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EXTRAPOLATION METHOD FOR IMPROVING THE SOLUTION OF FUZZY INITIAL VALUE PROBLEMS

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Abstract - In this paper we apply extrapolation to increase the accuracy of approximations to the solution of the fuzzy initial value problems, based on the standard Euler method and Midpoint method, [11]. The method in detail is discussed and is illustrated by solving some linear and nonlinear fuzzy Cauchy problems.
Keywords - Fuzzy Differential Equation, Extrapolation Method, Fuzzy Cauchy Problem.

1. INTRODUCTION

The fuzzy differential equation and the initial value problem were regularly treated by O. Kaleva in [5],[6] and by S. Seikkala in [7] and others. The numerical method for solving fuzzy differential equations is introduced by M. Ma, M. Friedman, A. Kandel in [11] by the standard Euler method. The 2[nd] Taylor method is applied by S. Abbasbandy and T. Allah viranloo in [14]. In section 2 some basic results on fuzzy numbers and definition of a fuzzy derivative ,which have been discussed by S. Seikkala in [11] are given. In section 3 we define the problem, this is a fuzzy Cauchy problem whose numerical solution and to increasing the accuracy of approximations is the main interest of this work. The extrapolation method is discussed in section 4. The proposed algorithm is illustrated by solving some examples in section 5 and conclusion is in section 6.

2. PRELIMINARIES

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)); & a \le t \le b, \\ y(a) = \alpha. \end{cases} \tag{1}$$

To apply extrapolation to solve initial value problems, a technique based on the Midpoint method is used

$$w_{i+1} = w_{i-1} + 2hf(t_i, w_i), \quad i \ge 1. \tag{2}$$

This technique requires two starting values, w_0 and w_1. As usual, one can use the initial condition for $w_0 = y(a) = \alpha$. To determine the second starting value, w_1, one can apply Euler's method. Subsequent approximations are obtained from (2). After a series of approximations of this type are generated ending at a value t, an endpoint correction is performed that involves the final two midpoint approximations. This produces an approximation $w(t,h)$ to $y(t)$ that has the form

$$y(t) = w(t,h) + \sum_{i=1}^{\infty} e_i h^{2i}, \tag{3}$$

where the e_i are constants related to the derivatives of the solutions $y(t)$. The important point is that the e_i do not depend on the step size h . Let us that we wish to approximate $y(t_1) = y(a+h)$. For the first extrapolation step we let $h_0 = \frac{h}{2}$ and use the Euler's method with $w(t_0) = \alpha$ to approximate

$$y(a+h_0) = y(a+\frac{h}{2}),$$

as

$$\begin{aligned} w_1 &= w_0 + h_0 f(a, w_0), \\ w_0 &= \alpha. \end{aligned} \tag{4}$$

Then we apply the Midpoint method with $t_0 = a$ and $t_1 = a + h_0 = a + \frac{h}{2}$ to produce a first approximation to $y(a+h) = y(a+2h_0)$,

$$w_2 = w_0 + 2h_0 f(a+h_0, w_1).$$

The endpoint correction is applied to obtain the final approximation to $y(a+h)$ for the stepsize h_0 This results is the $O(h_0^2)$ approximation for $y(t_1)$ and in this same way, that will be discussed and extended in section 4.

A triangular fuzzy number v is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $v(x)$,the membership function of the fuzzy number v , is a triangle with base on the interval $[a_1, a_3]$ and vertex at $x = a_2$.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level intervals. It means that if $v \in E$ then the r-level set

$$[v]_r = \{s \mid v(s) \geq r\}, \quad 0 < r \leq 1,$$

is a closed bounded interval which is denoted by

$$[v]_r = [v_1(r), v_2(r)].$$

Let I be a real interval. A mapping $y: I \to E$ is called a fuzzy process and its r-level set is denoted by

$$[y(t)]_r = [y_1(t;r), y_2(t;r)], \quad t \in I, \quad r \in (0,1].$$

Definition 2.1 - A function $s:[0,1] \to [0,1]$ is a reducing function if s is increasing and $s(0) = 0$ and $s(1) = 1$. We say that s is a regular reducing function if

$$\int_0^1 s(r)dr = \frac{1}{2}.$$

Definition 2.2 - If μ is a fuzzy number with r-cut representation, $(L(r),R(r))$, and if s is a reducing function then the value of μ (with respect to s) is defined by

$$val(\mu)=\int_0^1 s(r)(L(r)+R(r))dr.$$

Let be given the fuzzy numbers μ and v, we say that $\mu \leq v$ in case $val(\mu)\leq val(v)$.

Definition 2.3 - The fuzzy distance function on E, $\delta : E\times E \to E$, is defined by [13]

$$\delta\,(\mu,v)(z)=\sup_{|x-y|=z}\{\min(\mu(x),v(y)\}.$$

For notational simplicity we will let it by $\delta_{\mu v}$ for each pair of fuzzy numbers μ, v. It is not difficult to see that if $\mu, v\in E$, and if the r-cut representations of μ and v are $(a(r),b(r))$ and $(c(r),d(r))$, respectively, then the r-cut representation of $\delta_{\mu,v}$, $(L(r),R(r))$ is given by

$$L(r)=\begin{cases}\max\{c(r)-b(r),0\} & if \quad \frac{1}{2}(a(1)+b(1))\leq\frac{1}{2}(c(1)+d(1)),\\ \max\{a(r)-d(r),0\} & if \quad \frac{1}{2}(c(1)+d(1))<\frac{1}{2}(a(1)+b(1)),\end{cases} \tag{5}$$
$$R(r)=\max\{b(r)-c(r),d(r)-a(r)\}.$$

Lemma 2.1 - Let $v,w\in E$ and l scalar, then for any $r\in(0,1]$, [7],

$$[v+w]_r=[v_1(r)+w_1(r),v_2(r)+w_2(r)],$$
$$[v-w]_r=[v_1(r)-w_2(r),v_2(r)-w_1(r)],$$
$$[v.w]_r=[\min\{v_1(r).w_1(r),v_1(r).w_2(r),v_2(r).w_1(r),v_2(r).w_2(r)\},$$
$$\max\{v_1(r).w_1(r),v_1(r).w_2(r),v_2(r).w_1(r),v_2(r).w_2(r)\}],$$
$$[l.v]_r=l.[v]_r.$$

3. A FUZZY CAUCHY PROBLEM

Consider the fuzzy initial value problem

$$\begin{cases}y'(t)=f(t,y(t)),\ t\in I=[0,T],\\ y(0)=\alpha,\end{cases} \tag{6}$$

where f is a continuous mapping from $R_+\times R$ into R and $\alpha\in E$ with r-level intervals

$$[y_1(0;r),y_2(0;r)],\quad r\in(0,1].$$

The problem (6) is said to have a solution $y(t)$ on I if $y(t)$ is absolutely continuous and satisfies in (6). The extension principle of Zadeh leads to the following definition of $f(t,y)$ when $y=y(t)$ is a fuzzy number

$$f(t,y)(s)=\sup_{\tau\in f^{-1}(t,s)}\{y(\tau)\,|\,s=f(t,\tau)\},\quad s\in R.$$

It follows that

$$[f(t,y)]_r=[f_1(t,y;r),f_2(t,y;r)],\quad r\in(0,1],$$

where

$$\begin{aligned}f_1(t,y;r)&=\min\{f(t,u)\,|u\in[y_1(r),y_2(r)]\},\\ f_2(t,y;r)&=\max\{f(t,u)\,|u\in[y_1(r),y_2(r)]\}.\end{aligned}\tag{7}$$

Theorem 3.1 - Let f satisfy

$$|f(t,v)-f(t,\bar{v})|\le g(t,|v-\bar{v}|),\quad t\ge 0,\quad v,\bar{v}\in R,$$

where $g(\cdot):R_+\times R_+\to[0,+\infty)$ is a continuous mapping such that $r\to g(t,r)$ is nondecreasing, the initial value problem

$$u'(t)=g(t,u(t)),\quad u(0)=u_0,\tag{8}$$

has a solution on R_+ for $u_0>0$ and that $u(t)=0$ is the only solution of (8) for $u_0=0$. Then the fuzzy initial value problem (6) has a unique fuzzy solution.
Proof [7].

In this work we suppose (6) satisfies the hypothesis of theorem 3.1 From [11],

$$\begin{aligned}w_1(t+h;r)&=w_1(t;r)+hF(t,w_1(t;r),w_2(t;r)),\\ w_2(t+h;r)&=w_2(t;r)+hG(t,w_1(t;r),w_2(t;r)),\end{aligned}\tag{9}$$

$$\begin{aligned}y_1(t+h;r)&\approx y_1(t;r)+hF(t,y_1(t;r),y_2(t;r)),\\ y_2(t+h;r)&\approx y_2(t;r)+hG(t,y_1(t;r),y_2(t;r)),\end{aligned}\tag{10}$$

Since the difference methods for the exact solutions are not hold as equality, where

$$\begin{aligned}F(t,u,v)&=\min\{f(t,y)\,|\,y\in[u,v]\},\\ G(t,u,v)&=\max\{f(t,y)\,|\,y\in[u,v]\}.\end{aligned}\tag{11}$$

The following theorem will be applied to show convergence of these approximates ,i.e.,

$$\lim_{h\to 0}w_1(t;r,h)=y_1(t;r),$$
$$\lim_{h\to 0}w_2(t;r,h)=y_2(t;r).$$

Let

$$K=\{(t,u,v)\,|0\le t\le T,\ -\infty<v<\infty,-\infty<u\le v\}.$$

Theorem 3.2 - Let $F(t,u,v)$ and $G(t,u,v)$ be the functions belong to $C^1(K)$, where u and v are constants and $u\le v$. Then, for arbitrary fixed r, $0\le r\le 1$, the approximately solutions (6) converge to the exact solutions $y_1(t;r)$ and $y_2(t;r)$ uniformly in t.
Proof see [11].

4. EXTRAPOLATION METHOD

Let the exact solution $y(t;r)=[y_1(t;r),y_2(t;r)]$ of (6) is approximated by $w(t;r)=[w_1(t;r),w_2(t;r)]$. For $r\in(0,1]$, suppose

$$y_0=[\alpha_1(r),\alpha_2(r)]. \tag{12}$$

Let us assume the fixed step size h and we wish to approximate $y_i(t_1;r)=y_i(a+h;r)$, for any $r\in(0,1]$ and $i=1,2$. For the first extrapolation step we let $h_0=\frac{h}{2}$ and use (9) which approximate

$$y_i(a+h_0;r)=y_i(a+\frac{h}{2};r)$$

as

$$\begin{aligned} w_{i,1}(r)&=w_{i,0}(r)+h_0f_i(a,w_0;r), \\ w_{i,0}(r)&=\alpha_i(r), \end{aligned} \tag{13}$$

for any $r\in(0,1]$ and $i=1,2$, where $[w_j]_r=[w_{1,j}(r),w_{2,j}(r)]$ for any j see [11], and

$$\begin{aligned} f_1(t_n,w(t_n);r)&=\min\{f(t,u)\,|\,u\in[w_1(t_n;r),w_2(t_n;r)]\}, \\ f_2(t_n,w(t_n);r)&=\max\{f(t,u)\,|\,u\in[w_1(t_n;r),w_2(t_n;r)]\}. \end{aligned}$$

We then apply the Midpoint with $t_0=a$ and $t_1=a+h_0=a+\frac{h}{2}$ to produce a first approximation to $y_i(a+h;r)=y_i(a+2h_0;r)$,

$$w_{i,2}(r)=w_{i,0}(r)+2h_0f_i(a+h_0,w_1;r),$$

for any $r\in(0,1]$ and $i=1,2$. The endpoint correction is applied to obtain the final approximation to $y_i(a+h;r)$ for any $r\in(0,1]$ and $i=1,2$; and the stepsize h_0. This results in the $O(h_0^2)$ approximation to $y_i(t_1;r)$

$$y_{i,1}^1(r)=\frac{1}{2}[w_{i,2}(r)+w_{i,1}(r)+h_0f_i(a+2h_0,w_2;r)],$$

for any $r\in(0,1]$ and $i=1,2$; where $[y_1^1]_r=[y_{1,1}^1(r),y_{2,1}^1(r)]$. We save the approximation $y_{i,1}^1(r)$ and discard the intermediate results $w_{i,1}(r)$ and $w_{i,2}(r)$, for any $r\in(0,1]$ and $i=1,2$. To obtain the next approximation, $y_{i,2}^1(r)$, to $y(t_1)$ we let $h_1=\frac{h}{4}$ and use Euler's method initial values to obtain an approximation to $y_i(a+h_1;r)=y_i(a+\frac{h}{4};r)$, that we will call $w_{i,1}(r)$

$$\begin{aligned} w_{i,1}(r)&=w_{i,0}(r)+h_1f_i(a,w_0;r), \\ w_{i,0}(r)&=\alpha_i(r), \end{aligned} \tag{14}$$

for any $r\in(0,1]$ and $i=1,2$.

Next we produce approximations $w_{i,2}(r)$ to

$$y_i(a+2h_1;r)=y_i(a+\frac{h}{2};r) \text{ and } y_i(a+3h_1;r)=y_i(a+\frac{3h}{4};r)$$

given by

$$w_{i,2}(r)=w_{i,0}(r)+2h_1 f_i(a+h_1,w_1;r),$$

and

$$w_{i,3}(r)=w_{i,1}(r)+2h_1 f_i(a+2h_1,w_2;r),$$

for any $r\in(0,1]$ and $i=1,2$. Then we produce the approximation $w_{i,4}(r)$ to $y_i(a+4h_1;r)=y_i(t_1;r)$ given by

$$w_{i,4}(r)=w_{i,2}(r)+2h_1 f_i(a+3h_1,w_3;r)$$

for any $r\in(0,1]$ and $i=1,2$. The endpoint correction is now applied to $w_{i,3}(r)$ and $w_{i,4}(r)$ to produce the improved $O(h_1^2)$ approximation to $y_i(t_1;r)$,

$$y_{i,2}^1(r)=\frac{1}{2}[w_{i,4}(r)+w_{i,3}(r)+h_1 f_i(a+4h_1,w_4;r)],$$

for any $r\in(0,1]$ and $i=1,2$.

The two approximations to $y_i(a+h;r)$ have the property that

$$y_1(a+h;r)=y_{1,1}^1(r)+e_{1,1}(r)(\frac{h}{2})^2+e_{1,2}(r)(\frac{h}{2})^4+\ldots=y_{1,1}^1(r)+e_{1,1}(r)\frac{h^2}{4}+e_{1,2}(r)\frac{h^4}{16}+\ldots, \quad (15)$$

$$y_2(a+h;r)=y_{2,1}^1(r)+e'_{1,1}(r)(\frac{h}{2})^2+e'_{1,2}(r)(\frac{h}{2})^4+\ldots=y_{2,1}^1(r)+.e'_{1,1}(r)\frac{h^2}{4}+e'_{1,2}(r)\frac{h^4}{16}+\ldots, \quad (16)$$

$$y_1(a+h;r)=y_{1,2}^1(r)+e_{1,1}(r)(\frac{h}{4})^2+e_{1,2}(r)(\frac{h}{4})^4+=y_{1,2}^1(r)+e_{1,1}(r)\frac{h^2}{16}+e_{1,2}(r)\frac{h^4}{256}+\ldots, \quad (17)$$

$$y_2(a+h;r)=y_{2,2}^1(r)+e'_{2,1}(r)(\frac{h}{4})^2+e'_{2,2}(r)(\frac{h}{4})^4+\ldots=y_2^1(t_2;r)+e'_{2,1}(t;r)\frac{h^2}{16}+e'_{2,2}(t;r)\frac{h^4}{256}+..(18)$$

and

for any $r\in(0,1]$, where $[e_j]_r=[e_{1,j}(r),e_{2,j}(r)]$. We can eliminate the $O(h^2)$ portion of this truncation error by averaging these two formulas appropriately, for any $r\in(0,1]$. Specifically, if we subtract (15) from 4 times (17) and divide the result by 3 and repeat this process for (16) and (18) we have

$$y_1(a+h;r)=y_{1,2}^1(r)+\frac{1}{3}(y_{1,2}^1(r)-y_{1,1}(r)^1)-e_{2,1}(r)\frac{h^4}{64}+\ldots,$$
$$y_2(a+h;r)=y_{2,2}^1(r)+\frac{1}{3}(y_{2,2}^1(r)-y_{2,1}(r)^1)-e'_{2,2}(r)\frac{h^4}{64}+\ldots.$$

So the approximation

$$y_{i,2}^2(r)=y_{i,2}^1(r)+\frac{1}{3}(y_{i,2}^1(r)-y_{i,1}^1(r))$$

for any $r \in (0,1]$ and $i=1,2$, has error of order $O(h^4)$. Continuing in this manner, we next let $h_2 = \frac{h}{6}$ and apply Euler method once followed b. Then we use the endpoint correction to determine the h^2 approximation that we denote $y_{i,3}^1(r)$, to $y_i(a+h;r)$, this approximation can be averaged with $y_{i,2}^1(r)$ to produce a second $O(h^4)$ approximation that we denote $y_{i,3}^2(r)$ for $i=1,2$. Then $y_{i,3}^2(r)$ and $y_{i,2}^2(r)$ are averaged to eliminate the $O(h^4)$ error terms and produce an approximation with error of order $O(h^6)$. Higher-order formulas are generated by continuing the process. The error is controlled by requiring that the approximations $y_{i,1}^1(r), y_{i,2}^2(r), \ldots$ be computed until $|y_{1,i}^i(r) - y_{1,i-1}^{i-1}(r)|$ and $|y_{2,i}^i(r) - y_{2,i-1}^{i-1}(r)|$ is less than a given tolerance, for $i=1,2$. If $y_{1,i}^i(r)$ and $y_{2,i}^i(r)$ are found to be acceptable, then $w_{1,1}(r)$ and $w_{2,1}(r)$ is set to $y_{1,i}^i(r)$ and $y_{2,i}^i(r)$ respectively, and computations begin again to determine $[w_{1,2}(r), w_{2,2}(r)]$ which will approximate $y_{i,2} = y_i(a+2h;r)$ for any $r \in (0,1]$ and $i=1,2$. The process is repeated until $w_{1,N}(r)$ and $w_{2,N}(r)$ approximate $y_1(b;r)$ and $y_2(b;r)$, for any $r \in (0,1]$.

5. EXAMPLES

Example 5.1 - Consider the fuzzy initial value problem, [11],

$$\begin{cases} y' = y(t), \quad t \in I = [0,1], \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), \quad 0 < r \leq 1. \end{cases}$$

Using the Euler approximation with $N=10$ we obtain

$$w_1(1;r) = (0.75 + 0.25r)(1 + \frac{1}{10})^{10},$$

$$w_2(1;r) = (1.125 + 0.125r)(1 + \frac{1}{10})^{10}.$$

The exact solution is given by

$$y_1(t;r) = y_1(0;r)e^t \quad , \quad y_2(t;r) = y_2(0;r)e^t,$$

which at $t=1$

$$Y(1;r) = [(0.75 + 0.25r)e \quad , \quad (1.125 - 0.125r)e], \quad 0 < r \leq 1.$$

The distance between the Euler fuzzy number and the Exact fuzzy number is a fuzzy number that we called it μ and also the distance between the Extrapolation method fuzzy number and the Exact fuzzy number is a fuzzy number that we called it ν. One can see that

$$val(\mu) = 26.7841 \quad , \quad val(\nu) = 8.6646.$$

The exact and Euler and Extrapolation solutions are compared in Table 1.

Table 1, Tolerance $=0.001$, $h=0.5$

r	Euler	Extra	Exact
0.0	(1.687500,2.531250)	(2.038710,3.058064)	(2.03871,3.058067)
0.1	(1.743750,2.503125)	(2.106667,3.024086)	(2.106668,3.024089)
0.2	(1.800000,2.475000)	(2.174623,2.990107)	(2.174625,2.990110)
0.3	(1.856250,2.446875)	(2.242580,2.956129)	(2.242583,2.956131)
0.4	(1.912500,2.418750)	(2.310537,2.922150)	(2.310540,2.922153)
0.5	(1.968750,2.390625)	(2.378494,2.888172)	(2.378497,2.888174)
0.6	(2.025000,2.362500)	(2.446451,2.854193)	(2.446454,2.854196)
0.7	(2.081250,2.334375)	(2.514408,2.820215)	(2.514411,2.820217)
0.8	(2.137500,2.306250)	(2.582365,2.786236)	(2.582368,2.786239)
0.9	(2.193750,2.278125)	(2.650322,2.752258)	(2.650325,2.752260)
1	(2.250000,2.250000)	(2.718279,2.718279)	(2.718282,2.718282)

Example 5.2 - Consider the fuzzy initial value problem, [7]

$$\begin{cases} y'=[y(t)]^2, & t\geq 0, \\ y(0)=(r,1-\ln(r)), & 0<r\leq 1. \end{cases}$$

The exact fuzzy solution $y(t)$ is defined on $[0,t_\beta]$ with $t_\beta=\dfrac{1}{1-\ln(\beta)}$ by

$$[y(t)]_r=[y_1(t;r),y_2(t;r)], \quad \beta\leq r\leq 1,$$

Where

$$y_1(t;r)=\frac{r}{1-rt} \quad , \quad y_2(t;r)=\frac{1-\ln(r)}{1-(1-\ln(r))},$$

and $t_\beta\to 0$ as $\beta\to 0^+$, See [7]. The exact and Euler and Extrapolation solutions are compared in Table 2.

Table 2, $t_\beta = 0.2$, Tolerance $= 0.001$, $h = 0.1$

R	Euler	Extra	Exact
0.0	(0.000000,Inf)	(0.00000000,Inf)	(0.000000,Inf)
0.1	(0.102020,6.323393)	(0.102041,9.728272)	(0.102041,9.728279)
0.2	(0.208162,4.372998)	(0.208333,5.457783)	(0.208333,5.457792)
0.3	(0.318548,3.413183)	(0.319149,3.941250)	(0.319149,3.941258)
0.4	(0.433306,2.804949)	(0.434783,3.107118)	(0.434783,3.107120)
0.5	(0.552563,2.371791)	(0.555556,2.560058)	(0.555556,2.560058)
0.6	(0.676450,2.041527)	(0.681818,2.165019)	(0.681818,2.165019)
0.7	(0.805100,1.778117)	(0.813953,1.861859)	(0.813953,1.861864)
0.8	(0.938650,1.561196)	(0.952381,1.619256)	(0.952381,1.619261)
0.9	(1.077236,1.378229)	(1.097560,1.419077)	(1.097561,1.419079)
1	(1.221000,1.221000)	(1.249999,1.249999)	(1.250000,1.250000)

6. CONCLUSION

The order of convergence Euler's Method is $O(h)$, [11]. Now in this method we can get the higher-order of convergence, so that $| y_1^i(t_i;r) - y_1^{i-1}(t_{i-1};r)|$ and $| y_2^i(t_i;r) - y_2^{i-1}(t_{i-1};r)|$ are less than a given tolerance. Therefor we can improve the solution of FIVP arbitrary.

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