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ON THE PHASE DEPENDENCE OF THE CRAMÉR-RAO BOUNDS FOR DAMPED SINUSOIDS

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Abstract–The problem of estimating the parameters of superimposed damped sinusoids in noise is considered for both complex and real data cases. It is shown that for the complex case, the Cramér-Rao bounds on the variance of unbiased estimators of the signal parameters are independent of the phases of all the sinusoids, and that for the real case, the bounds for the parameters of a particular sinusoid are dependent on the phase of that sinusoid, but are independent of the phases of all the other sinusoids.

Keywords–Cramér-Rao bound, damped sinusoids, Fisher information matrix, close signals.

1. INTRODUCTION

It is well known that the Cramér-Rao (C-R) bound specifies a lower bound on the variance of any unbiased estimator. Accordingly, the C-R bound frequently is used to quantify the goodness of the parameter estimators for time-series data models.

The C-R bounds for the data models consisting of complex damped sinusoids in complex white Gaussian noise (the complex model) or real damped sinusoids in real white Gaussian noise (the real model) were recently derived by Yao and Pandit [1]. However, the expressions for the C-R bounds given in [1] do not show the dependence of the bounds on the phases of the sinusoids explicitly.

For the complex model, Hua and Sarkar [2] provided an interesting expression which exposes the dependence of the bounds on some signal parameters including the phases. But, the expression in [2] is not readily applicable to the real model. The real model is probably more common in practice. For the real model, the dependence of the C-R bounds on the signal phases is not emphasized in the current literature.

In this paper, we introduce an approach that reveals the dependence of the C-R bounds on the phases of the sinusoids for both complex and real models. In Section 2, we consider the complex model, and show that the bounds are independent of all the phases of the sinusoids in the model. In Section 3, we consider the real model, and show that the bounds for the parameters of a particular sinusoid in the model are functions of the phase of that sinusoid but are independent of the phases of the other sinusoids.

2. THE COMPLEX MODEL

The complex data model consists of M complex damped sinusoids in complex noise:

$$y(t)=\sum_{i=1}^{M}\alpha_i e^{-\beta_i t}e^{j(\omega_i t+\varphi_i)}+e(t),\ t=1,2,\ldots,N \tag{1}$$

where $j=\sqrt{-1}$, α_i is the amplitude, φ_i is the phase, β_i is the damping factor, ω_i is the frequency of the ith sinusoid, $e(t)$ is a complex white Gaussian noise with mean zero and variance σ^2, and N is the number of data samples. The signal parameter vector

$$\theta=[\alpha_1,\varphi_1,\beta_1,\omega_1,\alpha_2,\ldots,\omega_M]^T \tag{2}$$

is to be estimated from the data vector $Y=[y(1),y(2),\ldots y(N)]^T$. (The superscript "$T$" denotes the transpose.)

The logarithmic likelihood function for the estimation problem in (1) and (2) is

$$L(Y,\theta)=-N\log(\pi)-N\log(\sigma^2)-\frac{1}{\sigma^2}\sum_{t=1}^{N}e(t)e(t)^*$$

where "$*$" denotes the conjugate and

$$e(t)=y(t)-\sum_{i=1}^{M}\alpha_i e^{-\beta_i t}e^{j(\omega_i t+\varphi_i)}.$$

The Fisher information matrix for the problem is

$$J(\theta)=E\left\{\frac{\partial L(Y,\theta)}{\partial\theta}\left(\frac{\partial L(Y,\theta)}{\partial\theta}\right)^T\right\}$$

where $E\{\ \}$ denotes the expectation.

The C-R bounds on the variance of unbiased estimators of the parameters in θ are obtained from the corresponding diagonal entries of the inverse of the Fisher information matrix [3].

The order of the parameters in the θ vector is important, since different orderings will result in $J(\theta)$ matrices with different structures. Note that our choice of the order in (2) is different from that of [1]. We have tacitly changed the order to reveal the phase dependence of the C-R bounds.

It is easy to show that

$$J(\theta)=\frac{2}{\sigma^2}\sum_{t=1}^{N}\left[\bar{e}_\theta(t)\bar{e}_\theta(t)^T+\tilde{e}_\theta(t)\tilde{e}_\theta(t)^T\right]$$

where

$$e_\theta(t)=\left[e_\theta^{(1)}(t)^T, e_\theta^{(2)}(t)^T,\ldots,e_\theta^{(M)}(t)^T\right]^T$$

$$e_\theta^{(i)}(t)=\begin{bmatrix}\partial e(t)/\partial\alpha_i\\ \partial e(t)/\partial\varphi_i\\ \partial e(t)/\partial\beta_i\\ \partial e(t)/\partial\omega_i\end{bmatrix}=\begin{bmatrix}-e^{-\beta_i t}e^{j(\omega_i t+\varphi_i)}\\ -j\alpha_i e^{-\beta_i t}e^{j(\omega_i t+\varphi_i)}\\ \alpha_i te^{-\beta_i t}e^{j(\omega_i t+\varphi_i)}\\ -j\alpha_i te^{-\beta_i t}e^{j(\omega_i t+\varphi_i)}\end{bmatrix}$$

and $\bar{e}_\theta(t)=\mathrm{Re}\{e_\theta(t)\}$, $\tilde{e}_\theta(t)=\mathrm{Im}\{e_\theta(t)\}$.

Now, we introduce the following decomposition of the vectors $\bar{e}_\theta^{(i)}(t)$ and $\tilde{e}_\theta^{(i)}(t)$:

$$\bar{e}_\theta^{(i)}(t)=\begin{bmatrix}-e^{-\beta_i t}\cos(\omega_i t+\varphi_i)\\ \alpha_i e^{-\beta_i t}\sin(\omega_i t+\varphi_i)\\ \alpha_i te^{-\beta_i t}\cos(\omega_i t+\varphi_i)\\ \alpha_i te^{-\beta_i t}\sin(\omega_i t+\varphi_i)\end{bmatrix}$$

$$=\begin{bmatrix}1 & & & \\ & \alpha_i & & \\ & & \alpha_i & \\ & & & \alpha_i\end{bmatrix}\cdot\begin{bmatrix}\cos\varphi_i & \sin\varphi_i & & \\ -\sin\varphi_i & \cos\varphi_i & & \\ & & \cos\varphi_i & -\sin\varphi_i\\ & & \sin\varphi_i & \cos\varphi_i\end{bmatrix}\cdot\begin{bmatrix}-e^{-\beta_i t}\cos\omega_i t\\ e^{-\beta_i t}\sin\omega_i t\\ te^{-\beta_i t}\cos\omega_i t\\ te^{-\beta_i t}\sin\omega_i t\end{bmatrix}$$

$$\equiv D_i\cdot Q_i\cdot\bar{\varepsilon}_\theta^{(i)}(t) \tag{3}$$

$$
\begin{aligned}
\tilde{e}_\theta^{(i)}(t) &= \begin{bmatrix} -e^{-\beta_i t}\sin(\omega_i t+\varphi_i) \\ -\alpha_i e^{-\beta_i t}\cos(\omega_i t+\varphi_i) \\ \alpha_i t e^{-\beta_i t}\sin(\omega_i t+\varphi_i) \\ -\alpha_i t e^{-\beta_i t}\cos(\omega_i t+\varphi_i) \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \\ & \alpha_i & & \\ & & \alpha_i & \\ & & & \alpha_i \end{bmatrix} \cdot \begin{bmatrix} \cos\varphi_i & \sin\varphi_i & & \\ -\sin\varphi_i & \cos\varphi_i & & \\ & & \cos\varphi_i & -\sin\varphi_i \\ & & \sin\varphi_i & \cos\varphi_i \end{bmatrix} \cdot \begin{bmatrix} -e^{-\beta_i t}\sin\omega_i t \\ -e^{-\beta_i t}\cos\omega_i t \\ te^{-\beta_i t}\sin\omega_i t \\ -te^{-\beta_i t}\cos\omega_i t \end{bmatrix} \\
&\equiv D_i \cdot Q_i \cdot \tilde{\varepsilon}_\theta^{(i)}(t).
\end{aligned}
$$

Let

$$
\begin{aligned}
D &= \operatorname{diag}\{D_1, D_2, \ldots, D_M\} \\
Q &= \operatorname{diag}\{Q_1, Q_2, \ldots, Q_M\}.
\end{aligned}
$$

Then, the Fisher information matrix becomes

$$
J(\theta) = \frac{2}{\sigma^2} \cdot D \cdot Q \cdot I \cdot Q^T \cdot D \tag{4}
$$

where

$$
\begin{aligned}
I &= \sum_{t=1}^{N} \left[\bar{\varepsilon}_\theta(t) \bar{\varepsilon}_\theta(t)^T + \tilde{\varepsilon}_\theta(t) \tilde{\varepsilon}_\theta(t)^T \right] \\
\bar{\varepsilon}_\theta(t) &= \left[\bar{\varepsilon}_\theta^{(1)}(t)^T, \bar{\varepsilon}_\theta^{(2)}(t)^T, \ldots, \bar{\varepsilon}_\theta^{(M)}(t)^T \right]^T \\
\tilde{\varepsilon}_\theta(t) &= \left[\tilde{\varepsilon}_\theta^{(1)}(t)^T, \tilde{\varepsilon}_\theta^{(2)}(t)^T, \ldots, \tilde{\varepsilon}_\theta^{(M)}(t)^T \right]^T.
\end{aligned}
$$

Note that the entries of the matrix I do not depend on the amplitudes and the phases; they are functions of the damping factors, the frequencies, the number of data samples and the sampling instants.

Inverting (4) gives

$$
J^{-1}(\theta) = \frac{\sigma^2}{2} \cdot D^{-1} \cdot Q \cdot I^{-1} \cdot Q^T \cdot D^{-1} \tag{5}
$$

where we have used the property that Q is orthogonal. (The superscript "-1" denotes the inverse.)

It follows from (5) that the C-R bounds on the parameter estimation variances are

$$\mathrm{CRB}\{\alpha_i\}=\frac{\sigma^2}{4}\left\{\left[\left(I_i^{-1}\right)_{11}+\left(I_i^{-1}\right)_{22}\right]+\left[\left(I_i^{-1}\right)_{11}-\left(I_i^{-1}\right)_{22}\right]\cos 2\varphi_i+2\left(I_i^{-1}\right)_{12}\sin 2\varphi_i\right\}$$

$$\mathrm{CRB}\{\varphi_i\}=\frac{\sigma^2}{4\alpha_i^2}\left\{\left[\left(I_i^{-1}\right)_{11}+\left(I_i^{-1}\right)_{22}\right]-\left[\left(I_i^{-1}\right)_{11}-\left(I_i^{-1}\right)_{22}\right]\cos 2\varphi_i-2\left(I_i^{-1}\right)_{12}\sin 2\varphi_i\right\}$$

$$\mathrm{CRB}\{\beta_i\}=\frac{\sigma^2}{4\alpha_i^2}\left\{\left[\left(I_i^{-1}\right)_{33}+\left(I_i^{-1}\right)_{44}\right]+\left[\left(I_i^{-1}\right)_{33}-\left(I_i^{-1}\right)_{44}\right]\cos 2\varphi_i-2\left(I_i^{-1}\right)_{34}\sin 2\varphi_i\right\}$$

$$\mathrm{CRB}\{\omega_i\}=\frac{\sigma^2}{4\alpha_i^2}\left\{\left[\left(I_i^{-1}\right)_{33}+\left(I_i^{-1}\right)_{44}\right]-\left[\left(I_i^{-1}\right)_{33}-\left(I_i^{-1}\right)_{44}\right]\cos 2\varphi_i+2\left(I_i^{-1}\right)_{34}\sin 2\varphi_i\right\} \tag{6}$$

where I_i^{-1} is the ith diagonal submatrix of I^{-1} and $\left(I_i^{-1}\right)_{jk}$ is the (j,k)th entry of I_i^{-1}.

We now show that the C-R bounds actually do not depend on the phases φ_i. From Theorem 1 in [1] we know that

$$\begin{aligned}\mathrm{CRB}\{\alpha_i\}&=\mathrm{CRB}\{\varphi_i\}\cdot\alpha_i^2\\ \mathrm{CRB}\{\beta_i\}&=\mathrm{CRB}\{\omega_i\}\end{aligned} \tag{7}$$

for all values of the φ_i. Therefore, it must be true that

$$\begin{aligned}\left(I_i^{-1}\right)_{11}&=\left(I_i^{-1}\right)_{22}\\ \left(I_i^{-1}\right)_{12}&=0\\ \left(I_i^{-1}\right)_{33}&=\left(I_i^{-1}\right)_{44}\\ \left(I_i^{-1}\right)_{34}&=0.\end{aligned}$$

Replacing these in (6) shows that the right-hand sides in (6) are independent of the phases φ_i.

The dependence of the determinant of the inverse Fisher information matrix on the signal phases is also important [4]. We note from (5) that this determinant is also independent of all the phases.

Example 1: Consider two complex damped sinusoids with equal damping factors ($\beta_1=\beta_2$) and observed by $N=10$ data samples. It can be shown that the C-R bounds depend on the two sinusoid frequencies through their difference $\delta\omega=\omega_1-\omega_2$. The C-R amplitude bound $\mathrm{CRB}\{\alpha_1\}$ is illustrated in Figure 1 for $\beta_1=\beta_2=0.1,\ 0.5,\ 1$ and 2.

The vertical coordinate in the figure depicts the value of the product $\sigma^2 \cdot \mathrm{CRB}\{\alpha_1\}$. The horizontal coordinate in the figure depicts the value of $\delta\omega/\Omega$ where Ω denotes the Fourier resolution limit, $\Omega = 2\pi/N$. We see that the bound generally increases as the frequency difference $\delta\omega$ decreases, and that the increase in the bound becomes significant for $\delta\omega$ below one Fourier limit (which is the range of separations for which the two sinusoids are considered to be close). The bound also increases as the damping factors increase. Similar conclusions can also be drawn for the bounds for the other parameters of the sinusoids.

3. THE REAL MODEL

The real counterpart of the previous data model consists of M real damped sinusoids in real white Gaussian noise of mean zero and variance σ^2:

$$y(t) = \sum_{i=1}^{M} \alpha_i e^{-\beta_i t} \cos(\omega_i t + \varphi_i) + e(t),\ t = 1,2,\ldots,N. \tag{8}$$

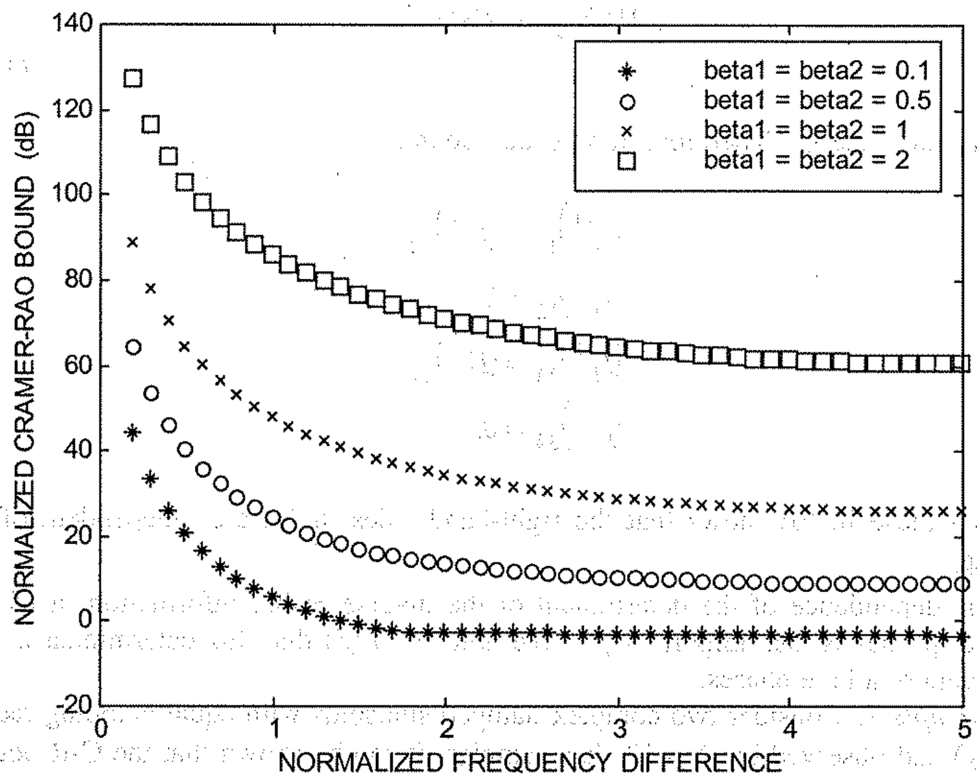

Figure 1: The C-R amplitude bound for two complex damped sinusoids observed by ten uniformly spaced samples. The damping factors $\beta_1 = \beta_2 = 0.1$, 0.5, 1 and 2.

The Fisher information matrix for estimation of the signal parameters in (8) reduces to

$$J(\theta)=\frac{1}{\sigma^2}\sum_{t=1}^{N} e_\theta(t) e_\theta(t)^T$$

where

$$e_\theta(t)=\left[e_\theta^{(1)}(t)^T, e_\theta^{(2)}(t)^T, \ldots, e_\theta^{(M)}(t)^T\right]^T$$

$$e_\theta^{(i)}(t)=\begin{bmatrix} \partial e(t)/\partial\alpha_i \\ \partial e(t)/\partial\varphi_i \\ \partial e(t)/\partial\beta_i \\ \partial e(t)/\partial\omega_i \end{bmatrix}=\begin{bmatrix} -e^{-\beta_i t}\cos(\omega_i t+\varphi_i) \\ \alpha_i e^{-\beta_i t}\sin(\omega_i t+\varphi_i) \\ \alpha_i t e^{-\beta_i t}\cos(\omega_i t+\varphi_i) \\ \alpha_i t e^{-\beta_i t}\sin(\omega_i t+\varphi_i) \end{bmatrix}.$$

Since $e_\theta^{(i)}(t)=D_i\cdot Q_i\cdot\bar{\varepsilon}_\theta^{(i)}(t)$ (c.f., (3)), the Fisher information matrix for the real model becomes

$$J(\theta)=\frac{1}{2\sigma^2}\cdot D\cdot Q\cdot I\cdot Q^T\cdot D \tag{9}$$

where the matrix I is now given by

$$I=\sum_{t=1}^{N} 2\bar{\varepsilon}_\theta(t)\bar{\varepsilon}_\theta(t)^T. \tag{10}$$

Again, inverting (9) to find the C-R bounds on the variance of unbiased estimators of the signal parameters, one similarly ends up with the same expressions in (6) with I replaced by (10) and the right-hand sides multiplied by 4. It can be shown that in general the expressions in (7) are no longer valid in the real data case.

Thus, for the real model, the C-R bounds for the parameters of the ith sinusoid are functions of the phase of that sinusoid, φ_i, but are independent of the phases of the other sinusoids, φ_j, $j\neq i$. Note that the determinant of the inverse Fisher information matrix is still independent of all the phases.

Some further comments on the results in (6) can be given. It follows from the expressions in (6) that the bounds for the parameters $(\alpha_i, \varphi_i, \beta_i, \omega_i)$ are periodic in φ_i with a period of π, and it thus suffices to consider the bounds in the interval, e.g., $A=\{\varphi_i:\varphi_i\in(-\pi/2,\pi/2]\}$. From the expressions for the C-R damping factor bound $\mathrm{CRB}\{\beta_i\}$ and the C-R frequency bound $\mathrm{CRB}\{\omega_i\}$, we see that the two bounds, considered as functions of the phase φ_i, are shifted versions of each other by an amount of $\pi/2$. In particular, their largest and smallest values are the same, i.e.

$$\max_{\varphi_i} \mathrm{CRB}\{\beta_i\} = \max_{\varphi_i} \mathrm{CRB}\{\omega_i\}$$
$$\min_{\varphi_i} \mathrm{CRB}\{\beta_i\} = \min_{\varphi_i} \mathrm{CRB}\{\omega_i\}.$$

Also, when one bound takes its largest value, the other bound takes its smallest value, i.e.

$$\arg\max_{\varphi_i \in A} \mathrm{CRB}\{\beta_i\} = \arg\min_{\varphi_i \in A} \mathrm{CRB}\{\omega_i\}$$
$$\arg\min_{\varphi_i \in A} \mathrm{CRB}\{\beta_i\} = \arg\max_{\varphi_i \in A} \mathrm{CRB}\{\omega_i\}.$$

Similar relations exist between the C-R amplitude bound $\mathrm{CRB}\{\alpha_i\}$ and the C-R phase bound $\mathrm{CRB}\{\varphi_i\}$.

Example 2: Consider a single real damped sinusoid observed by $N = 10$ data samples. The largest and the smallest values of the C-R amplitude bound $\mathrm{CRB}\{\alpha_1\}$ are illustrated in Figure 2 for the damping factor β_1 equals 0.1, 0.5, 1 and 2. The vertical coordinate in the figure depicts the value of $\sigma^2 \cdot \mathrm{CRB}\{\alpha_1\}$. The horizontal coordinate in the figure depicts the value of $\delta\omega/\Omega$ where $\delta\omega$ is the difference between the two signal frequencies present in the real sinusoid, $\delta\omega = 2 \cdot \omega_1$, and Ω denotes the Fourier limit. Note that the behavior of the bounds with respect to the changes in the $\delta\omega$ is the same as in the complex case. Also, note that the difference between the largest and the smallest values of the bound generally increases as $\delta\omega$ decreases, and that this increase is significant for $\delta\omega$ smaller than one Fourier limit. The difference between the two limits of the bound also increases as the damping factor of the sinusoid increases. Similar conclusions can also be drawn for the bounds for the other parameters of the sinusoid.

Note that the two data models considered in the examples differ only in their noise components; the imaginary part of the noise is absent in the real model. A comparison of the two figures indicates that parameter estimation accuracy may be poorer in the real case.

A recent paper [5] derived a fast algorithm for the computation of the Fisher information matrix for time-series data models for the case of colored Gaussian noise. It can be shown that our results on the phase dependence of the C-R bounds are still valid when the noise components of the models considered herein have a general covariance matrix.

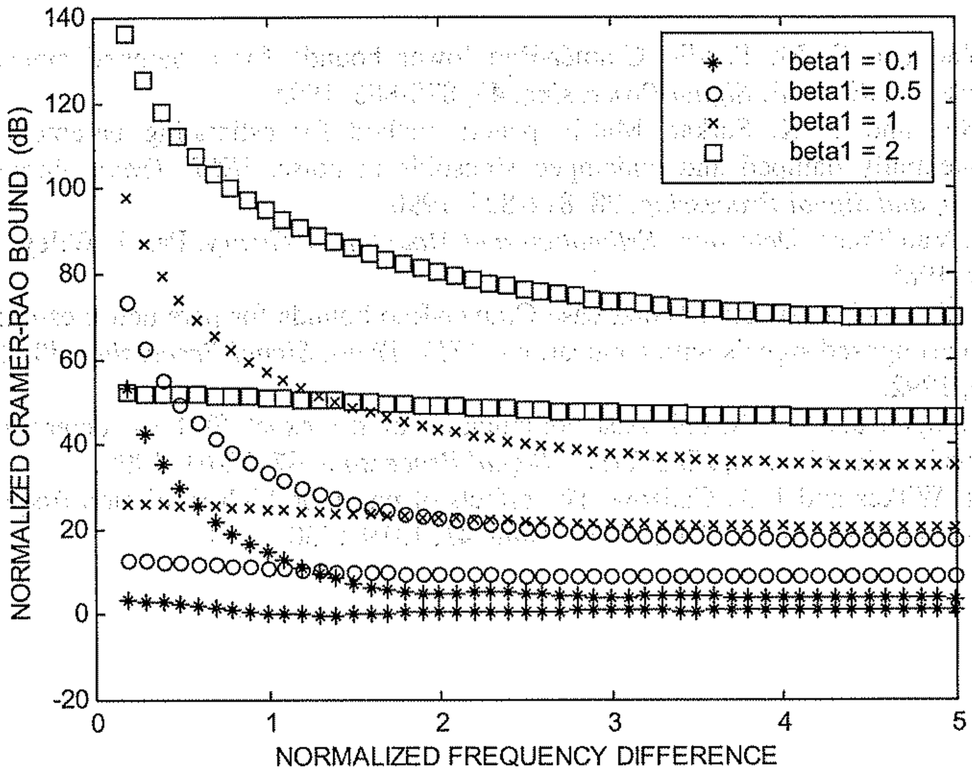

Figure 2: The largest and the smallest values of the C-R amplitude bound for a single real damped sinusoid observed by ten uniformly spaced samples. The damping factor β_1 equals 0.1, 0.5, 1 and 2.

4. CONCLUSIONS

We have studied the dependence of the C-R parameter bounds on the signal phases for complex and real valued time-series data models consisting of damped sinusoids buried in noise. For the complex model, we have shown that the bounds do not depend on the phases of the sinusoids. For the real model, we have shown that the C-R bounds for the parameters of a particular sinusoid depend on the phase of that sinusoid but do not depend on the phases of the other sinusoids in the model. This dependence has been examined for the case of a single real sinusoid via an example. It has been observed that as the frequency separation between the two signal components of the real sinusoid decreases or as the damping factor of the sinusoid increases the dependence of the bounds on the sinusoid phase becomes stronger.

We have also shown that for the damped sinusoidal cases the C-R bounds can be expressed explicitly as simple functions of the signal phases. It is known that for the cases in which the sinusoids are undamped (or the damping factors are known), the dependence of the C-R bounds on the phases is highly complicated [6].

REFERENCES

1. Y. Yao and S. M. Pandit, Cramér-Rao lower bounds for a damped sinusoidal process, *IEEE Trans. Signal Processing*, **43**, 878-885, 1995.
2. Y. Hua and T. K. Sarkar, Matrix pencil method for estimating parameters of exponentially damped and undamped sinusoids in noise, *IEEE Trans. Acoustics, Speech, and Signal Processing*, **38**, 814-824, 1990.
3. H. L. Van Trees, *Detection, Estimation and Modulation Theory*, Part I, Wiley, New York, 1968.
4. S. F. Yau and Y. Bresler, Worst case Cramér-Rao bounds for parametric estimation of superimposed signals with applications, *IEEE Trans. Signal Processing*, **40**, 2973-2986, 1992.
5. M. Ghogho and A. Swami, Fast computation of the exact FIM for deterministic signals in colored noise, *IEEE Trans. Signal Processing*, **47**, 52-61, 1999.
6. D. M. Wilkes and J. A. Cadzow, The effects of phase on high-resolution frequency estimators, *IEEE Trans. Signal Processing*, **41**, 1319-1330, 1993.