

# SOME BOUNDS ON $\ell_p$ MATRIX AND $\ell_p$ OPERATOR NORMS OF ALMOST CIRCULANT, CAUCHY-TOEPLITZ AND CAUCHY-HANKEL MATRICES

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**Abstract-** Let  $C_n$ ,  $T_n$  and  $H_n$  denote almost circulant, Cauchy-Toeplitz and Cauchy-Hankel matrices, respectively. We find some upper bounds for  $\ell_p$  matrix norm and  $\ell_p$  operator norm of this matrices. Moreover, we give some results for Kronecker products  $C_n \otimes T_n$  and  $C_n \otimes H_n$ .

**Keywords-** Circulant matrix, Cauchy-Toeplitz matrix, Cauchy-Hankel matrix, norm, Kronecker product.

## 1. INTRODUCTION AND PRELIMINARIES

Let  $C = [1/(x_i - y_j)]_{i,j=1}^n$  ( $x_i \neq y_j$ ) be a Cauchy matrix and  $T_n = [t_{j-i}]_{i,j=0}^n$  be a Toeplitz matrix. In generally Cauchy-Toeplitz matrices are being defined as

$$T_n = \left[ \frac{1}{g + (i-j)h} \right]_{i,j=1}^n \quad (1.1)$$

where  $h \neq 0$ ,  $g$  and  $h$  are some numbers and quotient  $g/h$  is not integer. Toeplitz matrices are precisely those matrices that one constant along all diagonals parallel to the main diagonal, and thus a Toeplitz matrix is determined by its first row and column.

Closely related to Toeplitz matrices are the so-called circulant matrices. An  $(n \times n)$  matrix  $C$  is called a circulant matrix if it is of the form

$$C_n = \begin{bmatrix} c_0 & c_1 & c_2 & \cdot & \cdot & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdot & \cdot & c_{n-3} & c_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_2 & c_3 & c_4 & \cdot & \cdot & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdot & \cdot & c_{n-1} & c_0 \end{bmatrix}.$$

For each  $i, j=1, \dots, n$  and  $k=0, 1, 2, \dots, n-1$ , all the elements  $(i, j)$  such that  $j-i \equiv k \pmod{n}$  have the same value  $c_k$ ; these elements form the so-called  $k$ th stripe of

C. Obviously, a circulant matrix is determined by its first row (or column). It is clear that every circulant matrix is a Toeplitz matrix, but the converse is not necessarily true.

Let  $H_n = [h_{i+j}]_{i,j=0}^n$  be Hankel matrix. A matrix

$$H_n = \left[ \frac{1}{g + (i+j)h} \right]_{i,j=1}^n \quad (1.2)$$

is called Cauchy-Hankel matrix. Hankel matrices are symmetric.

E. E. Tyrshnikov found a lower bound for the spectral norm of Cauchy-Toeplitz matrix such that  $h=1$  and  $g=1/2$ . D. Bozkurt has given an upper and lower bounds for the  $\ell_p$  matrix norm of almost Cauchy-Toeplitz matrix.

In this study, we have defined almost circulant matrix in the following form:

$$C_n = \begin{cases} a, & i = j \\ \frac{1}{k} & (j-i \equiv k(\text{mod } n)), i \neq j \end{cases} \quad (1.3)$$

where  $a \in \mathbb{R} - \{0\}$  ( $\mathbb{R}$  is the set of real numbers) and  $k=1,2,\dots,n-1$ .

In section 2, firstly we will establish upper bound for the  $\ell_p$  matrix and  $\ell_p$  operator norms of the matrix  $C_n$ . Secondly, we will establish upper bounds for the  $\ell_p$  matrix norm of Cauchy-Toeplitz and Cauchy-Hankel matrices defined with (1.1) and (1.2). Finally we will give results for  $\ell_p$  norms of Kronecker products  $C_n \otimes T_n$  and  $C_n \otimes H_n$ .

For  $1 \leq p < \infty$ , the  $\ell_p$  matrix norm of a matrix  $A = [a_{ij}]_{n \times n}$  is defined as

$$\|A\|_p = \left( \sum_{i,j=1}^n |a_{ij}|^p \right)^{1/p} \quad (1.4)$$

If  $p=\infty$ , then

$$\|A\|_\infty = \lim_{p \rightarrow \infty} \|A\|_p = \max_{i,j} |a_{ij}|.$$

In addition,  $\ell_p$  operator norm of an matrix  $A = [a_{ij}]_{n \times n}$  is defined as

$$\|A\|_p = \max \{ \|Ax\|_p : x \in \mathbb{C}^n, \|x\|_p = 1 \}. \quad (1.5)$$

Let  $A$  and  $B$  be arbitrary  $n \times m$  matrices. Kronecker product of this matrices given to be

$$A \otimes B = \begin{bmatrix} a_{11}B & . & . & . & a_{1m}B \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1}B & . & . & . & a_{nm}B \end{bmatrix}. \quad (1.6)$$

A function  $\Psi$  is called as psi (or digamma) function if  $\Psi(x) = \frac{d}{dx} \{\log[\Gamma(x)]\}$  where  $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ . It is called as Polygamma function the  $n$  th derivatives of  $\Psi$  function [4] i.e.

$$\begin{aligned}\Psi(n, x) &= \frac{d}{dx^n} \Psi(x) \\ &= \frac{d}{dx^n} \left[ \frac{d}{dx} \ln[\Gamma(x)] \right]\end{aligned}$$

If  $n=0$ , then  $\Psi(0, x) = \Psi(x) = \frac{d}{dx} \{\ln[\Gamma(x)]\}$ . In addition, if  $a > 0$  and  $b$  any number and  $n \in \mathbb{Z}^+$  is positive integer, then

$$\lim_{n \rightarrow \infty} \Psi(a, n+b) = 0. \quad (1.7)$$

In this study,  $\mathbb{Z}$  and  $\mathbb{R}$  will represent the sets of integers and real numbers, respectively.

## 2. $\ell_p$ MATRIX AND $\ell_p$ OPERATOR NORMS OF ALMOST CIRCULANT, CAUCHY-TOEPLITZ AND CAUCHY-HANKEL

**Theorem 2.1.** Let the matrix  $C_n$  be as in (1.3). Then

$$n^{-1/p} \|C_n\|_p \leq \left\{ |a|^p + \zeta(p) \right\}^{1/p} \quad (2.1)$$

is valid for the  $\ell_p$  matrix norm of the matrix  $C_n$  where  $2 \leq p < \infty$ ,  $a \in \mathbb{R} - \{0\}$  and  $\zeta$  is Riemann's zeta function.

**Proof.** From (1.4) we have

$$\|C_n\|_p^p = n \left( |a|^p + 1^p + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{(n-1)^p} \right) = n|a|^p + n \sum_{s=1}^{n-1} \frac{1}{s^p}. \quad (2.2)$$

If we divide both side of the (2.2) by  $n$ , then

$$\frac{1}{n} \|C_n\|_p^p = |a|^p + \sum_{s=1}^{n-1} \frac{1}{s^p}$$

can obtain. If we evaluate the right hand side of this equality, we have

$\lim_{n \rightarrow \infty} \sum_{s=1}^{n-1} \frac{1}{s^p} = \zeta(p)$ . Hence we obtain

$$\frac{1}{n} \|C_n\|_p^p \leq |a|^p + \zeta(p). \quad (2.3)$$

If we take  $1/p$  th power of inequality (2.3), then the proof is completed.

**Example 2.1.** Let  $a=1$  and  $p=2$  in Theorem 2.1. Then we have values listed given in the following table.

Table 2.1

$n$	$n^{-1/2}\ C_n\ _2$	$\sqrt{1+\zeta(2)}$	$n$	$n^{-1/2}\ C_n\ _2$	$\sqrt{1+\zeta(2)}$
1	1	1.626325327	20	1.610485406	1.626325327
2	1.414213562	"	30	1.615870976	"
3	1.500000000	"	40	1.618523699	"
4	1.536590743	"	50	1.620102689	"
5	1.556795142	"	60	1.621150125	"
6	1.569589472	"	70	1.621895748	"
7	1.578413409	"	80	1.622453578	"
8	1.584864994	"	90	1.622886626	"
9	1.589786794	"	100	1.623232547	"
10	1.593664874	"	150	1.624267560	"

**Corollary 2.1.** Since  $C_n C_n^{-1} = E_n$  ( $E$  denote  $n \times n$  unit matrix),  $\|C_n C_n^{-1}\|_p = n^{1/p}$

From this equality, we can write  $n^{1/p} = \|C_n C_n^{-1}\|_p \leq \|C_n\|_p \|C_n^{-1}\|_p$ . Hence  $1 \leq n^{-1/p} \|C_n\|_p \|C_n^{-1}\|_p$  and  $\frac{1}{\|C_n^{-1}\|_p} \leq n^{-1/p} \|C_n\|_p$ . Consequently, from (2.3) inequality we have

$$\frac{1}{\|C_n^{-1}\|_p} \leq [n|a|^p + \zeta(p)]^{1/p}.$$

**Theorem 2.2.** Let the matrix  $C_n$  be as in (1.3). For the  $\ell_p$  operator norm of matrix  $C_n$  is valid in the following:

$$\|C_n x\|_p^p \leq [n|a|^p + n\zeta(p)]^{1/p} n^{1/q} \|x\|_p^p \quad (2.4)$$

where  $1 < p < \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$  and  $a \in \mathbb{R} - \{0\}$ .

**Proof.** From (1.4) we have

$$\|C_n x\|_p^p = \sum_{i=1}^n \left| \sum_{j=1}^n c_{ij} x_j \right|^p \leq \sum_{i=1}^n \left( \sum_{j=1}^n |c_{ij}| |x_j| \right)^p = \sum_{i,j=1}^n |c_{ij}|^p \left( \sum_{j=1}^n |x_j| \right)^p.$$

Hence we can write

$$\|C_n x\|_p^p \leq \sum_{i,j=1}^n |c_{ij}|^p \left( \sum_{j=1}^n |x_j| \right)^p. \quad (2.5)$$

If we apply Hölder's inequality to (2.5) inequality, then we have

$$\left( \sum_{j=1}^n |x_j| \right)^p = \left( \sum_{j=1}^n |x_j| \cdot 1 \right)^p \leq \left[ \left( \sum_{j=1}^n |x_j|^p \right)^{1/p} \left( \sum_{j=1}^n 1^q \right)^{1/q} \right]^p = \|x\|_p^p n^{p/q} \quad (2.6)$$

where  $p > 1$  and  $q = \frac{p}{p-1}$ . For the other expression,

$$\sum_{i,j=1}^n |c_{ij}|^p = n|a|^p + n \sum_{s=1}^{n-1} \frac{1}{s^p}.$$

Since  $\sum_{s=1}^{n-1} \frac{1}{s^p} \leq \zeta(p)$  we have

$$\sum_{i,j=1}^n |c_{ij}|^p \leq n|a|^p + n\zeta(p) \quad (2.7)$$

where  $p > 1$ . Consequently from (2.6) and (2.7),

$$\|C_n x\|_p^p \leq [n|a|^p + n\zeta(p)] n^{p/q} \|x\|_p^p. \quad (2.8)$$

If we take  $1/p$  th power of inequality (2.8), then the proof is completed.

**Theorem 2.3.** Let  $T_m = \left[ \frac{1}{1/k + (i-j)} \right]_{i,j=1}^m$  ( $1 < k \in \mathbb{Z}^+$ ) be a Cauchy-Toeplitz

matrix. Then, for the  $\ell_p$  matrix norm of the matrix  $T_m$

$$m^{-1/p} \|T_m\|_p \leq \left\{ k^p + \frac{(-1)^p}{(p-1)!} [\Psi(p-1, 1+1/k) + \Psi(p-1, 1-1/k)] \right\}^{1/p} \quad (2.9)$$

is valid where  $2 \leq p < \infty$ .

**Proof.** From (1.4) we have

$$\begin{aligned} \|T_m\|_p^p &= m|k|^p + \sum_{s=1}^{m-1} (m-s) \left( \frac{1}{\left| \frac{1}{k} - s \right|^p} + \frac{1}{\left| \frac{1}{k} + s \right|^p} \right) \\ &= mk^p + \sum_{s=1}^{m-1} (m-s) \left( \frac{1}{(s-1/k)^p} + \frac{1}{(s+1/k)^p} \right). \end{aligned}$$

If we divide both side of this equality by  $n$ , then

$$\frac{1}{m} \|T_m\|_p^p = \left\{ k^p + \sum_{s=1}^{m-1} (1-s/m) \left( \frac{1}{(s-1/k)^p} + \frac{1}{(s+1/k)^p} \right) \right\}. \quad (2.10)$$

From the properties of polygamma functions the right hand side of this equality is written as the following:

$$\begin{aligned} \sum_{s=1}^{m-1} (1-s/m) \frac{1}{(s+1/k)^p} &= \frac{(-1)^{p-1} (km+1)}{(p-1)! mk} [\Psi(p-1, 1/k+m) - \Psi(p-1, 1/k+1)] \\ &\quad + \frac{(-1)^{p-1}}{(p-2)! m} [\Psi(p-2, 1/k+m) - \Psi(p-2, 1/k+1)] \end{aligned} \quad (2.11)$$

and

$$\sum_{s=1}^{m-1} (1-s/m) \frac{1}{(s-1/k)^p} = \frac{(-1)^{p-1} (km-1)}{(p-1)! mk} [\Psi(p-1, m-1/k) - \Psi(p-1, 1-1/k)] + \frac{(-1)^{p-1}}{(p-2)! m} [\Psi(p-2, m-1/k) - \Psi(p-2, 1-1/k)] \quad (2.12)$$

If we take limit as  $m \rightarrow \infty$  of (2.11) and (2.12) equalities, then we obtain

$$\lim_{m \rightarrow \infty} \sum_{s=1}^{m-1} (1-s/m) \frac{1}{(s+1/k)^p} = \frac{(-1)^p}{(p-1)!} \Psi(p-1, 1/k+1) \quad (2.13)$$

and

$$\lim_{m \rightarrow \infty} \sum_{s=1}^{m-1} (1-s/m) \frac{1}{(s-1/k)^p} = \frac{(-1)^p}{(p-1)!} \Psi(p-1, 1-1/k). \quad (2.14)$$

from (1.7). If we write (2.13) and (2.14) equalities into (2.10) equality, then we have

$$\frac{1}{m} \|T_m\|_p^p \leq \left\{ k^p + \frac{(-1)^p}{(p-1)!} [\Psi(p-1, 1/k+1) + \Psi(p-1, 1-1/k)] \right\}. \quad (2.15)$$

If we take  $1/p$  th power of this inequality, then the proof is completed.

**Example 2.2.** Let  $k=2$  and  $p=2$  in Theorem 2.3. Then we have values listed in the following table, where  $\Delta = \sqrt{2^2 + \Psi(1, 1+1/2) + \Psi(1, 1-1/2)}$ .

**Table 2.2**

$n$	$n^{-1/2} \ T_n\ _2$	$\Delta$	$n$	$n^{-1/2} \ T_n\ _2$	$\Delta$
1	2	3.141592654	20	3.045273482	3.141592654
2	2.494438258	"	30	3.073319650	"
3	2.676647988	"	40	3.088201132	"
4	2.774160003	"	50	3.097512511	"
5	2.835679523	"	60	3.103923804	"
6	2.878349332	"	70	3.108624672	"
7	2.909837335	"	80	3.112228486	"
8	2.934114510	"	90	3.115084744	"
9	2.953453583	"	100	3.117407717	"
10	2.969253825	"	150	3.124625178	"

**Theorem 2.4.** Let  $H_m = \left[ \frac{1}{1/k + (i+j)} \right]_{i,j=1}^m$  ( $1 < k \in \mathbb{Z}^+$ ) be a Cauchy-Hankel

matrix. Then, for the  $\ell_p$  matrix norm of the matrix  $H_m$

$$\|H_m\|_p \leq \left\{ k^p + \frac{(-1)^{p+1} [(p-1)\Psi(p-2, 1/k) + (k+1)\Psi(p-1, 1/k)]}{(p-1)!k} \right\}^{1/p} \quad (2.16)$$

is valid where  $3 \leq p < \infty$ .

**Proof.** From (1.4) we have

$$\begin{aligned} \|H_m\|_p^p &= \sum_{s=1}^m \frac{s}{\left| \frac{1}{k} + s + 1 \right|^p} + \sum_{s=1}^{m-1} \frac{m-s}{\left| \frac{1}{k} + s + m + 1 \right|^p} \\ &= \sum_{s=1}^m \frac{s}{(1/k + s + 1)^p} + \sum_{s=1}^{m-1} \frac{m-s}{(1/k + s + m + 1)^p} \end{aligned} \quad (2.17)$$

From the properties of polygamma functions the right hand side of this equality is wrote as the following:

$$\begin{aligned} \sum_{s=1}^m \frac{s}{(1/k + s + 1)^p} &= \frac{(-1)^p}{(p-2)!} \left\{ \frac{k+1}{(p-1)k} [\Psi(p-1, m+1 + (k+1)/k) - \Psi(p-1, 1 + (k+1)/k)] \right. \\ &\quad \left. + \Psi(p-2, m+1 + (k+1)/k) - \Psi(p-2, 1 + (k+1)/k) \right\} \end{aligned}$$

and

$$\begin{aligned} \sum_{s=1}^{m-1} \frac{(m-s)}{(1/k + m + s + 1)^p} &= \frac{(-1)^p}{(p-2)!} \left\{ \frac{2km + k + 1}{(p-1)k} [\Psi(p-1, 1 + (km + k + 1)/k) - \Psi(p-1, m + (km + k + 1)/k)] \right. \\ &\quad \left. + \Psi(p-2, 1 + (km + k + 1)/k) - \Psi(p-2, m + (km + k + 1)/k) \right\} \end{aligned}$$

If we take limit as  $m \rightarrow \infty$  of these equalities, then we have

$$\lim_{m \rightarrow \infty} \sum_{s=1}^m \frac{s}{(1/k + s + 1)^p} = k^p + \frac{(-1)^{p+1} [(p-1)\Psi(p-2, 1/k) + (k+1)\Psi(p-1, 1/k)]}{(p-1)!k} \quad (2.18)$$

and

$$\lim_{m \rightarrow \infty} \sum_{s=1}^{m-1} \frac{m-s}{(1/k + s + m + 1)^p} = 0. \quad (2.19)$$

from (1.7). If we write (2.18) and (2.19) equalities into (2.17) equality, then we have

$$\|H_m\|_p^p \leq \left\{ k^p + \frac{(-1)^{p+1} [(p-1)\Psi(p-2, 1/k) + (k+1)\Psi(p-1, 1/k)]}{(p-1)!k} \right\}. \quad (2.20)$$

If we take  $1/p$  th power of this inequality, then the proof is completed.

**Corollary 2.2.** For Kronecker product of  $C_n$  and  $T_m$  matrices defined by Theorem 2.1 and Theorem 2.3, following inequality is valid:

$$\frac{1}{(mn)^{1/p}} \|C_n \otimes T_m\|_p \leq \left\{ |a|^p + \xi(p) \right\}^{1/p} \left\{ k^p + \frac{(-1)^p}{(p-1)!} [\Psi(p-1, 1+1/k) + \Psi(p-1, 1-1/k)] \right\}^{1/p}$$

**Proof.** Since

$$\|C_n \otimes T_m\|_p = \|C_n\|_p \|T_m\|_p$$

the proof is trivial from (2.3) and (2.15).

**Corollary 2.3.** For Kronecker product of  $C_n$  and  $H_m$  matrices defined by Theorem 2.1 and Theorem 2.4, following inequality is valid.

$$\frac{1}{n^{1/p}} \|C_n \otimes H_m\|_p \leq \left\{ |a|^p + \xi(p) \right\}^{1/p} \left\{ k^p + \frac{(-1)^{p+1} [(p-1)\Psi(p-2, 1/k) + (k+1)\Psi(p-1, 1/k)]}{(p-1)k} \right\}^{1/p}$$

**Proof.** Since

$$\|C_n \otimes H_m\|_p = \|C_n\|_p \|H_m\|_p$$

the proof is trivial from (2.3) and (2.20).

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