Mathematical & Computational Applications, Vol. 9, No. 2, pp. 173-182, 2004

CROSSED MODULES OF ALGEBRAS

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Abstract - In this paper we will give some algebraic results of crossed modules of algebras.
Keywords - Commutative Algebras, Crossed Modules and Koszul Complexes.

1.INTRODUCTION

Crossed modules were invented by J.H.L.Whitehead in his work on combinatorial homotopy theory [14]. They have found important roles in many areas of mathematics (including homotopy theory, homology and cohomology of groups, algebraic K-theory, cyclic homology, combinatorial group theory, and differential geometry). Possible crossed modules should now be considered one of the fundamental algebraic structures.

Crossed modules of algebras are generalisation of both modules and ideals any rings (algebras) is a crossed module, so it is of interest to see generalisation of ring (algebra) theoretic concepts and structures to crossed modules. The commutative algebra version of crossed modules has been used, in essence rather than in name by Lichtenbaum-Schlessinger [8] also work of Gerstenhaber [6] essentially involves the notion of crossed modules in associative and commutative algebras, c.f. Lue [9]. The free crossed modules are related to Koszul complex construction (c.f. Porter [11], [12] and Arvasi and Porter [3]) and higher dimensional analogues have been proposed by Ellis [5] for use in homotopical and homological algebras.

Although the general theory of crossed modules of algebras (not necessarily commutative) does not exist in print yet, except Nizar, [10], our aim in this notes is to introduce the algebraic results in this theory. We will look at the substructures and ideals of crossed modules. To form factor crossed modules we need to work that out with some conditions on the ideals. This paper also contains the factorisation theorem of morphisms between crossed modules.

2. CROSSED MODULES AND EXAMPLES

J.H.C.Whitehead (1949) [14] described crossed modules in various contexts especially in his investigations into the algebraic structure of relative homotopy groups. In this section, we recall the definition and elementary theory of crossed modules of commutative algebras given by T.Porter, [11]. More details about this may be found in [10] and [5].

We recall that if M and R are algebra, a map

$$R\times M \longrightarrow M$$

$$(r,m) \longrightarrow r\cdot m,$$

is a left action if and only if

1. $k(r\cdot m)=(kr)\cdot m=r\cdot(km)$,
2. $r\cdot(m+m')=r\cdot m+r\cdot m'$,
3. $(r+r')\cdot m=r\cdot m+r'\cdot m$,
4. $r\cdot(mm')=(r\cdot m)m'=m(r.m')$,

5. $(rr')\cdot m = r(r'\cdot m)$,

for all $k \in \mathbf{k}$, $m, m' \in M$, $r, r' \in R$. A right action can be defined in a similar way.
Throughout this thesis we denote a left action and a right action of $r \in R$ on $m \in M$ by $r\cdot m$ and by $m \cdot r$ respectively.
Let R be a $\mathbf{k}$-algebra with identity. A pre-crossed module of algebras is an R-algebra C, together with an action of R on C and an R-algebra morphism

$$\partial : C \to R,$$

such that for all $c \in C$, $r \in R$

CM1) $\partial(r\cdot c) = r\partial(c)$, and $\partial(c\cdot r) = \partial(c)r$

This is a crossed R-module if in addition, for all $c, c' \in C$,

CM2) $\partial c \cdot c' = cc'$, and $c \cdot \partial c' = cc'$

The last condition is called the Peiffer identity. We denote such a crossed module by (C, R, ∂). Clearly any crossed module is a pre-crossed module.
A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of $\mathbf{k}$-algebra morphisms,

$$\theta : C \to C', \quad \psi : R \to R'$$

such that

$$\theta(r.c) = \psi(r)\cdot\theta(c), \theta(c\cdot r) = \theta(c)\cdot\psi(r) \text{ and } \partial'\theta(c) = \psi\partial(c).$$

In this case, we shall say that θ is a crossed R-module morphism if $R = R'$ and ψ is the identity. We therefore can define the category of crossed modules denoting it as **Xmod.**
Clearly the composition of two maps of crossed modules over R is a map of crossed R-modules. Thus we get a subcategory **XMod**$/R$ of **XMod**

2.1. Examples

1. Let I be any ideal of a k-algebra R. Consider an inclusion map

$$\text{inc.} : I \to R.$$

Then $(I, R, inc.)$ is a crossed module. Conversely given any crossed R-module $\partial : C \to R$, one can easily verify that $\partial C = I$ is an ideal in R.

2. Let M be any R-bimodule. It can be considered as an R-algebra with zero multiplication, and then $\mathbf{0} : M \to R$ is a crossed R-module by $\mathbf{0}(c)\cdot c' = 0c' = 0 = cc'$, for all $c, c' \in M$.
Conversely, given any crossed module $\partial : C \to R$, then $\text{Ker}\,\partial$ is an $R/\partial C$-module. For this, see Proposition 1

3. A simplicial algebra $\mathbf{E}$ consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps

$$d_i = d_i^n : E_n \to E_{n-1}, \ 0 \le i \le n, (n \ne 0) \text{ and } s_i = s_i^n : E_n \to E_{n+1}, \quad 0 \le i \le n,$$

satisfying the usual simplicial identities given in [3] for example. It can be completely described as a functor $\mathbf{E} : \Delta^{op} \to \mathbf{Alg}_k$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \ldots < n\}$ and increasing maps.
Given a simplicial algebra $\mathbf{E}$ and a simplicial ideal $\mathbf{I}$. The inclusion

$$\text{inc.}:\mathbf{I}\longrightarrow\mathbf{E},$$

induces a map

$$\partial:\pi_0(\mathbf{I})\to\pi_0(\mathbf{E}),$$

and $\mathbf{E}$ acting on $\mathbf{I}$ by multiplication induces an action of $\pi_0(\mathbf{E})$ on $\pi_0(\mathbf{I})$. Then $(\pi_0(\mathbf{I}),\pi_0(\mathbf{E}),\partial)$ is a crossed module.

4. Suppose that R is the algebra $\mathrm{Aut}(C)$ of automorphisms of some algebras C. Then the homomorphism $C\to R$ which sends an element $x\in C$ to the inner automorphism $C\to C,\ c\to xc$ is a crossed module.

3. SOME BASIC ALGEBRAIC PROPERTIES OF CROSSED MODULES

The following results prove consequences of the definition of crossed modules and state some properties of those algebras. (see [2])

Proposition 1 *[2] If (C,R,∂) is a crossed $R-$module, then*
i) $Ker\partial$ is a central ideal of C,
ii) both C/C^2 and $Ker\partial$ have natural $R/\partial C-$module structure.

Proof. [2] i) Since, for $c\in C, a\in Ker\partial$,

$$ac=\partial a\cdot c=0c=0=c0=c\cdot\partial a=ca$$

as required.

ii) It is enough to show that ∂C acts trivially on $\mathrm{Ker}\partial$ and C/C^2.

For $a\in Ker\partial,\partial c\in\partial C,$ by $\partial c\cdot a=ca=c\cdot\partial a=c0=0,$ ∂C acts trivially on $\mathrm{Ker}\partial$.

For $\partial c\in\partial C, c'+C\in C/C^2$, we obtain the following

$$\begin{aligned}\partial c\cdot(c'+C^2)&=\partial c\cdot c'+C^2\\&=cc'+C^2\\&=0,\end{aligned}$$

so ∂C acts trivially on C/C^2. Hence we can unambiguously define maps

$$\begin{array}{ll}R/\partial C\times Ker\partial\to Ker\partial & R/\partial C\times C/C^2\to C/C^2\\ (r+\partial c,a)\mapsto ra & (r+\partial c,c+C^2)\mapsto rc+C^2\end{array}$$

and it is routine to check that this turns the abelian groups $Ker\partial$ and C/C^2 into $R/\partial C-$modules. Thus $\mathrm{Ker}\partial$ and C/C^2 have $R/\partial C-$module structure.

Let M',M and M'' be algebras. Two maps

$$M'\xrightarrow{f}M\xrightarrow{g}M'$$

are exact at M if $\mathrm{Im}\,f=Kerg$. A sequence of maps (perhaps infinitely long)

$$\cdots\longrightarrow M_{n+1}\xrightarrow{f_{n+1}}M_n\xrightarrow{f_n}M_{n-1}\longrightarrow\cdots$$

is exact if each adjacent pair of maps is exact.

As a result of the previous result we have two exact sequences:

$$0\to Ker\partial\to C\to\mathrm{Im}\partial\to 0$$

and

$$0\to\mathrm{Im}\partial\to R\to R/\mathrm{Im}\partial\to 0$$

Proposition 2 *The exact sequence*

$$0\to Ker\partial\to C\to\mathrm{Im}\partial\to 0$$

induces the following exact sequence

$$Ker\partial \to C/C^2 \to I/I^2 \to 0$$

where $I = Im\partial$.

Proof. To prove the above sequence is exact we need to show that: (i) the morphism $\bar{\partial}: C/C^2 \to I/I^2$ is onto, (ii) Ker∂ maps onto the kernel of $\bar{\partial}$ i.e. each element $c + C^2$ in kernel $\bar{\partial}$ is of the form $k + C^2$, for some $k \in$ Ker∂ We know that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}\partial \arrow[r] & C \arrow[r, "\partial"] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\ & & C/C^2 \arrow[r, "\bar{\partial}"'] & I/I^2 \arrow[r] & 0 \end{tikzcd}$$

is commutative, and ∂ is onto, then $\bar{\partial}$ is onto, and the image of Ker$\partial \to C/C^2$ is contained in Ker$\bar{\partial}$

(ii) if $c + C^2 \in$ Ker$\bar{\partial}$, then

$$\bar{\partial}(c + C^2) = \partial(c) + I^2 = I^2$$

and $\partial(c) \in I^2$. Thus $\partial(c) = \partial(b)\partial(b') = \partial(bb')$, for some $b, b' \in C$. This implies that $(c - bb') \in Ker\partial$, i.e. $(c - bb') = k$, for some $k \in Ker\partial$, but then $c + C^2 = k + C^2$, so Ker∂ mapped onto Ker$\bar{\partial}$.

Proposotion 3 *Let* $\psi : (C, R, \partial) \to (B, R, \beta)$ *be a morphism of crossed* $R-$*modules. Then* (C, B, ψ) *is a crossed* $B-$*module, where* B *acts on* C *via* β.

Proof. We have the following commutative diagram

$$\begin{tikzcd} C \arrow[rr, "\varphi"] \arrow[dr, "\partial"'] & & B \arrow[dl, "\beta"] \\ & R & \end{tikzcd}$$

where ψ is a morphism of $R-$algebras and B acts on C via β, i.e. for $c \in C$ and $b \in B$ we have

$cb = c \cdot \beta(b)$ and $bc = \beta(b) \cdot c$.

Now we need to check that ψ is a morphism of $B-$algebras, and satisfies the conditions CM1, and CM2. Let $c \in C$, and $b \in B$ then we have

$$\begin{aligned}\psi(c\cdot b) &= \psi(c\beta(b))\\ &= \psi(c)\beta(b)\\ &= \psi(c)b.\end{aligned}$$

Also for $c,c'\in C$,

$$\begin{aligned}\varphi c\cdot c' &= \beta\varphi(c)\cdot c'\\ &= \partial(c)\cdot c'\\ &= cc'.\end{aligned}$$

Similarly $c\cdot\psi c' = cc'$. Thus the axioms of a crossed modules are satisfied.

Thus by 2.1 $\psi(C)$ is an ideal in B.

Proposotion 4 *Let (C,B,∂) be a crossed $B-$module and (B,R,β) be a crossed $R-$module such that R acts on C where the action is compatible with $B-$action on C, then $(C,R,\beta\partial)$ is a crossed $R-$module.*

Proof. The only thing we need to check is the Peiffer identity. If $c,c'\in C$, then $c\cdot(\beta\partial c') = c\cdot\beta(\partial c') = c\cdot\partial c' = cc'$.

Thus ∂ is a crossed module. Similarly $(\beta\partial c)\cdot c' = cc'$.

4. SUBCROSSED MODULES, CROSSED IDEALS AND FACTOR CROSSED MODULES

In regard to mathematical structures, the substructures, subgroups, subfields and subspaces of topological spaces, generally play an important role. In the investigation of crossed modules the subcrossed modules and crossed modules and crossed ideals, both of which are about to be defined to be defined, are correspondingly important.

A subcrossed module of a crossed $R-$module (C,R,∂) is a crossed $R-$module (C',R,∂') such that C' is a subalgebra of C, and $\partial' = \partial|_{C'}: C'\to R$, the restriction of ∂ to C'.

4.1. Remarks

1. Let C' be a subcrossed module of C, then for $c\in C$ and $x\in C', cx = \partial c\cdot x\in C'$ similarly, $xc\in C'$. Therefore C' is an ideal in C. Thus a subcrossed module is an ideal C' of C with restriction $\partial' = \partial|_{C'}: C'\to R.$

2. A subcrossed module should be a subobject in the categorical sense. For this we need to check that the inclusion of into is a monomorphism of crossed module. This is easy and obvious from definition.

4.2. Examples

Any ideal I of the ring R will give a subcrossed module (I,R,inc) of the crossed module (R,R,id_R).

The submodules of any module M over R, considered as a crossed module $(M,R,0)$ are subcrossed modules of M.

Our aim is to recall the definition of the "normal" subcrossed modules of a crossed module C in **XMod/R**. We will call them "crossed ideals". It is well-known that in ring theory kernels and ideals are the same, i.e., each ideal I of a ring R is the kernel of the canonical homomorphism $\nu: R\to R/I$ of rings, and each kernel of a ring homomorphism is an ideal. Suppose here we define a crossed ideal, to be a subcrossed module C' of C which is a kernel of the morphism

$$\begin{tikzcd} C \arrow[rr, "\nu"] \arrow[dr, "\partial"'] & & C/C' \arrow[dl, "\bar{\partial}"] \\ & R & \end{tikzcd}$$

of crossed $R-$modules, where $\nu(c)=c+C'$ and $\bar{\partial}(c+C')=\partial(c)$, for $c\in C$.

Clearly $\bar{\partial}$ is well defined if and only if C' is contained in $\operatorname{Ker}\partial$ since, if $c+C'=c'+C'$, then $(c-c')\in C'$ thus $c=c'+x$ for some $x\in C'$ and so $\partial(c)=\partial(c')$ if and only if $x\in Ker\partial$ Therefore the crossed ideals of the crossed module $\partial: C\longrightarrow R$ are all those subcrossed modules C' of C which are contained in $Ker\partial$. Also from 2.1 and 2.3 , the kernel of any morphism

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\partial"'] & & B \arrow[dl, "\beta"] \\ & R & \end{tikzcd}$$

of crossed modules is a subcrossed module of C and is contained in $Ker\partial$ since $Bf=\partial$.

Thus we get following result:

Proposition 5 *Let* (C,R,∂) *be a crossed* $R-$*modules and* (C',R,∂') *be a subcrossed module of* (C,R,∂) *, then* $(C/C',R,\bar{\partial})$ *is a factor crossed* $R-$*module in* $\mathbf{XMod}/R$ *if and only if* C' *is contained in* $Ker\partial$ *where* $\bar{\partial}$ *is given by,* $\bar{\partial}(c+C')=\partial_c$ *for* $c\in C$.

Note that factor crossed modules are not always defined in $\mathbf{XMod}/R$ e.g in the following commutative

$$\begin{tikzcd} C \arrow[rr, "\mathrm{id}_C"] \arrow[dr, "\partial"'] & & C \arrow[dl, "\partial"] \\ & R & \end{tikzcd}$$

the factor C/C is not a crossed $R-$module.

Also if $f:C \longrightarrow B$ is a morphism of crossed R-modules such that $(B/\mathrm{Im}\, f, \overline{\beta})$ is a factor crossed module of B modulo $\mathrm{Im}\, f$ then $\mathrm{Im}\, f \subseteq Ker\beta$ and thus $\partial = 0$ so C will be an $R-$module.

Proposition 6 *Each morphism of crossed R-modules in* ***XMod/R*** *can be uniquely factorised as a regular epimorphism followed by monomorphism.*

Proof. Let $f:C \longrightarrow B$ be a morphism of crossed $R-$modules. Since $Kerf \subseteq Ker\partial$ therefore we can define the morphism $\overline{\partial}: C/Kerf \longrightarrow R$ by $\overline{\partial}(c + Kerf) = \partial_c$ which gives $C/Kerf$ the structure of a crossed $R-$module. Thus we have a canonical morphism $p: C \longrightarrow C/Kerf$ of crossed $R-$modules. We will show that this morphism is a regular epimorphism. Let

$$T = C \times Kerf = \{(c,m) : c \in C\} \text{ and } m \in Ker\partial$$

be the semidirect product, we will define a crossed $R-$module structure on T. Define the multiplication as follows, for any (c,m) and $(c',m') \in T$

$$\begin{aligned}(c,m)(c',m') &= (cc', cm' + mc')\\ &= (cc',0), \text{ since } mc' = \partial_m \cdot c = 0 = cm'\end{aligned}$$

The ring R acts on T by obvious way, and the morphism $\tau: T \longrightarrow R$ given by $\tau(cm) = \partial_c$ satisfied the rules CM1 and CM2. Since

$$\begin{aligned}\tau(c,m).(c',m') &= \partial c(c',m')\\ &= (\partial c \cdot c', \partial c \cdot m')\\ &= (cc',0)\\ &= (c,m)(c',m').\end{aligned}$$

Thus (T,τ) is a crossed $R-$module.

Define two morphism $s,t: T \longrightarrow C$ by $s(c,m) = c$ and $t(c,m) = c + m$ These two morphism are morphisms of crossed R-modules such that $ps = pt$

Note that $s((c,m)(c',m')) = s(cc',0) = cc' = s(c,m)s(c',m')$ and

$$\begin{aligned}t((c,m)(c',m')) &= t(cc',0) = cc'\\ &= cc' + mm' + cm' + mc'\\ &= (c+m)(c'+m')\\ &= t(c,m)t(c',m').\end{aligned}$$

Thus the morphisms s and t are $R-$algebra morphisms.

Suppose there is a morphism $g: C \longrightarrow D$ of crossed R-modules, such that $gs = gt$ then there is a unique morphism $g': C/Kerf \longrightarrow D$ given by $g'(c + Kerf) = g(c)$.

Note that the morphism $f: C \longrightarrow B$ satisfies the condition $fs = ft$ therefore there is a unique morphism $\mu: C/Kerf \longrightarrow B$ given by $\mu(c + Kerf) = g(c)$

We will now check that the morphism μ is a monomorphism. Suppose there are two morphism $h,h': X \longrightarrow C/Kerf$ of crossed $R-$modules such that $\mu h = \mu h'$ Suppose

that $h \neq h'$ then there exists $x \in X$ such that $h(x) = x + Kerf \neq h'(x) = x' + Kerf'$ for some $x, x' \in X$ but $\mu h(x) = \mu h'(x)$ i.e. $\mu(x + Kerf) = \mu(x' + Kerf)$ that is Thus $(x - x') \in Kerf$ so $x = x' + m$ for some $m \in Kerf$
hence $x + Kerf = (x' + m) + x' + Kerf$ Hence μ is a monomorphism. Thus $f = \mu p$.

Theorem 7 *Let $f . C \longrightarrow B$ be a morphism of crossed $R-$modules in* **XMod**$/R$ *. Then*

$$C/Kerf \cong \operatorname{Im} f$$

Proof. The isomorphism can be defined as follows

$$\theta \ : \ C/Kerf \longrightarrow \operatorname{Im} f$$
$$c + Kerf \mapsto f(c) .$$

4.3 Examples

(1) The crossed ideals of the crossed $R-$modules $(R, R, 0)$ are all the ideals, I, of the ring R considered as crossed module $(I, R, 0)$ with the inclusion $inc. : I \longrightarrow R$ as a morphism of crossed $R-$modules.

(2) The submodules of an $R-$module M considered as a crossed $R-$module $(M, R, 0)$ are crossed ideals.

(3) There are subcrossed modules which are not crossed ideals. Consider the crossed module $(\mathbf{Z}, id_Z)$ and $(n\mathbf{Z}, inc.)$ be any subcrossed module with $n \in \mathbf{Z}$ The following diagram

$$\begin{tikzcd} n\mathbf{Z} \arrow[rr, "\text{inc}"] \arrow[dr, "\partial"'] & & \mathbf{Z} \arrow[dl, "\beta"] \\ & \mathbf{Z} & \end{tikzcd}$$

commutes. Also $n\mathbf{Z}$ by no way could be a kernel of a morphism of crossed $\mathbf{Z}-$modules.

A subcrossed module of a crossed module (C, R, ∂) in **XMod** consists of

(i) a subalgebra C' of C and a subring R' of R

(ii) an action of R' on C' induced by the action of R on C

(iii) (C', R', ∂') is a crossed $R-$module

(iv) The following diagram of morphisms of crossed modules in **XMod**

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "\partial'"'] & C \arrow[d, "\partial"] \\ R' \arrow[r, "v"'] & R \end{tikzcd}$$

commutes, where u and v are the inclusions.
A subcrossed module (C',R',∂') of a crossed module (C,R,∂) will be called a crossed ideal in **XMod** if
(i) $C'C \cup CC' \subseteq C'$ and R' is an ideal in R
(ii) $CR' \cup R'C \subseteq C'$,
(ii) C' is closed under the action of R i.e., $RC', C'R \subseteq C'$.

4.4. **Examples**

1. The crossed module (R,R,inc) has subcrossed modules given by all pairs (I,J) where I is an ideal, J is a subring of R which contains I.

2. Let I be any ideal in the ring R, then (I,I, inc.) is a crossed ideal of (R,R,inc).

3. If I be two sided ideals of R then we can consider them as two crossed modules, (I,R,inc_1) and (I',R,inc_2) then $((I \cap I'),I,v)$ and $((I \cap I'),I',v')$ are ideals in (I',R,inc_2) and (I',R,inc_1) respectively.

Proposition 7 *The intersection of any family of subcrossed modules (respectively crossed ideals) $\{(I_i,J_i,\partial_i)\}$ of a crossed module (C,R,∂) is a subcrossed module (respectively crossed ideal) of (C,R,∂).*

Proof. Let Λ be a set of index and let

$$(I,J,S)=\bigcap_{i\in\Lambda}(I_i,J_i,\partial_i).$$

If $x,y\in I$ and $c\in C$ then $x-y\in J$ and $xc\in C$ as $x,y\in J_i$ and $J_i \leq R$ for all $i\in\Lambda$. Similarly once can eaisly shown that I is a sub algebra of C.

4.5. **Factor crossed modules**

The definition of factor crossed modules holds as in the case of factor ring module some ideal J of R.

Let (C',R',∂) be an ideal in (C,R,∂) then the ring R acts on C/C' and R' acts trivially on C/C' since $r'(c+C')=r'c+C$ which is the trivial element of C/C' since $r'c\in C'$ therefore the factor ring R/R' acts on the ring C/C' and hence ∂ induced a morphism

$$\bar{\partial}: C/C' \longrightarrow R/R'$$

given by $\bar{\partial}(c+C')=\partial(c)+R'$ for $c\in C$

The following result can be easily proved.

Proposition 8 *Let $(C/C',R/R',\bar{\partial})$ be a crossed module defined above. Then the universal property holds for crossed modules. That is there is a morphism $(p_C,p_R): C \longrightarrow C/C'$ of crossed modules, such that given any morphism*

$$(\psi,\varphi):(C,R,\partial)\longrightarrow(B,S,\beta)$$

of crossed modules, with $\psi(c')=0$ and $c'\in C', r'\in R'$ for any there is a unique morphism $(f,g): C/C' \longrightarrow B$ of crossed modules with $fp_C=\varphi$ and $gp_C=\psi$.

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