

GENERAL NONLINEAR MIXED VARIATIONAL-LIKE INEQUALITY PROBLEM INVOLVING RELAXED LIPSCHITZ OPERATORS

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Abstract-In this article, we use the auxiliary principle technique to prove the existence, uniqueness theorem of solutions for the general nonlinear mixed variational-like inequality and special cases, which can be obtained from our main result are also discussed.

Keywords-general nonlinear mixed variational-like inequalities, relaxed Lipschitz operator, strongly η -monotone operator, Lipschitz operator, Hausdorff metric.

1. INTRODUCTION

The theory of monotone (nonlinear) operators in general and the variational inequality in particular have generalized a tremendous interest amongst mathematicians. This is because of the wide applicability of the variational inequality in nonlinear elliptic boundary value problems, obstacle problems, complementarity problems, mathematical programming, mathematical economics and many others area, papers which concern differential equation and variational inequality are too many to cite. We will only cite [1], [6], [7], [12] and [13]. In recent years variational inequality problem has been extended and generalized in many different directions.

Variational-like inequalities introduced by Parida and Sen [11] is one of the generalized form of variational inequality problem. For further detail one may see [2, 3, 4, 10, 14, 16, 17, 18] and the references therein. There is already literature a substantial number of numerical methods for computing the numerical solution of the variational inequality and complementarity problems. We note that the projection technique and its variant forms can no longer be applied to suggest the iterative type algorithms for variational-like inequalities.

This fact motivates us to use the auxiliary principle technique of Glowinski, Lions and Tremoliers [5] and earlier developed by Noor [8] to prove the existence of this unique solution of the GNMVLIP.

Let H be a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let $K \subset H$ be a nonempty closed convex subset, and the form $b(\cdot, \cdot): H \times H \rightarrow H$ be a non-differentiable and satisfy the following properties:

- (i) $b(u, v)$ is linear with respect to u .
- (ii) $b(u, v)$ is bounded, that is, there exists a constant $\gamma > 0$, such that

$$|b(u, v)| \leq \gamma \|u\| \|v\|, \text{ for all } u, v \in H. \quad (1.1)$$
- (iii) $b(u, v) - b(u, w) \leq b(u, v - w)$, for all $u, v, w \in H$. (1.2)

We need the following concepts.

Definition 1.1 [15]: For all $u_1, u_2 \in H$, the operator $N(\cdot, \cdot): H \times H \rightarrow H$ is said to be relaxed Lipschitz continuous, Lipschitz continuous with respect to first argument, if there exist constants $\kappa \geq 0$, $r > 0$, such that

$$\langle N(x_1, \cdot) - N(x_2, \cdot), u_1 - u_2 \rangle \leq -\kappa \|u_1 - u_2\|^2, \text{ for all } x_i \in A(u_i), i=1,2.$$

and

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq r \|u_1 - u_2\|.$$

In a similar way, we define the Lipschitz continuity with respect to second argument of the operator $N(\cdot, \cdot)$.

An operator $A: H \rightarrow C(H)$ is M -Lipschitz continuous, if there exists a constant $\nu > 0$ such that for all x_1 in $A(u_1)$ and x_2 in $A(u_2)$, and

$$\|x_1 - x_2\| \leq M(A(u_1), A(u_2)) \leq \nu \|u_1 - u_2\|,$$

where $M(\cdot, \cdot)$ is Hausdorff metric on H .

Definition 1.2 [9]: The operator $\eta(\cdot, \cdot): H \times H \rightarrow H$ is said to be

- (i) strongly monotone, if there exists a constant $\sigma > 0$, such that

$$\langle \eta(v, u), v - u \rangle \geq \sigma \|v - u\|^2, \text{ for all } u, v \in H;$$
- (ii) Lipschitz continuous, if there exists a constant $\omega > 0$ such that

$$\|\eta(v, u)\| \leq \omega \|v - u\|, \text{ for all } u, v \in H.$$

Definition 1.3 [9]: Let the operator $\eta(\cdot, \cdot): H \times H \rightarrow H$ and operator $g: H \rightarrow H$, then

- (i) g is said to be strongly η -monotone with constant $\alpha > 0$ such that

$$\langle g(u_1) - g(u_2), \eta(u_1, u_2) \rangle \geq \alpha \|u_1 - u_2\|^2, \text{ for all } u_1, u_2 \in H;$$
- (ii) g is said to be Lipschitz continuous with constant $\beta > 0$ such that $u_1, u_2 \in H$

$$\|g(u_1) - g(u_2)\| \leq \beta \|u_1 - u_2\|.$$

Assumption 1.1: The operator $\eta(\cdot, \cdot): H \times H \rightarrow H$ satisfies the condition

$$\eta(u, v) + \eta(v, u) = 0, \text{ for all } u, v \in H.$$

Clearly $\eta(u, u) = 0$, for all $u \in H$, an assumption used in [2, 11, 17] to study the existence of a solution of variational-like inequalities problems.

Let $g: H \rightarrow H$ be a single-valued mapping and $A, T: H \rightarrow C(H)$ be the multivalued mappings, where $C(H)$ be the family of nonempty compact subset of H . Given operators $N, \eta: H \times H \rightarrow H$, we consider the problem of finding $u \in H, x \in A(u), y \in T(u)$ such that

$$\langle g(u) - N(x, y), \eta(v, u) \rangle + b(u, v) - b(u, u) \geq 0, \text{ for all } v \in H, \quad (1.3)$$

which is called the general nonlinear mixed variational-like inequality and has many applications in engineering, optimization and structural analysis, see [1, 2, 6, 7, 13, 16] and the references therein.

Special cases:

If $N(x, y) = x - y$, and $\eta(v, u) = v - g(u)$ where $g: K \rightarrow K$ and $b(u, u) = b(v, u) \equiv 0$, then problem (1.3) is equivalent to finding $u \in H, x \in A(u), y \in T(u)$ such that $g(u) \in K$

$$\langle g(u) - (x - y), v - g(u) \rangle \geq 0, \text{ for } v \in H, \quad (1.4)$$

which is called generalized variational inequality problem considered by Verma [15].

2. MAIN RESULT

In this section, we prove the existence of a solution of the general nonlinear mixed variational-like inequality problem (1.3) by using the auxiliary principle technique of Glowinski, Lions and Tremoliers [4].

Theorem 2.1: Let the operator $N(\cdot, \cdot): H \times H \rightarrow H$ be relaxed Lipschitz continuous with $\kappa \geq 0$ and Lipschitz continuous with constant $r > 0$ with respect to the first argument. Let the operator $N(\cdot, \cdot)$ be a Lipschitz continuous with constant $s > 0$ with respect to the second argument, and $A, T: H \rightarrow C(H)$ be M -Lipschitz continuous with constant $\nu > 0$ and $\delta > 0$, respectively. Let $\eta: H \times H \rightarrow H$ be strongly monotone with constant $\sigma > 0$ and

Lipschitz continuous with constant $\omega > 0$; let $g: H \rightarrow H$ be a Lipschitz continuous with constant $\beta > 0$ and strongly η -monotone with respect to η with constant $\alpha > 0$. Let the form $b(.,.)$ satisfy the condition (i) – (iii), if Assumption 1.1, holds that

$$0 < \rho < 2(q - \kappa)(q^2 - r^2 v^2)^{-1},$$

$$\rho q < 1 \text{ and } \kappa < r v \leq q, \quad (2.1)$$

where $q = (\beta + r v)(1 - 2\sigma + \omega^2)^{1/2} + s\delta\omega + \gamma + \beta$,

then there exists a solution set $u \in H, x \in A(u), y \in T(u)$ satisfying the problem (1.3).

Proof:

Existence: We used the auxiliary principle technique of Glowinski, Lions and Tremoliers [5] to prove the existence of solution of general nonlinear mixed variational-like inequality problem (1.3). For a given $u \in H$, we consider the problem of finding $z \in H, x \in A(u), y \in T(u)$ satisfying the auxiliary variational-like inequality

$$\langle z, v - z \rangle \geq \langle u, v - z \rangle - \rho \langle g(u) - N(x, y), \eta(v, z) \rangle + \rho b(u, z) - \rho b(u, v), \quad (2.2)$$

for all $v \in H$, where $\rho > 0$ is a constant.

The relation (2.2) define a mapping $u \rightarrow z$. In order to prove the existence of solution of (2.2), it is sufficient to show that the mapping $u \rightarrow z$ define by (2.2) has a fixed-point belonging to H satisfying (1.3). In other words, it is enough to show that for a well chosen $\rho > 0$,

$$\|z_1 - z_2\| \leq \theta \|u_1 - u_2\| \text{ with } 0 < \theta < 1,$$

where θ is independent of u_1 and u_2 .

Let $z_1, z_2 \in H, z_1 \neq z_2$ be two solutions of (2.2) related to $u_1, u_2 \in H$, respectively. Taking $v = z_2$ (respectively z_1) related to u_1 (respectively u_2), we have

$$\langle z_1, z_2 - z_1 \rangle \geq \langle u_1, z_2 - z_1 \rangle - \rho \langle g(u_1) - N(x_1, y_1), \eta(z_2, z_1) \rangle + \rho b(u_1, z_1) - \rho b(u_1, z_2).$$

and

$$\langle z_2, z_1 - z_2 \rangle \geq \langle u_2, z_1 - z_2 \rangle - \rho \langle g(u_2) - N(x_2, y_2), \eta(z_1, z_2) \rangle + \rho b(u_2, z_2) - \rho b(u_2, z_1).$$

Using Assumption 1.1, adding the above inequalities, using (ii)-(iii), and rearranging the terms, we obtain

$$\begin{aligned} \langle z_1 - z_2, z_1 - z_2 \rangle &\leq \langle u_1 - u_2, z_1 - z_2 \rangle - \rho \langle g(u_1) - g(u_2) - (N(x_1, y_1) - N(x_2, y_2)), \eta(z_1, z_2) \rangle \\ &\quad + \rho b(u_1 - u_2, z_2 - z_1) \\ &\leq \langle u_1 - u_2, z_1 - z_2 \rangle - \rho \langle g(u_1) - g(u_2), \eta(z_1, z_2) \rangle + \rho \langle N(x_1, y_1) - N(x_2, y_2), \eta(z_1, z_2) \rangle \\ &\quad + \rho b(u_1 - u_2, z_2 - z_1) \\ &\leq \langle u_1 - u_2 + \rho(N(x_1, y_1) - N(x_2, y_1)), z_1 - z_2 \rangle - \rho \langle N(x_1, y_1) - N(x_2, y_1), z_1 - z_2 \rangle \\ &\quad - \eta(z_1, z_2) \rangle + \rho \langle N(x_2, y_1) - N(x_2, y_2), \eta(z_1, z_2) \rangle + \rho b(u_1 - u_2, z_2 - z_1) \\ &\quad - \rho \langle g(u_1) - g(u_2), z_1 - z_2 \rangle + \rho \langle g(u_1) - g(u_2), z_1 - z_2 - \eta(z_1, z_2) \rangle, \end{aligned}$$

from which it follows that

$$\begin{aligned} \|z_1 - z_2\|^2 &\leq \|u_1 - u_2 + \rho(N(x_1, y_1) - N(x_2, y_1))\| \|z_1 - z_2\| \\ &\quad + \rho \|N(x_1, y_1) - N(x_2, y_1)\| \|z_1 - z_2 - \eta(z_1, z_2)\| \\ &\quad + \rho \|N(x_2, y_1) - N(x_2, y_2)\| \|\eta(z_1, z_2)\| + \rho \gamma \|u_1 - u_2\| \|z_1 - z_2\| \\ &\quad + \rho \|g(u_1) - g(u_2)\| \|z_1 - z_2\| + \rho \|g(u_1) - g(u_2)\| \|z_1 - z_2 - \eta(z_1, z_2)\|. \end{aligned} \quad (2.3)$$

Since the operator $N(.,.)$ is relaxed Lipschitz continuous and Lipschitz continuous with respect to the first argument and $A: H \rightarrow C(H)$ is M -Lipschitz continuous, so

$$\begin{aligned} \|u_1 - u_2 + \rho (N(x_1, y_1) - N(x_2, y_1))\|^2 &\leq \|u_1 - u_2\|^2 \\ &+ 2\rho \langle N(x_1, y_1) - N(x_2, y_1), u_1 - u_2 \rangle + \rho^2 \|N(x_1, y_1) - N(x_2, y_1)\|^2 \\ &\leq [1 - 2\kappa\rho + \rho^2 v^2 r^2] \|u_1 - u_2\|^2. \end{aligned} \quad (2.4)$$

Using the Lipschitz continuity of $N(\cdot, \cdot)$ with respect to the second argument and M -Lipschitz continuity of T , we have

$$\begin{aligned} \|N(x_2, y_1) - N(x_2, y_2)\| &\leq s \|y_1 - y_2\| \\ &\leq s M(T(u_1), T(u_2)) \\ &\leq s \delta \|u_1 - u_2\|. \end{aligned} \quad (2.5)$$

Again, η is strongly η -monotone and Lipschitz continuous, so

$$\begin{aligned} \|z_1 - z_2 - \eta(z_1, z_2)\|^2 &\leq \|z_1 - z_2\|^2 - 2 \langle \eta(z_1, z_2), z_1 - z_2 \rangle \\ &+ \|\eta(z_1, z_2)\|^2 \\ &\leq [1 - 2\sigma + \omega^2] \|z_1 - z_2\|^2. \end{aligned} \quad (2.6)$$

And g is strongly monotone, we have

$$\|g(u_1) - g(u_2)\| \leq \beta \|u_1 - u_2\|. \quad (2.7)$$

From (2.3) - (2.7) and using the Lipschitz continuity of η , we have

$$\begin{aligned} \|z_1 - z_2\| &\leq [(1 - 2\kappa\rho + \rho^2 v^2 r^2)^{1/2} + \rho \{(\beta + r v)(1 - 2\sigma + \omega^2)^{1/2} + s\delta\omega + \gamma + \beta\}] \\ &\quad \|u_1 - u_2\| \\ &= [t(\rho) + q\rho] \|u_1 - u_2\|, \text{ using (2.1)} \\ &= \theta \|u_1 - u_2\|, \end{aligned} \quad (2.8)$$

where $\theta = t(\rho) + q\rho$, $t(\rho) = (1 - 2\kappa\rho + \rho^2 v^2 r^2)$, and

$$q = (\beta + r v)(1 - 2\sigma + \omega^2)^{1/2} + s\delta\omega + \gamma + \beta.$$

We have to show that $\theta < 1$. It is clear that $t(\rho)$ assumes its minimum value for

$$\rho^* = (\kappa r^2 v^2)^{-1} \text{ with } t(\rho^*) = (1 - (\kappa r v)^2)^{1/2}.$$

For $\rho = \rho^*$, $t(\rho) + q\rho < 1$ implies that $q\rho < 1$. Thus it follows that $\theta < 1$, for all ρ with

$$0 < \rho < 2(q - \kappa)(q^2 - r^2 v^2)^{-1}, \quad \rho q < 1 \text{ and } \kappa < r v \leq q.$$

Since $0 < \theta < 1$, so the map defined by (2.2) is a contraction map and consequently, it has a fixed-point belonging to H satisfying the problem (1.3).

Uniqueness: Let $u_1, u_2 \in H$, $u_1 \neq u_2$ be two solutions of the problem (1.3), that is

$$\langle g(u_1) - N(x_1, y_1), \eta(v, u_1) \rangle + b(u_1, v) - b(u_1, u_1) \geq 0, \text{ for all } v \in H \quad (2.9)$$

and

$$\langle g(u_2) - N(x_2, y_2), \eta(v, u_2) \rangle + b(u_2, v) - b(u_2, u_2) \geq 0, \text{ for all } v \in H \quad (2.10)$$

Taking $v = u_2$ in (2.9) and $v = u_1$ in (2.10), using Assumption 1.1, and adding the resultant inequalities, we obtain

$$\langle g(u_1) - g(u_2) - (N(x_1, y_1) - N(x_2, y_2)), \eta(u_1, u_2) \rangle \leq b(u_1 - u_2, u_2 - u_1)$$

which can be written as

$$\begin{aligned} \langle g(u_1) - g(u_2), \eta(u_1, u_2) \rangle &\leq \langle N(x_1, y_1) - N(x_2, y_2), \eta(u_1, u_2) \rangle + b(u_1 - u_2, u_2 - u_1) \\ &= \langle N(x_1, y_1) - N(x_2, y_1), u_1 - u_2 \rangle + \langle N(x_2, y_1) - N(x_2, y_2), \eta(u_1, u_2) \rangle \\ &\quad + \langle N(x_2, y_1) - N(x_1, y_1), u_1 - u_2 - \eta(u_1, u_2) \rangle + b(u_1 - u_2, u_2 - u_1) \end{aligned}$$

Now using the strongly η -monotonicity of g , the Cauchy-Schwartz inequality and the relation (1.1).

$$\alpha \|u_1 - u_2\|^2 \leq \|N(x_1, y_1) - N(x_2, y_1)\| \|u_1 - u_2\| + \|N(x_2, y_1) - N(x_1, y_1)\| \|u_1 - u_2 - \eta(u_1, u_2)\| + \gamma \|u_1 - u_2\|^2 + \|N(x_2, y_1) - N(x_2, y_2)\| \|\eta(u_1, u_2)\|. \quad (2.11)$$

Combining (2.5), (2.6), (2.11), and using the Lipschitz continuity of $N(.,.)$ with respect to first argument and Lipschitz continuity of A, η , we obtain

$$\alpha \|u_1 - u_2\|^2 \leq \{r v (1 + (1 - 2\sigma + \omega^2)^{1/2}) + s \delta \omega + \gamma\} \|u_1 - u_2\|^2, \\ = p \|u_1 - u_2\|^2,$$

which implies that

$$(\alpha - p) \|u_1 - u_2\|^2 \leq 0,$$

from which, it follows that $u_1 = u_2$, the uniqueness of the solution, since $p < \alpha$.

REFERENCES

- [1] F. E. Browder, Nonlinear monotone operators and convex sets in Banach spaces, *Bull. Amer. Math. Soc.* **71**, 780-785, 1965.
- [2] X. P. Ding, General algorithm for nonlinear variational-like inequalities in reflexive Banach spaces, *Indian J. Pure Appl. Math.* **29**(2), 109-120, 1998.
- [3] X. P. Ding and E. Tarafdar, Existence and uniqueness of solution for a general nonlinear variational inequalities, *Appl. Math. Lett.* **8**, 31-36, 1995.
- [4] X. P. Ding and E. Tarafdar, Generalized nonlinear variational inequalities with non-compact monotone set-valued mappings, *Appl. Math. Lett.* **7**, 5-11, 1994.
- [5] R. Glowinski, J. L. Lions and Tremoliers, *Numerical analysis of variational inequalities*, North-Holland, Amsterdam, 1981.
- [6] P. Hartman and G. Stampacchia, On some nonlinear elliptic differentiable functional equations, *Acta Mathematica* **115**, 271-310, 1966.
- [7] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.
- [8] M. A. Noor, General auxiliary principle for variational inequalities, *Pan Amer. Math. J.* **4**, 27-44, 1994.
- [9] M. A. Noor, Generalized variational like inequalities, *Math. Comput. Modelling* 1997.
- [10] M. A. Noor, Auxiliary principle for generalized mixed variational like-inequalities, *J. Math. Anal. Appl.* **215**, 75-85, 1999.
- [11] J. Parida and A. Sen, Variational-like inequalities for multifunctions with applications, *J. Math. Anal. Appl.* **124**, 73-81, 1987.
- [12] D. Pascali and S. Sburlan, Nonlinear mappings of monotone type, The Netherland 1978.
- [13] R. T. Rockafeller, *Convex analysis*, Princeton 1970.
- [14] A. H. Siddiqi, Q. H. Ansari and R. Ahmad, On generalized variational-like inequalities, *Indian J. Pure Appl. Math.* **26**, 1135-1141, 1995.
- [15] R. U. Verma, On general variational inequalities involving relaxed Lipschitz and relaxed monotone operators, *J. Math. Anal. Appl.* **23**, 383-392, 1997.
- [16] J. C. Yao, A basic theorem of complementarity for the generalized variational inequalities, *J. Math. Anal. Appl.* **158**, 124-138, 1991.
- [17] J. C. Yao, On the general variational inequalities, *J. Math. Anal. Appl.* **174**, 550-555, 1993.

- [18] S. J. Yu and J. C. Yao, On generalized variational inequalities, *J. Math. Anal. Appl.* **198**, 178-193, 1996.