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CHEBYSHEV COLLOCATION METHOD FOR SOLVING LINEAR DIFFERANTIAL EQUATIONS

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Abstract-A matrix method, which is called the Chebyshev-matrix method, for the approximate solution of linear differential equations in term of Chebyshev collocations is presented. The method is based on first taking the truncated Chebyshev series of the functions in equation and then substituting their matrix forms into the given equation. Thereby the equation reduces to a matrix equation, which corresponds to a system of linear algebraic equations with unknown Chebyshev coefficients. To illustrate the method, it is applied to certain linear differential equation under the given conditions and the results are compared.
Keywords- Chebyshev Polynomials and Series, Chebyshev-Matrix Method

1. INTRODUCTION

Chebyshev polynomials were discovered almost a century ago by the Russian Mathematician Chebyshev. Their important for practical computation was rediscovered fifty years ago by C. Lanczos[3] and then this has been extended by C. W. Clenshaw[1] to differential equations and recently by Sezer and Kaynak to Chebyshev-Matrix Methods [4]. The coming of the digital computer gave further emphasis to this development.

In this study, a Chebyshev collocation method is presented to find the approximate solutions of differentials equations with variable coefficients of the second order to complex conditions, in terms of Chebyshev polynomials. Here, Chebyshev-Matrix method[4] is developed by means of Chebyshev collocations in the interval $-1\leq x\leq 1$, equations is transformed to a matrix equations or a algebraic system which is based on Chebyshev collocations points.

2. METHOD OF SOLUTION

In this study, let's consider second-order differential equations

$$P(x)y''+Q(x)y'+R(x)y=f(x) \tag{1}$$

under the prescribed conditions which will be given in the illustrations. Here, P, Q, R and f are defined in the range $-1\leq x\leq 1$. We assume a series expansion,

$$y(x)=\sum_{r=0}^{N}{}' a_r T_r(x) \tag{2}$$

or

$$[y(x)]=\mathbf{TA} \tag{3}$$

to find the solutions of this equation with Chebyshev collocation point.

$$\mathbf{T}(x)=\begin{bmatrix}T_0(x) & T_1(x) & \cdots & T_N(x)\end{bmatrix},\ \mathbf{A}=\begin{bmatrix}\tfrac{1}{2}a_0 & a_1 & \cdots & a_N\end{bmatrix}^t$$

Here,$\sum'$ denotes a sum whose fist term is halved, $T_r(x)$ denotes Chebyshev polynomial of the first kind of degree r[3], defined by

$$T_r(x)=\cos\left(r\cos^{-1}x\right) \qquad -1\leq x\leq 1$$

and a_r ,r=0,1,…,N, are the Chebyshev coefficient to be determined. x_j ,j=0,1,...,N, denotes the Chebyshev collocation points, defined by

$$x_j = \cos\frac{(N-j)\pi}{N} \qquad -1 \le x \le 1 \tag{4}$$

Substituting this collocation points in the equations (1), we can right matrix form following as;

$$\mathbf{P\,Y''+Q\,Y'+R\,Y=F} \tag{5}$$

These matrices can be defined by;

$$\mathbf{P}=\begin{bmatrix} P(x_0) & 0 & \cdots & 0 \\ 0 & P(x_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P(x_N) \end{bmatrix},\ \mathbf{Q}=\begin{bmatrix} Q(x_0) & 0 & \cdots & 0 \\ 0 & Q(x_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & Q(x_N) \end{bmatrix},$$

$$\mathbf{R}=\begin{bmatrix} R(x_0) & 0 & \cdots & 0 \\ 0 & R(x_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & R(x_N) \end{bmatrix},\ \mathbf{F}=\begin{bmatrix} f(x_0) & f(x_1) & \cdots & f(x_N) \end{bmatrix}^t,$$

$$\mathbf{Y}=\begin{bmatrix} y(x_0) & y(x_1) & \cdots & y(x_N) \end{bmatrix}^t$$

on the other hand, the function y(x) and it's nth derivative with respect to $x=x_j$, respectively, can be expanded in Chebyshev series

$$y(x_j)=\sum_{r=0}^{N}{}' a_r T_r(x_j) \tag{6}$$

$$y^{(n)}(x_j)=\sum_{r=0}^{N}{}' a_r^{(n)} T_r(x_j) \tag{7}$$

where ,j=0,1,...,N, matrix form of equations (6) and (7) is defined as,

$$\mathbf{Y=T\,A} \tag{8}$$

$$\mathbf{Y'=T\,A^{(1)}} \text{ and } \mathbf{Y''=T\,A^{(2)}} \tag{9}$$

Here, **T** and $\mathbf{A}^{(n)}$ matrices are defined as,

$$\mathbf{T}=\begin{bmatrix} T_0(x_0) & T_1(x_0) & \cdots & T_N(x_0) \\ T_0(x_1) & T_1(x_1) & \cdots & T_N(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ T_0(x_N) & T_1(x_N) & \cdots & T_N(x_N) \end{bmatrix},\ A^{(n)}=\begin{bmatrix} \frac{1}{2}a_0^{(n)} & a_1^{(n)} & \cdots & a_N^{(n)} \end{bmatrix}^t$$

Substituting equations (8) and (9) into equations (4), we obtain

$$\mathbf{P\,T\,A^{(2)}+Q\,T\,A^{(1)}+R\,T\,A=F}$$

Where $\mathbf{A^{(2)}=4\,M^2\,A}$, $\mathbf{A^{(1)}=2\,M\,A}$. for odd and even M are;

for odd M:

$$\mathbf{M}=\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & \frac{N}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

for even M:

$$\mathbf{M}=\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Consequently, equation (1) is transformed as;

$$(4\,\mathbf{P}\,\mathbf{T}\,\mathbf{M}^2+2\,\mathbf{Q}\,\mathbf{T}\,\mathbf{M}+\mathbf{R}\,\mathbf{T})\mathbf{A}=\mathbf{F} \tag{10}$$

Briefly, we can write this equation in the form

$$\mathbf{W}\,\mathbf{A}=\mathbf{F} \tag{11}$$

So that

$\mathbf{W}=[\mathbf{W}_{ij}]=4\,\mathbf{P}\,\mathbf{T}\,\mathbf{M}^2+2\,\mathbf{Q}\,\mathbf{T}\,\mathbf{M}+\mathbf{R}\,\mathbf{T}$, where $i,j=0,1,\ldots,N$.

Then the augmented matrix of (11) becomes $[\mathbf{W};\mathbf{F}]$ or

$$[\mathbf{W};\mathbf{F}]=\begin{bmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,N} & ; & f(x_0) \\ w_{1,0} & w_{1,1} & \cdots & w_{1,N} & ; & f(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{N-1,0} & w_{N-1,1} & \vdots & w_{N-1,N} & ; & f(x_{N-1}) \\ w_{N,0} & w_{N,1} & \vdots & w_{N,N} & ; & f(x_N) \end{bmatrix} \tag{12}$$

If the initial conditions are given as

$$\begin{aligned} &\sum_{i=0}^{1}\left[a_i y^{(i)}(a)+b_i y^{(i)}(1)+c_i y^{(i)}(c)\right]=\lambda \\ &\sum_{i=0}^{1}\left[\alpha_i y^{(i)}(a)+\beta_i y^{(i)}(1)+\gamma_i y^{(i)}(c)\right]=\mu \end{aligned} \tag{13}$$

the functions $y^{(0)}(x)$ and $y^{(1)}(x)$ are to obtained in points $x=-1$, $x=1$ and $x=c$, defined by

$$y^{(0)}(-1)=\sum_{r=0}^{N}{}' a_r T_r(-1) \Rightarrow [y^{(0)}(-1)]=\mathbf{T_x(-1)\,A}$$

$$y^{(0)}(1)=\sum_{r=0}^{N}{}' a_r T_r(1) \Rightarrow [y^{(0)}(1)]=\mathbf{T_x(1)\,A}$$

$$y^{(0)}(c)=\sum_{r=0}^{N}{}' a_r T_r(c) \Rightarrow [y^{(0)}(c)]=\mathbf{T_x(c)\,A}$$

$$y^{(1)}(-1)=\sum_{r=0}^{N}{}' a_r T_r(-1) \Rightarrow [y^{(1)}(-1)]=2\,\mathbf{T_x(-1)\,M\,A}$$

$$y^{(1)}(1)=\sum_{r=0}^{N}{}' a_r T_r(1) \Rightarrow [y^{(1)}(1)]=2\,\mathbf{T_x(1)\,M\,A}$$

$$y^{(1)}(c)=\sum_{r=0}^{N}{}' a_r T_r(c) \Rightarrow [y^{(1)}(c)]=2\,\mathbf{T_x(c)\,M\,A}$$

Substituting the expressions $y^{(i)}(-1)$, $y^{(i)}(1)$ and $y^{(i)}(c)$ into conditions (13), Briefly, we can obtain the row matrix in the form

$$\begin{bmatrix} u_0 & u_1 & \cdots & u_N \end{bmatrix}\mathbf{A}=[\lambda] \text{ or } \mathbf{U}\,\mathbf{A}=[\lambda]$$

$$\begin{bmatrix} v_0 & v_1 & \cdots & v_N \end{bmatrix}\mathbf{A}=[\mu] \text{ or } \mathbf{V}\,\mathbf{A}=[\mu]$$

Then the augmented matrix conditions (13) becomes

$$[\mathbf{U}\ ;\ \lambda] \text{ and } [\mathbf{V}\ ;\ \mu] \tag{14}$$

Consequently, replacing the row matrices (14) by the last two rows of augment matrix (12), we have the new augment matrix

$$[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=\begin{bmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,N} & ; & f(x_0) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{N-2,0} & w_{N-2,1} & \cdots & w_{N-2,N} & ; & f(x_{N-2}) \\ u_0 & u_1 & \cdots & u_N & ; & \lambda \\ v_0 & v_1 & \cdots & v_N & ; & \mu \end{bmatrix} \tag{15}$$

or more clearly

$$\tilde{\mathbf{W}}\,\mathbf{A}=\tilde{\mathbf{F}} \tag{16}$$

where

$$\tilde{\mathbf{W}}=\begin{bmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,N} \\ w_{1,0} & w_{1,1} & \cdots & w_{1,N} \\ \vdots & \vdots & \vdots & \vdots \\ w_{N-2,0} & w_{N-2,1} & \cdots & w_{N-2,N} \\ u_0 & u_1 & \cdots & u_N \\ v_0 & v_1 & \cdots & v_N \end{bmatrix},\ \tilde{\mathbf{F}}=\begin{bmatrix} f(x_0) & f(x_1) & \cdots & f(x_{N-2}) & \lambda & \mu \end{bmatrix}^t$$

If rank $\tilde{\mathbf{W}}$=rank$[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]$=N+1 or det $\tilde{\mathbf{W}}\neq 0$, we can write of the equation matrix (16)

$$\mathbf{A}=\tilde{\mathbf{W}}^{-1}.\tilde{\mathbf{F}} \tag{17}$$

and thus the matrix **A** (thereby the coefficient a_r) is uniquely determined. Because of, the differential equation (1) has a unique solution for conditions (13). This solution as;

$$[y(x)]=\mathbf{TA} \text{ or } y(x)=\sum_{r=0}^{N}{}' a_r T_r(x) \tag{18}$$

Also we can easily check the accuracy of this solution as follows. Since the truncated Chebyshev series (18) or the corresponding polynomial expansion is an approximate solution of equation (1), when the solution y(x) with the derivates $y^{(1)}(x)$ and $y^{(2)}(x)$ is substituted in equation (1), the resulting equation must be satisfied approximately; that is, for $x=x_i\in[-1,1]$, $i=0,1,\dots,M$

$$D(x_i)=\left|P(x_i)y^{(2)}(x_i)+Q(x_i)y^{(1)}(x_i)+R(x_i)y(x_i)-F(x_i)\right|\cong 0 \tag{19}$$

or

$D(x_i)\leq 10^{-k_i}$, k_i is any positive integer

If max $(10^{-k_i})=10^{-k}$ $(k\in Z^+)$ is prescribed, then the truncation limit N is increased until the difference $D(x_i)$ at each of the points becomes smaller than the prescribed 10^{-k} [2].

3. ILLUSTRATIONS

The Chebyshev-Matrix method present in this method is useful in finding the approximate solution of linear differential equations under the given conditions, in terms of Chebyshev collocations. We illustrate it by the following example.

Example 1. Let us consider the problem [3]

$$\begin{aligned} & y''+xy'+xy=1+x+x^2,\ -1\leq x\leq 1 \\ & y(0)=1,\ y'(0)+2y(1)-y(-1)=-1 \end{aligned} \tag{20}$$

and approximate the solution y(x) by the truncated Chebyshev series

$$y(x)=\sum_{r=0}^{4}{}' a_r T_r(x)\,,\ -1\le x\le 1 \tag{20}$$

which has Chebyshev polynomial of degree four, where N=4. Since P(x) =1, Q(x) =x, R(x) =x and F(x) $=1+x+x^2$, we have form relation (4),

$$x_0=-1,\quad x_1=-\frac{\sqrt{2}}{2},\quad x_2=0,\quad x_3=\frac{\sqrt{2}}{2},\quad x_4=1\,,$$

and

$$P=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}Q=\begin{bmatrix}-1&0&0&0&0\\0&-\frac{\sqrt{2}}{2}&0&0&0\\0&0&0&0&0\\0&0&0&\frac{\sqrt{2}}{2}&0\\0&0&0&0&1\end{bmatrix}R=\begin{bmatrix}-1&0&0&0&0\\0&-\frac{\sqrt{2}}{2}&0&0&0\\0&0&0&0&0\\0&0&0&\frac{\sqrt{2}}{2}&0\\0&0&0&0&1\end{bmatrix}F=\begin{bmatrix}1\\\frac{3-\sqrt{2}}{2}\\1\\\frac{3+\sqrt{2}}{2}\\3\end{bmatrix} \tag{21}$$

Then, for N=4, the matrix equation (10) becomes

$$(\mathbf{4\,P\,T\,M^2+2\,Q\,T\,M+R\,T)A=F} \tag{22}$$

Substituting the expressions (21) into equation (22), we have the augmented matrix from equation (12)

$$[\mathbf{W;F}]=\begin{bmatrix}-1&0&7&-32&95&;&1\\-0.7071&-0.2071&6&-19.5918&32.7071&;&\frac{3}{2}-\frac{\sqrt{2}}{2}\\0&0&4&0&-16&;&1\\0.7071&1.2071&6&18.5918&31.2928&;&\frac{3}{2}+\frac{\sqrt{2}}{2}\\1&2&9&34&97&;&3\end{bmatrix} \tag{23}$$

For the first condition y(0) =1 and the second condition y'(0)+2y(1)-y(-1)=-1, respectively, the augment matrices become

$$[1\ \ 0\ \ -1\ \ 0\ \ 1\ \ ;\ \ 1] \text{ and } [1\ \ 4\ \ 1\ \ 0\ \ 1\ \ ;\ \ -1] \tag{24}$$

from (14). Thus, substituting the required elements of matrices (23) and (24) in expression (15), we find the new augmented matrix

$$[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=\begin{bmatrix}-1&0&7&-32&95&;&1\\-0.7071&-0.2071&6&-19.5918&32.7071&;&\frac{3}{2}-\frac{\sqrt{2}}{2}\\0&0&4&0&-16&;&1\\1&0&-1&0&1&;&1\\1&4&1&0&1&;&-1\end{bmatrix}$$

From the solution of this system, we obtain the coefficients approximately as

$$\frac{1}{2}a_0=1.29203481\quad a_1=-0.65302322\quad a_2=0.30604642\quad a_3=0.03691853\quad a_4=0.0140116$$

which are the results in [3] and [4]. Thus the desired solution of problem (20)

$$y=1.292T_0(x)-0.65302T_1(x)+0.306T_2(x)+0.03691T_3(x)+0.01401T_4(x)$$

or

$$y=1-0.76377881x+0.50000004x^2+0.14767412x^3+0.1120928x^4$$

Thus all solutions are given in Table-1.

Table-1

x	Chebyshev-Collocation Method				Taylor-Matrix Method N=9
	N=4	N=7	N=8	N=9	
0	0.99999999	0.99999997	1.00000003	1.00000001	1
0.2	0.86860497	0.8822512	0.88206831	0.8781107	0.882055
0.4	0.78680919	0.81090085	0.81054078	0.80273319	0.810520
0.6	0.76815756	0.79265061	0.79199326	0.78050312	0.791965
0.8	0.83049933	0.83221871	0.83137963	0.8207936	0.831360
1	0.99598814	0.92923398	0.93029483	0.94976188	0.930458

Example 2. Let us now consider the problem

$$y'' + y\sin x = e^{x}, \quad -1 \le x \le 1$$
$$y(0)=1, \quad y(1)=0 \tag{25}$$

and seek the solution in the form

$$y(x)=\sum_{r=0}^{4}{}' a_r T_r(x), \ -1\le x\le 1$$

So that

$N=4$, $P(x)=1$, $Q(x)=0$, $R(x)=\sin(x)$, $F(x)=e^{x}$.

Taking N=4 in the relations (4), we find

$$x_0=-1,\ x_1=-\frac{\sqrt{2}}{2},\ x_2=0,\ x_3=\frac{\sqrt{2}}{2},\ x_4=1$$

and

$$\mathbf{P}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{R}=\begin{bmatrix} -0.8414 & 0 & 0 & 0 & 0 \\ 0 & -0.6496 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6496 & 0 \\ 0 & 0 & 0 & 0 & 0.8414 \end{bmatrix}, \mathbf{F}=\begin{bmatrix} 0.36787 \\ 0.49306 \\ 1 \\ 2.02811 \\ 2.71828 \end{bmatrix} \tag{26}$$

Then, for N=4, the matrix equation (10) becomes

$$\mathbf{(4\,P\,T\,M^2+2\,Q\,T\,M+R\,T)A=F} \tag{27}$$

Substituting the expressions (22) into equation (27), we have the augmented matrix from equation (12)

$$[\mathbf{W;F}]=\begin{bmatrix} -0.841 & 0.841 & -0.841 & 0.841 & -0.841 & ; & 0.367 \\ -0.649 & 0.459 & 4 & -17.429 & 32.649 & ; & 0.493 \\ 0 & 0 & 4 & 0 & -16 & ; & 1 \\ 0.649 & 0.459 & 4 & 16.511 & 31.350 & ; & 2.028 \\ 0.841 & 0.841 & 4.841 & 24.841 & 80.841 & ; & 2.718 \end{bmatrix} \tag{28}$$

Additionally, for the conditions y(0)=1 and y(1)=1, the augment matrices are obtained as,

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & ; & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & ; & 0 \end{bmatrix} \tag{29}$$

Consequently, the new augment matrix, from matrices (28) and (29), is obtained as

$$[\tilde{W};\tilde{F}]=\begin{bmatrix} -0.841 & 0.841 & 3.158 & -23.158 & 79.158 & ; & 0.367 \\ -0.649 & 0.459 & 4 & -17.429 & 32.649 & ; & 0.493 \\ 0 & 0 & 4 & 0 & -16 & ; & 1 \\ 1 & 0 & -1 & 0 & 1 & ; & 1 \\ 1 & 1 & 1 & 1 & 1 & ; & 0 \end{bmatrix}$$

The solution of this system is approximately

$$\frac{1}{2}a_0=1.3175 \quad a_1=-1.6786 \quad a_2=0.34003 \quad a_3=-0.0014 \quad a_4=0.0225$$

which are the results in [3] and [4]. Thereby the solution of the problem (25) can be found as

$$y=1.3175T_0(x)-1.67864T_1(x)+0.34003T_2(x)-0.0014T_3(x)+0.0225T_4(x)$$

or

$$y=1-0.67720652x+0.31752858x^2-0.00143484x^3+0.02250952x^4$$

Thus all solutions are given in Table-2.

Table-2

x	Chebyshev-Collocation Method				Taylor-Matrix Met. N=7
	N=4	N=7	N=8	N=9	
0	0.99999999	0.99999998	0.99999989	1.00000001	1
0.2	0.68537483	0.68779142	0.68782361	0.68776746	0.687955
0.4	0.41450789	0.41947375	0.41955784	0.419457	0.419830
0.6	0.19749607	0.20451011	0.2048864	0.20473467	0.205277
0.8	0.05135119	0.05853383	0.05883086	0.05860542	0.059194
1	0	-0.0000167	-0.0000001	-0.000000007	0

Example 3. Let us consider the differential equation

$$y''+xy'+y=e^x, \quad -1\le x\le 1 \qquad y(0)=0, \quad y(1)=0 \tag{30}$$

and seek the solution in the form

$$y(x)=\sum_{r=0}^{4}{}' a_r T_r(x), \; -1\le x\le 1$$

So that

$$N=4, \qquad P(x)=1, \qquad Q(x)=x, \qquad R(x)=1, \qquad F(x)=e^x .$$

Taking N=4 in the relations (4), we find

$$x_0=-1, \quad x_1=-\frac{\sqrt{2}}{2}, \quad x_2=0, \quad x_3=\frac{\sqrt{2}}{2}, \quad x_4=1$$

and

$$P=R=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\quad Q=\begin{bmatrix}-1&0&0&0&0\\0&-\frac{\sqrt{2}}{2}&0&0&0\\0&0&0&0&0\\0&0&0&\frac{\sqrt{2}}{2}&0\\0&0&0&0&1\end{bmatrix}\quad F=\begin{bmatrix}0.3678\\0.493\\1\\2.0281\\2.7182\end{bmatrix}\tag{31}$$

Then, for N=4, the matrix equation (10) becomes

$$(4\,\mathbf{P}\,\mathbf{T}\,\mathbf{M}^2+2\,\mathbf{Q}\,\mathbf{T}\,\mathbf{M}+\mathbf{R}\,\mathbf{T})\mathbf{A}=\mathbf{F}\tag{32}$$

Substituting the expressions (31) into equation (32), we have the augmented matrix from equation (12)

$$[\mathbf{W};\mathbf{F}]=\begin{bmatrix}1&-2&9&-34&97&;&0.3678\\1&-1.4142&6&-18.3847&31&;&0.493\\1&0&3&0&-15&;&1\\1&1.4142&6&18.3847&31&;&2.0281\\1&2&9&34&97&;&2.7182\end{bmatrix}\tag{33}$$

Thus, for the conditions y(0)=0 and y(1)=0, the augment matrices are obtained as,

$$[1\ \ 0\ \ -1\ \ 0\ \ 1\ \ ;\ \ 0]\ \text{and}\ [1\ \ 1\ \ 1\ \ 1\ \ 1\ \ ;\ \ 0]\tag{32}$$

Consequently, the new augment matrix, from matrices (28) and (29), is obtained as

$$[\tilde{\mathbf{W}};\tilde{\mathbf{F}}]=\begin{bmatrix}1&-2&9&-34&97&;&0.3678\\1&-1.4142&6&-18.3847&31&;&0.493\\1&0&3&0&-15&;&1\\1&0&-1&0&1&;&0\\1&1&1&1&1&;&0\end{bmatrix}$$

The solution of this system is approximately

$$\frac{1}{2}a_0=0.33046243\quad a_1=-0.93984409\quad a_2=0.35728323\quad a_3=0.22527761\quad a_4=0.02682081$$

which are the results in [3] and [4]. Thereby the solution of the problem (25) can be obtained as

$$y=0.3304T_0(x)-0.9398T_1(x)+0.3572T_2(x)+0.2252T_3(x)+0.0268T_4(x)$$

or

$$y=1.10^{-8}-1.61567692x+0.49999998x^2+0.90111044x^3+0.21456648x^4$$

Thus all solutions are given in Table-3.

Table-3

x	Chebyshev-Collocation Method				Taylor-Matrix Method N=9
	N=4	N=7	N=8	N=9	
0	0.000000009	0.00000003	0.00088493	-0.00000001	0
0.2	-0.29558318	-0.13551389	-0.13535137	-0.13545331	-0.135379
0.4	-0.50310679	-0.21286515	-0.21381239	-0.21274396	-0.212602
0.6	-0.56695848	-0.21909065	-0.22016033	-0.2189601	-0.218764
0.8	-0.42328656	-0.14805032	-0.14798978	-0.14792585	-0.147714
1	-0.00000001	0.00000002	-0.00088494	0.00000002	-0.0000006

4. CONCLUSIONS

Differential equations with variable coefficients are usually difficult to solve analytically. In many cases, it is required to approximate solution. For this purpose, the present method can be proposed. A considerable advantage of the method is that the solution is expressed as a truncated Chebyshev series and thereby a Taylor polynomial about x=0. Furthermore, after calculation of the series coefficients, the solution y(x) can be easily valuated for arbitrary values of x at low computational effort.

If the function P(x), Q(x), R(x) and f(x) can be expanded to the Chebyshev series in $-1 \leq x \leq 1$, then there exists the solution y(x); otherwise the solution may not exist. On the other hand, it would appear that this method shows to best advantages when the know functions in equation can be expanded to Taylor series about x=0 which converge rapidly.

To get the best approximating solution of the equation, the truncation limit N must be chosen large enough. For computational efficiency, some estimate for N, the degree of the approximating polynomial (the truncating limit of Taylor series) to y(x), should be available. Because the choice of N determines the precision of the solution y(x). If N is chosen too large, unnecessary labour may be done; but if N is taken a small value, the solution will not be sufficiently accurate. Therefore N must be chosen sufficiently large to get a reasonable approximation. In addition, N can be determined by using computer by means of $D(x_r)$.

When the problem is defined in a finite range $a \leq x \leq b$, by means of the linear transformation

$$x = \tfrac{1}{2}(b-a)t + \tfrac{1}{2}(b+a)$$

this range can be transformed to the range $-1 \leq t \leq 1$, which is the domain of the Chebyshev polynomial $T_r(x)$.

The method can be developed for the problem defined in the domain $0 \leq t \leq 1$, to obtain the solution in terms of the Chebyshev polynomials $T^*_r(x)$ [3].

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