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SIMILARITY ANALYSIS OF LONG WAVE EQUATIONS WITH VARIABLE BOTTOM FRICTION

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Abstract- Long wave equations propagating along an open channel with bottom friction are considered. The equations are cast into a non-dimensional form. Scaling and translational symmetries of the equations are calculated for the cases of constant bottom friction coefficient and variable bottom friction coefficient. Using the symmetries, the partial differential equations are transformed into ordinary differential equations. Analytical and numerical solutions of the ordinary differential equations are found and solutions are contrasted with each other.

Keywords- Long Waves, Bottom Friction, Similarity Analysis

1. INTRODUCTION

Modeling of long waves such as swells and tidal waves in a horizontal infinite channel are investigated by many researchers. The literature is vast and a partial review can be found in [1]. Most of the researchers did not consider the influence of the bottom friction. A model incorporating the bottom friction is proposed in [1]

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+g\frac{\partial H^*}{\partial x^*}=-\frac{ku^*}{H^*} \tag{1}$$

$$\frac{\partial H^*}{\partial t^*}+H^*\frac{\partial u^*}{\partial x^*}+u^*\frac{\partial H^*}{\partial x^*}=0 \tag{2}$$

where $u^*(x,t)$ is the horizontal component of water velocity, $H^*=h+\zeta^*(x,t)$ with h and $\zeta^*(x,t)$ representing respectively the undisturbed water depth and the water surface elevation, g denotes gravitational acceleration, k is the drag coefficient, x^* and t^* are horizontal coordinate and time respectively. The variables with asterisk are dimensional and the non-dimensional quantities are defined as follows

$$x=\frac{x^*}{L},\qquad t=\frac{t^*}{L/\sqrt{gh}},\qquad u=\frac{u^*}{\sqrt{gh}},\qquad \zeta=\frac{\zeta^*}{h},\qquad H=\frac{H^*}{h} \tag{3}$$

where L is a reference length and $\sqrt{gh}$ is some reference velocity. The dimensionless velocity is the well known Froude number in fluid mechanics. Inserting (3) into equations (1) and (2) yields

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial \zeta}{\partial x}=-\beta\frac{u}{1+\zeta} \tag{4}$$

$$\frac{\partial \zeta}{\partial t}+(1+\zeta)\frac{\partial u}{\partial x}+u\frac{\partial \zeta}{\partial x}=0 \tag{5}$$

The dimensionless friction coefficient is defined as

$$\beta = \frac{kL}{h\sqrt{gh}} \tag{6}$$

Under the assumption of $\zeta << 1$, the term

$$\frac{1}{1+\zeta} \cong 1-\zeta \tag{7}$$

can be approximated to first order. In ref. [1], the second term is not included in the approximation. With this approximation, the equations read

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial \zeta}{\partial x} = -\beta u(1-\zeta) \tag{8}$$

$$\frac{\partial \zeta}{\partial t} + (1+\zeta)\frac{\partial u}{\partial x} + u\frac{\partial \zeta}{\partial x} = 0 \tag{9}$$

In the following chapters, first the scaling and translational symmetries of the equations will be determined and then the partial differential system will be transformed into an ordinary differential system via similarity transformations. Analytical and numerical solutions of the ordinary differential system will be presented and solutions will be contrasted with each other.

2. SCALING AND TRANSLATIONAL SYMMETRIES

In this part, scaling and translational symmetries which are special Lie Group transformations will be determined. For a general treatment of Lie Groups applied to differential equations, the reader is referred to [2, 3]. The theory is involved and a simple treatment of the special Lie Group transformations can be found in [4].

2.1. Translational Symmetries

Translational symmetries will be searched both for constant friction coefficient and variable friction coefficient.

Constant Friction Coefficient

For this case, $\beta = \beta_0$, a constant is selected. The translational transformations are

$$x^* = x + \in a, \quad t^* = t + \in b, \quad u^* = u + \in c, \quad \zeta^* = \zeta + \in d \tag{10}$$

Inserting the transformations into the original equations and requiring that the equations remain invariant under this transformation yields

$$a = a, \quad b = b, \quad c = 0, \quad d = 0 \tag{11}$$

Note that equations admit coordinate translations in x and t, but not permit translations in the dependent variables. These symmetries yield the following determining equations for similarity transformations

$$\frac{dx}{a} = \frac{dt}{b} = \frac{du}{0} = \frac{d\zeta}{0} \tag{12}$$

Solving the system, one finally obtains the similarity variable and functions

$$\mu = x - mt, \quad u = u(\mu), \quad \zeta = \zeta(\mu) \tag{13}$$

Variable Friction Coefficient

For this case, the friction coefficient is assumed to vary with the spatial variable as $\beta = \beta(x)$. The translational transformations are

$$x^* = x + \in a, \quad t^* = t + \in b, \quad u^* = u + \in c, \quad \zeta^* = \zeta + \in d, \quad \beta^* = \beta + \in e \tag{14}$$

Inserting the transformations into the original equations with variable friction coefficient yields

$$a = a, \quad b = b, \quad c = 0, \quad d = 0, \quad e = 0 \tag{15}$$

The determining equations are

$$\frac{dx}{a} = \frac{dt}{b} = \frac{du}{0} = \frac{d\zeta}{0} = \frac{d\beta}{0} \tag{16}$$

which yields the similarity variable and functions

$$\mu = x - mt, \quad u = u(\mu), \quad \zeta = \zeta(\mu), \quad \beta = \beta(\mu) \tag{17}$$

The initial assumption is that the friction coefficient depends on x, but as seen in (17) the friction coefficient depends on both x and t, a violation of the assumption. Therefore, one can conclude that β can only be a constant. But this case was treated before. As a conclusion, variable friction coefficient equations do not admit translational symmetries.

2.2. Scaling Symmetries

Again, scaling symmetries of the equations will be searched for constant and variable friction coefficients.

Constant Friction Coefficient

For this case, $\beta = \beta_0$, a constant is selected. The scaling transformations are

$$x^* = e^{\in a} x, \quad t^* = e^{\in b} t, \quad u^* = e^{\in c} u, \quad \zeta^* = e^{\in d} \zeta \tag{18}$$

Inserting the transformed variables into the original equations and requiring that the equations remain invariant under the transformation yields

$$a - b - c = 0, \quad a + c - b - d = 0, \quad b = 0, \quad d = 0, \quad a + d - b - c = 0 \tag{19}$$

Solving the equations yields a = b = c = d = 0. Therefore for constant friction coefficient, the equations do not admit scaling symmetries.

Variable Friction Coefficient

Again, $\beta = \beta(x)$ is assumed. The transformations are

$$x^* = e^{\in a} x, \quad t^* = e^{\in b} t, \quad u^* = e^{\in c} u, \quad \zeta^* = e^{\in d} \zeta, \quad \beta^* = e^{\in e} \beta \tag{20}$$

Inserting the transformations into the original equations and requiring that the equations remain invariant under this transformation yields

$$a - b - c = 0, \quad a + c - b - d = 0, \quad a + d - b - c = 0, \quad b + e = 0, \quad b + e + d = 0 \tag{21}$$

Solving the equations, one has

$$a = b, \quad e = -b, \quad c = d = 0 \tag{22}$$

The determining equations for this case are

$$\frac{dx}{bx} = \frac{dt}{bt} = \frac{du}{0} = \frac{d\zeta}{0} = \frac{d\beta}{-b\beta} \tag{23}$$

The similarity variable and functions are

$$\mu = \frac{x}{t}, \qquad \beta = \frac{\beta_0}{x}, \qquad u = u(\mu), \qquad \zeta = \zeta(\mu) \tag{24}$$

Note that scaling symmetry is admitted by the equations if and only if the friction coefficient takes the special form $\beta = \beta_0 / x$. $\beta = 0$ is a sub case of this form, for which also scaling symmetries exist. Similarity solutions will be constructed in the next section.

3. SIMILARITY SOLUTIONS

In this chapter, similarity transformations corresponding to the symmetries given previously will be employed for transforming the partial differential system to an ordinary differential system. Analytical solutions of the ordinary differential system will be given.

3.1. Translational Symmetry

In chapter 2, it is shown that translational symmetry exists only for constant friction coefficient. Expressing the original equations in terms of the similarity variable and functions given in (13), one finally obtains

$$-mu' + uu' + \zeta' = -\beta_o u(1-\zeta) \tag{25}$$

$$-m\zeta' + (1+\zeta)u' + u\zeta' = 0 \tag{26}$$

Integrating the second equation and solving ζ

$$\zeta = \frac{k-u}{u-m} \tag{27}$$

Substituting this into the first equation and selecting the special case of m=k, the final solutions are

$$u - m\ln u = -2\beta_o \mu + c, \qquad \zeta = -1 \tag{28}$$

In terms of the original variables, the solution is

$$u - m\ln u = -2\beta_o (x - mt) + c, \qquad \zeta = -1 \tag{29}$$

This solution can be valid for a very specific set of conditions.

3.2. Scaling Symmetry

In chapter 2, it is shown that scaling symmetry exists only for the specific form of the friction coefficient $\beta = \beta_0 / x$. For this specific choice, using the similarity variable and functions given in (24), the original partial differential system transforms into the following ordinary differential system

$$-\mu u' + uu' + \zeta' = -\frac{\beta_o u}{\mu}(1-\zeta) \tag{30}$$

$$-\mu\zeta' + (1+\zeta)u' + u\zeta' = 0 \tag{31}$$

An approximate series solution of the equations can be found as follows

$$u = \frac{1}{2}\mu + \frac{1}{40\beta_o}(2\beta_o^2 - 1)\mu^2 + \frac{(4\beta_o^2 + 13)(2\beta_o^2 - 1)}{4000\beta_o^2}\mu^3 + \ldots \tag{32}$$

$$\zeta = -1 - \beta_o\mu - \frac{3}{20}(2\beta_o^2 - 1)\mu^2 - \frac{(17\beta_o^2 - 1)(2\beta_o^2 - 1)}{500\beta_o}\mu^3 + \ldots \tag{33}$$

In terms of the original variables, the solutions are

$$u = \frac{1}{2}\frac{x}{t} + \frac{1}{40\beta_o}(2\beta_o^2 - 1)\left(\frac{x}{t}\right)^2 + \frac{(4\beta_o^2 + 13)(2\beta_o^2 - 1)}{4000\beta_o^2}\left(\frac{x}{t}\right)^3 + \ldots \tag{34}$$

$$\zeta = -1 - \beta_o\left(\frac{x}{t}\right) - \frac{3}{20}(2\beta_o^2 - 1)\left(\frac{x}{t}\right)^2 - \frac{(17\beta_o^2 - 1)(2\beta_o^2 - 1)}{500\beta_o}\left(\frac{x}{t}\right)^3 + \ldots \tag{35}$$

4. NUMERICAL SOLUTIONS

In this chapter, numerical integration of the ordinary differential system corresponding to scaling transformation, namely equations (30) and (31) is performed. Runge-Kutta algorithm is used. First, the equations are cast into a more suitable form as follows

$$u' = -\frac{(\mu - u)\beta_o u}{\mu}\frac{1-\zeta}{1+\zeta-(\mu-u)^2} \tag{36}$$

$$\zeta' = -\frac{\beta_o}{\mu}u\frac{1-\zeta^2}{1+\zeta-(\mu-u)^2} \tag{37}$$

Solutions are sought for the conditions

$$u(0) = 0, \qquad \zeta(0) = -1 \tag{38}$$

The numerical algorithm is checked for the specific choice of $\beta_o = 1/\sqrt{2}$. This choice yields degenerate linear solutions as can be checked from the series solutions. Comparison of the series solutions and the numerical solutions are presented in Figures 1-4.

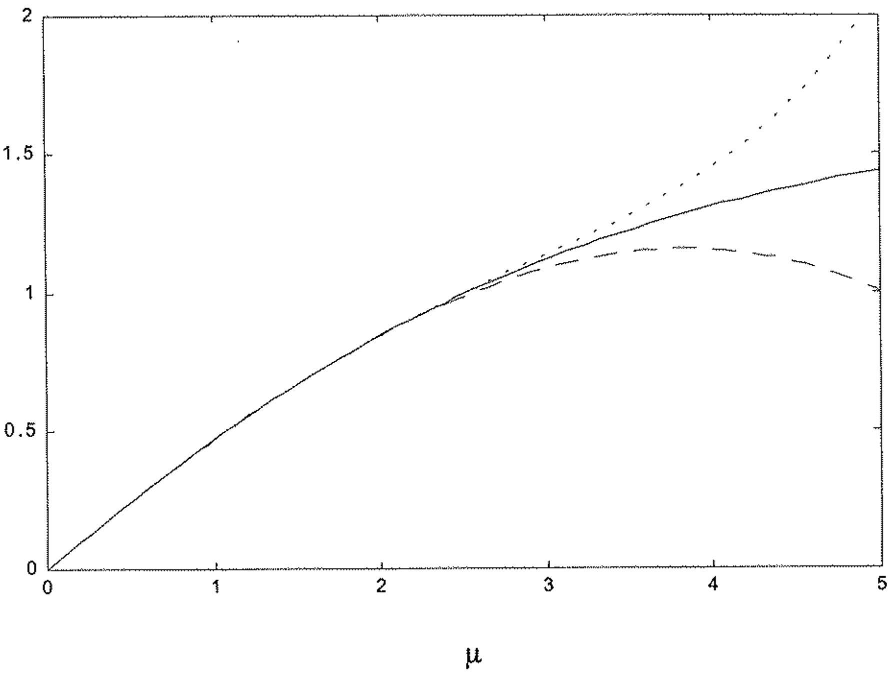

Figure 1. Variation of non-dimensional velocity with similarity variable for $\beta_0 = 0.5$ (____ numerical solution, - - - -3 term analytical solution, - . - . 7 term analytical solution)

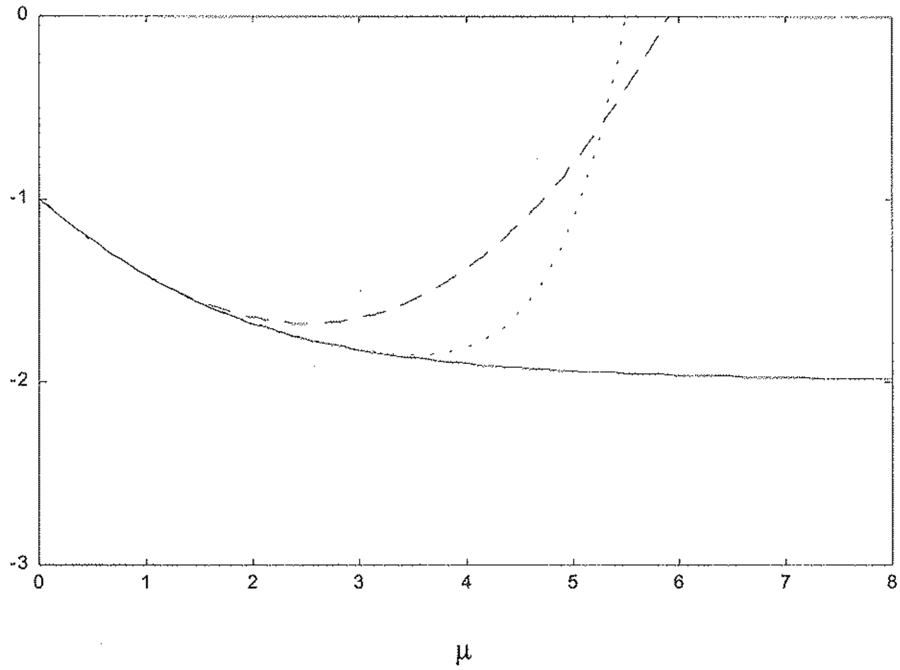

Figure 2. Variation of non-dimensional water height with similarity variable for $\beta_0 = 0.5$ (___ numerical solution, - - - -3 term analytical solution, - . - . 7 term analytical solution)

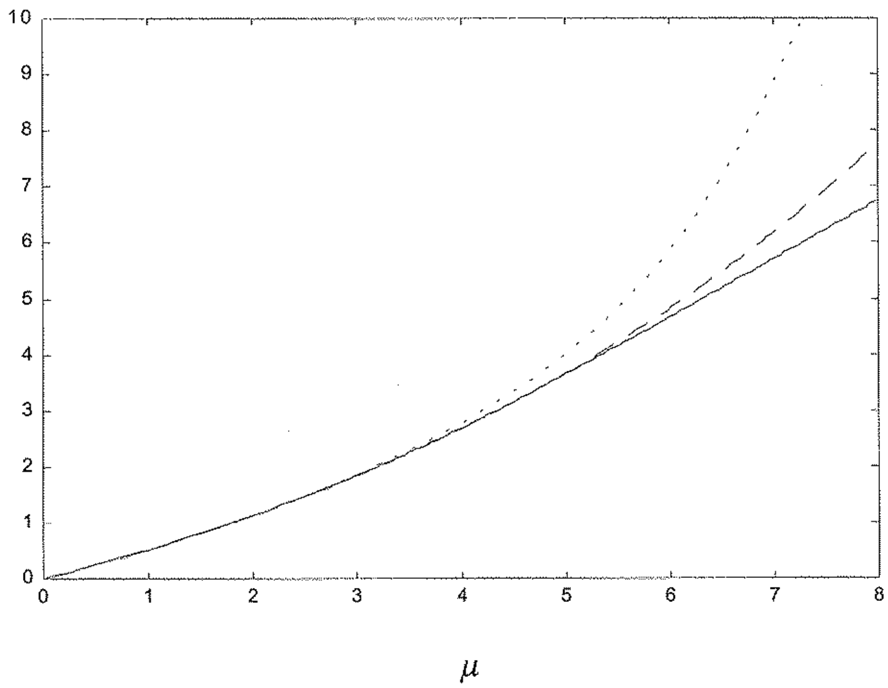

Figure 3. Variation of non-dimensional velocity with similarity variable for $\beta_0 = 1$ (___ numerical solution, - - - -3 term analytical solution, - . - . 7 term analytical solution)

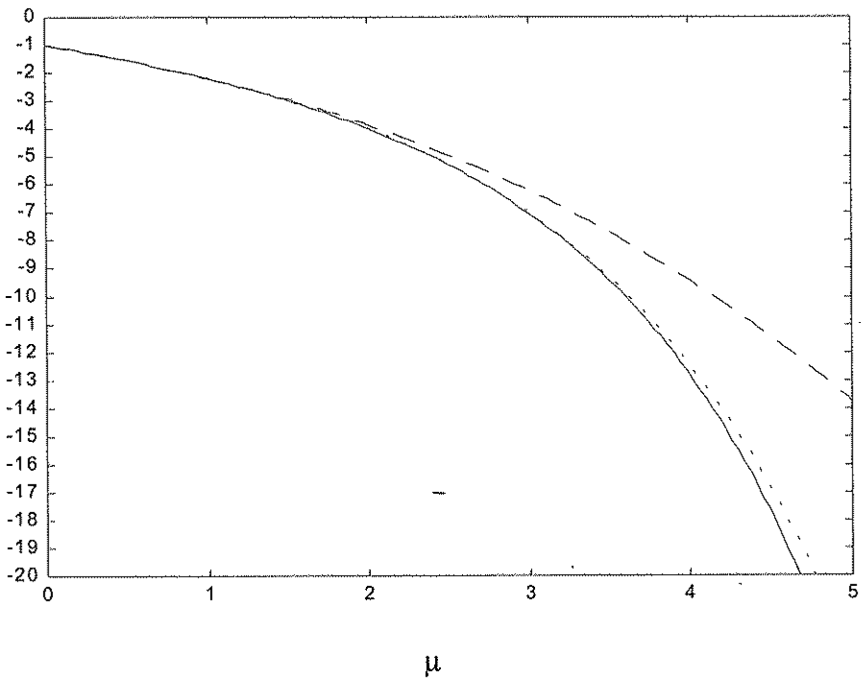

Figure 4. Variation of non-dimensional water height with similarity variable for $\beta_0 = 1$ (___ numerical solution, - - - -3 term analytical solution, - . - . 7 term analytical solution)

From the comparisons, one can conclude that series solutions have a limited range of applicability.

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