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A DESIGNED ADI SOFTWARE FOR SOLVING POISSON'S EQUATION

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Abstract- In This paper, a software has been designed to perform the alternating direction implicit, ADI, method for the case of two dimensional flow represented by Poisson's equation. A square mesh with varying mesh size has been adopted. The results came out very much complying with the analytical solutions asserting the rigour of the software.

Key Words- alternating direction implicit, ADI, method, Poisson's equation.

1.INTRODUCTION

In many important scientific and engineering problems we meet the Poisson's equation. In the field of heat transfer, this equation governs the temperature distribution with a heat source. In fluid mechanics and as a result of a change of variables, we are able to separate the mixed elliptic-parabolic 2-D incompressible N-S equations to one parabolic equation (vorticity transport-diffusion equation) and one elliptic equation (the stream function-Poisson's equation) (See [1], [2], [3] for details of the procedure).

The Poisson equation and the Laplace equation are both arise in many various fields of science, and represent boundary value problems about which many studies have been done in the literature [4], [5], [6].

In the numerical solution of a boundary value problem over a region R we first choose the grid spacing, h, and introduce a grid consisting of equidistant horizontal and vertical straight lines of distance h . We use a difference equation approximating the given partial differential equation by which we relate the unknown values of some function u at the mesh points in R to each other and to the given boundary values.

These yield a system of linear algebraic equations that we solve by using the implicit schemes, which are preferred because of their properties of stability, it is recommended that one select method leading to the solution of a tridiagonal algebraic system [7], [8].

The alternating direction implicit (ADI) method was introduced by Peaceman and Rachford [10] allows the construction of very efficient implicit schemes. The ADI scheme primates the use of centered difference approximation for the convective terms, which gives a truncation error of higher order than the one results from one- sided order difference approximation used to stabilize some explicit schemes. The ADI method presents the following advantages: Second order accuracy in time and space, Good stability properties and Easy solution by inversion of tridigonal matrices. Many works have been published on the subject [9], [12].

The ADI schemes are considered as a good alternative to the fully implicit methods. These fully implicit methods involve solving a large, $(M-1)\times(N-1)$, say, of simultaneous equations each time step. The basic idea is to apply an implicit scheme is only one of the co-ordinates x, y and to alternate between the two.

This will result in a tridiagonal system of simultaneous equations, $(M-1)$ and $(N-1)$ in each of the two half time steps which is easy to solve. Therefore, alternating between rows and columns derives a solution. This alternating provides an unconditionally stable scheme, however, each half time step is unstable and leads to unacceptable growth of error if repeated indefinitely (See Bontoux [11]).

In the case of circular boundary conditions the grids cannot be interchanged freely as it is the case in rectangular grids. Therefore, our applications will be confined to rectangular grids only.

2. GOVERING EQUATION AND DISCRETIZATION

The equation which governs the two dimensional flow is Poisson equation which is given in Cartesian coordinates as follows:

$$\nabla^2 u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \tag{1}$$

for (x,y)$\in$R and u(x,y)=g(x,y) for (x,y) $\in$S

where $$R = \{(x,y), a \prec x \prec b,\ c \prec y \prec d\},$$

and S denotes the boundary of R. For the Laplace equation ,f(x,y)=0, equation (1) becomes

$$\nabla^2 u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

Adopting the finite-difference method for equation (1), we get

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right] u_{i,j} - \left(u_{i+1,j} + u_{i-1,j}\right) - \left(\frac{h}{k}\right)^2 \left(u_{i,j+1} + u_{i,j-1}\right) = -h^2 f(x_i, y_j) \tag{3}$$

We assume the domain to be a square (a=b , c=d) and take the grid spacing, h as follows $h = \dfrac{b-a}{N}$. Then the finite-difference equation (3) becomes:

$$-4u_{i,j} + \left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right] = h^2 f_{i,j} \tag{4}$$

and for Laplace equation

$$-4u_{i,j} + \left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right] = 0 \tag{5}$$

Equation (4) can be obtained by using the star-shaped pattern (6)

$$\begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix} u = h^2 f(x, y) \tag{6}$$

3. ADI METHOD

The pattern in (6) shows that we could obtain a tridiagonal matrix if there were only the three points in a row (or only the three points in a column). This suggests writing (4) in the form

$$u_{i-1,j} - 4u_{i,j} + u_{i+1,j} = -u_{i,j-1} - u_{i,j+1} + h^2 f_{i,j} \tag{7a}$$

so that the left-hand side belongs to y-row j and the right-hand side to x-column i. of course, we can also write (4) in the form

$$u_{i,j-1} - 4u_{i,j} + u_{i,j+1} = -u_{i-1,j} - u_{i+1,j} + h^2 f_{i,j} \tag{7b}$$

so that the left-hand side belongs to column i and the right-hand side to row j. In the ADI method we proceed by iteration. At every mesh point we choose an arbitrary starting value u_{ij}^{0}. In each step we compute new values at all mesh points. In one step we use an iterative formula resulting from (7a) and in the next step an iteration formula resulting from (7b), and so on in alternating order. In details, suppose that the approximations $u_{ij}^{(m)}$ have been computed. Then, to obtain the next approximations $u_{ij}^{(m+1)}$, we substitute $u_{ij}^{(m)}$ on the right-hand side of (7a) and solve all the $u_{ij}^{(m+1)}$ on the left-hand side; that is, we use

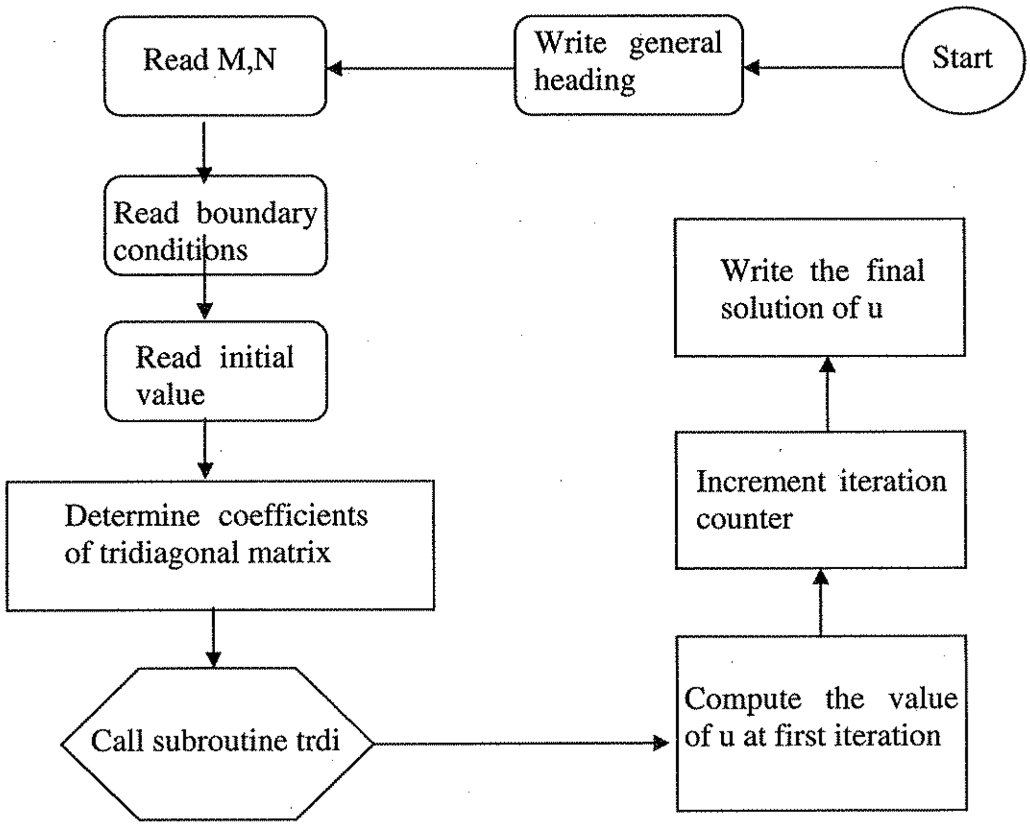
$$u_{i-1,j}^{(m+1)} - 4u_{ij}^{(m+1)} + u_{i+!,j}^{(m+1)} = -u_{i,j-1}^{(m)} - u_{i,j+1}^{(m)} + h^2 f_{i,j} \tag{8a}$$

We use this for a fixed j, that a fixed row j, and for all internal mesh points in this row. This gives a system of N linear algebraic equations in N unknowns, the new approximations of u at these mesh points. Equation (8a) involves not only approximations computed in the previous step but also given boundary values. We solve the system (8a), tridiagonal system, . Then we go to the next row, obtain another of N equations and solve it and so on until all rows are done. In the next step we alternate direction, that is, we compute the next approximations $u_{ij}^{(m+2)}$ column by column from the $u_{ij}^{(m+1)}$ and the given boundary values, using a formula obtained from (7b) by substituting the $u_{ij}^{(m+1)}$ on the right:

$$u_{i,j-1}^{(m+2)} - 4u_{ij}^{(m+2)} + u_{i,j+1}^{(m+2)} = -u_{i-1,j}^{(m+1)} - u_{i+1,j}^{(m+1)} + h^2 f_{i,j} \tag{8b}$$

for each fixed i, that is, for each column. This system of M equations in M unknowns, which is a tridiagonal system. Then we go to the next column, and so on until all columns are done.

3.1. Flow chart of ADI Program



4. COMPUTATIONAL RESULTS AND CONCLUSION

To demonstrate the validity of our program we compute the numerical solution to three different problems whose exact solutions are known, and present a comparison between the computed and the exact solutions.

Example (1):
We solve Poisson's equation with a source function $f(x,y) = -2\pi^2 \sin(\pi x)\sin(\pi y)$ and the exact solution is $u(x,y) = \sin(\pi x)\sin(\pi y)$.

Example (2):
We solve Poisson's equation with a source function $f(x,y) = -2\pi^2 \sin(\pi x)\sin(\pi y)$ and the exact solution is $u(x,y) = y + \sin(\pi x)\sin(\pi y)$.

Example (3):
We solve Poisson's equation with a source function $f(x,y) = -2\pi^2 \sin(\pi x)\sin(\pi y)$ and the exact solution is $u(x,y) = xy + \sin(\pi x)\sin(\pi y)$.

The results show that the method described in this paper gives accurate results over the whole square and these results are very acceptable compared to those obtained by using exact solution.

The results in Table (1) show that for the different examples, the maximum absolute error between the numerical solution and the exact analytic one is at most of the order of 10^{-4}. The numerical solutions are shown graphically as contour maps in Figs.(1) through (3).

Table (1) : Maximum absolute error for different examples

1	1.705885×10^{-4}
2	5.424023×10^{-5}
3	1.87186×10^{-4}

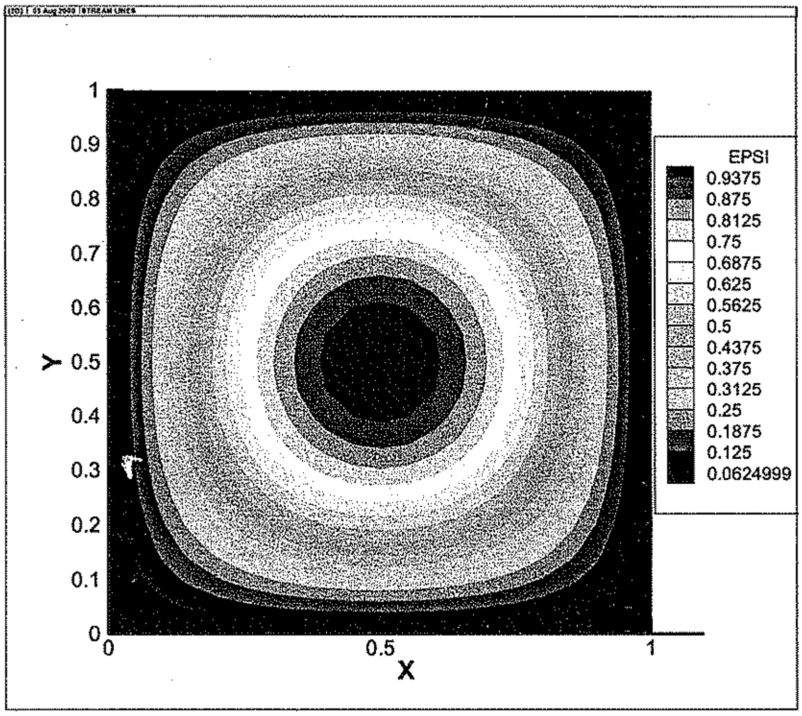


Fig.(1): u-contour lines for example (1)

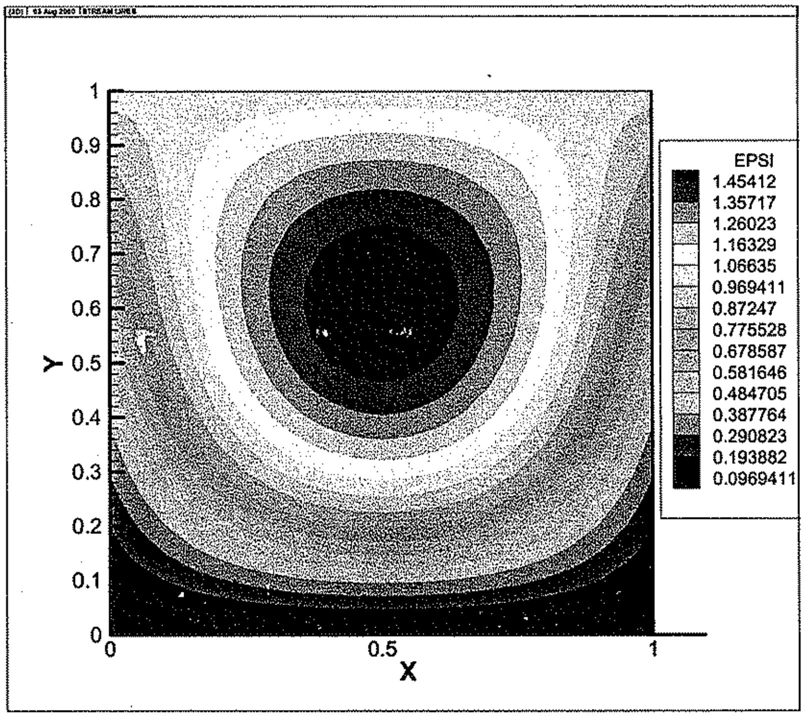


Fig. (2): u-contour lines for example (2)

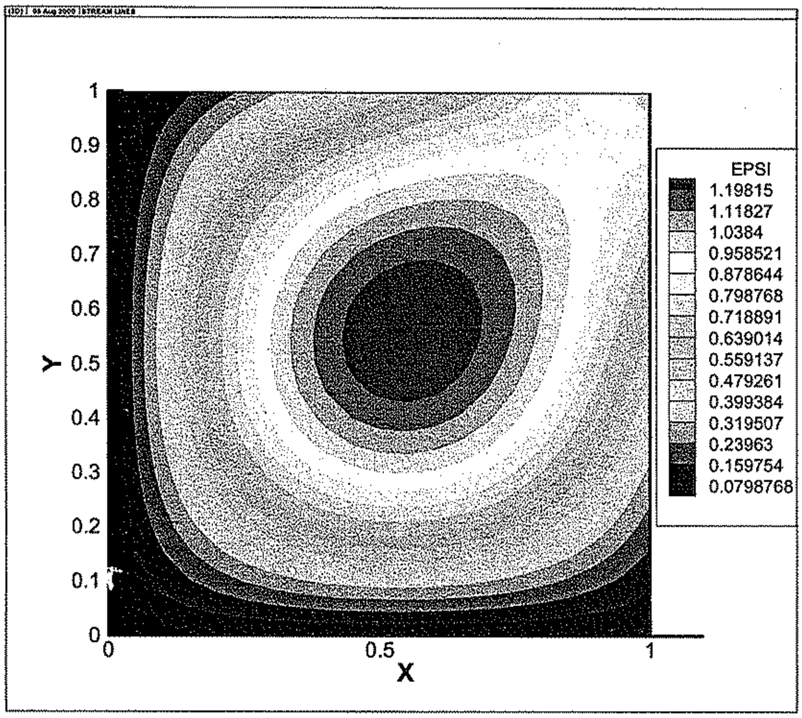


Fig. (3): u-contour lines for example (3)

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