VIBRATIONS OF A SIMPLY SUPPORTED BEAM WITH A NON-IDEAL SUPPORT AT AN INTERMEDIATE POINT

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Abstract– A simply supported Euler-Bernoulli beam with an intermediate support is considered. The concept of non-ideal boundary conditions is applied to the beam problem. In accordance, the intermediate support is assumed to allow small deflections. Approximate analytical solution of the problem is found using the Method of Multiple Scales, a perturbation technique. Ideal and non-ideal frequency response curves are contrasted.

Keywords– nonlinear beam vibrations, non-ideal boundary conditions, perturbation methods

1. INTRODUCTION

In vibration of continuous systems, types of support conditions are important and have direct effect on the solutions and natural frequencies. Boundary conditions of real systems are idealized by different types of supports such as simply supported, clamped, sliding, free, etc. The real system is modeled by choosing one of the nearest ideal boundary conditions. It is always assumed that those ideal conditions are satisfied exactly. However, small deviations from ideal conditions in real systems indeed occur. For example, a beam connected at its ends to rigid supports by pins is modeled using simply supported boundary conditions which require deflections and moments to be zero. However, the hole and pin assembly may have small gaps and/or friction which may introduce small deflections and/or moments at the ends. Similarly, a real built-in beam may have very small variations in deflection and/or slope. These types of boundary conditions with small deviations from the ideal conditions are defined here as non-ideal boundary conditions.

Non-ideal boundary conditions are modeled using perturbations. The idea is applied to a simple-simple beam having a non-ideal simple support at an intermediate point. Effect of non-ideal boundary condition at the intermediate point on the natural frequencies and frequency-response relations are examined using the Method of Multiple Scales. The pioneering work on non-ideal boundary conditions was due to Pakdemirli and Boyacı [1] for linear beam problems and Pakdemirli and Boyacı [2] for a non-linear beam problem.

2. PROBLEM FORMULATION AND SOLUTION

The simple-simple Euler-Bernoulli beam considered here has a simple support at an intermediate point at $x=\eta$ ($0<\eta<1$), where x is the spatial co-ordinate (See Fig.1). The equations of motion and the boundary conditions are,

$$\ddot{w}_1 + w_1^{iv} = -2\bar{\mu}\dot{w}_1 + \bar{F}_1 \cos \Omega t, \quad \ddot{w}_2 + w_2^{iv} = -2\bar{\mu}\dot{w}_2 + \bar{F}_2 \cos \Omega t \quad (1, 2)$$

$$w_1(0, t) = w_1''(0, t) = w_2(1, t) = w_2''(1, t) = 0 \quad w_1(\eta, t) = w_2(\eta, t) = \varepsilon \alpha(t) \quad (3)$$

$$w_1'(\eta, t) = w_2'(\eta, t) \quad w_1'''(\eta, t) = w_2'''(\eta, t)$$

where w_1 is the left hand side deflection and w_2 is the right hand side deflection, t is the time variable, $\bar{\mu}$ is the damping coefficient and, $\bar{F}_i$'s and Ω are the magnitudes and the frequency of the external excitations. $(\cdot)$ denotes derivation with respect to time variable t and $(\cdot)'$ denotes derivation with respect to spatial variable x . ε is a small perturbation parameter. All variables are dimensionless. Here, small deflections at the intermediate point η are permitted to indicate deviations from the ideal boundary condition.

Fig. 1- A simply supported beam with a non-ideal boundary condition at an intermediate point

Now, approximate solutions of equations (1) and (2) with associated boundary conditions (3) are sought. The Method of Multiple Scales (a perturbation technique) [3] is applied directly to the partial differential systems and boundary conditions. Expansions are assumed of the forms

$$w_1(x, t; \varepsilon) = w_{11}(x, T_0, T_1) + \varepsilon w_{12}(x, T_0, T_1),$$

$$w_2(x, t; \varepsilon) = w_{21}(x, T_0, T_1) + \varepsilon w_{22}(x, T_0, T_1) \quad (4, 5)$$

where $T_0 = t$ is the fast time scale and $T_1 = \varepsilon t$ is the slow time scale. Only primary resonance case is considered and hence, the forcing and damping terms are ordered as

$$\bar{\mu} = \varepsilon \mu \quad \text{and} \quad \bar{F}_i = \varepsilon F_i \quad (6)$$

The time derivatives are written as

$$(\cdot) = D_0 + \varepsilon D_1, \quad (\cdot\cdot) = D_0^2 + 2\varepsilon D_0 D_1, \quad D_n = \partial / \partial T_n \quad (7)$$

Inserting equations (4)–(7) into equations (1)–(3) and equating coefficients of like powers of ε , one obtains at order 1,

$$D_0^2 w_{11} + w_{11}^{iv} = 0, \quad D_0^2 w_{21} + w_{21}^{iv} = 0 \quad (8, 9)$$

$$w_{11}(0, T_0, T_1) = w_{11}''(0, T_0, T_1) = w_{21}(1, T_0, T_1) = w_{21}''(1, T_0, T_1) = 0$$

$$w_{11}(\eta, T_0, T_1) = w_{21}(\eta, T_0, T_1) = 0 \quad w_{11}'(\eta, T_0, T_1) = w_{21}'(\eta, T_0, T_1) \quad (10)$$

$$w_{11}'''(\eta, T_0, T_1) = w_{21}'''(\eta, T_0, T_1)$$

and, at order ε ,

$$D_0^2 w_{12} + w_{12}^{iv} = -2D_0 D_1 w_{11} - 2\mu D_0 w_{11} + F_1 \cos \Omega T_0,$$

$$D_0^2 w_{22} + w_{22}^{iv} = -2D_0 D_1 w_{21} - 2\mu D_0 w_{21} + F_2 \cos \Omega T_0 \quad (11, 12)$$

$$w_{12}(0, T_0, T_1) = w_{12}''(0, T_0, T_1) = w_{22}(1, T_0, T_1) = w_{22}''(1, T_0, T_1) = 0$$

$$w_{12}(\eta, T_0, T_1) = w_{22}(\eta, T_0, T_1) = \alpha(T_0, T_1) \quad w_{12}'(\eta, T_0, T_1) = w_{22}'(\eta, T_0, T_1) \quad (13)$$

$$w_{12}'''(\eta, T_0, T_1) = w_{22}'''(\eta, T_0, T_1)$$

At order 1, a solution of the form

$$w_{11} = (A(T_1)e^{i\omega T_0} + cc)Y_1(x) \quad w_{21} = (A(T_1)e^{i\omega T_0} + cc)Y_2(x) \quad (14, 15)$$

is assumed, where cc stands for the complex conjugate of the preceding terms. Substituting equations (14) and (15) into equations (8)–(10), one has

$$Y_1^{iv} - \omega^2 Y_1 = 0, \quad Y_2^{iv} - \omega^2 Y_2 = 0 \quad (16, 17)$$

$$Y_1(0) = Y_1''(0) = Y_2(1) = Y_2''(1) = Y_1(\eta) = Y_2(\eta) = 0, \quad (18)$$

$$Y_1'(\eta) = Y_2'(\eta) \quad Y_1''(\eta) = Y_2''(\eta)$$

Solving equations (16)–(18) exactly yields the mode shapes,

$$Y_1(x) = C \sin \beta(1-\eta) \left[\sin \beta x - \frac{\sin \beta \eta}{\sinh \beta \eta} \sinh \beta x \right], \quad (19, 20)$$

$$Y_2(x) = C \sin \beta \eta \left[\sin \beta(1-x) - \frac{\sin \beta(1-\eta)}{\sinh \beta(1-\eta)} \sinh \beta(1-x) \right]$$

and natural frequencies ω which satisfy the transcendental equation

$$\sin \beta \eta \sinh \beta \eta [\cos \beta(1-\eta) \sinh \beta(1-\eta) - \sin \beta(1-\eta) \cosh \beta(1-\eta)] \\ + \sin \beta(1-\eta) \sinh \beta(1-\eta) [\cos \beta \eta \sinh \beta \eta - \sin \beta \eta \cosh \beta \eta] = 0 \quad (21)$$

where

$$\beta = \sqrt{\omega} \quad (22)$$

Equation (21) is solved numerically for the first seven modes and results are given in Table-1 for different η values. Due to the symmetry of the problem, results are given up to $\eta=0.5$. Inspecting equation (21), one finds that for some specific β values $\sin \beta \eta$ and $\sin \beta(1-\eta)$ vanishes and for those degenerate cases the mode shapes take the simpler form

$$Y_1(x) = C \sin \beta x, \quad Y_2(x) = -C \sin \beta(1-x) \quad (23, 24)$$

Because the homogeneous equations (8)–(10) have a non-trivial solution, the non-homogeneous problem (11)–(13) will have a solution only if a solvability condition [3] is satisfied. To determine this condition, the secular and non-secular terms are separated by assuming a solution of the form

$$w_{12} = \varphi_1(x, T_1)e^{i\omega T_0} + W_1(x, T_0, T_1) + cc, \quad w_{22} = \varphi_2(x, T_1)e^{i\omega T_0} + W_2(x, T_0, T_1) + cc \quad (25, 26)$$

Substituting this solution into equations (11)–(13), secular and non-secular terms separate (up to this order, only the secular ones are of interest)

$$\varphi_1^{iv} - \omega^2 \varphi_1 = -2i\omega D_1 A Y_1 - 2\mu i \omega A Y_1 + \frac{F_1}{2} e^{i\sigma T_1}, \quad (27, 28)$$

$$\varphi_2^{iv} - \omega^2 \varphi_2 = -2i\omega D_1 A Y_2 - 2\mu i \omega A Y_2 + \frac{F_2}{2} e^{i\sigma T_1}$$

In obtaining these equations, the order 1 solutions (14) and (15) are substituted into equations (11)–(13). It is also assumed that the external excitation frequency is close to one of the natural frequencies of the system such that;

$$\Omega = \omega + \varepsilon \sigma \quad (29)$$

Here σ is a detuning parameter of order 1. After algebraic manipulations, the solvability conditions for equations (27) and (28) are obtained

$$2i\omega(D_1 A + \mu A) + kA[Y_2'''(\eta) - Y_1''(\eta)] - \frac{1}{2} f e^{i\sigma T_1} = 0 \quad (30)$$

where

$$f = \int_0^\eta F_1 Y_1 dx + \int_\eta^1 F_2 Y_2 dx \quad (31)$$

In obtaining equation (30) the normalization condition $\int_0^\eta Y_1^2 dx + \int_\eta^1 Y_2^2 dx = 1$ is employed.

The amplitude of the deflection allowed at the intermediate point is assumed to be of the same form as that of order 1 solution, namely

$$\alpha(T_0, T_1) = kA(T_1)e^{i\omega T_0} + cc \quad (32)$$

where k is an arbitrary constant of order 1. Equation (30) determines the modulations in the complex amplitudes. The polar form

$$A = \frac{1}{2} a(T_1) e^{i\theta(T_1)} \quad (33)$$

is to be used to calculate real amplitudes and phases. After separating real and imaginary parts, one obtains

$$\omega\alpha' = -\mu\omega\alpha + \frac{1}{2} f \sin\gamma, \quad \omega\alpha\gamma' = \omega\alpha\sigma - K\alpha + \frac{1}{2} f \cos\gamma \quad (34, 35)$$

where γ and K are defined as

$$\gamma = \sigma T_1 - \theta, \quad K = \frac{k}{2} [Y_2'''(\eta) - Y_1''(\eta)] \quad (36)$$

In the steady state case, $\alpha' = \gamma' = 0$ and solving for the detuning parameter yields

$$\sigma = \frac{K}{\omega} \pm \sqrt{\frac{f^2}{4\omega^2 a^2} - \mu^2} \quad (37)$$

For free undamped vibrations, non-ideal natural frequencies are obtained from

$$\omega_{ni} = \omega + \varepsilon \frac{K}{\omega} \quad (38)$$

In Table 1, first five ideal and non-ideal natural frequencies are calculated and given for different η up to 0.5. The non-ideal frequencies may increase, decrease or remain unchanged depending on the position parameter η and number of modes.

Table 1– Ideal and non-ideal natural frequencies ($\varepsilon k/2 = 0.1$)

	ω_{1i}	ω_{1ni}	ω_{2i}	ω_{2ni}	ω_{3i}	ω_{3ni}	ω_{4i}	ω_{4ni}	ω_{5i}	ω_{5ni}
$\eta = 0.1$	17.8622	15.1822	58.2367	61.4112	122.1138	118.4668	209.6487	213.7117	320.8512	316.4768
$\eta = 0.2$	21.3289	19.5473	70.418	72.6689	147.9077	145.6414	246.7401	242.2972	319.4266	312.1935
$\eta = 0.3$	26.3352	24.7986	86.0614	87.2882	138.7773	134.8397	204.0458	208.9341	338.7358	335.0162
$\eta = 0.4$	33.4385	32.1416	76.8753	79.3723	127.9821	124.2754	246.7401	251.183	300.3159	293.3388
$\eta = 0.5$	39.4784	39.4784	61.6728	64.8165	157.9137	157.9137	199.8594	205.5143	355.3058	355.3058

One can also obtain amplitude-excitation frequency relation from (37) and (29)

$$\Omega = \omega + \varepsilon \frac{K}{\omega} \pm \sqrt{\frac{\varepsilon^2 f^2}{4\omega^2 a^2} - (\varepsilon \mu)^2} \quad (39)$$

In Figure 2, four different plots corresponding to the first four natural frequencies are given for the parameter values, $\eta = 0.3$, $\varepsilon \mu = 0.01$, $\varepsilon k/2 = 0.1$ and $\varepsilon f = 1$.

Figure 2- Frequency-response graphs; a) 1st mode, b) 2nd mode,
c) 3rd mode, d) 4th mode (--- ideal, — non-ideal)

Beam deflections to the first order approximation are given as

$$w_1 = a \cos(\Omega t - \gamma) Y_1(x) + O(\varepsilon), \quad w_2 = a \cos(\Omega t - \gamma) Y_2(x) + O(\varepsilon) \quad (40, 41)$$

3. CONCLUDING REMARKS

Non-ideal boundary conditions are defined and formulated using perturbation theory. A simply supported beam with a non-ideal simple support at an intermediate point is treated. Approximate analytical solution of the problem is presented using the Method of Multiple Scales. Ideal and non-ideal natural frequencies are given for different intermediate support locations. Depending on the mode numbers and locations, the frequencies may increase, decrease or remain unchanged. Mode shapes are also affected by the non-ideal boundary condition. To determine those variations, calculations should be carried up to order ε^2 .

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