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ASYMPTOTIC EVALUATION OF THE EDGE DIFFRACTION IN CYLINDRIC PARABOLOIDAL REFLECTOR ANTENNAS

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Abstract-Diffraction fields on the edge of a cylindrical parabolic reflector antenna are examined. Firstly canonical problem is defined for the edge of the antenna and study is reduced to a half plane scattering problem. Coordinate system is defined as the origin being the edge of the reflector, so the feed became offset. Space is divided into two parts according to ρ and edge diffracted fields are calculated for the region where $\rho \leq \rho_0$. Obtained poorly divergent series are changed into complex integrals for $k \to \infty$ and these integrals are solved in the same conditions. Uniform expression of edge diffracted field is calculated by using Fresnel integrals.

Keywords-Edge diffraction, Asymptotic Evaluation, parabolic reflector antenna, Fresnel integral

1. INTRODUCTION

Let S be a half plane defined by Cartesian Coordinates $x > 0$, $y = 0$, $z \in (-\infty,\infty)$ in a homogenous and isotropic media. Such a plane disturbs the homogeneity of the space and affects the propagating wave. This affect is called as edge diffraction and the problems, studying this affect, are named as half plane problem.

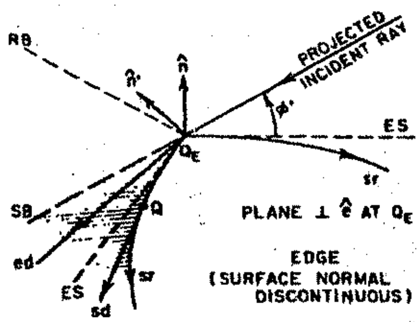

Figure 1. Geometry of incident, diffracted and reflected fields on an edge

Figure 1 can be considered for a model of any edge diffraction problem [1]. Normal of the surface is discontinuous at the edge point. Solution is developed according to the Geometrical Theory of Diffraction (GTD) [2], [3], [4]. For this reason, we are interested in the dyadic diffraction coefficients. According to GTD, the incident high frequency wave forms reflected, edge diffracted and edge stimulated wave on the edge. Q_E is the diffraction point for the geometry in Figure 1. The ray, incident to the edge, forms *ed* edge diffracted field and *sr* surface diffracted field according to the Keller's generalized Fermat's principle. Surface ray causes surface diffracted ray *sd*, which scatters from all Q points in the surface, for convex surfaces. *ES* is the boundary between edge diffracted rays and the surface diffracted rays and tangent to the surface on point Q_E. *SB* and *RB* are the shadow boundaries of incident and reflected field, respectively. If all of the surfaces are illuminated, there will not be any shadow boundary in the edge; instead there will be two reflection boundaries. There are transient regions in the shadow and reflection boundaries and field changes very rapidly for these regions.

For the half plane problem, there will be no *ES* boundary. Total electric field can be written as

$$\vec{E} = \vec{E}_i u_i + \vec{E}_r u_r + \vec{E}_d \tag{1.1}$$

where $\vec{E}_i$ is the incident field for empty media, $\vec{E}_r$ is the reflected wave when there is only surface in the space and $\vec{E}_d$ is the edge diffracted field. u_i and u_r represents unit step functions of incident and reflected fields, respectively.

A significant point of the research activity is devoted to develop diffraction coefficients within the framework of the Geometrical Theory of Diffraction (GTD), and its uniform version (UTD), in order to broaden the class of practical problems that can be treated by ray techniques. Whenever possible, uniform diffraction coefficients are rigorously derived from the exact solution of the canonical problem that locally approximates the actual structure. However, only few canonical solutions are known in an appropriate analytical form; thus, in order to extend the applicability of UTD one needs to resort to approximate solutions based on high-frequency assumptions. To this end, different canonical problems have been recently considered, including perfectly electric conducting (PEC), loaded, and periodic structures.

2. CANONICAL PROBLEM CONCEPT AND ADAPTION TO PARABOLIC REFLECTOR ANTENNAS

Practically geometry of many bodies is too complex to be expressed by easy formulas. The calculation of the scattered field of such bodies will be too hard. Asymptotic expression of problems which have complex geometry for finding the definite solution can be calculated with the help of Keller's fourth hypothesis. This assumption states that diffraction of high frequency waves in space is a local phenomenon. A canonical problem is defined by using this hypothesis. For a diffraction point Q we will suppose that the physical properties of space and geometrical properties of the scatter for a small neighborhood of the point Q don't change and the solution will represent the exact solution for Q. So we can model a complex body with the simple problems that can be solved easily.

In order to find the scattered field at the edge of the cylindrical reflector, we can benefit from a perfect conductor infinite half plane which is located at the diffraction point. This will be a first order canonical problem because only the tangent of the diffraction point is taken into account. If the curvature of the surface at this point was evaluated, a second order canonical problem would be defined.

3. RADIATION FROM A LINEAR CURRENT SOURCE LOCATED AT THE FOCUS OF REFLECTOR ANTENNA

An infinitely long linear current source can be used as the feed of the cylindrical parabolic reflector antenna. If the edge of the reflector is taken as the origin of the coordinate system, the feed will be offset for this system. This representation can be seen in Figure 2. In the figure linear feed is parallel to the z axis. ρ_0 represents the geometrical formula of the parabola in polar coordinates and is a function of ϕ_0. We can write (x,y) instead of (x_1, y_1).

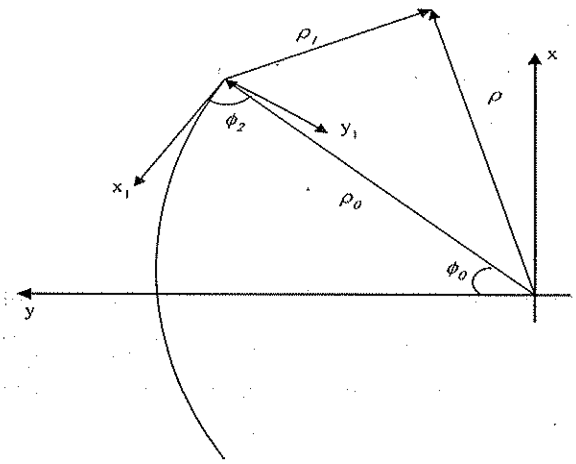

Figure 2. Geometry of a cylindrical parabolic reflector antenna fed by a current source

Current density can be written by using delta functions for polar coordinates. Since the electric current source is parallel to the z axis and the current flows in this direction, there will be only the E_z component of the field according to the Maxwell-Ampere Equation. This component must prove the Helmholtz Equation with source as

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial E_z}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2}+k^2 E_z=-j\omega\mu I_0\frac{1}{\rho}\delta(\rho-\rho_0)\delta(\phi-\phi_0) \tag{3.1}$$

where differentiation with respect to z is zero because of symmetry. This expression is a inhomogeneous partial differential equation. The solution of such an equation can be made in two steps. First homogenous solution is found by making the source term zero. Then exact solution is obtained by including the source in terms of homogenous solution. There are different methods for the second step. In this study we will evaluate the source as a boundary condition of a homogenous equation. We can suppose easily that the solution of the homogenous part will be an infinite series of eigen values and eigenvectors. Series representation can also be found by solving the homogenous differential equation with the separation of variables method. In every aspect, the solution will give a Green's Function. We can write the series representation as

$$E_z=\sum_{\nu} C_\nu u_\nu(\rho) sin\,\nu\phi \tag{3.2}$$

where C_ν is a function of ϕ_0. When we put (3.2) in (3.1) and solve the homogenous part for ρ, two separate functions of $u(\rho)$ will be obtained for two different regions. For $\rho \le \rho_0$, the solution must be a Bessel Function and for $\rho \ge \rho_0$, it must be a Hankel Function of the second kind for a time factor of $e^{j\omega t}$. There will be integration constants, which are the functions of ρ_0, with these solutions. So two separate solutions of E_z is found for the homogenous equation. We will define boundary conditions in order to obtain the exact solution. This solution will give the incident, reflected and diffracted fields in space.

Two boundary conditions can be written for $\rho = \rho_0$. Tangential components of the electrical field will be continuous at the boundary. So two homogenous solutions must be equal to each other for $\rho = \rho_0$. The integration constants, which are function of ρ_0, will be obtained from this condition.

As a second step, the source at $\rho = \rho_0$ will be included as a boundary condition. We can model this source as

$$\vec{J}_{es}(\vec{r}) = \frac{I_0}{\rho}\delta(\phi - \phi_0)\vec{e}_z \tag{3.3}$$

where $\vec{J}_{es}$ is the surface current density. Difference of the tangential components of the magnetic field at $\rho = \rho_0$ will be equal to this current density. If the rotational of the homogenous solutions is taken according to Maxwell-Faraday equation and the surface current density is written as a Fourier Sinus Series, C_ν can be evaluated as

$$C_\nu = \frac{\omega\mu I_0}{2} \sin \nu\phi_0 \tag{3.4}$$

which is a function of ϕ_0. The exact solution of (3.1) can be written as

$$E_z = \frac{\omega\mu I_0}{2}\sum_\nu \begin{matrix} H_\nu^{(2)}(k\rho_0)J(k\rho)\sin\nu\phi_0 \sin\nu\phi & , \rho \le \rho_0 \\ J_\nu(k\rho_0)H_\nu^{(2)}(k\rho)\sin\nu\phi_0 \sin\nu\phi & , \rho \ge \rho_0 \end{matrix} \tag{3.5}$$

where ν will be found from the boundary conditions of the half plane. The tangential component of the electrical field will be zero at the perfectly conducting surface for $\phi = 0$ and $\phi = 2\pi$. We can find $\nu = n/2$ by using the field expression where $\rho \le \rho_0$.

4. ASYMPTOTIC EVALUATION OF THE ELECTRICAL FIELD

The series representation of the electrical field in (3.5) is poorly convergent. For this reason, evaluation of the reflected and diffracted fields is highly hard. Practically, the series can be transformed to complex integrals for $k\rho \to \infty$. The solutions of these integrals will give the geometrical optics terms of incident, reflected and diffracted fields, which forms the total field. If asymptotic expansion of the Hankel Function in (3.5) for $\rho \leq \rho_0$ and $k\rho_0 \to \infty$ is used,

$$E_z = \sqrt{\frac{2}{\pi k\rho_0}} \frac{\omega\mu I_0}{2} e^{-jk\rho_0} e^{j\frac{\pi}{4}} \sum_{n=1}^{\infty} J_{n/2}(k\rho) e^{jn\frac{\pi}{4}} \sin\frac{n}{2}\phi_0 \sin\frac{n}{2}\phi \tag{4.1}$$

term will be obtained. The series can be represented by symbol I. Infinite series of I converges rapidly for the small terms of $k\rho$. We can easily show that all of the functions in I can be expressed as exponential functions. There will be a complex integral coming from the Bessel Function. According to the sign of ϕ, we can write two separate integral representations for two different contours in the complex plane. So there will be two complex integrals in a series expression. All of the terms in the series can be written as exponential functions of n which is the variable of the series. In this way, summation of the exponential terms can be calculated by using different identities [5]. As a result, we will have the form

$$I = \int_{C_1 - C} e^{jk\rho \cos z} \cot g\left[\frac{z + (\phi - \phi_0)}{4}\right] dz - \int_{C_1 - C} e^{jk\rho \cos z} \cot g\left[\frac{z + (\phi + \phi_0)}{4}\right] dz \tag{4.2}$$

where $C_1 - C$ shows the difference of the contour integrals of Bessel Functions in Figure 3.

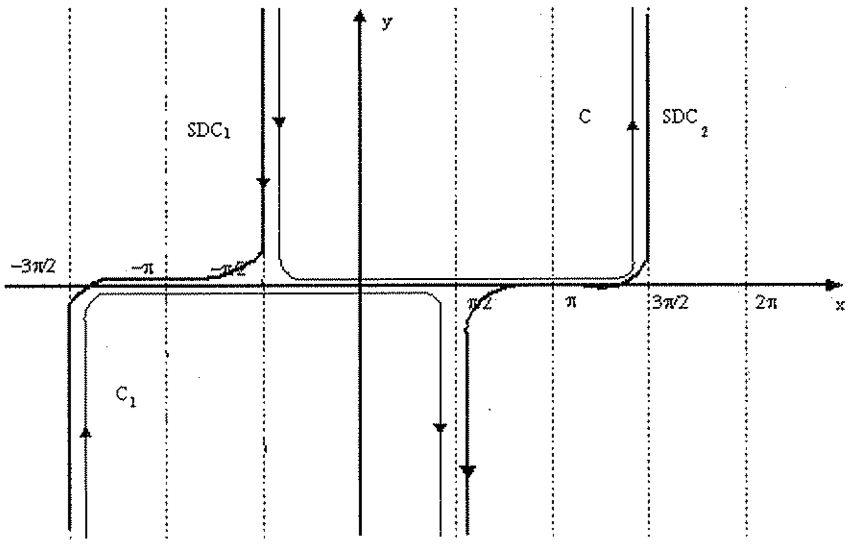

Figure 3. Integration contours for the electric field.

If C_T is the total contour, we can express $C_1 - C$ as

$$C_1 - C = C_T - (SDIC_1 - SDIC_2) \tag{4.3}$$

from Figure 3. The contour to be determined is the difference of the total contour and steepest-descent integral contours. There will be poles of I in C_T and the integral part concerning the total contour can be calculated by using the residue theorem for the poles. Steepest descent integral contours can be obtained by evaluating the integrals in I asymptotically for $k\rho \to \infty$. $SDIC_1$ and $SDIC_2$ are the steepest descent integral contours passing from the saddle points at -π and π, respectively. C_T can be calculated by using the residue theorem as

$$\int_{C_T} f(z)e^{jk\rho g(z)}dz = 2\pi j \sum_n Re\,z(z = z_n) \tag{4.4}$$

where z_n is the pole of f(z). The poles of f(z) which are in C_T are evaluated. In 4.2 integrals, f(z) is the cotangent function of ϕ and z, so poles will be the zeros of the sinus function. In the region of the contour, we can write

$$z_k = -(\phi \pm \phi_0) \tag{4.5}$$

for the poles. From 4.5 we can see that n will be equal to two in 4.4. By using the residue theorem for I, the total contour integral can be evaluated as

$$\int_{C_T} e^{jk\rho\cos z} \cot g\left[\frac{z+(\phi \pm \phi_0)}{4}\right]dz = 2\pi j e^{jk\rho\cos(\phi \pm \phi_0)} \tag{4.6}$$

with the poles. This equation represents the incident and reflected waves of geometrical optics. Steepest descent integration contour method [6], can be applied to the 4.2 integrals at the saddle point. Saddle points will be found by differentiating phase function of 4.4 and equating it to zero. The solution of the equation will give the saddle points as

$$z_{s1} = -\pi,\ z_{s2} = \pi \tag{4.7}$$

for the region in C_T. Complex variable z can be thought of being formed from real and complex parts. Phase function g(z) is expressed as real and complex functions by writing z in such a manner. Steepest descent integration contour can be found by equating the real part of $g(z)$ to the real part of $g(z_s)$. The convergence region will be determined from the $Img(z)>0$ expression. $SDIC_1$ and $SDIC_2$ are obtained as a result of these operations. The phase and amplitude functions ($g(z)$ and $f(z)$) are expressed by Taylor Series at the saddle point. If we are contended with the first two terms of $g(z)$ and the first term of $f(z)$, the remaining part of the integral can be evaluated easily. The total diffracted field can be written as

$$E_d = 2\sqrt{2\pi}\frac{e^{-jk\rho}}{\sqrt{k\rho}}\frac{\sin(\phi/2)\sin(\phi_0/2)}{\cos\phi + \cos\phi_0} \tag{4.8}$$

which gives the edge diffracted field of geometrical optics. The incident and reflected diffraction functions of the (4.8) form are referred to as Keller's diffraction functions and possess singularities along the incident and reflection shadow boundaries.

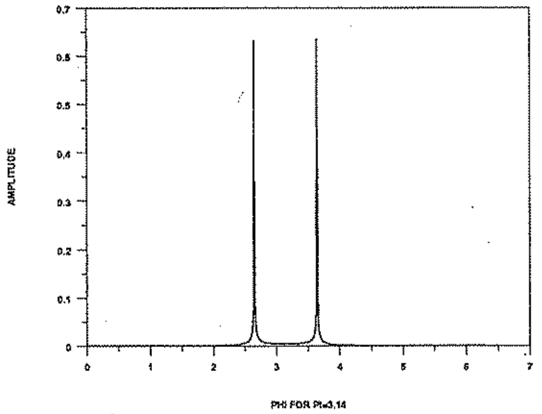

Figure 4. Non-uniform edge diffracted field for $\phi_0 = 30^0$.

The field diverges for $\pi \pm \phi_0$ as can be seen from Figure 4. This divergence can also be observed for angles near transition regions. Solution is non-uniform for these reasons.

5. UNIFORM ASYMPTOTIC EXPRESSION OF THE EDGE DIFFRACTED FIELD

4.2 integrals must be evaluated in a different way at the saddle points in order to obtain a continuous asymptotic expression for the transition regions. The divergence at the transition regions arises because of the zeros of *f(z)* in 4.4. For this reason, the solution must be found in a manner to compensate the poles in the concerned regions. The aim is to express integrals in 4.2 as Fresnel functions, which are continuous at the transition regions. We can write –z instead of z and divide the integrals by two after adding them [7]. After expressing trigonometric functions of z by a variable, the integrals can be written as

$$e^{-jk\rho} \int_T \frac{e^{jk\rho 2u^2}}{\cos\frac{\phi \pm \phi_0}{2} - u} du \tag{5.1}$$

where T represents $C_1 - C$ after variable transform. The method is expressing the integrals as the known identity

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi} \tag{5.2}$$

and reducing them to a standard Fresnel integral form. As a result I can be obtained as

$$I = -j2\sqrt{\pi} e^{j\frac{\pi}{4}} e^{-jk\rho} \left[F\left(\sqrt{\frac{k\rho}{2}} \cos\frac{\phi - \phi_0}{2} \right) + F\left(\sqrt{\frac{k\rho}{2}} \cos\frac{\phi + \phi_0}{2} \right) \right] \tag{5.3}$$

which is a sum of two Fresnel integrals. It is observed from Figure 5 that the divergence in the transition regions diminishes. The reason is that the integrals were evaluated in

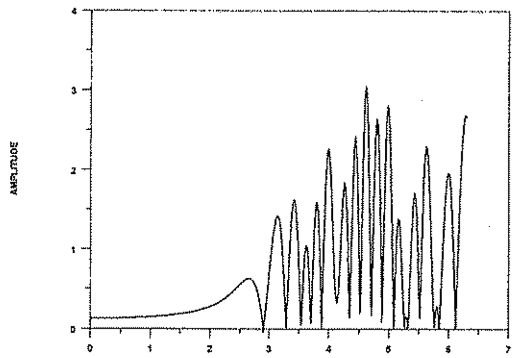

Figure 5. Uniform asymptotic expression of diffracted field.

order to compensate the divergence when they were being calculated asymptotically. The uniform expression of the diffracted field, which changes continuously from point to point, is obtained.

An other method for the evaluation of the uniform expression is to multiply and divide amplitude function, f(z), by $(z - z_0)$ and calculation of the integral by the steepest descent method at $\pm\pi$ saddle points [8].

6. CONCLUSION

As can be seen from Figures 4 and 5 the diffracted field at the shadow boundaries gives finite values. This shows the evaluation of the diffraction integral with Fresnel functions give the uniform field terms this type of evaluation is more exact from the Keller's diffraction coefficients which take infinite values form shadow boundaries and at caustics.

In this study a canonical approximation for the edge of a parabolic reflector antenna is made. So edge diffraction coefficient for a perfectly conducting edge was found easily. The exact solution for a half plane problem is considered. It was shown that the geometrical optics fields and the diffracted fields can be obtained from the asymptotic evaluation of the poor convergent series by asymptotic techniques.

REFERENCES

[1] R. G. Kouyoumjian and P. Pathak, A Uniform Geometrical Theory of Diffraction for an Edge in a Perfectly Conducting Surface, *Proc. IEEE*, Vol. 62, pp. 1448, 1974.
[2] J. B. Keller, Geometrical Theory of Diffraction, *J. Opt. Soc. of America*, Vol. 52/2, pp. 116-130, 1962.
[3] J. B. Keller, *J. Apply Physics*, Vol. 28,pp. 426, 1957.
[4] S. N. Karp and J. B. Keller, *Optica Acta*, Vol. 8, pp. 61, 1961.
[5] C. A. Balanis, *Advanced Engineering Electro-magnetics*, John Wiley & Sons, Inc., New York, 1989.
[6] H. E. Bayrakçı, *Engineering Mathematics of Linear Systems*, Caglayan Press, Istanbul, 1991.
[7] G. Uzgören and A. Büyükaksoy, *First Order Canonic Problems in Geometrical Theory of Diffraction*, Yıldız University Press, Istanbul, 1987.
[8] P. H. Pathak, *Radio Science*, Vol. 14, pp. 419, 1979.