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# TWO STRIPS PROBLEM RESTING ON AN ELASTIC FOUNDATION

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**Abstract**-In this paper, a layered composite made of two materials with different elastic constants and heights resting on an elastic foundation is studied according to theory of Elasticity. Symmetrical distributed load whose length is 2a is applied to the upper elastic layer. Gravity forces of the layered composite are neglected. The components of the stresses and displacements at a point located in the layered composite are obtained by using Integral Transform technique. Singular terms are subtracted from expressions of the stresses and their closed integral forms are added. The stresses and the displacements in any point of the layered composite are investigated and their graphics are plotted.

**Keywords**-Elasticity, elastic layer, strip, elastic foundation, Fourier transform, singularity.

## 1. INTRODUCTION

Layer problems resting on an elastic half-plane or elastic foundation have attracted attention of several researchers due to their wide application. Continuous foundation beams, runways of airport, foundation grillages, rigid pavements, and railway ballast resting on the ground or elastic foundation are some examples of such beams and plates.

Material properties, intensity of the applied force and height of the plate or beam play a very important role in the formation of the stresses and the displacements distribution on the layered composites. Several studies are done about an elastic layer supported by foundation $[1-10]$. However, there are few studies about layered composite made of different elastic materials resting on elastic foundation or an elastic semi-infinite plane $[11]$.

In the present study, a layered composite made of two bonded layers with different elastic constants and heights resting on an elastic foundation is examined by using theory of Elasticity (see Fig. 1). The upper elastic layer is subjected to a symmetric distributed load and gravity forces of the strips are neglected. The length of the distributed load is taken to be 2a and the heights of the strips are taken as $h_1$ and $h_2$. In the solution, displacements are explained in terms of Fourier transformation of an unknown function. After the expressions of the displacements are obtained, the expressions of the stresses are founded by substituting the displacements into constitutive equations. Singular terms which is formed while width of the uniformly applied load is sufficiently small, i.e. $a/h<0.10$, are subtracted from expressions of the normal stresses and their closed integral forms are added. The stress and the displacement distributions on the y symmetrical cross section, contact surfaces of the layers, and at $y=-h_2$ are calculated by giving different numerical values of spring constant, ratio of material constants and strip heights. Finally, numerical results are analyzed and conclusions are drawn.

## 2. GENERAL EXPRESSIONS FOR STRESSES AND DISPLACEMENTS

In the absence of body forces, the two dimensional Navier equations may be written as in the following form for considered an infinite layered composite consisting of two elastic layers and resting on elastic foundation as shown in Fig. 1.

$$\mu_i \nabla^2 u_i + \frac{2\mu_i}{\kappa_i - 1} \frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) = 0, \tag{1.a}$$

$$\mu_i \nabla^2 v_i + \frac{2\mu_i}{\kappa_i - 1} \frac{\partial}{\partial y}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) = 0, \qquad (i=1,2). \tag{1.b}$$

where $u_i$ and $v_i$ are the x and y-components of the displacement vector. $\mu_i$ and $\kappa_i$ (i =1,2) represent shear modules and constants of the elastic layers, respectively. $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for plane stress and $\kappa_i = 3-4\nu_i$ for plane strain. $\nu_i$ is the Poisson's ratio of layer. Subscript i (i = 1,2) indicates the values are related to the layer.

Observing that $x=0$ is a plane symmetry, it is sufficient to consider the problem in the region $0 \le x < \infty$ only. Using the symmetry consideration, the following expressions may be written,

$$u_i(x,y) = -u_i(-x,y), \tag{2.a}$$

$$v_i(x,y) = v_i(-x,y), \tag{2.b}$$

$$u_i(x,y) = \frac{2}{\pi}\int_0^\infty \Phi_i(\alpha,y)\sin(\alpha x)\,d\alpha, \tag{3.a}$$

$$v_i(x,y) = \frac{2}{\pi}\int_0^\infty \Psi_i(\alpha,y)\cos(\alpha x)\,d\alpha, \tag{3.b}$$

where $\Phi_i$ and $\Psi_i$ (i =1,2) functions are inverse Fourier transforms of $u_i$ and $v_i$, respectively. Taking necessary derivatives of equations (3.a) and (3.b), substituting them into equations (1.a) and (1.b), and solving second order differential equations, the following expressions may be obtained for displacements

***For strip one*** *($0 \le y \le h_1$ and $0 \le x \le \infty$);*

$$u_1(x,y) = \frac{2}{\pi}\int_0^\infty \left[\left(A_1(\alpha) + A_2(\alpha)y\right)e^{-\alpha y} + \left(A_3(\alpha) + A_4(\alpha)y\right)e^{\alpha y}\right]\sin(\alpha x)\,d\alpha, \tag{4}$$

$$v_1(x,y)=\frac{2}{\pi}\int_0^{\infty}\left\{\left[A_1(\alpha)+\left(\frac{\kappa_1}{\alpha}+y\right)A_2(\alpha)\right]e^{-\alpha y}\right.\qquad(5)$$
$$\left.+\left[-A_3(\alpha)+\left(\frac{\kappa_1}{\alpha}-y\right)A_4(\alpha)\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha,$$

***For strip two*** *($-h_2 \le y \le 0$ and $0 \le x \le \infty$);*

$$u_2(x,y)=\frac{2}{\pi}\int_0^{\infty}\left[\left(B_1(\alpha)+B_2(\alpha)y\right)e^{-\alpha y}+\left(B_3(\alpha)+B_4(\alpha)y\right)e^{\alpha y}\right]\sin(\alpha x)d\alpha,\qquad(6)$$

$$v_2(x,y)=\frac{2}{\pi}\int_0^{\infty}\left\{\left[B_1(\alpha)+\left(\frac{\kappa_2}{\alpha}+y\right)B_2(\alpha)\right]e^{-\alpha y}\right.\qquad(7)$$
$$\left.+\left[-B_3(\alpha)+\left(\frac{\kappa_2}{\alpha}-y\right)B_4(\alpha)\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha,$$

where $A_j(\alpha)$, $B_j(\alpha)$, $(j = 1,2,3,4)$ are unknown coefficients which will be determined from boundary conditions of the problem. Using Hooke's law and equations (4-7), the expressions of the stress for each strip may be expressed as follows:

$$\frac{1}{2\mu_1}\sigma_{x_1}(x,y)=\frac{2}{\pi}\int_0^{\infty}\left\{\left[\alpha\left(A_1(\alpha)+A_2(\alpha)y\right)-\frac{3-\kappa_1}{2}A_2(\alpha)\right]e^{-\alpha y}\right.$$
$$\left.+\left[\alpha\left(A_3(\alpha)+A_4(\alpha)y\right)+\frac{3-\kappa_1}{2}A_4(\alpha)\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha,\qquad(8)$$

$$\frac{1}{2\mu_1}\sigma_{y_1}(x,y)=\frac{2}{\pi}\int_0^{\infty}\left\{-\left[\alpha\left(A_1(\alpha)+A_2(\alpha)y\right)+\frac{1+\kappa_1}{2}A_2(\alpha)\right]e^{-\alpha y}\right.$$
$$\left.+\left[-\alpha\left(A_3(\alpha)+A_4(\alpha)y\right)+\frac{1+\kappa_1}{2}A_4(\alpha)\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha,\qquad(9)$$

$$\frac{1}{2\mu_1}\tau_{xy_1}(x,y)=\frac{2}{\pi}\int_0^{\infty}\left\{-\left[\alpha\left(A_1(\alpha)+A_2(\alpha)y\right)+\frac{\kappa_1-1}{2}A_2(\alpha)\right]e^{-\alpha y}\right.$$
$$\left.+\left[\alpha\left(A_3(\alpha)+A_4(\alpha)y\right)-\frac{\kappa_1-1}{2}A_4(\alpha)\right]e^{\alpha y}\right\}\sin(\alpha x)d\alpha,\qquad(10)$$

$$\frac{1}{2\mu_2}\sigma_{x_2}(x,y)=\frac{2}{\pi}\int_0^{\infty}\left\{\left[\alpha\left(B_1(\alpha)+B_2(\alpha)y\right)-\frac{3-\kappa_2}{2}B_2(\alpha)\right]e^{-\alpha y}\right.$$
$$\left.+\left[\alpha\left(B_3(\alpha)+B_4(\alpha)y\right)+\frac{3-\kappa_2}{2}B_4(\alpha)\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha,\qquad(11)$$

$$\frac{1}{2\mu_2}\sigma_{y_2}(x,y)=\frac{2}{\pi}\int_0^{\infty}\{\ -\left[\alpha\left(B_1(\alpha)+B_2(\alpha)y\right)+\frac{1+\kappa_2}{2}B_2(\alpha)\right]e^{-\alpha y}$$
$$+\left[-\alpha\left(B_3(\alpha)+B_4(\alpha)y\right)+\frac{1+\kappa_2}{2}B_4(\alpha)\right]e^{\alpha y}\ \}\cos(\alpha x)d\alpha, \tag{12}$$

$$\frac{1}{2\mu_2}\tau_{xy_2}(x,y)=\frac{2}{\pi}\int_0^{\infty}\{\ -\left[\alpha\left(B_1(\alpha)+B_2(\alpha)y\right)+\frac{\kappa_2-1}{2}B_2(\alpha)\right]e^{-\alpha y}$$
$$+\left[\alpha\left(B_3(\alpha)+B_4(\alpha)y\right)-\frac{\kappa_2-1}{2}B_4(\alpha)\right]e^{\alpha y}\ \}\sin(\alpha x)d\alpha, \tag{13}$$

## 3. BOUNDARY CONDITIONS AND SOLUTION OF PROBLEM

The layered composite consisting of two bonded elastic layers and resting on an elastic foundation shown in Fig.1 will be analyzed. It is subjected to a symmetric distributed load and heights of the strips are $h_1$ and $h_2$, respectively.

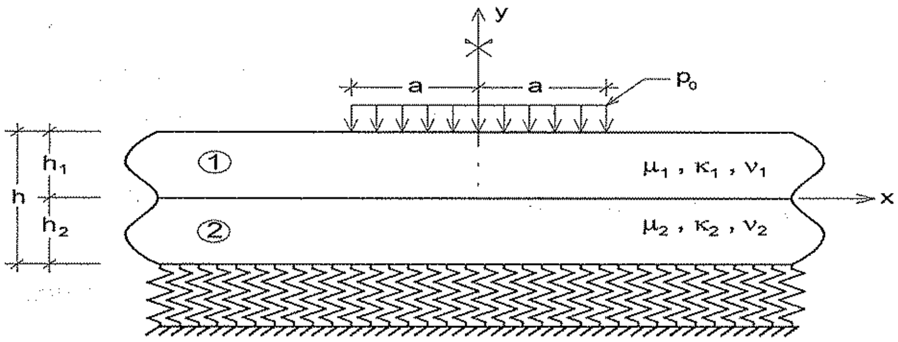


**Figure 1.** Loading condition and geometry of the problem.

In this problem, the boundary conditions may be expressed as,

$$\sigma_{y_1}(x,h_1)=-p_0, \qquad 0\le x\le a, \tag{14.a}$$
$$\sigma_{y_2}(x,-h_2)=k_0 v_2(x,-h_2), \qquad 0\le x<\infty, \tag{14.b}$$
$$\sigma_{y_1}(x,0)=\sigma_{y_2}(x,0), \qquad 0\le x<\infty, \tag{14.c}$$
$$\tau_{xy_1}(x,h_1)=0, \qquad 0\le x<\infty, \tag{14.d}$$
$$\tau_{xy_2}(x,-h_2)=0, \qquad 0\le x<\infty, \tag{14.e}$$
$$\tau_{xy_1}(x,0)=\tau_{xy_2}(x,0), \qquad 0\le x<\infty, \tag{14.f}$$
$$u_1(x,0)=u_2(x,0), \qquad 0\le x<\infty, \tag{14.g}$$
$$v_1(x,0)=v_2(x,0), \qquad 0\le x<\infty, \tag{14.h}$$

where, $k_0$ and $p_0$ are the elastic spring constant and the magnitude of the uniformly applied load, respectively.

Substituting equations (4-13) into the boundary conditions which are given equations (14.a-h) and using inversion Fourier transform, eight linear equations which are depended on $A_j(\alpha)$ and $B_j(\alpha)$ ($j=1,2,3,4$) are obtained following form.

$$-2\alpha A_1(\alpha)-(1+\kappa_1+2\alpha h_1)A_2(\alpha)-2\alpha e^{2\alpha h_1}A_3(\alpha)+(1+\kappa_1-2\alpha h_1)e^{2\alpha h_1}A_4(\alpha)$$
$$=-p_0\frac{e^{\alpha h_1}}{\mu_1}\frac{\sin(\alpha a)}{\alpha}, \tag{15}$$

$$-2\alpha A_1(\alpha)-(-1+\kappa_1+2\alpha h_1)A_2(\alpha)+2\alpha e^{2\alpha h_1}A_3(\alpha)+(1-\kappa_1+2\alpha h_1)e^{2\alpha h_1}A_4(\alpha)=0, \tag{16}$$

$$-2\alpha A_1(\alpha)-(1+\kappa_1)A_2(\alpha)-2\alpha A_3(\alpha)+(1+\kappa_1)A_4(\alpha)+2\alpha\beta B_1(\alpha)+(1+\kappa_2)\beta B_2(\alpha)$$
$$+2\alpha\beta B_3(\alpha)-(1+\kappa_2)\beta B_4(\alpha)=0, \tag{17}$$

$$-2\alpha A_1(\alpha)-(-1+\kappa_1)A_2(\alpha)+2\alpha A_3(\alpha)-(-1+\kappa_1)A_4(\alpha)+2\alpha\beta B_1(\alpha)+(-1+\kappa_2)\beta B_2(\alpha)$$
$$-2\alpha\beta B_3(\alpha)+(-1+\kappa_2)\beta B_4(\alpha)=0, \tag{18}$$

$$\alpha A_1(\alpha)+\kappa_1 A_2(\alpha)-\alpha A_3(\alpha)+\kappa_1 A_4(\alpha)-\alpha B_1(\alpha)-\kappa_2 B_2(\alpha)+\alpha B_3(\alpha)-\kappa_2 B_4(\alpha)=0, \tag{19}$$

$$A_1(\alpha)+A_3(\alpha)-B_1(\alpha)-B_3(\alpha)=0, \tag{20}$$

$$-2\alpha B_1(\alpha)-(-1+\kappa_2-2\alpha h_2)B_2(\alpha)+2\alpha e^{-2\alpha h_2}B_3(\alpha)-(-1+\kappa_2+2\alpha h_2)e^{-2\alpha h_2}B_4(\alpha)=0, \tag{21}$$

$$-(2\alpha+k)B_1(\alpha)+(-1-k\kappa_2/\alpha-\kappa_2+2\alpha h_2+kh_2)B_2(\alpha)+(-2\alpha+k)e^{-2\alpha h_2}B_3(\alpha)$$
$$+(1-k\kappa_2/\alpha+\kappa_2+2\alpha h_2-kh_2)e^{-2\alpha h_2}B_4(\alpha)=0, \tag{22}$$

where,

$$k=k_0/\mu_2 \quad \text{and} \quad \beta=\mu_2/\mu_1.$$

Solving above equations, $A_j(\alpha)$ and $B_j(\alpha)$ ($j = 1,\dots,4$) coefficients are calculated in terms of $\alpha$. These coefficients are given by (A.1-A.8) in Appendix. Substituting the values of the coefficients into the expression of displacements and stresses given equations (4-13), the displacements and the stresses of at any point of the layered composite resting on an elastic foundation may be calculated.

## 4. SINGULAR TERMS

In the case of the distributed load width being sufficiently small, i.e. $a/h<0.10$, while y reaches $h_1$, there are spoils in the kernel of normal stresses $\sigma_{x_1}$ and $\sigma_{y_1}$. These singular terms are existed in the expressions while there is $e^{-\alpha(h_1-y)}$ as a multiple. The singular terms may be defined following form.

For case $y\to h_1$;

$$\sigma_{x_1 s}(x,y)=\frac{2}{\pi}\frac{p_0}{a}\int_0^{\infty}\left(-\frac{1}{\alpha}+h_1-y\right)e^{-\alpha(h_1-y)}\sin(\alpha a)\cos(\alpha x)d\alpha, \tag{23.a}$$

$$\sigma_{y_1 s}(x,y)=\frac{2}{\pi}\frac{p_0}{a}\int_0^{\infty}\left(-\frac{1}{\alpha}-h_1+y\right)e^{-\alpha(h_1-y)}\sin(\alpha a)\cos(\alpha x)d\alpha, \tag{23.b}$$

Their closed integral forms may be written as follows [12],

$$\sigma_{x_1k}(x,y)=\frac{p_0}{2a}\left\{-\tan^{-1}\left(\frac{a+x}{h_1-y}\right)-\tan^{-1}\left(\frac{a-x}{h_1-y}\right)\right.$$
$$\left.+(h_1-y)\left[\frac{a+x}{(h_1-y)^2+(a+x)^2}+\frac{a-x}{(h_1-y)^2+(a-x)^2}\right]\right\}, \qquad (24.a)$$

$$\sigma_{y_1k}(x,y)=\frac{p_0}{4a}\left\{-\tan^{-1}\left(\frac{a+x}{h_1-y}\right)-\tan^{-1}\left(\frac{a-x}{h_1-y}\right)\right.$$
$$\left.-(h_1-y)\left[\frac{a+x}{(h_1-y)^2+(a+x)^2}+\frac{a-x}{(h_1-y)^2+(a-x)^2}\right]\right\}, \qquad (24.b)$$

Subtracting the singular terms from kernel of the normal stresses and adding to their closed integral forms, the expressions of the normal stresses may be defined as

$$\sigma^*_{x_1}(x,y)=\sigma_{x_1}(x,y)-\sigma_{x_1s}(x,y)+\sigma_{x_1k}(x,y), \qquad (25.a)$$

$$\sigma^*_{y_1}(x,y)=\sigma_{y_1}(x,y)-\sigma_{y_1s}(x,y)+\sigma_{y_1k}(x,y), \qquad (25.b)$$

## 5. NUMERICAL SOLUTIONS

Some of the calculated results obtained from substituting the coefficients of $A_j(\alpha)$ and $B_j(\alpha)$, (j = 1,...,4), into the expressions of the stress and the displacement for various
dimensionless quantities such as $a/h$, $h_2/h$, $\beta$, and $k$ are shown in Table 1 and Figs. 2-7.

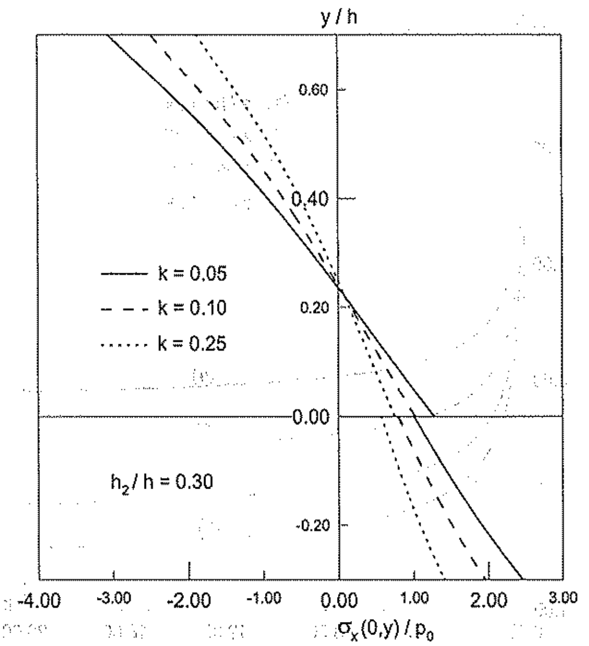


**Figure 2.** $\sigma_x(0,y)$ normal stress distribution for various values of k $(a/h=0.50,\ \beta=0.85)$

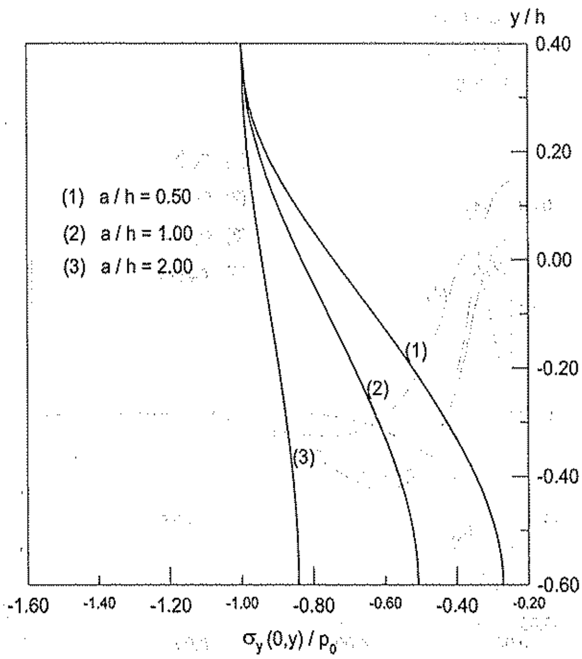


**Figure 3.** $\sigma_y(0,y)$ stress distribution for various values of $a/h$ $(h_2/h=0.6,\beta=0.85,k=0.1)$

Fig. 2 shows that $\sigma_x(0,y)$ normal stress distribution along the symmetry plane for various values of k taking $a/h$ and $\beta$ 0.50 and 0.85, respectively. When k increases, the stress of $\sigma_x(0,y)$ decreases and discontinuity occurs in $\sigma_x(0,y)$ stress on the contact surface between the layers due to different elastic constants of materials. Normal stress $\sigma_y(0,y)$ distribution is shown in Fig. 3 for various values of $a/h$, where $h_2/h=0.60$, $\beta=0.85$, and $k=0.10$. As shown in the Fig. 3, when load width increases, the normal stress $\sigma_y(0,y)$ increases. Fig. 4 shows that normal and shear stresses distributions on the contact surface of the elastic layers for $a/h=1.0$, $k=0.10$ and $h_2/h=0.60$. As seen from Fig. 4, the maximum values of the normal stresses occur on the plane of symmetry, $x=0$. Fig. 5 shows that variation of the point where the contact stress on the bonded surface changes sign to (+) with k for various values of $a/h$. The variations of maximum shear stresses on the contact surface of two elastic layers and their distances from plane of symmetry with k for various values of $a/h$ are given in Table 1.

Fig. 6 shows that variation of the displacements $v_2(x,-h_2)$ between the layered composite and the elastic foundation with $x/h$ for various values of k and Fig. 7 shows that variation of maximum displacements $v_2(0,-h_2)$ with k for various values of $a/h$.

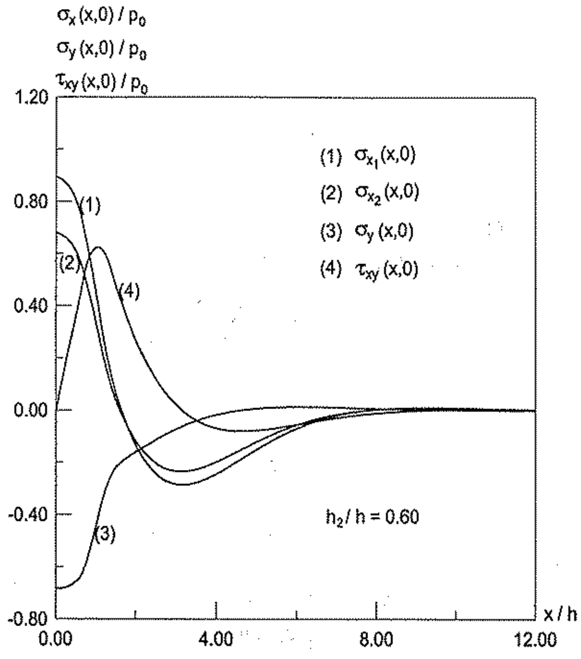


**Figure 4.** Normal and shear stresses distribution between two bonded elastic layers ($a / h = 1.00$, $k = 0.10$, $h_2 / h = 0.60$)

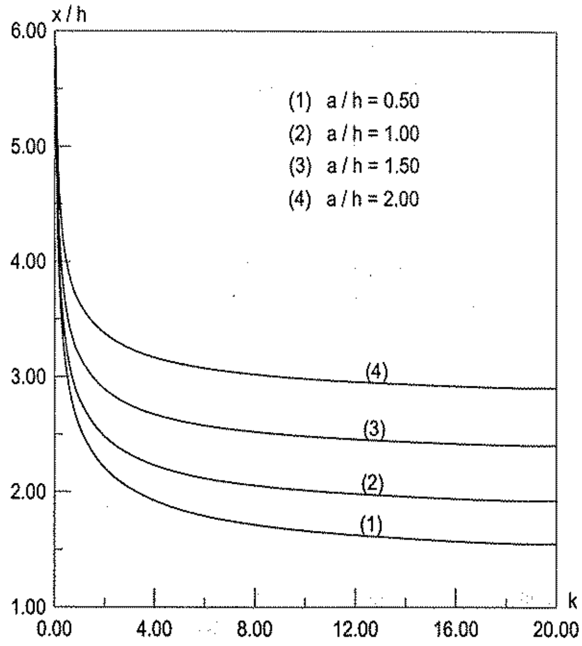


**Figure 5.** Variations of the points where $\sigma_y(x,0)$ contact stress on the bonded surface change sign with k ($h_2 / h = 0.50$, $\beta = 0.85$)

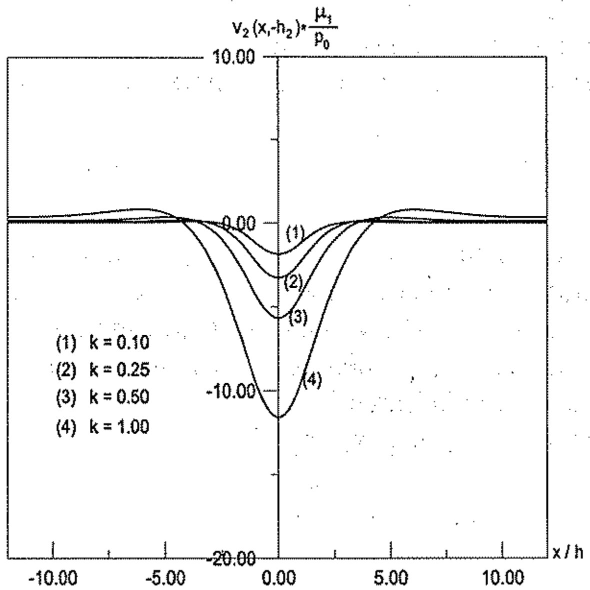


**Figure 6.** Variations of displacement $v_2(x,-h_2)$ between the layered composite and the elastic foundation with x for various values of k ($a / h = 1.00$, $h_2 / h = 0.60$, $\beta = 0.85$)

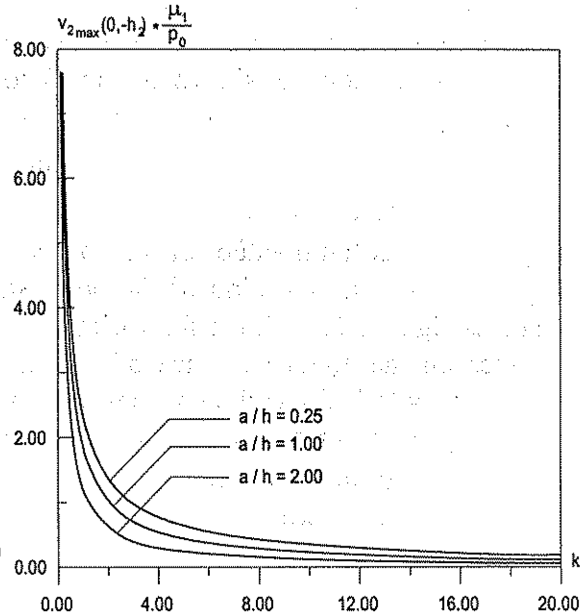


**Figure 7.** Variations of maximum displacement $v_2(0,-h_2)$ with k for various values of a / h ($h_2 / h = 0.70$, $\beta = 0.85$)

**Table 1.** Variations of maximum shear stresses on the contact surface of the layers and their distances from symmetry plane with k for various values of a/h ($h_2/h=0.50$, $\beta=0.85$).

| k ⇓ | a/h=0.50 | | a/h=1.00 | | a/h=2.00 | |
|---|---|---|---|---|---|---|
| | x/h | $\tau_{xy_{max}}(x,0)/p_0$ | x/h | $\tau_{xy_{max}}(x,0)/p_0$ | x/h | $\tau_{xy_{max}}(x,0)/p_0$ |
| 0.05 | 0.639 | 0.5112 | 1.058 | 0.7858 | 1.992 | 0.8915 |
| 0.10 | 0.619 | 0.4735 | 1.037 | 0.6876 | 1.986 | 0.7061 |
| 0.25 | 0.592 | 0.4176 | 1.014 | 0.5522 | 1.986 | 0.5004 |
| 0.50 | 0.572 | 0.3716 | 0.999 | 0.4517 | 1.990 | 0.3821 |
| 1.00 | 0.553 | 0.3246 | 0.989 | 0.3597 | 1.995 | 0.2954 |
| 5.00 | 0.517 | 0.2297 | 0.985 | 0.2095 | 2.000 | 0.1808 |
| 25.00 | 0.501 | 0.1802 | 0.988 | 0.1502 | 2.000 | 0.1369 |
| ∞ | 0.495 | 0.1609 | 0.990 | 0.1307 | 2.000 | 0.1214 |

## 6. CONCLUSIONS

The following conclusions can be drawn from the analysis of a layered system on an elastic foundation.

1) Maximum values of the normal stresses and the displacement occur on the symmetry plane, $x=0$. These values gradually decrease apart from symmetry plane and disappear at an infinite distance away.

2) The stresses and the displacements decrease, with increasing elastic spring constant.

3) Discontinuity occurs on the contact surface between the layers for normal stress $\sigma_x$. This discontinuity highly increases when the difference of the elastic constants of layers are getting more.

4) With increasing load width (a/h), the normal stresses increase but maximum displacements on the symmetry plane decrease.

5) Maximum shear stress on the contact surface of the layers occurs approximately at the equal distance to the load width. This distance decreases when the elastic spring constant increases. In addition, the maximum shear stress increases, with increasing a/h for small values of k, but for big values of k, with increasing a/h, the maximum shear stress decreases.

6) With increasing a/h, the point which the contact stress pass from compression to tension is away from the symmetry plane and it approaches towards a constant value with increasing of rigidity of spring.

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## APPENDIX

The coefficients $A_j(\alpha)$ and $B_j(\alpha)$ $(j = 1,...,4)$ are given following expressions:

$$\begin{aligned}\alpha A_1(\alpha) = -P(\alpha)\Big\{4\alpha\big\{(\kappa_1 - 1 + 2\alpha h_1)Y1A(\alpha)e^{-2\alpha h_1} + \{Z1 + 2\alpha h_1.Z2 + e^{-2\alpha h_2}[Z3 + 2\alpha h_2.Z4 \\ +4\alpha^2 h_2^2.Z5 + e^{-2\alpha h_2}.Z1 + 2\alpha h_1(Z6 + 2\alpha h_2.Z7 + 4\alpha^2 h_2^2.Z8 + e^{-2\alpha h_2}.Z2)]\}\big\} \\ + k\big\{(\kappa_1 - 1 + 2\alpha h_1)Y1K(\alpha)e^{-2\alpha h_1} + \{Z9 + 2\alpha h_1.Z10 + e^{-2\alpha h_2}[Z11 + 4\alpha h_2.Z12 \\ - e^{-2\alpha h_2}.Z9 + 2\alpha h_1(Z13 + 4\alpha h_2.Z14 - e^{-2\alpha h_2}.Z10)]\}\big\}\Big\}\end{aligned} \tag{A.1}$$

$$\begin{aligned}A_2(\alpha) = P(\alpha)\{&8\alpha\{Y1A(\alpha)e^{-2\alpha h_1} + \{Z15.(1+2\alpha h_1) + e^{-2\alpha h_2}[Z16 - 2\alpha h_2.Z7 + 4\alpha^2 h_2^2.Z17\\ &+e^{-2\alpha h_2}.Z15 + 2\alpha h_1(Z16 + 4\alpha^2 h_2^2.Z17 + e^{-2\alpha h_2}.Z15)]\}\}\\ &+2k\{Y1K(\alpha).e^{-2\alpha h_1} + \{(1+2\alpha h_1).Z18 + e^{-2\alpha h_2}[-Z13 + 4\alpha h_2.Z19 - e^{-2\alpha h_2}.Z18\\ &+2\alpha h_1(4\alpha h_2.Z19 - e^{-2\alpha h_2}.Z18)]\}\}\}\end{aligned}, \quad (A.2)$$

$$\begin{aligned}\alpha A_3(\alpha) = -P(\alpha)\{&4\alpha\{(\kappa_1 - 1 - 2\alpha h_1)Y2A(\alpha) + e^{-2\alpha h_1}\{Z1 - 2\alpha h_1.Z2 + e^{-2\alpha h_2}[Z3 - 2\alpha h_2.Z4\\ &+4\alpha^2 h_2^2.Z5 + e^{-2\alpha h_2}.Z1 + 2\alpha h_1(-Z6 + 2\alpha h_2.Z7 - 4\alpha^2 h_2^2.Z8 - e^{-2\alpha h_2}.Z2)]\}\}\\ &+k\{(\kappa_1 - 1 - 2\alpha h_1)Y2K(\alpha) + e^{-2\alpha h_1}\{Z9 - 2\alpha h_1.Z10 + e^{-2\alpha h_2}[-Z11 + 4\alpha h_2.Z12\\ &-e^{-2\alpha h_2}.Z9 + 2\alpha h_1(Z13 - 4\alpha h_2.Z14 + e^{-2\alpha h_2}.Z10)]\}\}\}\end{aligned} \quad (A.3)$$

$$\begin{aligned}A_4(\alpha) = P(\alpha)\{&8\alpha\{-Y2A(\alpha) + e^{-2\alpha h_1}\{Z15(-1+2\alpha h_1) + e^{-2\alpha h_2}[-Z16 - 2\alpha h_2 Z7 - 4\alpha^2 h_2^2 Z17\\ &-e^{-2\alpha h_2}.Z15 + 2\alpha h_1(Z16 + 4\alpha^2 h_2^2.Z17 + e^{-2\alpha h_2}.Z15)]\}\}\\ &+2k\{-Y2K(\alpha) + e^{-2\alpha h_1}\{(-1+2\alpha h_1).Z18 + e^{-2\alpha h_2}[-Z13 - 4\alpha h_2.Z19 + e^{-2\alpha h_2}Z18\\ &+2\alpha h_1(4\alpha h_2.Z19 - e^{-2\alpha h_2}.Z18)]\}\}\}\end{aligned} \quad (A.4)$$

$$\begin{aligned}\alpha B_1(\alpha) = -P(\alpha)\{&4\alpha(1+\kappa_1)e^{-2\alpha h_2}\{-R8 - R9 - 2\alpha h_2 R7 + 4\alpha^2 h_2^2 R11 + 2\alpha h_1(2\alpha h_2 R4 - R12)\\ &+e^{-2\alpha h_2}(R9 + 2\alpha h_1.R10) + e^{-2\alpha h_1}[R8 - R7 - 2\alpha h_2.R9 + R3(2\alpha h_1 - 4\alpha^2 h_2^2)\\ &+2\alpha h_1(2\alpha h_2.R10 + 4\alpha^2 h_2^2.R3) + e^{-2\alpha h_2}(R7 + 2\alpha h_1.R4)]\} + k.e^{-2\alpha h_2}\{(1+\kappa_1).\\ &.\{R13 - 4\alpha h_2 R5 + 2\alpha h_1 R14 + e^{-2\alpha h_2}(R16 + 2\alpha h_1 R17) + e^{-2\alpha h_1}[2\alpha h_1 R17 - R16\\ &+4\alpha h_2 R6(1 - 2\alpha h_1) + e^{-2\alpha h_2}(2\alpha h_1 R14 - R13)]\} + R15(1 + e^{-2\alpha h_2})(1 - e^{-2\alpha h_1})\}\}\end{aligned} \quad (A.5)$$

$$\begin{aligned}B_2(\alpha) = P(\alpha).(1+\kappa_1)\{&8\alpha e^{-2\alpha h_2}\{-R4.(1+2\alpha h_1) + 2\alpha h_2.R1 - e^{-2\alpha h_2}.R3.(1+2\alpha h_1)\\ &+e^{-2\alpha h_1}[-R2 + 2\alpha h_2 R3(1 - 2\alpha h_1) - e^{-2\alpha h_2}R1]\} + 2ke^{-2\alpha h_2}\{R5 + e^{-2\alpha h_2}R6.(1+2\alpha h_1)\\ &+e^{-2\alpha h_1}[R6(2\alpha h_1 - 1) - e^{-2\alpha h_2}.R5]\}\}\end{aligned} \quad (A.6)$$

$$\begin{aligned}\alpha B_3(\alpha) = -P(\alpha)\{&4\alpha(1+\kappa_1).\{R7 - 2\alpha h_1.R4 + e^{-2\alpha h_2}[R8 - R7 + 2\alpha h_2.R9 - 4\alpha^2 h_2^2.R3\\ &+2\alpha h_1(-R3 + 2\alpha h_2 R10 - 4\alpha^2 h_2^2 R3)] + e^{-2\alpha h_1}\{R9 - 2\alpha h_1 R10 + e^{-2\alpha h_2}[-R8 - R9\\ &+2\alpha h_2.R7 + 4\alpha^2 h_2^2.R11 + 2\alpha h_1(2\alpha h_2 R4 + R12)]\}\} + k.\{(1+\kappa_1)\{R13 + 2\alpha h_1 R14\\ &+e^{-2\alpha h_2}[2\alpha h_1.R17 + R16 + 4\alpha h_2.R6(1+2\alpha h_1)] + e^{-2\alpha h_1}[2\alpha h_1.R17 - R16\\ &+e^{-2\alpha h_2}(-R13 - 4\alpha h_2.R5 + 2\alpha h_1.R14)]\} + R15.(1 + e^{-2\alpha h_2}).(1 - e^{-2\alpha h_1})\}\}\end{aligned} \quad (A.7)$$

$$\begin{aligned}B_4(\alpha) = P(\alpha).(1+\kappa_1)\{&8\alpha.\{R1 + e^{-2\alpha h_2}[R2 + 2\alpha h_2.R3.(1+2\alpha h_1)]\\ &+e^{-2\alpha h_1}\{R3.(1-2\alpha h_1) + e^{-2\alpha h_2}[R4.(1-2\alpha h_1) + 2\alpha h_2.R1]\}\}\\ &+2k\{-R5 + e^{-2\alpha h_2}[-R6(2\alpha h_1 + 1)] + e^{-2\alpha h_1}[R6.(1-2\alpha h_1) + e^{-2\alpha h_2}.R5]\}\}\end{aligned}, \quad (A.8)$$

where,

$$P(\alpha)=\frac{p_0}{\mu_1}\frac{\sin(\alpha a)}{\alpha}\frac{e^{-\alpha h_1}}{\Delta(\alpha)}$$

$$\Delta(\alpha)=16\alpha\Big\{\ Y1A(\alpha)e^{-4\alpha h_1}+Y2A(\alpha)+e^{-2\alpha h_1}\big\{Z20+e^{-2\alpha h_2}(Z21-4\alpha h_1.2\alpha h_2.Z7+4\alpha^2h_2^2.Z22$$
$$+e^{-2\alpha h_2}.Z20)+4\alpha^2h_1^2\big[Z15+e^{-2\alpha h_2}(Z16+4\alpha^2h_2^2.Z17+e^{-2\alpha h_2}.Z15)\big]\big\}\Big\}$$
$$+4k\Big\{Y1K(\alpha)e^{-4\alpha h_1}+Y2K(\alpha)+e^{-2\alpha h_1}\big\{Z23+e^{-2\alpha h_2}\big[4\alpha h_2.Z24+4\alpha h_1.Z25$$
$$-e^{-2\alpha h_2}.Z23\ \big]+4\alpha^2h_1^2\big[Z18+e^{-2\alpha h_2}(4\alpha h_1.Z19-e^{-2\alpha h_2}.Z18)\big]\big\}\Big\}$$

$Z1=2\kappa_2-2\kappa_1\kappa_2+\beta[-4\kappa_1+4\kappa_1\kappa_2+2\beta(\kappa_1-\kappa_1^2)]$, $\quad Z2=-2\kappa_1\kappa_2+2\beta(\kappa_1\kappa_2-\kappa_1+\kappa_1\beta)$

$Z3=2-2\kappa_1+2\kappa_2^2-2\kappa_1\kappa_2^2+4\beta[2\kappa_1-2\kappa_1\kappa_2+\beta(\kappa_1^2-\kappa_1)]$, $\quad Z4=2\beta(1+\kappa_2-\kappa_1^2\kappa_2-\kappa_1^2)$

$Z5=2-2\kappa_1+2\beta[4\kappa_1+\beta(\kappa_1^2-\kappa_1)]$, $\quad Z6=-2\kappa_1-2\kappa_1\kappa_2^2+4\beta(\kappa_1-\kappa_1\kappa_2+\kappa_1\beta)$

$Z7=2\beta(1+\kappa_2+\kappa_1+\kappa_1\kappa_2)$, $\quad Z8=-2\kappa_1+2\kappa_1\beta(2-\beta)$

$Z9=2\kappa_2(1+\kappa_2-\kappa_1\kappa_2-\kappa_1)+2\kappa_1\beta[-2+2\kappa_2^2+\beta(1+\kappa_2-\kappa_1\kappa_2-\kappa_1)]$

$Z10=-2\kappa_1\kappa_2(1+\kappa_2)+2\kappa_1\beta[\kappa_2^2-1+\beta(1+\kappa_2)]$, $\quad Z11=2\beta[\kappa_1^2-1+\kappa_2(2\kappa_1^2+\kappa_1^2\kappa_2-2-\kappa_2)]$

$Z12=2\{-1+\kappa_1(1+\kappa_2)-\kappa_2+2\kappa_1\beta[-4-4\kappa_2+\beta(1-\kappa_1+\kappa_2-\kappa_1\kappa_2)]\}$

$Z13=-2\beta[1+\kappa_1(1+2\kappa_2)+\kappa_2(2+\kappa_2+\kappa_1\kappa_2)]$, $\quad Z14=2\kappa_1(1+\kappa_2)-2\kappa_1\beta[2+\kappa_2-\beta(1+\kappa_2)]$

$Z15=-2\kappa_2+2\beta(\kappa_2-1+\beta)$, $\quad Z16=-2-2\kappa_2^2+4\beta(1-\kappa_2-\beta)$, $\quad Z17=-2+2\beta(2-\beta)$

$Z18=-2\kappa_2-2\kappa_2^2+2\beta[\kappa_2^2-1+\beta(1+\kappa_2)]$, $\quad Z19=2+2\kappa_2+2\beta[-2-2\kappa_2+\beta(1+\kappa_2)]$

$Z20=2\beta[\kappa_1-1+\kappa_2-\kappa_1\kappa_2+\beta(1+\kappa_1^2)]-4\kappa_2$, $\quad Z21=4\{\beta[1-\kappa_1-\kappa_2(1-\kappa_1)-\beta(1+\kappa_1^2)]-1-\kappa_2^2\}$

$Z22=-4+2\beta[2-2\kappa_1-\beta(1+\kappa_1^2)]$,

$Z23=-4\kappa_2-4\kappa_2^2+2\beta[-1+\kappa_1+\kappa_2^2-\kappa_1\kappa_2^2+\beta(1+\kappa_2+\kappa_1^2+\kappa_1^2\kappa_2)]$

$Z24=4+4\kappa_2+2\beta[2(\kappa_1-1-\kappa_2+\kappa_1\kappa_2)+\beta(1+\kappa_2+\kappa_1^2+\kappa_1^2\kappa_2)]$

$Z25=2\beta[1+\kappa_1(1+2\kappa_2)+2\kappa_2+\kappa_2^2(1+\kappa_1)]$, $\quad Z26=2\kappa_2+2\beta(-\kappa_2-\kappa_1+\kappa_1\beta)$

$Z27=2+2\kappa_2^2+2\beta(\kappa_1-1+\kappa_2-\kappa_1\kappa_2-2\kappa_1\beta)$, $\quad Z28=2+2\beta(\kappa_1-1-\kappa_1\beta)$

$Z29=2\kappa_2+2\beta(1+\kappa_1\kappa_2+\kappa_1\beta)$, $\quad Z30=2\kappa_2+2\kappa_2^2-2\beta[\kappa_1+\kappa_2+\kappa_1\kappa_2+\kappa_2^2-\beta(\kappa_1+\kappa_1\kappa_2)]$

$Z31=-2-2\kappa_2+2\beta[1+\kappa_2-\kappa_1-\kappa_2\kappa_1+\beta(\kappa_1+\kappa_2\kappa_1)]$

$Z32=2\kappa_2+2\kappa_2^2+2\beta[1+\kappa_2+\kappa_1\kappa_2+\kappa_1\kappa_2^2+\beta(\kappa_1+\kappa_1\kappa_2)]$

$R1=-2-2\kappa_1\beta$, $\quad R2=-2\kappa_2+2\kappa_1\beta$, $\quad R3=2-2\beta$, $\quad R4=2\kappa_2+2\beta$

$R5=2+2\kappa_2+2\beta(\kappa_1+\kappa_1\kappa_2)$, $\quad R6=2+2\kappa_2-2\beta(1+\kappa_2)$, $\quad R7=2\beta(-1+\kappa_1\kappa_2)$

$R8=-2+2\kappa_2^2$, $\quad R9=2\beta(\kappa_2-\kappa_1)$, $\quad R10=-2\kappa_2+2\kappa_2\beta$, $\quad R11=2+2\kappa_1\beta$

$R12=2\kappa_2^2+2\kappa_2\beta$, $\quad R13=2\beta(-1+\kappa_1\kappa_2^2)$, $\quad R14=-2\kappa_2-2\kappa_2^2-2\beta(1+\kappa_2)$

$R15=2\beta(-\kappa_2+\kappa_2\kappa_1^2)$, $\quad R16=2\beta(\kappa_1-\kappa_2^2)$, $\quad R17=2\kappa_2+2\kappa_2^2-2\beta(\kappa_2+\kappa_2^2)$

$$Y1A(\alpha)=Z26+e^{-2\alpha h_2}(Z27+4\alpha^2h_2^2.Z28+e^{-2\alpha h_2}.Z29)$$

$$Y2A(\alpha)=Z29+e^{-2\alpha h_2}(Z27+4\alpha^2h_2^2.Z28+e^{-2\alpha h_2}.Z26)$$

$$Y1K(\alpha)=Z30+e^{-2\alpha h_2}(4\alpha h_2.Z31-e^{-2\alpha h_2}.Z32)$$

$$Y2K(\alpha)=Z32+e^{-2\alpha h_2}(4\alpha h_2.Z31-e^{-2\alpha h_2}.Z30)$$