepsilon(t)}/\sqrt{t^{\chi_{T\tilde{T}}}},$$

$$\varepsilon(z)=\frac{1}{2\pi i}\int_L\frac{\ln\left(\tau^{-\chi_{T\tilde{T}}}v(\tau)\right)}{\tau-z}d\tau,\qquad v(t)=\frac{\tilde{m}(t)-\tilde{n}(t)}{\tilde{m}(t)+\tilde{n}(t)},$$

and

$$\tilde{m}^2(t)-\tilde{n}^2(t)=1.$$

Consequently, the solution of the equation (4.1) has the form:

$$u(t)+\int_L N_{T\tilde{T}}(t,\tau)u(\tau)d\tau=\tilde{h}(t) \tag{4.6}$$

with kernel

$$N_{T\tilde{T}}(t,\tau)=\tilde{m}(t)K_{T\tilde{T}}(t,\tau)-\frac{\tilde{n}(t)\tilde{Z}(t)}{\pi i}\int_L\frac{K_{T\tilde{T}}(\tau,\tau_1)}{\tilde{Z}(\tau_1)(\tau_1-t)}d\tau_1$$

which is weakly singular kernel (see,[14]), where

$$K_{T\tilde{T}}(t,\tau)=\frac{B_{T\tilde{T}}(t,\tau)}{(\tau-t)^{\gamma}}\quad(0<\gamma<1)$$

Lemma 4.3 [4]. The homogeneous Fredholm integral equation

$$u(t)+\int_L N_{T\tilde{T}}(t,\tau)u(\tau)d\tau=0, \tag{4.7}$$

has no eigen functions if and only if

$$\alpha(T\tilde{T})=\chi_{T\tilde{T}}, \tag{4.8}$$

where $\alpha(T\tilde{T})$ is the number of linearly independent solutions of the equation $T\tilde{T}u=0$.

Remark 4.1. From Lemma 4.3, we deduce that the solution of equation (4.6) is equivalent to the following NSIE with shift:

$$Du(t)=u(t)=\tilde{h}(t)-\int_L R_{T\tilde{T}}(t,\tau)\tilde{h}(\tau)d\tau \tag{4.9}$$

where the function $R_{T\tilde{T}}(t,\tau)$ is the resolvent of the kernel $N_{T\tilde{T}}(t,\tau)$. Since, the functional properties of the resolvent are the same as those of the kernel. Thus, according to Lemma 2.3 the operator

$$(Mu)(t) = \int_L R_{T\tilde{T}}(t,\tau)u(\tau)d\tau$$

is compact on $H_L(\omega)$.

From compactness of the operator Mu for every $u(t) \in H_L^{R,K}(\omega)$, we have

$$\|Mu\|_{H_L(\omega)} \le T_\omega \|u\|_{H_L(\omega)} \le (R+K)T_\omega, \tag{4.10}$$

where T_ω is the norm of the linear operator M . Putting

$$\begin{aligned} Du(t) = {}& \tilde{m}(t)[(\Lambda_F \tilde{T}u)(t) + (W\Lambda_G \tilde{T}u)(t)] - \\ & -\frac{\tilde{n}(t)\tilde{Z}(t)}{\pi i}\int_L \frac{(\Lambda_F \tilde{T}u)(\tau) + (W\Lambda_G \tilde{T}u)(\tau)}{\tilde{Z}(\tau)(\tau - t)}d\tau + \tilde{n}(t)\tilde{Z}(t)P_{\chi_{T\tilde{T}}-1}(t) - \\ & -\int_L R_{T\tilde{T}}(t,\tau)\left\{\begin{aligned} & \tilde{m}(\tau)[(\Lambda_F \tilde{T}u)(\tau) + (W\Lambda_G \tilde{T}u)(\tau)] - \\ & \quad -\frac{\tilde{n}(\tau)\tilde{Z}(\tau)}{\pi i}\int_L \frac{(\Lambda_F \tilde{T}u)(\tau_1) + (W\Lambda_G \tilde{T}u)(\tau_1)}{\tilde{Z}(\tau_1)(\tau_1 - \tau)}d\tau_1 + \\ & + \tilde{n}(\tau)\tilde{Z}(\tau)P_{\chi_{T\tilde{T}}-1}(\tau) \end{aligned}\right\}d\tau \end{aligned} \tag{4.11}$$

Hence, with aid of Lemmas 2.1,2.2 and Theorem 2.2 we have the following lemma

Lemma 4.4 [14]. Let Γ(t) be a function in $H_L^{R_{\tilde{T}},K_{\tilde{T}}}(\omega)$. Then $\xi(t) = (D\Gamma)(t)$ is a function in the class $H_L^{R,K}(\omega)$.

Let us endow the class of $H_L^{R,K}(\omega)$ with the metric of the space of continuous functions defined by:

$$\rho_{H_L}(u_1,u_2) = \rho_{c(L)}(u_1,u_2) = \max_{t\in\Gamma}|u_1(t) - u_2(t)| = \|u_1 - u_2\|_c. \tag{4.12}$$

Notice that the class $H_L^{R,K}(\omega)$ is made into a closed, convex and compact metric space.

Lemma 4.5 [14]. Let all assumptions of Theorem 2.1, Lemmas 2.3, 2.4, and inequalities (2.1)- (2.3) are satisfied. Then the operator D which maps from $H_L^{R,K}(\omega)$ into itself is continuous.

Proof. If $u \in H_L^{R,K}(\omega)$, then $\Gamma \in H_L^{R\tilde{T},K\tilde{T}}(\omega)$, where $\Gamma(t) = (\tilde{T}u)$, $t \in L$, hence from assumptions of the lemma it follows that $(Du)(t) \in H_L^{R,K}(\omega)$, $t \in L$.

Now, we prove the continuity of the operator D. Since S is continuous in the metric (4.12), hence

$$\|Su\|_c \le \|S\|_c \|u\|_c.$$

Putting

$$f_1(t,\tau,u(\tau)) = F(t,\tau,u(\tau)) - F(\tau,\tau,u(\tau))$$

where

$$|f_1(t,\tau,u_1) - f_1(t,\tau,u_2)| \le 2A_f \omega^*(|\tau - t|),$$
$$|f_1(t,\tau,u_1) - f_1(t,\tau,u_2)| \le 2c_f(|u_1(\tau) - u_2(\tau)|)$$

For $u_1,u_2 \in H_L(\omega)$ and every fixed positive number υ in the interval (0,1), the following inequality holds (see [5])

$$|f_1(t,\tau,u_1(\tau)) - f_1(t,\tau,u_2(\tau))| \le 2A_f^{1-\upsilon}c_f^{\upsilon}(\omega^*(|\tau - t|))^{1-\upsilon}|u_1 - u_2|^{\upsilon}, \tag{4.13}$$

we have, for $\alpha^* \in (0,1)$ such that $\omega^*(r)/r^{\alpha^*}$ is almost increasing for every $r \in (0,l)$. Consequently,

$$\omega^*(r) \le c_* r^{\alpha^*} \tag{4.14}$$

Where c_* is a positive constant. By the relation (4.14), we get

$$\int_L \frac{[\omega^*(|\tau - t|)]^{1-\upsilon}}{|\tau - t|}|d\tau| \le \int_0^l \frac{(c_* r^{\alpha^*})^{1-\upsilon}}{r} m^* dr = m^* c_*^{1-\upsilon} \int_0^l r^{\alpha^*(1-\upsilon)-1} dr \le \frac{m^* c_*^{1-\upsilon}}{\alpha^*(1-\upsilon)} l^{\alpha^*(1-\upsilon)} \tag{4.15}$$

Then, we get

$$(\Lambda_F u)(t) = \lambda_1 \int_L \frac{f_1(t,\tau,u(\tau))}{\tau - t} d\tau + \lambda_1 \int_L \frac{F(\tau,\tau,u(\tau))}{\tau - t} d\tau$$

Therefore for every $u_1,\ u_2 \in H_L(\omega)$, the following inequality holds

$$\|(\Lambda_F u_1)(t) - (\Lambda_F u_2)(t)\|_c = \left\| \lambda_1 \int_L \frac{f_1(t,\tau,u_1(\tau)) - f_1(t,\tau,u_2(\tau))}{\tau - t} d\tau \right\|_c +$$
$$+ \left\| \lambda_1 \int_L \frac{F(\tau,\tau,u_1(\tau)) - F(\tau,\tau,u_2(\tau))}{\tau - t} d\tau \right\| = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &\le |\lambda_1| \int_L \frac{|f_1(t,\tau,u_1) - f_1(t,\tau,u_2)||d\tau|}{|\tau - t|} \le \\ &\le 2|\lambda_1| A_f^{1-\upsilon} c_f^{\upsilon} |u_1 - u_2|^{\upsilon} \int_L \frac{[\omega^*(|\tau - t_1|)]^{1-\upsilon} |d\tau|}{|\tau - t|} \le \\ &\le 2|\lambda_1| A_f^{1-\upsilon} c_f^{\upsilon} \|u_1 - u_2\|^{\upsilon} c_*^{1-\upsilon} m^* \frac{l^{\alpha^*(1-\upsilon)}}{\alpha^*(1-\upsilon)} = \\ &= l_{1F}(\lambda_1)[\rho_{H_l(\omega)}(u_1,u_2)]^{\upsilon}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} I_2 &= \pi|\lambda_1| \|S\|_c \|F(\tau,\tau,u_1) - F(\tau,\tau,u_2)\|_c \le \\ &\le \pi|\lambda_1| \|S\|_c c_f \|u_1 - u_2\|_c = \\ &= l_{2F}(\lambda_1) \rho_{H_L(\omega)}(u_1,u_2), \end{aligned} \tag{4.17}$$

and

$$l_{1F} = 2|\lambda_1| A_f^{1-\upsilon} c_f^{\upsilon} c_*^{1-\upsilon} m^* \frac{l^{\alpha^*(1-\upsilon)}}{\alpha^*(1-\upsilon)},$$
$$l_{2F} = |\lambda_1| \pi \|S\|_c c_f.$$

Hence, we have

$$\rho_{H_L}(\Lambda_F u_1, \Lambda_F u_2) \le l_{1F}(\lambda_1)(\rho_{H_L}(u_1,u_2))^{\upsilon} + l_{2F}(\lambda_2)\rho_{H_L}(u_1,u_2).$$

Similarly,

$$\rho_{H_L}(\Lambda_G u_1, \Lambda_G u_2) \le l_{1G}(\lambda_1)(\rho_{H_L}(u_1,u_2))^{\upsilon} + l_{2G}(\lambda_2)\rho_{H_L}(u_1,u_2),$$

Furthermore, the following estimate holds

$$\|\tilde{T}u\|_c \le \tilde{T}_c \|u\|_c \cdot$$

where

$$\tilde{T}_c = \|a\|_c + \|b\|_c + \|c\|_c \|S\|_c + \|d\|_c \|S\|_c$$

It is obvious that the linear operator W is continuous in the metric (4.12) in $H_L(\omega)$. Finally we obtain

$$\rho_{H_L}(Du_1, Du_2) \le (1+M_R)\left(\|\tilde{m}\|_c + \|\tilde{n}\|_c \|\tilde{Z}\|_c \|S\|_c / m_z\right) \times \\ \times \left\{ \begin{array}{l} (l_{1F}(\lambda_1) + l_{1G}(\lambda_2)) \tilde{T}_c^{\,v} (\rho_{H_L}(u_1,u_2))^{v} + \\ \qquad + (l_{2F}(\lambda_1) + l_{2G}(\lambda_2)) \tilde{T}_c \rho_{H_L}(u_1,u_2) \end{array} \right\}.$$

This inequality shows that the operator D maps continuously $H_L^{R,K}(\omega)$ into itself. Thus the lemma is proved.

From the above results, we introduce the following theorem

Theorem 4.2. Let the assumptions of Theorem 2.1, Lemmas 2.3, 4.4 and the inequalities (2.1)-(2.3) are satisfied. Then the equation (0.1) has at least one solution in the generalized Holder space $H_L(\omega)$.

Proof. From Lemmas 2.1-2.3,4.3,4.5 and Shouder's fixed-point theorem, it follows that the equation (4.9) and hence the equation (4.1) has at least a solution $u \in H_L^{R,K}(\omega)$. If $\Gamma(t) = (\tilde{T}u)(t)$, $t \in L$ then $\Gamma(t)$ is a solution of the equation (0.1) and belongs to the class $H_L^{R\tilde{T},K\tilde{T}}(\omega)$. Thus the theorem is proved.

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