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FREDHOLM-VOLTERRA INTEGRAL EQUATION IN THE CONTACT PROBLEM

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Abstract-In this paper, under certain condition the Fredholm-Volterra integral equation of the first kind is solved. The existence and uniqueness of the solution is considered. The Fredholm integral equation of the second kind is established from the work, and its solution is also obtained.

Keywords-Fredholm-Volterra integral equation, Logarithmic kernel, Legendre polynomial.

1. INTRODUCTION

Singular integral equations arises in many problems of mathematical physics, in the theory of elasticity, viscoelasticity, hydrodynamics, biological problems, population, genetics, and others. On the other hand, many authors are interested in studying the different methods for solving the singular integral equation, (see [1,19,25]).

Over the past thirty years substantial progress has been made in developing innovative approximate analytical and purely numerical solution to a large class of singular integral equation. The solution of these problems can be obtained analytically, using the theory developed by Muskhelishvili [15,16]. The books edited by Green [2], Hochstad [7], Popov [5] and Tricomi [4] contained many different methods to solve the integral equation analytically. The books edited by Golberg [13,14] contain extensive literature surveys on approximate, analytical and purely numerical techniques. The interested reader should consult the fine expositions by Atkinson [10], Delves and Mohammed [12] and Linz [27] for numerical methods solution. Kovalenko [3] obtained a Fredholm integral equation of the first kind with singular kernel, when the mixed problems of continuous media with boundary conditions specified on a circle is studied.

In [16] Abdou solved the Fredholm integral equation of the second kind, which is investigated from the semi-symmetric Hertz problem for two different elastic materials in three dimensions. The solution of the three dimensional contact problem of two different elastic material obtained, and the structure resolvent is, also, established in the domain of contact $\Omega = (x, y) : \sqrt{x^2 + y^2} \leq a \ , z = 0$ in the work of Abdou [15]. Kauthen in his work [9] used a collection method to solve the Fredholm-Volterra integral equation numerically. In [11] a projection method based on the Fourier theory is used to obtain an approximate solution for linear Volterra-Fredholm integral equation of the second kind. Burner, in his work [6], used collection method to obtain, numerically, the solution of nonlinear Volterra-Fredholm integral equation.

In this work, the Fredholm-Volterra integral equation is considered under certain conditions, and the solution in series form of a Legendre polynomial is obtained. The existence and uniqueness of the solution is taken in mind. The convergence of the series is discussed.

2. FORMULATION OF THE PROBLEM

Consider an elastic material of strip (G, υ) of thickness h. Occupying the region $0 \le y \le h$, $|x| \prec \infty$, lies without friction on rigid elastic layer surface h(x) describing it. Consider a rigid rectangular stamp of width 2a is impressed into the boundary of the strip y=h, through a function of time F(t), $0 \le t \le T \prec \infty$, by a constant force $p \prec \infty$, whose eccentricity of application e. Here υ and G are the Poisson's coefficient and the displacement module of each material, respectively.
Assume the frictional forces in the contact area between the stamp and the strips are so small that it can be neglected.
It is known [15] that such a problem leads us to the following boundary value problem:

$$\lambda \int_{-1}^{1} K\left(\frac{\xi - x}{\lambda_1}\right)\phi(\xi,t)\,d\xi + \lambda\int_{0}^{t} F(\tau)\phi(x,\tau)d\tau = \pi[\delta(t) + \alpha(t)x - h(x)] = \pi f(x,t) \tag{2.1}$$

$|x| \le 1,\ t \in [o,T]$, λ is a constant $\lambda_1 \in [(0,\infty)]$.

$$K(y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh S(u)}{S(u)} e^{iuy}\,du \quad , \quad \left(y = \frac{\xi - x}{\lambda_1}\right) \tag{2.2}$$

under the conditions:

$$p(t) = \int_{-1}^{1} \phi(\xi,t)d\xi \quad , \qquad M(t) = \int_{-1}^{1} \xi\phi(\xi,t)d\xi$$

Here δ(t) is the rigid displacement under the action of a variable force P(t) and α(t) are the twisting of M(t), $0 \le t \prec T \prec \infty$.

3. THE SINGULAR KERNEL OF THE PROBLEM

Formula (2.2) can be written in the form (see [5])

$$K(y) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh S(u)}{S(u)} e^{iuy}\,du = -\ln\left|\tanh\frac{\pi y}{4}\right| , \quad y = \frac{\xi - x}{\lambda_1} \tag{3.1}$$

when $\lambda_1 \longrightarrow \infty$, y is sufficiently small, that it satisfies $\tanh y \approx y$, we have

$$\ln \left| \tanh \frac{\pi(\xi - x)}{4\lambda_1} \right| = \ln|\xi - x| - d \,, \quad \left(d = \ln \frac{4\lambda_1}{\pi} \right) \tag{3.2}$$

Substituting from (3.1) and (3.2) in equation (2.1), then we have:

$$\lambda \int_{-1}^{1} \left[-\ln|\xi - x| + d \right] \phi(\xi, t) d\xi + \lambda \int_{0}^{t} F(\tau)\, \phi(x, \tau) d\tau = \pi f(x, t) \tag{3.3}$$

4. METHOD OF SOLUTION

If we divide the interval [0,T], $o \le t \le T \prec \infty$, in the following form $o \le t_o \prec t_1 \prec t_2 .... \prec t_n = T$, i.e $t = t_k$, $k = 1, 2, ...N.$,
the second integral of (3.3) with the quadratic formula u_j (j=1,2,....k) gives

$$\int_{o}^{t_k} F(\tau)\phi(x,\tau)d\tau = \sum_{j=0}^{k} u_j F(t_j)\phi(x,t_j) + 0(\hbar^p) \quad (p \ge k) \tag{4.1}$$

where

$$\hbar_k = \max_{0 \le t \le k} h_j, \quad h_j = t_{j+1} - t_j$$

The characteristic points of (4.1) with the quadratic coefficients u_j are explained in [12]. Hence (3.3) becomes

$$\lambda \int_{-1}^{1} \left[-\ln|\xi - x| + d \right] \phi(\xi, t_k) d\xi + \lambda \sum_{j=0}^{k} u_j F(t_j) \phi(x, t_j) = \pi f(x, t_k) \tag{4.2}$$

$$\mu_k \phi_k(x) + \lambda \int_{-1}^{1} \left[-\ln|\xi - x| \phi_k + d \right] \phi_k(\xi) d\xi + \lambda \sum_{j=0}^{k-1} u_j F_j \phi_j(x) = \pi f_k(x) \tag{4.3}$$

The formula (4.2) can be adapted in the form
where

$$\mu_k = \lambda u_k F_k$$

The general solution of (4.3) can be obtained by using the recurrence relation, for example at k = 0, we have the integral equation

$$\lambda \int_{-1}^{1} \left[-\ln|\xi - x| + d \right] \phi_0(\xi) d\xi = \pi f_0(x) \tag{4.4}$$

Many different methods are used to solve equation (4.4) Abdou and Hassan [18] used potential theory method to obtain the function $\phi_0(x)$ which is the solution of (4.4) when the given function takes a Chebychev polynomial form. Also in [17] Abdou and Ezz-Eldin, solved the integral equation (4.4) when the free function takes a Chebychev polynomial form by using Krein's method. In [16] Abdou obtained and derived many spectral relations in the form of eigenvalues and eigenfunction of the formula (4.4) the leader work for the different methods to solve (4.4) can be found in [26] in the form:

$$\phi_0(x) = \frac{1}{\pi\lambda_1\sqrt{1-x^2}}\left[P + \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\tau^2}{\tau - x}}f_0'(\tau)d\tau\right] \tag{4.5}$$

where

$$f_0'(y) = \frac{df_0(y)}{dy}, \quad P = \int_{-1}^{1}\phi_0(x)dx$$

Let k=1 , in (4.3), we have

$$\mu_1\,\phi_1\,(x) + \lambda\int_{-1}^{1}\left[-\ln|\xi - x| + d\right]\phi\,(\xi)\,d\xi + \lambda\,u_0\,F_0\phi_0(x) = \pi f_1\,(x) \tag{4.6}$$

to solve the integral equation (4.5), we differentiate it with respect to x, to get

$$\mu_1\frac{d\phi_1}{dx} - \lambda\int_{-1}^{1}\frac{\phi_1(\xi)d\xi}{\xi - x} = \pi g_1(x)$$

where

$$g_1(x) = f_1(x) - \frac{\lambda}{\pi}u_0\,F_0\phi_0(x) \tag{4.7}$$

Assume the solution of (4.7) in the Legendre polynomials form

$$\phi_1(x) = \sum_{n=0}^{\infty}C_n^{(1)}P_n(x) \tag{4.8}$$

Where $C_n^{(1)}$ are constants and $P_n(x)$ are the Legendre polynomials. If $\phi_1(x) \in L_2(-1,1)$, the polynomial series of (4.8) is also convergent.

The Legendre polynomials satisfies the orthogonal relation form (see [8])

$$\int_{-1}^{1}P_n\,(x)\,P_m\,(x)dx = \begin{cases} 0 & m \neq n \\ \dfrac{2}{2n+1} & m = n \end{cases} \tag{4.9}$$

Differentiating (4.8) with respect to x, we have

$$\phi_1'(x) = \sum_{n=0}^{\infty}C_n^{(1)}P_n^{(1)}(x)\,.\,(1-x^2)^{-\frac{1}{2}} \tag{4.10}$$

Also, the free term $g_1(x)$ can be represented in the form

$$g_1(x)=\sum_{n=0}^{\infty} g_n^{(1)} P_n^{(1)}(x).(1-x^2)^{-1/2} \tag{4.11}$$

Where g_n^1 are constant terms, which can be determined by using the orthogonal relation. If $g(x)\in L_2(-1,1)$, the polynomial series (4.11) is also convergent to g(x) in $L_2(-1,1)$. While $P_n^{(m)}(x)$ is the associated Legendre polynomial of the first kind, which satisfies the orthogonal relation.

$$\int_{-1}^{1} P_n^k(x)P_m^k(x)dx=\begin{cases} 0 & ;n\neq m \\ \dfrac{2}{2n+1}\bullet\dfrac{(n+k)!}{(n-k)!} & ;n=m \end{cases} \tag{4.12}$$

(see [8] pp. 808)

Hence, the integral term of (4.7) can be written in the form

$$\int_{-1}^{1}\frac{\phi_1(\xi)}{x-\xi}d\xi=2\sum_{n=0}^{\infty} C_n^1 Q_n(x) \tag{4.13}$$

(See [8] pp. 835)

Where

$$Q_n(x)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(y)}{x-y}dy \tag{4.14}$$

In view of equation (4.11), the values of $g_n^{(1)}$ of (4.11) takes the form

$$g_n^{(1)}=\frac{-(2n+1)}{2n(n+1)}\int_{-1}^{1} g_1(x)\sqrt{1-x^2}\,P_n^1(x)dx \tag{4.15}$$

Hence, substituting from equations. (4.10), (4.11) and (4.13) in the integral equation (4.7), we have

$$2\lambda\sqrt{1-x^2}\sum_{n=0}^{\infty} C_n^{(1)} Q_n(x)=\sum_{n=0}^{\infty}\left(\pi g_n^{(1)}+\mu_1 C_n^{(1)}\right)P_n^1(x) \tag{4.16}$$

Multiplying both sides of equation (4.16) by $P_m^1(x)dx$ then integrating form –1 to 1 and with the aid of the following integral relation (see [8] pp. 807)

$$\int_{-1}^{1}\sqrt{1-x^2}\,Q_n(x)\,P_m^1(x)dx=\begin{cases}0 & (n=m\neq 1)\\ \dfrac{-2(m+1)[1+(-1)^{n+m}]}{(m-n-1)(m-n+1)(m+n)(m+n+2)} & (n\neq m\neq 1)\end{cases} \quad (4.17)$$

The formula (4.16) takes the form

$$\mu_1 C_m^{(1)}+2\sum_{n=0}^{\infty}\frac{(2m+1)\left[1+(-1)^{n+m}\right]C_n^{(1)}}{(m-n-1)(m-n+1)(m+n)(m+n+2)}=\pi g_m^{(1)} \quad (4.18)$$

The infinite system (4.18) can be represented in the two infinite system of even and odd case,

where

$$\mu_1 X_{2m}-\sum_{n=1}^{\infty} b_{2m,2n}X_{2n}=\pi G_{2m}-b_{2m,0}P \quad (4.19)$$

(for even function)

$$\mu_1 X_{2m-1}-\sum_{n=1}^{\infty} b_{2m+1,2n-1}X_{2n-1}=\pi G_{2m-1}$$

(for odd function)

where

$$b_{m,n}=\frac{2(2n+1)\left[1+(-1)^{n+m}\right]}{(m-n-1)(m-n+1)(m+n)(m+n+2)}$$

$$b_{m,m-1}=b_{m,m+1}=o$$

For the convergence of the infinite series (4.19) we must have the inequality (see [8]).

$$\sum_{n=1}^{\infty}\left|b_{m,n}\right|\prec S \qquad (m\geq 1)$$

Where

$$S=\frac{3}{4}-4\sum_{n=3}^{\infty}\frac{2n+1}{n(n-2)(n+1)(n+3)} \quad (4.20)$$

when $n\rightarrow\infty$ in (4.20), we have $S_\infty=\frac{3}{4}$ (4.21)

Similarly, at k=2, we have an integral equation:

$$\mu_2 \phi_2 + \lambda \int_{-1}^{1}\left[-\operatorname{Ln}|\xi - x| + d\right]\phi_1(\xi)\, d\xi = \pi f_1(x) - u_0 F_0 \phi_0(x) - u_1 F_1 \phi_1(x) \tag{4.22}$$

Differentiating equation (4.22), we have

Let $\phi_2(x) = \sum_{n=0}^{\infty} C_n^{(2)} P_n(x)$ (4.23)

By following the same previous method, we get

$$2\sqrt{1-x^2}\sum_{n=0}^{\infty} C_n^{(2)} Q_n(x) = \sum_{n=0}^{\infty}\left(\pi g_n^{(2)} + \mu_1 C_n^{(2)} - u_0 F_0 C_n^{(1)}\right) P_n^{(1)}(x) \tag{4.24}$$

Multiplying both sides of equation (4.24) by $P_n^{(1)}$ dx and integrating through the interval [-1,1], then using the relation (4.16), we obtain

$$\mu_2 C_m^{(2)} - u_1 F_1 C_m^{(1)} + 2\sum_{n=0}^{\infty} \frac{(2m+1)\left[1+(-1)^{n+m}\right] C_n^{(2)}}{(m-n-1)(m-n+1)(m+n)(m+n+2)} = \pi g_m^{(2)} \tag{4.25}$$

where $C_m^{(0)}$ is given by (4.17)

In general, we get for $(m \geq 1)$, the following:

$$\mu_k C_m^{(k)} - \sum_{j=1}^{k-1} U_j F_j C_m^{(j)} + 2\sum_{n=0}^{\infty} \frac{(2m+1)\left[1+(-1)^{n+m}\right] C_n^{(k)}}{(m-n-1)(m-n+1)(m+n)(m+n+2)} = \pi g_m^{(k)}, \quad (k \neq 0) \tag{4.26}$$

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